Journal of Statistical Physics (2024) 191:61
https://doi.org/10.1007/s10955-024-03277-w

Mixing by Statistically Self-similar Gaussian Random Fields

Michele Coti Zelati[1] · Theodore D. Drivas[2] · Rishabh S. Gvalani[3]

Received: 28 September 2023 / Accepted: 26 April 2024 / Published online: 15 May 2024

Abstract
We study the passive transport of a scalar field by a spatially smooth but white-in-time incompressible Gaussian random velocity field on $\mathbb{R}^d$. If the velocity field u is homogeneous, isotropic, and statistically self-similar, we derive an exact formula which captures non-diffusive mixing. For zero diffusivity, the formula takes the shape of $\mathbb{E}\ \|\theta_t\|^2_{\dot{H}^{-s}} = \mathrm{e}^{-\lambda_{d,s}t}\|\theta_0\|^2_{\dot{H}^{-s}}$ with any $s \in (0, d/2)$ and $\frac{\lambda_{d,s}}{D_1} := s(\frac{\lambda_1}{D_1} - 2s)$ where $\lambda_1/D_1 = d$ is the top Lyapunov exponent associated to the random Lagrangian flow generated by u and D_1 is small-scale shear rate of the velocity. Moreover, the mixing is shown to hold *uniformly* in diffusivity.

Keywords Mixing · Scalar transport · Stochastic flows · Turbulence · Transport noise

1 Passive Scalar Transport by Gaussian Random Fields

We study passive scalar $\theta_t^\kappa(x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ transport with diffusivity $\kappa \geq 0$ on $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$

$$\mathrm{d}\theta_t^\kappa + \mathrm{d}u_t \circ \nabla\theta_t^\kappa = \kappa\Delta\theta_t^\kappa\ \mathrm{d}t + \mathrm{d}f_t, \tag{1}$$

$$\theta_t^\kappa|_{t=0} = \theta_0, \tag{2}$$

$$\int_{\mathbb{R}^d} \theta_0 \mathrm{d}x = 0, \tag{3}$$

Communicated by Denis Bernard.

✉ Theodore D. Drivas
tdrivas@math.stonybrook.edu

Michele Coti Zelati
m.coti-zelati@imperial.ac.uk

Rishabh S. Gvalani
rgvalani@ethz.ch

1 Department of Mathematics, Imperial College London, London SW7 2AZ, UK

2 Department of Mathematics, Stony Brook University, 11790 Stony Brook, NY, USA

3 Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland

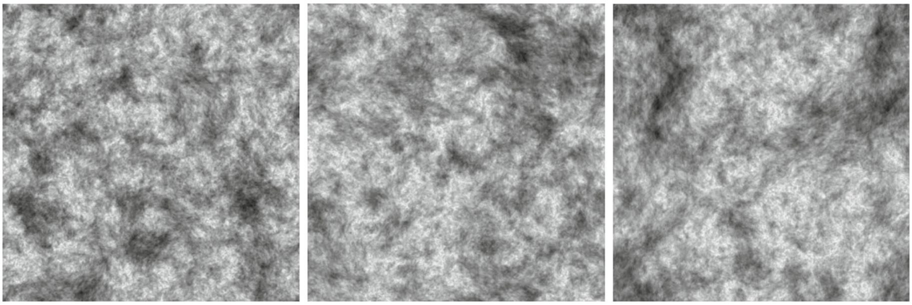

Fig. 1 Snapshots in space at various times of a white-in-time Gaussian random velocity

where $u_t := u(x,t)$ is a white-in-time, incompressible Gaussian random field with mean and covariance

$$\mathbb{E}\left[u^i(x,t)\right] = 0, \tag{4}$$

$$\mathbb{E}\left[u^i(x,t)u^j(x',t')\right] = D^{ij}(x,x')\delta(t-t'), \tag{5}$$

We shall consider random fields which are homogeneous and divergence-free

$$D^{ij}(x,x') := D^{ij}(x-x'), \tag{6}$$

$$\partial_{x_i} D^{ij}(x,x') = \partial_{x_j} D^{ij}(x,x') = 0. \tag{7}$$

For visualization of a Gaussian velocity whose covariance mimics inertial range turbulence, see Fig. 1.

The homogenous Gaussian forcing $f_t := f(x,t)$ is taken independent and defined by the covariance

$$\mathbb{E}\left[f(x,t)f(x',t')\right] = F(x-x')\delta(t-t'). \tag{8}$$

Equation (1) is interpreted in the Stratonovich sense (denoted by $\circ$). The Itô form of equation (1) reads

$$\mathrm{d}\theta_t^\kappa + \mathrm{d}u_t \cdot \nabla\theta_t^\kappa = \tfrac{1}{2}D(0) : \nabla \otimes \nabla\theta_t^\kappa \mathrm{d}t + \kappa\Delta\theta_t^\kappa \,\mathrm{d}t + \mathrm{d}f_t, \tag{9}$$

$$\theta_t^\kappa|_{t=0} = \theta_0. \tag{10}$$

The setup is called the Kraichnan model [7, 14, 23], reviewed in the wonderful lecture notes of Gawędzki [17–19]. The velocity field can be written concretely as

$$\mathrm{d}u_t(x) = \sum_{n=0}^{\infty} \sigma^{(n)}(x)\mathrm{d}W_t^{(n)}, \qquad \sigma^{(n)}(x) := \sqrt{\lambda_n}e_n(x), \tag{11}$$

where $W_t^{(n)}$ are independent one-dimensional Brownian motions and where λ_n and $e_n(x)$ for $n = 0, 1, 2, \ldots$ are the eigenvalues and eigenfunctions of the positive, trace-class operator with kernel $D(x-x')$ acting on $L^2(\mathbb{R}^d, \mathbb{R}^d)$. The kernel D can be represented by

$$D^{ij}(x-x') = \sum_{n=0}^{\infty} \sigma_i^{(n)}(x)\sigma_j^{(n)}(x'). \tag{12}$$

