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ISLANDS IN STABLE FLUID EQUILIBRIA

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ABSTRACT. We prove that stable fluid equilibria with trivial homology on curved, reflection-symmetric periodic channels must posses "islands", or cat's eye vortices. In this way, arbitrarily small disturbances of a flat boundary cause a change of streamline topology of stable steady states.

Given a smooth periodic function $h : \mathbb{T} \to \mathbb{R}$ such that $|h| < 1$, consider an annular domain D_h with reflection symmetry across the centerline,

$$D_h = \{(x, y) \colon x \in \mathbb{T},\ -1 - h(x) \le y \le 1 + h(x)\}. \tag{1}$$

We are interested in the structure of ideal fluid equilibria, i.e. steady solutions of the Euler equation, on D_h

$$u \cdot \nabla u = -\nabla p \qquad \text{in} \quad D_h, \tag{2}$$

$$\nabla \cdot u = 0 \qquad \text{in} \quad D_h, \tag{3}$$

$$u \cdot \hat{n} = 0 \qquad \text{on} \quad \partial D_h. \tag{4}$$

Since u is divergence-free and tangent to the boundaries, there exists a stream-function $\psi : D_h \to \mathbb{R}$ so that $u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi)$. Of particular interest are Arnol'd stable solutions [1, Ch. II, §4], which satisfy

$$\omega = F(\psi) \quad \text{for} \quad -\lambda_1 < F' < 0 \quad \text{or} \quad F' > 0 \tag{5}$$

for Lipschitz $F : \mathbb{R} \to \mathbb{R}$, where $\lambda_1 = \lambda_1(D_h)$ is the smallest eigenvalue of the Dirichlet Laplacian $-\Delta$ on D_h. It turns out that all Arnol'd stable equilibria $u = (u_1, u_2)$ on D_h with trivial homology (trivial projection onto harmonic vector fields) must conform to the symmetry of the domain (see Lemma 1), i.e.

$$u_1(x, y) = -u_1(x, -y), \quad u_2(x, y) = u_2(x, -y). \tag{6}$$

When $h' = 0$, all stable equilibria are shears, e.g. their streamlines are all straight (topologically, non-contractable loops), see [13,14] and [2, Proposition 1.1]. A well-known example is Couette flow $u(x, y) = (y, 0)$. Note that, since all harmonic vector fields are parallel to $(1, 0)$, Couette has trivial homology as $\int_{D_0} u_1 \, \mathrm{d}\, x = 0$. On the other hand, we prove here that if $h' \neq 0$, *all* stable solutions with trivial homology must also possess streamlines that are contractible loops, e.g. they must have "islands" or cat's eye vortices. See Figure 1.

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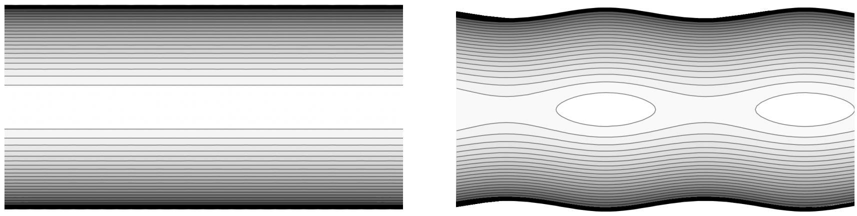

FIGURE 1. Stable equilibria (constant vorticity) on straight and curved channel

Theorem 1. *Let $u \in C^1(D_h)$ be a steady Euler solution with trivial homology and Lipschitz vorticity profile with $F' > -\lambda_1$. Then either $h' = 0$ or u possesses a contractable streamline, i.e. it has an island.*

Remark 1. Since they are stable with trivial homology, the velocities discussed in Theorem 1 will always have stagnation points, or even lines, with Couette flow being a prime example. The conclusions of the theorem apply more generally to any reflection-symmetric solution, regardless of the (Lipschitz) profile F as the assumption that $F' > -\lambda_1$ is used only to establish reflection symmetry via Lemma 1.

