

Sparse Structure Design for Stochastic Linear Systems via a Linear Matrix Inequality Approach

Yi Guo¹, *Member, IEEE*, Ognjen Stanojev², *Student Member, IEEE*, Gabriela Hug³, *Senior Member, IEEE*, and Tyler Holt Summers⁴, *Member, IEEE*

Abstract—We propose a sparsity-promoting feedback control design for stochastic linear systems with multiplicative noise. The objective is to identify an optimal sparse control architecture and optimize the closed-loop performance while stabilizing the system in the mean-square sense. Our approach approximates the nonconvex combinatorial optimization problem by minimizing various matrix norms subject to the linear matrix inequality (LMI) stability condition. We present two design problems to reduce the number of actuators and the number of sensors via a low-dimensional output. A regularized linear quadratic regulator with multiplicative (LQRm) noise optimal control problem and its convex relaxation are presented to demonstrate the tradeoff between the suboptimal closed-loop performance and the sparsity degree of control structure. Case studies on power grids for wide-area frequency control show that the proposed sparsity-promoting control can considerably reduce the number of sensors and actuators without significant loss in system performance. The sparse control architecture is robust to substantial system-level disturbances while achieving mean-square stability.

Index Terms—Multiplicative noise, sparsity-promoting optimal structure design, stochastic linear systems, stochastic optimal control.

I. INTRODUCTION

DYNAMICAL systems with multiplicative noise provide rich models for many practical applications, including optimal frequency control of power grids, deployment of robot agent teams, optimal control of segmented mirrors in extremely large telescopes, and other applications in biological movement systems and aerospace engineering systems [1], [2], [3], [4]. Especially in large-scale systems, substantial system-level disturbances and uncertainties may lead to oscillation and possibly instability. One of the major challenges is to design efficient, high-performance, and robust control architectures that limit the number of actuators, sensors, and actuator-sensor communication links to reduce the complexity and cost.

Sparse control architectures are crucial to manage complexity in emerging complex systems but require a solution

to complicated mixed combinatorial optimization problems. Several recent efforts have demonstrated that sparse controller architectures can successfully balance closed-loop performance and controller complexity [5], [6], [7], [8], [9], [10], [11], [12], [13]. One line of research formulated convex structured optimal control problems for controller design, such as symmetric modifications [5], [14], [15], diagonal modifications for optimal sensor and actuator selection [16], and a linear matrix inequality (LMI) approach with ℓ_1 -optimization [17]. Another line of research employed an algorithmic approach to solve the convex problems, such as the alternating direction method of multipliers [6], [16], the proximal gradient, and Newton methods [18], and also the second-order method of multipliers for efficiently identifying the controller structure and its structured feedback synthesis [19]. However, none of these works consider the multiplicative noise, which are normally caused by model-based time-varying stochasticity or the inherent uncertainties within input–output communication channels. The system-level disturbances inherently appear on the system parameters themselves and have fundamentally different effects on the state evolution than additive noise. Particularly, a noise-ignorant classical optimal linear-quadratic controller may destabilize a stochastic system with multiplicative noise in the mean-square sense [20], [21].

The stochastic linear system model with multiplicative noise has shown its advantages in the controller design with robustness to the inherent state model-based disturbances, and input channel-based uncertainties [22], [23]. Many recent works proposed various analyses on controlling and filtering the systems with multiplicative noise, including LMI approach [24], the Riccati difference equation method [25], and the game theory approach [26]. However, all of this work uses fully centralized control architectures, which become impractical and expensive as scale and complexity increase. These limitations of fully centralized architectures motivate optimal control architecture design in stochastic linear systems with multiplicative noise.

In this brief, we propose a sparsity-promoting optimal controller design for stochastic linear systems with multiplicative noise. Instead of performing a computationally expensive combinatorial search, our approach leverages the convexity of various sparsity-promoting matrix norms to encourage the feedback control matrix, while stabilizing the systems via LMI constraints. We first present two sparsity exploration problems to reduce the number of actuators and the number of sensors via a low-dimensional output. The solutions of these two problems identify a small number of sensors and actuators required to stabilize the stochastic system with multiplicative noise. We then formulate a regularized linear quadratic regulator with multiplicative (LQRm) noise optimal control problem and

Manuscript received 9 December 2023; accepted 9 March 2024. Date of publication 20 March 2024; date of current version 25 June 2024. This work was supported in part by the ETH Zürich Postdoctoral Fellowship, in part by the United States Air Force Office of Scientific Research under Grant FA2386-19-1-4073 and Grant FA9550-23-1-0424, and in part by the National Science Foundation under Award ECCS-2047040. Recommended by Associate Editor N. Quijano. (*Corresponding author: Yi Guo.*)

Yi Guo, Ognjen Stanojev, and Gabriela Hug are with the Department of Information Technology and Electrical Engineering, ETH Zürich, 8092 Zürich, Switzerland (e-mail: guo@eeh.ee.ethz.ch; stanojev@eeh.ee.ethz.ch; hug@eeh.ee.ethz.ch).