Here the $\sigma^{(n)}$ are divergence-free vector fields since D is divergence-free in each index. We will consider a model of the viscous–convective range for the scalar in which the velocity if Lipschitz, the so-called Batchelor regime for the model. The rough version of the model, mimicking a turbulent inertial range, has been a topic of a great deal of study [5, 27, 28]. See also discussion in [12, 13] regarding the effects of molecular fluctuations on the phenomenology of scalar advection below the viscous–convective range. In this work, we will specialize our covariance even further

Definition 1 [Self-similar isotropic covariance] We say D is self-similarly isotropic if it takes the form

$$D(0) - D(r) = D_1 \left[I + \left(\frac{2}{d-1} \right) \left(I - \hat{r} \otimes \hat{r} \right) \right] |r|^2 \tag{13}$$

for some constant.[1] $D_1 > 0$ and $D(0) = D_0 I$ for $D_0 > 0$.

Example 1 A prototypical example of a self-similarly isotropic random field arises in the following context. Consider a Gaussian field defined by the covariance

$$D_{ij}(r) = \bar{D}_0 \int \frac{P_{ij}(k)}{(|k|^2 + m^2)^{(d-\zeta)/2}} \mathrm{e}^{ik\cdot r} \mathrm{d}k \tag{14}$$

where $\zeta > 2$. The constant m is an infrared cutoff for the velocity and P_{ij} is the projection onto the divergence-free subspace. At short distances, one can compute that

$$D_0 \delta^{ij} - D^{ij}(r) = D_1 \left[\delta^{ij} + \left(\frac{2}{d-1} \right) \left(\delta^{ij} - \hat{r}^i \hat{r}^j \right) \right] r^2 + O((m|r|)^2), \tag{15}$$

where

$$D_0 = \frac{\overline{D}_0}{m^2} \frac{d-1}{d(4\pi)^{d/2}\Gamma\left(\frac{d+2}{2}\right)}, \qquad D_1 = \overline{D}_0 \frac{d-1}{(d+2)(4\pi)^{d/2}\Gamma\left(\frac{d+2}{2}\right)}. \tag{16}$$

Thus, taking the infrared limit $m \to 0$ gives a self-similarly isotropic random field.

The forcing in (1) is also represented by a statistically independent collection of one-dimensional independent Brownian motions $\{B_t^{(k)}\}_{k\in\mathbb{N}}$ and of scalar functions $\{q^{(k)}\}_{k\in\mathbb{N}}$, as

$$\mathrm{d} f_t(x) = \sum_{k=0}^{\infty} q^{(k)}(x) \mathrm{d} B_t^{(k)}, \qquad q^{(k)}(x) := \sqrt{\mu_k} u_k(x), \tag{17}$$

where $B_t^{(k)}$ are independent Brownian motions and where μ_k and $u_k(x)$ for $k = 0, 1, 2, \ldots$ are the eigenvalues and eigenfunctions of the positive, trace-class operator with kernel $F(x - x')$ acting on $L^2(\mathbb{R}^d, \mathbb{R}^d)$. We denote averages over the random velocity *and* the forcing by $\mathbb{E}[\cdot]$.

In this note, we are interested in the mixing properties of Gaussian random fields. In the context of stochastic transport, exponential mixing estimates have been obtained in [3, 4] for velocities generated by the stochastic Navier–Stokes equations, in [21] for Kraichnan-type models with noise satisfying a general Hrmander condition, and in [6, 9] for alternating shear flows with either random phases or random switching times. On the other hand, deterministic constructions of exponentially mixing flows can be found in [1, 10, 11, 32]. The purpose of

[1] The physical dimensions of D_1 are inverse time and it can be regarded as a proxy for the shear rate at small scales. See the discussion of Kraichnan, e.g. [23, Eq. (3.5)] This shows that D_1 is essentially a square of the fine-scale turbulent shear-rate times a Lagrangian correlation time of the shear rate. We note that our notation differs slightly different from that used in recent literature which replaces $D_1 \to (d-1)D_1$.

this note is to give a new and simple, yet explicit and quantitative proof of mixing for (1), in terms of the usual homogeneous Sobolev norms [29], defined on the Fourier side as

$$\|\varphi\|_{\dot{H}^r}^2 = \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|^{2r}} \mathrm{d}\xi, \qquad r \in \mathbb{R}.$$

Our main result can be stated as follows.

Theorem 1 *[Scalar Mixing Identity] Fix $s \in [0, d/2)$. Suppose that D is an incompressible, homogeneous and self-similarly isotropic correlation function* (13)*. Then we have the identity*

$$\mathbb{E}\|\theta_t^\kappa\|_{\dot{H}^{-s}}^2 = \mathrm{e}^{-\lambda_{d,s}t}\|\theta_0\|_{\dot{H}^{-s}}^2 + \frac{F_s}{\lambda_{d,s}}\left(1 - \mathrm{e}^{-\lambda_{d,s}t}\right) - \kappa \int_0^t \mathrm{e}^{-\lambda_{d,s}(t-\tau)}\mathbb{E}\|\theta_\tau^\kappa\|_{\dot{H}^{1-s}}^2 \mathrm{d}\tau \quad (18)$$

where $\lambda_{d,s} := 2D_1 s(d-2s)$ and $F_s = \sum_{k=0}^{\infty} \|q^{(k)}\|_{\dot{H}^{-s}}^2$.