These flows possessing contractable streamlines have what are called cat's eyes, of which the Kelvin-Stuart vortex is an explicit example [9, 11]. These structures appear also in the plasma literature where u represents the magnetic field and contractable streamlines are termed "magnetic islands" [5, 12]. The Kelvin-Stuart vortex has been shown to be nonlinearly stable [8, 10] and are thus dynamically persistent. Our result shows that, in a certain sense, they are a ubiquitous feature of stable fluid equilibria.[1]

Remark 2 (Islands are destroyed by current). The assumption in Theorem 1 on trivial homology can, in general, not be dropped. Indeed, there exist smooth Arnol'd stable steady states in any D_h with $h' \neq 0$ sufficiently small having non-trivial homology such that all are non-contractable loops, i.e. there are no islands. This follows directly from the results of [2]. Specifically, consider the velocity $u = \nabla^\perp \psi$ with $\psi(x, y) = -\frac{1}{2}y^2 + 1.01y$ on the channel $D_0 = \mathbb{T} \times [-1, 1]$, which has the property that $|\nabla\psi| > 0.01$ on D_0. Then by [2, Theorem 1.2], for small but arbitrary

[1]We note that Arnol'd's condition (5) is sufficient for stability but not necessary, at least on the *straight channel* D_0. For instance, Poiseuille flow $u = (y^2 - \frac{1}{3}, 0)$ is nonlinearly stable, has trivial homology, but lacks reflection symmetry. Here, $\psi = G(\omega)$ with $G(\omega) = \frac{\omega}{24}(4 - \omega^2)$. Thus $G'(\omega) = \frac{1}{8}(\frac{4}{3} - \omega^2)$ which vanishes $\omega|_{y=\pm\frac{1}{\sqrt{3}}} = \frac{2}{\sqrt{3}}$, e.g. along the stagnant streamlines $\{\psi = \pm - \frac{2}{9\sqrt{3}}\}$, violating both of Arnol'd's conditions. As such, $F := G^{-1}$ is only Hölder continuous near the critical levels and Lemma 1 does not apply. On the other hand, by adding a sufficiently large mean flow, $\mathsf{U}e_1$ with $\mathsf{U} > \frac{1}{3}$, one destroys the stagnation set and $u = (y^2 - \frac{1}{3} + \mathsf{U}, 0)$ becomes stable in the sense of Arnol'd (5). However, this changes the homology of the flow $\mathbf{P}_{\mathcal{H}_1} u = \mathsf{U}e_1$ and, again, Lemma 1 does not apply. In a sense, the reason why stable flows *not* satisfying Arnol'd's condition can exist at all on the channel is that the harmonic vector fields $\propto e_1$ happen to be simultaneously Killing vector fields for the Euclidean metric on D_0 and tangent to the boundary. This leads to a "Galilean symmetry group" mapping all solutions with a given mean flow to solutions with any other and hence stability is transferred despite the breaking of condition (5), see [4, Section 2.2].

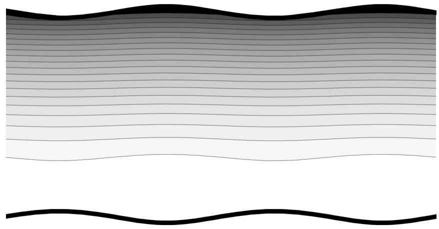

FIGURE 2. Steady states with no islands with non-trivial homology

deformations h of the boundary there exists a diffeomorphism φ taking the original domain to the perturbed domain such that $\psi \circ \varphi$ is the streamfunction of a stable stationary solution on the new domain. The resulting solution has the same topology as ψ. See Figure 2.

We now prove Theorem 1. We begin by establishing symmetry under the stated hypotheses.

Lemma 1. *Suppose that $u = \nabla^\perp \psi$ has trivial homology (projection onto the space of harmonic fields) and*

$$\Delta\psi = F(\psi) \quad \text{in} \quad D_h, \tag{7}$$

$$\psi = \text{const} \quad \text{on} \quad \partial D_h, \tag{8}$$

for $F' > -\lambda_1$. Then u has reflection symmetry, i.e. it satisfies (6).

Proof. The space $\mathcal{H}_1(D_h)$ of harmonic vector fields tangent to the boundary is

$$\mathcal{H}_1(D_h) = \{\nabla^\perp q \colon q \in C^\infty(\overline{D}_h),\ \Delta q = 0 \text{ on } D_h,\ q = c_j \text{ on } \Gamma_j,\ j = 1, 2\} \tag{9}$$

where c_j are arbitrary constants and Γ_j are the two connected components of ∂D_h. This space is one-dimensional as the only free parameter is the difference in boundary values $c_2 - c_1$, since the streamfunction q is defined up to an additive constant. The L^2 orthogonal projection $\mathbf{P}_{\mathcal{H}_1}$ onto $\mathcal{H}_1(D_h)$ is given by

$$(\mathbf{P}_{\mathcal{H}_1} u, q)_{L^2} = \int_{D_h} u \cdot \nabla^\perp q \, \mathrm{d}\, x = \psi|_{\Gamma_2} \int_{\Gamma_2} \partial_n q - \psi|_{\Gamma_1} \int_{\Gamma_1} \partial_n q = \big(\psi|_{\Gamma_2} - \psi|_{\Gamma_1}\big) \int_{\Gamma_1} \partial_n q \tag{10}$$

where q generates $\mathcal{H}_1(D_h)$ and has unit L^2 norm, and where we used that $\int_{\Gamma_2} \partial_n q - \int_{\Gamma_1} \partial_n q = \int_{D_h} \Delta q = 0$. As a result, $\mathbf{P}_{\mathcal{H}_1} u = 0$ implies that ψ takes the same value on both connected components of ∂D_h.

We now show that any such streamfunction has an even symmetry about the line $\{y = 0\}$. To this end, let $\psi_{\mathsf{R}}(x, y) := \psi(x, -y)$. The function $\phi = \psi - \psi_{\mathsf{R}}$ satisfies

$$\Delta\phi = G(\psi)\phi, \qquad G(\psi) = \begin{cases} \frac{F(\psi) - F(\psi_{\mathsf{R}})}{\psi - \psi_{\mathsf{R}}}, & \psi \neq \psi_{\mathsf{R}}, \\ F'(\psi), & \psi = \psi_{\mathsf{R}}. \end{cases} \tag{11}$$

Since $\phi = 0$ on ∂D_h, integrating by parts yields $\int_{D_h} |\nabla\phi|^2 = -\int_{D_h} G(\psi)\phi^2$. On the other hand, by Poincaré's inequality, $\lambda_1 \int_{D_h} \phi^2 \le \int_{D_h} |\nabla\phi|^2$ and so combining the above we find

$$\int_{D_h} (\lambda_1 + G(\psi))\, \phi^2 \le 0. \tag{12}$$

By the mean value theorem, we have $G(\psi) = F'(\xi)$ for some $\xi = \xi(x,y)$ lying between $\psi(x,y)$ and $\psi_{\mathsf{R}}(x,y)$. Since $F' > -\lambda_1$, the inequality (12) forces $\phi = 0$. $\square$

The reflection symmetry, together with the assumed C^1 smoothness, forces $u_1|_{y=0} = 0$. We now show

Proposition 1. *Under the conditions of Theorem 1, $u_2|_{y=0} \neq 0$ on $\{y=0\}$ unless $h' = 0$ or $u = 0$.*

Proof. We shall show that, under our hypotheses, demanding $u_2|_{y=0} = 0$ forces ψ to solve an overdetermined elliptic problem and this results in ψ being constant unless $h' = 0$. This is similar to the rigidity encountered in stationary free boundary fluid problems [3, 6]. In this case, the rigidity will follow from a unique continuation argument via the following Carleman estimate.

Lemma 2. *Let D be a bounded domain and $w \in H^2(D)$ satisfy $w, \nabla w = 0$ on ∂D. Let $\varphi_0 \in C^4(D)$ and suppose that $\varphi_0 \geq 0$ and $|\partial_y \varphi_0| > 0$ in D. Then, for $C, \mathsf{m}_0, \lambda_0 > 0$ depending only on D and $\|\varphi_0\|_{C^4(D)}$, $(\inf_D |\nabla \varphi_0|)^{-1}$, if $|\mathsf{m}| > \mathsf{m}_0$ and $\lambda > \lambda_0$, we have*

$$\int_D |e^{\mathsf{m}\varphi} \Delta e^{-\mathsf{m}\varphi} w|^2 \,\mathrm{d}\, x \geq C\mathsf{m}^2 \int_D |w|^2 \,\mathrm{d}\, x, \qquad \text{where } \varphi := e^{\lambda \varphi_0}. \tag{13}$$