We note that the work of Haynes and Vanneste [22] provides a different, spectral theoretic, argument for the uniform in diffusivity mixing which is implied by our main identity. We remark also that when $m = 0$ the homogeneous isotropic Gaussian field on $\mathbb{R}^d$ has infinite energy pointwise and thus the transport equation is not naively well defined. Our identity should thus be interpreted as a *universal* limiting relation as $m \to 0$, since all quantities depend only on differences $D_0\delta^{ij} - D^{ij}(r)$ and not D_0 itself.

Remark 1 [Intermittency] We note that the quadratic dependence of $\lambda_{d,s}$ on s, suggesting that mixing is rather intermittent. Intermittency in Batchelor range decay laws for scalar L^p norms with $\kappa > 0$ has been discussed by Son [31] who proved for $d = 3$ that

$$\mathbb{E}\|\theta_t^\kappa\|_{L^p}^p \sim C_\kappa \mathrm{e}^{-\gamma_p t} \qquad \gamma_p = \frac{D_1}{4}\begin{cases} p(6-p) & 0 < p < 3 \\ 9 & p \geq 3 \end{cases}. \quad (19)$$

Note that, curiously, $\gamma_{4s} = \lambda_{d,s}$. We expect more generally that

$$\mathbb{E}\|\theta_t^\kappa\|_{L^p}^p \sim C_{\kappa,d} \mathrm{e}^{-\gamma_p t} \qquad \gamma_p = \frac{D_1}{4}\begin{cases} p(2d-p) & 0 < p < d \\ d^2 & p \geq d \end{cases}. \quad (20)$$

See Balkovsky and Fouxon [2] for a generalization to the time correlated case.

In particular, when there is no forcing ($q^{(k)} \equiv 0$), we obtain mixing in expectation $\mathbb{E}\,\|\theta_t\|_{H^{-s}}^2 \to 0$ with an exponentially fast rate, with rate uniform in the diffusivity κ. The identity (18) may look surprising, since mixing estimates typically require to pay derivatives on the initial datum to gain decay. This would indeed be the case for a pathwise estimate, since in general solutions can mix and unmix and hence a time reversal in a $\dot{H}^{-s} \mapsto \dot{H}^{-s}$ would be in conflict with a decay estimate. The point is that (18) is an identity *in expectation*, hence the possible realizations in which growth and unmixing happen are averaged out.

2 Adapted Mixing Identity and Proof of Theorem 1

We establishes an exact balance of "mixing norms" tailored in a particular way to the covariance of the velocity field D. A similar identity appeared recently in the work of Coghi and Maurelli, where it was used to improve the local wellposedness theory for stochastic Euler with multiplicative noise [8].

Lemma 1 *[Adapted Mixing Identity] Let D be homogenous and divergence-free. Suppose that a function $G : \mathbb{R}^d \to \mathbb{R}$ can be found to satisfy*

$$\big(D(0) - D(r)\big) : \nabla_r \otimes \nabla_r G = -\lambda G, \tag{21}$$

pointwise in space for some $\lambda \in \mathbb{R}$. Then the solution θ_t^κ of (1) satisfies the following "mixing" identity,

$$\mathbb{E}\,\langle \theta_t^\kappa, G * \theta_t^\kappa\rangle_{L^2} = \mathrm{e}^{-\lambda t}\langle \theta_0, G * \theta_0\rangle_{L^2} + \frac{1}{\lambda}F_G\Big(1 - \mathrm{e}^{-\lambda t}\Big) - \kappa\int_0^t \mathrm{e}^{-\lambda(t-\tau)}\mathbb{E}\langle \nabla\theta_\tau^\kappa, G * \nabla\theta_\tau^\kappa\rangle_{L^2}\mathrm{d}\tau \tag{22}$$

where the source is $F_G := 2\int_{\mathbb{R}^d} G(r)F(r)\mathrm{d}r$.

Proof of Lemma 1 First, by definition, we have

$$\langle \theta_t^\kappa, G * \theta_t^\kappa\rangle_{L^2} = \int_{\mathbb{R}^d} \theta_t^\kappa(x)\int_{\mathbb{R}^d} G(r)\theta_t^\kappa(x + r)\mathrm{d}r\mathrm{d}x. \tag{23}$$

Recall (leaving the sum over n implicit) that

$$\begin{aligned}
\mathrm{d}\theta_t^\kappa + \sigma^{(n)}\cdot\nabla\theta_t^\kappa\mathrm{d}W_t^{(n)} &= \tfrac{1}{2}D(0) : \nabla\otimes\nabla\theta_t^\kappa\mathrm{d}t + \kappa\Delta\theta_t^\kappa\mathrm{d}t + q^{(n)}\mathrm{d}B_t^{(n)},\\
\mathrm{d}(G * \theta_t^\kappa) + G * (\sigma^{(n)}\cdot\nabla\theta_t^\kappa)\mathrm{d}W_t^{(n)} &= \tfrac{1}{2}D(0) : \nabla\otimes\nabla(G * \theta_t^\kappa)\mathrm{d}t\\
&\quad + \kappa\Delta(G * \theta_t^\kappa)\mathrm{d}t + (G * q^{(n)})\mathrm{d}B_t^{(n)}.
\end{aligned}$$

Thus, by Ito's product rule, we have

$$\begin{aligned}
\mathrm{d}\left(\theta_t^\kappa\; G * \theta_t^\kappa\right) &= (G * \theta_t^\kappa)\;\mathrm{d}\theta_t^\kappa + \theta_t^\kappa\;\mathrm{d}(G * \theta_t^\kappa) + \mathrm{d}[\theta_t^\kappa, G * \theta_t^\kappa]\\
&= -(G * \theta_t^\kappa)\sigma^{(n)}\cdot\nabla\theta_t^\kappa\mathrm{d}W_t^{(n)} - \theta_t^\kappa(G * (\sigma^{(n)}\cdot\nabla\theta_t^\kappa))\mathrm{d}W_t^{(n)}\\
&\quad + (G * \theta_t^\kappa)q^{(n)}\mathrm{d}B_t^{(n)} + \theta_t^\kappa(G * q^{(n)})\mathrm{d}B_t^{(n)}\\
&\quad + \tfrac{1}{2}(G * \theta_t^\kappa)D(0) : \nabla\otimes\nabla\theta_t^\kappa\mathrm{d}t + \tfrac{1}{2}\theta_t^\kappa D(0) : \nabla\otimes\nabla(G * \theta_t^\kappa)\mathrm{d}t\\
&\quad + \kappa\theta_t^\kappa\Delta(G * \theta_t^\kappa) + \kappa(G * \theta_t^\kappa)\Delta\theta_t^\kappa + \sigma^{(n)}\cdot\nabla\theta_t^\kappa\;(G * (\sigma^{(n)}\cdot\nabla\theta_t^\kappa))\mathrm{d}t.
\end{aligned}$$

Upon integrating over space and using the fact that mollification is self-adjoint in L^2, we have

$$\begin{aligned}
\mathrm{d}\langle \theta_t^\kappa, G * \theta_t^\kappa\rangle_{L^2} &= -2\langle G * \theta_t^\kappa, \sigma^{(n)}\cdot\nabla\theta_t^\kappa\rangle_{L^2}\mathrm{d}W_t^{(n)} + 2\langle \theta_t^\kappa, (G * q^{(n)})\rangle_{L^2}\mathrm{d}B_t^{(n)}\\
&\quad + (D(0) + \kappa I) : \langle \theta_t^\kappa, \nabla\otimes\nabla(G * \theta_t^\kappa)\rangle_{L^2}\\
&\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \theta_t^\kappa(x)\sigma^{(n)}(x)\otimes\sigma^{(n)}(x + r) : \nabla\otimes\nabla(G * \theta_t^\kappa)\mathrm{d}x\mathrm{d}r\mathrm{d}t\\
&\quad + 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(r)F(x, x + r)\mathrm{d}r\mathrm{d}x\\
&= -2\langle G * \theta_t^\kappa, \sigma^{(n)}\cdot\nabla\theta_t^\kappa\rangle_{L^2}\mathrm{d}W_t^{(n)} + 2\langle \theta_t^\kappa, (G * q^{(n)})\rangle_{L^2}\mathrm{d}B_t^{(n)}\\
&\quad + \int_{\mathbb{R}^d}(D(0) - D(r) + \kappa I) : \langle \theta_t^\kappa(\cdot), G(r)(\nabla\otimes\nabla\theta_t^\kappa)(\cdot + r)\rangle_{L^2}\mathrm{d}r\mathrm{d}t\\
&\quad + 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(r)F(x, x + r)\mathrm{d}r\mathrm{d}x.
\end{aligned}$$

Thus, we obtain

$$\mathrm{d}\langle \theta_t^\kappa, G * \theta_t^\kappa\rangle_{L^2} = -2\langle G * \theta_t^\kappa, \sigma^{(n)}\cdot\nabla\theta_t^\kappa\rangle_{L^2}\mathrm{d}W_t^{(n)} + 2\langle \theta_t^\kappa, (G * q^{(n)})\rangle_{L^2}\mathrm{d}B_t^{(n)}$$

$$+\int_{\mathbb{R}^d}(D(0)-D(r)+\kappa I):(\nabla_r\otimes\nabla_r G)(r)\langle\theta_t^\kappa(\cdot),\theta_t^\kappa(\cdot+r)\rangle_{L^2}\mathrm{d}r\mathrm{d}t$$
$$+2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(r)F(x,x+r)\mathrm{d}r\mathrm{d}x.$$

In the above calculation, we repeatedly made use of the fact that the $\sigma^{(n)}$ are divergence-free, integrated by parts and changed some x to r derivatives. Appealing to the defining equation (21) for the kernel G, we find

$$\begin{aligned}\mathrm{d}\langle\theta_t^\kappa,G*\theta_t^\kappa\rangle_{L^2}&=-\lambda\langle\theta_t^\kappa,G*\theta_t^\kappa\rangle_{L^2}-2\langle G*\theta_t^\kappa,\sigma^{(n)}\cdot\nabla\theta_t^\kappa\rangle_{L^2}\mathrm{d}W_t^{(n)}\\&\quad+2\langle\theta_t^\kappa,(G*q^{(n)})\rangle_{L^2}\mathrm{d}B_t^{(n)}+F_G+\kappa\langle\Delta\theta_t^\kappa,G*\theta_t^\kappa\rangle_{L^2},\end{aligned}\tag{24}$$

where we have introduced the notation F_G from the statement of the lemma. The result follows upon taking expectation, using the mean zero property of the martingale term and subsequently integrating. □

We now demonstrate a solution G to (21) if it covariance function is self-similarly isotropic.