Proof. The proof is motivated by arguments that can be found, e.g. in [7]. We first claim that, regardless of the specific form of φ, if $|\partial_y \varphi| > 0$ for any m we have the following inequality,

$$\int_D |e^{\mathsf{m}\varphi} \Delta e^{-\mathsf{m}\varphi} w|^2 \,\mathrm{d}\, x \geq \int_D |Aw|^2 \,\mathrm{d}\, x + \int_D K[w, \varphi] \,\mathrm{d}\, x, \tag{14}$$

where $A := \Delta + \mathsf{m}^2 |\nabla \varphi|^2$ and where, writing $\operatorname{Hess} \varphi = \nabla \otimes \nabla \varphi$,

$$K[w,\varphi] := 4\mathsf{m}(\operatorname{Hess}\varphi)(\nabla w, \nabla w) + 4\mathsf{m}^3 (\operatorname{Hess}\varphi)(\nabla\varphi, \nabla\varphi)|w|^2 - \mathsf{m}\Delta^2\varphi |w|^2. \tag{15}$$

Proof of inequality (14). Write

$$e^{\mathsf{m}\varphi} \Delta e^{-\mathsf{m}\varphi} w = Aw + Bw, \text{ where } B = -\mathsf{m}\left(2\nabla\varphi \cdot \nabla + \Delta\varphi\right).$$

Then

$$\begin{aligned} \int_D |Aw + Bw|^2 \,\mathrm{d}\, x &= \int_D \left(|Aw|^2 + |Bw|^2 + 2AwBw\right) \mathrm{d}\, x \\ &\geq \int_D \left(|Aw|^2 + 2AwBw\right) \mathrm{d}\, x. \end{aligned} \tag{16}$$

To deal with the last term, we claim that

$$2AwBw = -2\mathsf{m}\left(\Delta w + \mathsf{m}^2 |\nabla\varphi|^2 w\right)\left(2\nabla\varphi \cdot \nabla w + (\Delta\varphi) w\right) = \operatorname{div} T + K, \tag{17}$$

where K is as in (15) and where

$$T = -2\mathsf{m}\left(\nabla w \nabla\varphi \cdot \nabla w - \tfrac{1}{2}\nabla\varphi |\nabla w|^2 + \nabla w w \Delta\varphi - \tfrac{1}{2}|w|^2 \nabla\Delta\varphi + \mathsf{m}^2 |\nabla\varphi|^2 \nabla\varphi |w|^2\right) \tag{18}$$

has the property that $\int_{\partial D} T \cdot \widehat{n} = 0$ under our assumptions on w. The bound (14) then follows immediately from the identities (17) and (18). To prove this identity, we first write

$$2\Delta w \nabla\varphi \cdot \nabla w = \operatorname{div}\left(2\nabla w \nabla\varphi \cdot \nabla w - \nabla\varphi |\nabla w|^2\right) + \Delta\varphi |\nabla w|^2 - 2(\operatorname{Hess}\varphi)(\nabla w, \nabla w).$$

We also have

$$\begin{aligned}\Delta\varphi\Delta w w &= \Delta\varphi \ \mathrm{div}(\nabla w w) - \Delta\varphi|\nabla w|^2\\ &= \mathrm{div}\left(\Delta\varphi\nabla w w - \tfrac{1}{2}|w|^2\nabla\Delta\varphi\right) + \tfrac{1}{2}\Delta^2\varphi|w|^2 - \Delta\varphi|\nabla w|^2\end{aligned}$$

as well as

$$\begin{aligned}\mathsf{m}^2|\nabla\varphi|^2 w\,(2\nabla\varphi\cdot\nabla w + \Delta\varphi w)\\ = \mathrm{div}\left(\mathsf{m}^2|\nabla\varphi|^2\nabla\varphi|w|^2\right) - 2\mathsf{m}^2(\mathrm{Hess}\,\varphi)(\nabla\varphi,\nabla\varphi)|w|^2,\end{aligned} \tag{19}$$

and adding up the previous three lines and multiplying by $-2\mathsf{m}$ we arrive at (17).