Lemma 2 *Suppose that D is self-similarly isotropic. Then for any $s\in(0,d/2)$, the Riesz potential*

$$G(r)=I_s(|r|),\qquad I_s(\rho)=\frac{1}{c_{d,s}}\frac{1}{\rho^{d-2s}},\qquad c_{d,s}=\pi^{d/2}2^{2s}\frac{\Gamma(s)}{\Gamma((d-2s)/2)}\tag{25}$$

solves equation (21) with $\lambda=2D_1 s(d-2s)$.

Proof of Lemma 2 We will prove something more general here by studying instead the covariance

$$D(0)-D(r)=D_1\left[I+\left(\frac{\zeta}{d-1}\right)\left(I-\hat r\otimes\hat r\right)\right]|r|^\zeta\tag{26}$$

with the statement of the lemma following as a corollary with $\zeta=2$. Let $z=|r|$. We seek a radial kernel $G(r):=\mathcal{G}(z)$. On such a function, we have

$$\nabla_r\otimes\nabla_r G=\nabla_r\otimes(\hat r\mathcal{G}'(z))=\frac{1}{z}(I-\hat r\otimes\hat r)\mathcal{G}'(z)+\hat r\otimes\hat r\mathcal{G}''(z).$$

Note that $(I-\hat r\otimes\hat r):I=(I-\hat r\otimes\hat r):(I-\hat r\otimes\hat r)=d-1$ so

$$\begin{aligned}\big(D(0)-D(r)\big):(I-\hat r\otimes\hat r)&=(d-1+\zeta)D_1|r|^\zeta,\\\big(D(0)-D(r)\big):\hat r\otimes\hat r&=D_1|r|^\zeta.\end{aligned}$$

Thus we have

$$\big(D(0)-D(r)\big):\nabla_r\otimes\nabla_r\mathcal{G}=D_1\Big[((d-1)+\zeta)z\mathcal{G}'(z)+z^2\mathcal{G}''(z)\Big]z^{\zeta-2}.$$

Thus, equating this to $-\lambda\mathcal{G}$, we find that the kernel must solve

$$z^2\mathcal{G}''(z)+((d-1)+\zeta)z\mathcal{G}'(z)+\frac{\lambda}{D_1}z^{2-\zeta}\mathcal{G}(z)=0.\tag{27}$$

We seek a solution of the type $\mathcal{G}(z)=z^\alpha\ln^\beta(z)$. Demanding the ansatz be a solution, we require

$$0=\left[\alpha^2+\left(d+\zeta-2\right)\alpha+\frac{\lambda}{D_1}z^{2-\zeta}\right]\ln^2(z)+\beta\Big(d+\zeta-2+2\alpha\Big)\ln(z)+\beta(\beta-1).$$

Note that for this ansatz to be a solution, we require $\beta = 0$ or $\beta = 1$. Thus, we have the cases

(a) If $\lambda = 0$, then

$$\alpha = 2 - d - \zeta, \qquad \beta = 0, \tag{28}$$

giving a statistically conserved quantity.

(b) If $\beta = 0$ and $\lambda \neq 0$ then

$$\zeta = 2, \qquad \alpha = -\frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 - \frac{\lambda}{D_1}}, \tag{29}$$

provided that $\frac{\lambda}{D_1} \leq \left(\frac{d}{2}\right)^2$.

(c) If $\beta = 1$ then

$$\zeta = 2, \qquad \alpha = -\frac{d}{2}, \qquad \frac{\lambda}{D_1} = \left(\frac{d}{2}\right)^2. \tag{30}$$

The stated conclusion follows easily as a special case of (b). □

Recall the characterization of the $\dot{H}^{-s}$ semi-norm, which is a consequence of Plancherel's identity:

Lemma 3 *[[25, 30]] Let $d \geq 2$, $s \in (0, d/2)$. Then $\langle h, I_s * h\rangle_{L^2} = \|h\|^2_{\dot{H}^{-s}}$ where I_s is the Riesz potential.*

With this, Theorem 1 follows immediately from Lemmas 1, 2 and 3.

2.1 Two-Particle Dispersion

We now show that similar arguments as those used in Lemma 1 can be used to prove a kind of decay estimate for the law of the two-point Lagrangian flow associated to (1), i.e.

$$\begin{aligned} \mathrm{d}X_t(x) &= \sum_{n=0}^{\infty} \sigma^{(n)}(X_t(x)) \circ \mathrm{d}W_t^{(n)}, \\ \mathrm{d}X_t(y) &= \sum_{n=0}^{\infty} \sigma^{(n)}(X_t(y)) \circ \mathrm{d}W_t^{(n)}, \end{aligned} \tag{31}$$

with $X_0(x) = x$, $X_0(y) = y$. Since the velocity field is spatially homogeneous, we need only study the law $\mu_t = \mathrm{Law}(R_t(r))$ of the separation between particles $R_t(r) = X_t(x) - X_t(y)$ with $r = x - y$. Using the spatial homogeneity of the velocity field u, one can check that μ_t is a solution of the following PDE

$$\partial_t \mu_t = (D(0) - D(r)) : \nabla_r \otimes \nabla_r \mu_t\,, \tag{32}$$

with $\mu_0 = \delta_r$. Given the above characterisation of μ_t, we have the following result.

Proposition 1 *Assume there exists a G as in the statement of Lemma 1. Then, we have*

$$\int_{\mathbb{R}^d} G(\bar{r})\, d\mu_t(\bar{r}) = \mathrm{e}^{-\lambda t} G(r)\,. \tag{33}$$

Proof The proof follows directly by testing (32) with G, integrating by parts using the fact that D is divergence-free, and then using the definition of G. □

If D is self-similarly isotropic, in view of Lemma 2, this means that for any $s \in (0, d/2)$, we have

$$\mathbb{E}[|X_t(x) - X_t(y)|^{2s-d}] = \mathrm{e}^{-\lambda_{d,s}t}|x-y|^{2s-d}, \qquad \lambda_{d,s}/D_1 = 2s(d-2s). \tag{34}$$

Thus, we obtain exact decay laws for (inverse) particle dispersion, capturing the average dispersion. In the case $s = 0$, this relation becomes the statistical conservation law $\mathbb{E}[|X_t(x) - X_t(y)|^{-d}] = |x-y|^{-d}$ derived in the works [15, 33]. This fact can be understood as akin to the statement that in a given, *linear, isotropic and incompressible* velocity field, the average $⨍_{S^{d-1}} |X_t(x) - X_t(x+r)|^{-d} \mathrm{d}\omega(\hat{r})$ is a constant of motion. See the work of Frishman et al [16].

Remark 2 [Connections to large deviations approach] Note that the quadratic dependence of the mixing rate $\lambda_{d,s}$ on s can be explained by the time-asymptotic behaviour of the distribution of finite-time stretching factor h which is defined as

$$h(t) = \frac{1}{t} \ln \frac{|X_t(x) - X_t(y)|}{|x-y|}. \tag{35}$$

It was observed in [2] that if there is enough decorrelation in time the above quantity can be thought of as a sum of i.i.d random variables and its law therefore satisfies a large deviations principle in the limit $t \to \infty$. By formal computation, one can check that

$$\mathbb{E}|X_t(x) - X_t(y)|^m \sim e^{F(m)t}, \tag{36}$$

where F is the Legendre transform of the rate function associated to the LDP. It follows that $F(2s-d) \sim \lambda_{d,s}$. Furthermore, for the Kraichnan model F is known to be quadratic (see [18, Section 2.4.3]). We refer the reader to [22, Section II] and [20] where this is discussed in more detail.

3 Top Lyapunov Exponent in the Kraichnan Model

In this section, we explicitly compute the leading Lyapunov exponent of the Lagrangian flow associated to (1). This expression was first derived by Le Jan in [26] in his study of isotropic Brownian flows. For completeness, we present this calculation below.

Proposition 2 *Consider the Lagrangian flow $\varphi_t(x)$ associated to (1), i.e. the following Stratonovich SDE*

$$\mathrm{d}\varphi_t(x) = \sum_{n=0}^{\infty} \sigma^{(n)}(\varphi_t(x)) \circ \mathrm{d}W_t^{(n)}, \tag{37}$$

with $\varphi_t(x) = x$. Furthermore, assume that D is self-similarly isotropic in the sense of Definition 1. Then

$$\log(\|D\varphi_t\|) = dD_1 t + M_t, \tag{38}$$

where M_t is the martingale defined by (46). The top Lyapunov exponent, defined almost surely, is

$$\lambda_1 = \lim_{t\to\infty} \frac{\log(\|D\varphi_t\|)}{t} = dD_1. \tag{39}$$

Remark 3 [Log-normal statistics of tracer separations] Equation (38) shows that $\|D\varphi_t\|$ follows a log-normal distribution. This fact is well-known and in fact goes back to Kraichnan's seminal work [24]. It is here derived as a consequence of the martingale central-limit theorem.

Proof Let us define by $A_{t,x} := (D\varphi_t)(x)$. It is straightforward to check that

$$\mathrm{d}A_{t,x} = \sum_{n=0}^{\infty}(D\sigma^{(n)})(\varphi_t(x)) \cdot A_{t,x}\mathrm{d}W_t^{(n)} . \tag{40}$$

Indeed, since $\sigma^{(n)}$ are divergence-free, the Itô and Stratonovich form of SDE (37) coincide. Pick $v \in \mathbb{S}^{d-1}$ and define $v_{t,x} := A_{t,x}v$. We then have that

$$\mathrm{d}v_{t,x} = \sum_{n=0}^{\infty}(D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x}\,\mathrm{d}W_t^{(n)} . \tag{41}$$

Applying Itô's formula, we obtain

$$\mathrm{d}|v_{t,x}|^2 = \sum_{n=0}^{\infty}|(D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x}|^2\,\mathrm{d}t + 2\sum_{n=0}^{\infty}\langle v_{t,x}, (D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x}\rangle\,\mathrm{d}W_t^{(n)} . \tag{42}$$

Note that the drift term in the above expression can be simplified as follows

$$\begin{aligned}\sum_{n=0}^{\infty}|(D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x}|^2 &= \sum_{n=0}^{\infty}\sum_{i=1}^{d}\left|\sum_{j=1}^{d}\partial_j\sigma_i^{(n)}v_{t,x}^j\right|^2 = \sum_{n=0}^{\infty}\sum_{i=1}^{d}\sum_{j,k=1}^{d}\partial_j\sigma_i^{(n)}\partial_k\sigma_i^{(n)}v_{t,x}^j v_{t,x}^k\\ &=\langle v_{t,x}, C \cdot v_{t,x}\rangle ,\end{aligned} \tag{43}$$

with the matrix C given by $C_{jk}(x) := \sum_{i=1}^{d}\partial_{x_j}\partial_{x_k'}D^{ii}(x-x')\Big|_{x=x'}$. Using the expression for D given by (13), we obtain $\partial_{x_j}\partial_{x_k'}D^{ii}(x-x') = 2D_1\delta_{jk}\left(\frac{d+1}{d-1} - \frac{2}{d-1}\delta_{ik}\right)$. Using the above, (42) reduces to

$$\mathrm{d}|v_{t,x}|^2 = 2D_1(d+2)|v_{t,x}|^2\,\mathrm{d}t + 2\sum_{n=0}^{\infty}\langle v_{t,x}, (D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x}\rangle\,\mathrm{d}W_t^{(n)} . \tag{44}$$