Proof of inequality (13). With $\varphi = e^{\lambda\varphi_0}$, we compute

$$\begin{aligned}\nabla\varphi &= \lambda\nabla\varphi_0\varphi,\\ \mathrm{Hess}\,\varphi &= \lambda\left(\mathrm{Hess}\,\varphi_0 + \lambda\nabla\varphi_0\otimes\nabla\varphi_0\right)\varphi,\\ \Delta\varphi &= \lambda\left(\Delta\varphi_0 + \lambda|\nabla\varphi_0|^2\right)\varphi\end{aligned} \tag{20}$$

and so K takes the form

$$\begin{aligned}K &= 4\mathsf{m}\lambda^2|\nabla\varphi_0\cdot\nabla w|^2\varphi^2 + 4\mathsf{m}^3\lambda^4|\nabla\varphi_0|^4|w|^2\varphi^3\\ &+ 4\mathsf{m}\lambda(\mathrm{Hess}\,\varphi_0)(\nabla w,\nabla w)\varphi + 4\mathsf{m}^3\lambda^3(\mathrm{Hess}\,\varphi_0)(\nabla\varphi_0,\nabla\varphi_0)|w|^2\varphi^3 - \mathsf{m}\Delta^2\varphi|w|^2.\end{aligned} \tag{21}$$

Bounding $|\Delta^2\varphi| \le C\lambda^4\varphi$ where C depends on $\|\varphi_0\|_{C^4}$ and noting the positivity of the terms on the first line, since $|\nabla\varphi_0|$ is bounded below in D, there are $\lambda_0 > 0$ and $\mathsf{m}_0 > 0$ depending on $(\inf|\nabla\varphi_0|)^{-1} > 0$ and $\|\varphi_0\|_{C^2(D)}$ so that if $\lambda > \lambda_0$ and $\mathsf{m} > \mathsf{m}_0$ we have

$$\begin{aligned}4\mathsf{m}^3\lambda^4|\nabla\varphi_0|^4|w|^2\varphi^3 + 4\mathsf{m}^3\lambda^3(\mathrm{Hess}\,\varphi_0)(\nabla\varphi_0,\nabla\varphi_0)|w|^2\varphi^3 - \mathsf{m}\Delta^2\varphi|w|^2\\ \ge C\mathsf{m}^3\lambda^4|w|^2\varphi^3,\end{aligned} \tag{22}$$

and for such λ we find

$$K[w,\varphi] \ge C\left(\mathsf{m}\lambda^2|\nabla\varphi_0\cdot\nabla w|^2\varphi^2 + \mathsf{m}^3\lambda^4|w|^2\varphi^3\right) - C'\mathsf{m}\lambda|\nabla w|^2\varphi, \tag{23}$$

and returning to (14), we have the inequality

$$\begin{aligned}&\int_D |e^{\mathsf{m}\varphi}\Delta e^{-\mathsf{m}\varphi}w|^2 \,\mathrm{d}\,x\\ &\ge C\int_D\left(|Aw|^2 + \mathsf{m}\lambda^2|\nabla\varphi_0\cdot\nabla w|^2\varphi^2 + \mathsf{m}^3\lambda^4|w|^2\varphi^3\right)\mathrm{d}\,x - C'\mathsf{m}\lambda\int_D|\nabla w|^2\varphi\,\mathrm{d}\,x.\end{aligned} \tag{24}$$

To control the last term here, we start by getting a lower bound for $\|Aw\|_{L^2}$. For this, we write

$$\begin{aligned}&\int_D|\nabla w|^2\,\varphi\,\mathrm{d}\,x\\ &= -\int_D Aww\,\varphi\,\mathrm{d}\,x + \int_D\Big(\mathsf{m}^2|\nabla\varphi|^2|w|^2\,\varphi - \nabla\varphi\cdot\nabla w w\Big)\,\mathrm{d}\,x\\ &= -\int_D Aww\,\varphi\,\mathrm{d}\,x + \int_D\mathsf{m}^2\lambda^2|\nabla\varphi_0|^2|w|^2\varphi^3\,\mathrm{d}\,x - \int_D\lambda(\nabla\varphi_0\cdot\nabla w)w\,\varphi\,\mathrm{d}\,x\\ &\le \frac{1}{2\delta}\|Aw\|_{L^2}^2 + C\Big(\delta + \mathsf{m}^2\lambda^2\Big)\|w\varphi^{3/2}\|_{L^2}^2 + \lambda\|\nabla\varphi_0\cdot\nabla w\|_{L^2}\|w\varphi\|_{L^2}\end{aligned}$$

for any $\delta > 0$, where $C = C(\|\varphi_0\|_{C^1(D)})$, $L^2 = L^2(D)$. Here, we used $\varphi, \mathsf{m}, \lambda \geq 1$. In particular,

$$\frac{1}{2}\|Aw\|_{L^2}^2 \geq \delta\|(\nabla w)\varphi^{1/2}\|_{L^2}^2 - C\Big(\delta^2 + \delta\mathsf{m}^2\lambda^2\Big)\|w\varphi^{3/2}\|_{L^2}^2 - C\lambda\delta\|\nabla\varphi_0\cdot\nabla w\|_{L^2}\|w\varphi\|_{L^2}.$$

Returning to (24), we have shown that for λ, m sufficiently large and any $\delta > 0$,

$$\begin{aligned}\|e^{\mathsf{m}\varphi}\Delta e^{-\mathsf{m}\varphi}w\|_{L^2}^2 \geq{}& C\delta\|(\nabla w)\varphi^{1/2}\|_{L^2}^2 + C\mathsf{m}^3\lambda^4\|w\varphi^{3/2}\|_{L^2}^2 + C\mathsf{m}^2\lambda^2\|(\nabla\varphi_0\cdot\nabla w)\varphi\|_{L^2}^2\\ &- C\mathsf{m}\lambda\|(\nabla w)\varphi^{1/2}\|_{L^2}^2 - C\Big(\delta^2 + \delta\mathsf{m}^2\lambda^2\Big)\|w\varphi^{3/2}\|_{L^2(D_h)}^2\\ &- C\lambda\delta\|\varphi_0\cdot\nabla w\|_{L^2}\|w\varphi\|_{L^2}.\end{aligned}$$

We now want to choose $\delta > 0$ so that we can absorb the terms on the second line into the terms on the first line. For this we take $\delta = \delta'\mathsf{m}\lambda^2$ for small $\delta' > 0$, and the above becomes

(25)

$$\begin{aligned}\|e^{\mathsf{m}\varphi}\Delta e^{-\mathsf{m}\varphi}w\|_{L^2}^2 \geq{}& C\mathsf{m}\lambda^2\left(\delta'\lambda - C'\right)\|(\nabla w)\varphi^{1/2}\|_{L^2}^2\\ &+ C\mathsf{m}^2\lambda^4\left((1 - C'\delta')\mathsf{m} - C'(\delta')^2\right)\|w\varphi^{3/2}\|_{L^2(D_h)}^2\\ &+ C\mathsf{m}^2\lambda^2\|(\nabla\varphi_0\cdot\nabla w)\varphi\|_{L^2}^2 - C\delta'\mathsf{m}\lambda^3\|\nabla\varphi_0\cdot\nabla w\|_{L^2}\|w\varphi\|_{L^2}\end{aligned}$$

where all the above constants depend only on $\|\varphi_0\|_{C^4(D)}$. Taking δ' small enough that the coefficient of the second term is bounded below by $C\mathsf{m}^3\lambda^3/2$ whenever $\mathsf{m} \geq 1$, and then taking λ large enough that the coefficient of the first term is bounded below by $C\mathsf{m}\lambda^3/2$, and finally taking m large enough that for this choice of λ, δ', we can absorb the last term into the first term on the last line and the second term on the first line, we have arrived at the estimate

$$\begin{aligned}\|e^{\mathsf{m}\varphi}\Delta e^{-\mathsf{m}\varphi}w\|_{L^2}^2\\ \geq C\mathsf{m}\lambda^3\|(\nabla w)\varphi^{1/2}\|_{L^2}^2 + C\mathsf{m}^3\lambda^4\|w\varphi^{3/2}\|_{L^2}^2 + C\mathsf{m}^2\lambda^2\|(\nabla\varphi_0\cdot\nabla w)\varphi\|_{L^2}^2,\end{aligned} \tag{26}$$

for large enough λ, m, which is more than sufficient for the estimate (13) since $\varphi \geq 1$.