Applying Itô's formula to the above expression again, we arrive at

$$\mathrm{d}\log|v_{t,x}| = \frac{1}{2}\mathrm{d}\log|v_{t,x}|^2 = D_1(d+2)\,\mathrm{d}t - \frac{1}{|v_{t,x}|^4}\sum_{n=0}^{\infty}\left|\langle v_{t,x}, (D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x}\rangle\right|^2\,\mathrm{d}t + \mathrm{d}M_t , \tag{45}$$

where the martingale term M_t is given by

$$M_t = \int_0^t \frac{1}{|v_{t,x}|^2}\sum_{n=0}^{\infty}\langle v_{t,x}, (D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x}\rangle\,\mathrm{d}W_t^{(n)} . \tag{46}$$

We first simplify the drift term in (45) as follows

$$\begin{aligned}\sum_{n=0}^{\infty}\left|\langle v_{t,x}, (D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x}\rangle\right|^2 &= \sum_{n=0}^{\infty}\left|\sum_{j,k=1^d} v_{t,x}^k\partial_j\sigma_k^{(n)}v_{t,x}^j\right|^2\\ &= \sum_{n=0}^{\infty}\sum_{j,k,\ell,m=1}^{d} v_{t,x}^k v_{t,x}^\ell\partial_j\sigma_k^{(n)}\partial_\ell\sigma_m^{(n)}v_{t,x}^j v_{t,x}^m .\end{aligned}$$

Note now that $\sum_{n=0}^{\infty} \partial_j \sigma_k^{(n)} \partial_\ell \sigma_m^{(n)} = \partial_{x_j} \partial_{x'_\ell} D^{km}(x - x')\Big|_{x=x'}$. Using (13) again, we obtain

$$\partial_{x_j} \partial_{x'_\ell} D^{km}(x - x') = 2D_1 \left(\delta_{km}\delta_{j\ell} \left(\frac{d+1}{d-1} \right) - \frac{1}{d-1} (\delta_{kj}\delta_{m\ell} + \delta_{mj}\delta_{k\ell}) \right). \tag{47}$$

From (45), we obtain $\mathrm{d} \log |v_{t,x}| = D_1 d \, \mathrm{d}t + \mathrm{d}M_t$, whence (38) follows. For (39), we note that

$$\mathbb{E}(M_t^2) = \sum_{n=0}^{\infty} \int_0^t \mathbb{E} \left(\frac{1}{|v_{t,x}|^4} |\langle v_{t,x}, (D\sigma^{(n)})(\varphi_t(x)) \cdot v_{t,x} \rangle|^2 \right) \mathrm{d}t = 2D_1 t. \tag{48}$$

It follows then that $t^{-1} M_t$ is a supermartingale, and so by Doob's martingale convergence theorem and the above bound, it converges almost surely to 0. This clearly implies (39), since for any $v \in \mathbb{S}^{d-1}$, we have

$$\lim_{t \to \infty} \frac{1}{t} \log |v_{t,x}| = D_1 d \,.$$

This completes the proof. □

Acknowledgements We thank G. Eyink, A. Frishman and S. Punshon-Smith and the anonymous referee for useful comments. The research of MCZ was partially supported by the Royal Society URF\R1\191492 and EPSRC Horizon Europe Guarantee EP/X020886/1. The research of TDD was partially supported by the NSF DMS-2106233 grant and NSF CAREER award #2235395. The research of RSG was partially supported by the Deutsche Forschungsgemeinschaft through the SPP 2410/1 *Hyperbolic Balance Laws in Fluid Mechanics: Complexity, Scales, Randomness*. RSG's research took place within the scope of the NCCR SwissMAP which was funded by the Swiss National Science Foundation (grant number 205607).

Declarations

Conflict of interest There are no conflict of interest.

References

1. Alberti, Giovanni, Crippa, Gianluca, Mazzucato, Anna L.: Exponential self-similar mixing by incompressible flows,: 0894–0347,1088–6834. J. Am. Math. Soc. **32**(2), 445–490 (2019). https://doi.org/10.1090/jams/913
2. Balkovsky, E., Fouxon, A.: Universal long-time properties of lagrangian statistics in the batchelor regime and their application to the passive scalar problem. Phys. Rev. E **60**(4), 4164 (1999)
3. Bedrossian, J., Blumenthal, A., Punshon-Smith, S.: Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection-diffusion by stochastic Navier-Stokes. Probab. Theory Relat. Fields **179**(3–4), 777–834 (2021). https://doi.org/10.1007/s00440-020-01010-8
4. Bedrossian, J., Blumenthal, A., Punshon-Smith, S.: Almost-sure exponential mixing of passive scalars by the stochastic Navier-Stokes equations,: 0091–1798,2168–894X. Ann. Probab. **50**(1), 241–303 (2022). https://doi.org/10.1214/21-aop1533
5. Bernard, D., Gawedzki, K., Kupiainen, A.: Slow modes in passive advection. J. Stat. Phys. **90**(3–4), 519–569 (1998). https://doi.org/10.1023/A:1023212600779
6. Blumenthal, A., Zelati, C., Michele, G., Rishabh, S.: Exponential mixing for random dynamical systems and an example of Pierrehumbert: exponential mixing for random dynamical systems and an example of Pierrehumbert. Ann. Probab. **51**(4), 1559–1601 (2023). https://doi.org/10.1214/23-aop1627
7. Cardy, J., Falkovich, G., Gawedzki, K., Nazarenko, S., Zaboronski, O.V.: Non-equilibrium statistical mechanics and turbulence, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2008, 355, 978-0-521-71514-0, Lectures from the London Mathematical Society (LMS) Summer School held as part of the Warwick Turbulence Symposium at the University of Warwick, Warwick, 2568424, (2006) https://doi.org/10.1017/CBO9780511812149
8. Coghi, M., Maurelli, M.: Existence and uniqueness by Kraichnan noise for 2D Euler equations with unbounded vorticity, arXiv preprint arXiv:2308.03216 (2023)