□

We use this to prove

Lemma 3. *Let $h \in C^2(\mathbb{R})$ and set $D_h^{\mathsf{up}} = \{(x, y) : 0 \leq y \leq 1 + h(x), x \in \mathbb{T}\}$. Let F be a Lipschitz function and suppose that $\psi \in H^2(\Omega)$ satisfies*

$$\Delta\psi = F(\psi) \qquad \text{in } D_h^{\mathsf{up}}, \tag{27}$$

$$\partial_x\psi = \partial_y\psi = 0 \qquad \text{on } \{y = 0\}. \tag{28}$$

Then $\partial_x\psi = 0$. Moreover, if ψ is constant at $\{y = 1 + h(x)\}$ then either $h' = 0$ or $\partial_y\psi = 0$ as well.

Proof. We first show that $\partial_x\psi = 0$. Since F is a Lipschitz function, it is differentiable almost everywhere and so by elliptic regularity, $\psi \in H^3(D_h^{\mathsf{up}})$. Now we note that $\partial_x\psi$ satisfies

$$\Delta\partial_x\psi = F'(\psi)\partial_x\psi \qquad \text{in } D_h^{\mathsf{up}}, \tag{29}$$

$$\partial_x\psi = \partial_y\partial_x\psi = 0 \qquad \text{on } \{y = 0\}. \tag{30}$$

Here, the fact that $\partial_y \partial_x \psi|_{y=0}$ is well-defined follows from the trace theorem and the fact that $\psi \in H^3$. Now define a smooth increasing cutoff function $\chi_0(z)$ so that $\chi_0(z) = 1$ when $z \geq 1$ and $\chi_0(z) = 0$ when $z \leq 1/2$. Let $\varphi_0(x,y) = 1 + h(x) - y$ and set $\varphi(x,y) = e^{\lambda \varphi_0(x,y)}$ where $\lambda > \lambda_0$ with λ_0 as in Lemma 2. Then $D_h^{\mathsf{up}} = \{(x,y) \in \mathbb{T} \times \mathbb{R} : \varphi(x,y) \geq 1, y \geq 0\}$ and $\varphi > 1$ in the interior. If we define $\chi_c(x,y) = \chi_0((\varphi(x,y) - 1)/c)$ and $\Omega_c = \chi_c^{-1}(1)$, then $1 + c \leq \varphi(x,y)$ on Ω_c. We also have that $\varphi(x,y) \leq 1 + c$ on the support of $\nabla \chi_c$. Moreover for $c > 0$, χ_c is zero near the boundary $\{y = 1 + h(x)\}$.

Under our hypotheses on h, the assumptions of Lemma 2 hold, and so taking m sufficiently large and applying the Carleman estimate (13) to $w = e^{\mathsf{m}\varphi} \chi_c \partial_x \psi$ with $c > 0$, we find

$$
\begin{aligned}
\|e^{\mathsf{m}\varphi} \chi_c \partial_x \psi\|_{L^2(D_h^{\mathsf{up}})} &\leq \frac{C}{\mathsf{m}} \|e^{\mathsf{m}\varphi} \Delta(\chi_c \partial_x \psi)\|_{L^2(D_h^{\mathsf{up}})} \\
&\leq \frac{C}{\mathsf{m}} \|e^{\mathsf{m}\varphi} [\Delta, \chi_c] \partial_x \psi\|_{L^2(D_h^{\mathsf{up}})} + \frac{C}{\mathsf{m}} \|e^{\mathsf{m}\varphi} F'(\psi) \partial_x \psi\|_{L^2(D_h^{\mathsf{up}})} \\
&\leq \frac{C}{\mathsf{m}} e^{\mathsf{m}(1+c)} \|\partial_x \psi\|_{H^1(D_h^{\mathsf{up}})} \\
&\quad + \frac{C}{\mathsf{m}} \|F'\|_{L^\infty(\mathrm{rang}(\psi))} \|e^{\mathsf{m}\varphi} \chi_c \partial_x \psi\|_{L^2(D_h^{\mathsf{up}})},
\end{aligned}
\tag{31}
$$

where C depends continuously on $1/c$. Here we used the equation for $\partial_x \psi$ and that $\varphi(x,y) \leq c + 1$ on the support of the commutator $[\Delta, \chi_c]$. We also used that $c > 0$ to justify the vanishing of w at the top boundary and that $\partial_x \psi, \nabla \partial_x \psi = 0$ when $y = 0$ to justify the vanishing at the bottom boundary. If we now take m so large that $\frac{C}{\mathsf{m}} \|F'\|_{L^\infty(\mathrm{rang}\psi)} \leq \frac{1}{2}$, we can absorb the second term on the right-hand side into the left, giving

$$
e^{\mathsf{m}(1+c)} \|\partial_x \psi\|_{L^2(\Omega_c)} \leq \|e^{\mathsf{m}\varphi} \chi_c \partial_x \psi\|_{L^2(D_h^{\mathsf{up}})} \leq \frac{C}{\mathsf{m}} e^{\mathsf{m}(1+c)} \|\partial_x \psi\|_{H^1(D_h^{\mathsf{up}})}, \qquad c > 0.
\tag{32}
$$