9. Cooperman, W.: Exponential mixing by shear flows, arXiv e-prints, arXiv:2206.14239, 2206.14239 (2022)
10. Elgindi, T.M., Liss, K., Mattingly, J.C.: Optimal enhanced dissipation and mixing for a time-periodic, Lipschitz velocity field on T^2, arXiv e-prints, arXiv:2304.05374, 2304.05374 (2023)
11. Elgindi, Tarek M., Zlatoš, A.: Universal mixers in all dimensions. Adv. Math. **356**(106807), 33 (2019). https://doi.org/10.1016/j.aim.2019.106807
12. Eyink, Gregory, Jafari, A.: High schmidt-number turbulent advection and giant concentration fluctuations. Phys. Rev. Res. **4**(2), 023246 (2022)
13. Eyink, G., Jafari, A.: The Kraichnan model and non-equilibrium statistical physics of diffusive mixing. In: Annales Henri Poincaré 2024 (Vol. 25, No. 1, pp. 497–516). Cham: Springer International Publishing
14. Falkovich, G., Gawedzki, K., Vergassola, M.: Particles and fields in fluid turbulence. Rev. Mod. Phys. **73**(4), 913–975 (2001). https://doi.org/10.1103/RevModPhys.73.913
15. Falkovich, G., Frishman, A.: Single flow snapshot reveals the future and the past of pairs of particles in turbulence. Phys. Rev. Lett. **110**(21), 214502 (2013)
16. Frishman, A., Boffetta, G., De Lillo, F., Liberzon, A.: Statistical conservation law in two-and three-dimensional turbulent flows. Phys. Rev. E **91**(3), 033018 (2015)
17. Gawedzki, K.: Turbulence under a magnifying glass. Quant. Fields Quant. Space Time, 123–150 (1997)
18. Gawedzki, K.: Easy turbulence. arXiv preprint chao-dyn, 9907024/ (1999)
19. Gawedzki, K.: Soluble models of turbulent advection, lectures given at the workshop “random media 2000”. Madralin by Warsaw. arXiv preprint nlin, 0207058/ (2000)
20. Cardy, J., Falkovich, G., Gawedzki, K.: Soluble models of turbulent transport, Non-equilibrium statistical mechanics and turbulence. London Math. Soc. Lecture Note Ser., 355, Cambridge University Press, Cambridge, 44–107:2498207 (2008)
21. Gess, B., Yaroslavtsev, I.: Stabilization by transport noise and enhanced dissipation in the Kraichnan model, arXiv e-prints. arXiv:2104.03949, 2104.03949 (2021)
22. Haynes, P.H., Vanneste, J.: What controls the decay of passive scalars in smooth flows? Phys. Fluids **17**(9), 097103 (2005)
23. Kraichnan, Robert H.: Small-scale structure of a scalar field convected by turbulence. Phys. Fluids **11**(5), 945–953 (1968)
24. Kraichnan, Robert H.: Statistical dynamics of two-dimensional flow. J. Fluid Mech. **67**(1), 155–175 (1975)
25. Landkof, N.S.: Foundations of modern potential theory, Springer-Verlag, New York-Heidelberg,: Translated from the Russian by A, p. 180. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band (1972)
26. Le Jan, Y.: On isotropic Brownian motions. Z. Wahrsch. Verw. Gebiete **70**(4), 609–620 (1985). https://doi.org/10.1007/BF00531870
27. Le Jan, Y., Raimond, O.: Integration, of Brownian vector fields. Ann. Probab. **30**(2), 826–873 (2002). https://doi.org/10.1214/aop/1023481009
28. Le Jan, Y., Raimond, O.: Flows, coalescence and noise. Ann. Probab. **32**(2), 1247–1315 (2004). https://doi.org/10.1214/009117904000000207
29. Oakley, B.W., Thiffeault, J.-L., Doering, C.R.: On mix-norms and the rate of decay of correlations. Nonlinearity **34**(6), 3762 (2021)
30. Silvestre, L.: Regularity of the obstacle problem for a fractional power of the Laplace operator: Comm. Pure Appl. Math. **60**(1), 67–112 (2007). https://doi.org/10.1002/cpa.20153
31. Son, D.T.: Turbulent decay of a passive scalar in the batchelor limit: exact results from a quantum-mechanical approach. Phys. Rev. E **59**(4), R3811 (1999)
32. Yao, Y., Zlatoš, A.: Mixing and un-mixing by incompressible flows,: 1435–9855,1435–9863. J. Eur. Math. Soc. (JEMS) **19**(7), 1911–1948 (2017). https://doi.org/10.4171/JEMS/709
33. Zel’Dovich, Y.B., Ruzmaikin, A.A., Molchanov, S.A., Sokoloff, D.D.: Kinematic dynamo problem in a linear velocity field. J. Fluid Mech. **144**, 1–11 (1984)