Dividing both sides by $e^{\mathsf{m}(1+c)}$ and taking $\mathsf{m} \to \infty$ we find that for each $c > 0$, $\partial_x \psi$ vanishes in the region Ω_c. Since $\Omega_c \to D_h^{\mathsf{up}}$ as $c \to 0$, taking $c \to 0$ we find that $\partial_x \psi$ vanishes everywhere in D_h^{up}.

This has the following consequence: ψ is a function of one variable, y and moreover is identically constant on the set $\mathsf{S} := \{(x,y) \colon x \in \mathbb{T},\ 1 + h_- \leq y \leq 1 + h(x)\}$ where $h_- := \min_{x \in \mathbb{T}} h(x)$. Now consider the rectangular domain $\mathsf{R} := \{(x,y) \colon x \in \mathbb{T},\ 0 \leq y \leq 1 + h_-\}$. On this domain, $\partial_y \psi$ satisfies

$$
\Delta \partial_y \psi = F'(\psi) \partial_y \psi, \qquad \text{in } \mathsf{R}, \tag{33}
$$

$$
\partial_y \psi = \partial_y^2 \psi = 0 \qquad \text{on } \{y = 1 + h_-\}. \tag{34}
$$

Indeed, the boundary condition (34) follows from the fact that $\Delta \psi = \partial_y^2 \psi = F(\psi)$, so $\partial_y^2 \psi \in C(\mathbb{R})$. Along $\{y = 1 + h_-\}$, the solution ψ must match its constant value in S and thus $\partial_y \psi = \partial_y^2 \psi = 0$. Provided $h' \neq 0$ so that $\mathsf{S} \neq \emptyset$, the result that $\partial_y \psi = 0$ follows by essentially the same argument as above. □

The proposition follows, since if $u_2|_{\{y=0\}} = 0$ then unless $h' = 0$ we must have $u = 0$ everywhere. □

To complete the proof of Theorem 1, we appeal to

Proposition 2. *Let u be a Lipschitz divergence-free vector field on D_h with reflection symmetry. Almost every point p on $\{y = 0\}$ such that $|u_2(p)| \neq 0$ lies on a closed streamline.*

Proof. Let $\mathsf{F} := \{p \in \{y = 0\} \colon |u_2(p)| \neq 0\}$ be the (open) set of non-stagnant points along the centerline. By incompressibility, almost every point $p \in \mathsf{F}$ is a regular point of ψ, namely the set $\{x \colon \psi(x) = \psi(p)\}$ contains no critical points. Otherwise there would exist an interval $I \subset \mathsf{F}$ such that for almost every $q \in I$, the Omega limit set of the orbit passing through q is a critical point. Since $|u_2||_I > \varepsilon_0 > 0$ as u is continuous, flowing for short time the interval, one can form a flowbox of positive area which would then (in infinite time) compress to a set of zero area, a contradiction. For any regular point p, it is easy to see that the level set $\{x \colon \psi(x) = \psi(p)\}$ is a simple closed curve. Moreover, since the streamfunction ψ has even symmetry about $\{y = 0\}$, any such curve must be contractible since (a) the orbit through p must return to $\{y = 0\}$ by uniqueness of integral curves and (b) once it returns, by symmetry, the integral curve is contractible. This completes the proof. $\square$

Remark 3. In fact, if ψ is a Morse function, *every* point p on $\{y = 0\}$ such that $|u_2(p)| \neq 0$ lies on a closed streamline. This follows from the Poincaré–Hopf index theorem, the fact that the Euler characteristic of the annulus is zero, and the Poincaré-Bendixson theorem together with incompressibility which ensures all critical points are either hyperbolic or elliptic.

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