Tyler Holt Summers is with the Department of Mechanical Engineering, The University of Texas at Dallas, Richardson, TX 75080 USA (e-mail: tyler.summers@utdallas.edu).

Digital Object Identifier 10.1109/TCST.2024.3377509

1063-6536 © 2024 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See <https://www.ieee.org/publications/rights/index.html> for more information.

write it as a semidefinite programming (SDP) problem. This formulation aims to yield the tradeoffs between the system performance and sparsity degree of the control architecture by different sparsity measures. Finally, we apply our approach to design a sparse wide-area frequency control structure in power grids. The numerical results on a four-bus system show that the control structure can be quite sparse at the expense of a slight loss in performance. We also test the computational affordability of our approach on an IEEE 39-bus network to visualize the tradeoffs under various strengths of multiplicative noise. The noise-aware sparse structure requires more actuators to stabilize the system in the mean-square sense than the noise-ignorant design, which emphasizes the necessity of having this optimal structure approach for dynamical systems with significant system-level disturbances.

II. PROBLEM FORMULATION

A. Stochastic Linear Systems With Multiplicative Noise

Consider stochastic linear systems with state- and input-dependent multiplicative noises¹

$$dx_t = (A_0 x_t + B_0 u_t)dt + \sum_{i=1}^k \sigma_i A_i x_t d\beta_{it} + \sum_{j=1}^{\ell} \rho_j B_j u_t d\delta_{jt} \quad (1)$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ denotes the control input vector, $\beta_{it} (i = 1, \dots, k)$, and $\delta_{jt} (j = 1, \dots, \ell)$ denote the disturbances. We assume these disturbances to be zero mean uncorrelated stationary normalized Wiener processes. The following properties hold:

$$\begin{aligned} \mathbf{E}[d\beta_{it}] &= 0, \quad \mathbf{E}[d\delta_{jt}] = 0, \quad \mathbf{E}[d\beta_{it}^2] = dt, \quad \mathbf{E}[d\delta_{jt}^2] = dt \\ \mathbf{E}[d\beta_{it} d\beta_{i_2 t}] &= 0 (i_1 \neq i_2), \quad \mathbf{E}[d\delta_{j_1 t} d\delta_{j_2 t}] = 0 (j_1 \neq j_2) \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}[d\beta_{it} d\delta_{jt}] &= 0 \\ \forall i, i_1, i_2 &= 1, \dots, k, \text{ and } \forall j, j_1, j_2 = 1, \dots, \ell. \end{aligned}$$

The scale factors σ_i and ρ_j indicate the intensities of the disturbances, which scale the unit variance of $d\beta_{it}$ and $d\delta_{jt}$. The constant system matrices are $A_0 \in \mathbb{R}^{n \times n}$ and $B_0 \in \mathbb{R}^{n \times m}$. The state diffusion term projects state-dependent noise $d\beta_{it}$ by matrix $A_i \in \mathbb{R}^{n \times n}$, and the input diffusion term projects input-dependent noise $d\delta_{jt}$ by matrix $B_j \in \mathbb{R}^{n \times m}$. Assuming the dynamic system (1) is open-loop mean-square unstable, we apply the sparse methodology in the design of optimal linear feedback control while stabilizing the system in the mean-square sense. We first present a LMI condition for the mean-square stability of (1) and then discuss how to explore a subset of actuators or sensors by trading off the system performance under various degrees of sparsity.

¹Notation, we use $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the sets of real numbers, nonnegative real numbers, and positive real numbers, respectively. Sets $\mathbb{S}^n$, $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$ collect all n -dimension symmetric matrices, semidefinite positive matrices and positive definite matrices, respectively. Given a matrix M , $M^\top$ denotes its transpose and $\text{Tr}(M)$ denotes its trace. We write $M \succeq 0$ ($M \succ 0$) to denote that M is semipositive definite (positive definite). For a given column vector $x \in \mathbb{R}^n$, we define $\|x\|_1 := \sum_i |x_i|$, $\|x\|_2 := \sqrt{x^\top x}$ and $\|x\|_\infty := \max_i |x_i|$. Further, $|\cdot|$ denotes the absolute value of a number, $\text{diag}(\cdot)$ constructs a diagonal matrix from a vector, and $\text{blkdiag}(\cdot)$ returns a block diagonal matrix. At last, I denotes the identity matrix of the appropriate dimension. We use the cardinality function $\text{card}(\cdot)$ to quantify the number of nonzero elements of a matrix.

Definition 1: The system (1) is *mean-square stable* if for every initial condition $\mathbf{E}[x_0 x_0^\top] = \Sigma_0$ the solution of (1) satisfies

$$\lim_{t \rightarrow +\infty} \mathbf{E}[x_t^\top x_t] = 0.$$

Note that the above mean-square stability condition will converge to a constant if the system (1) also has nonzero mean additive noise. We give the definition of mean-square stability for two feedback control schemes and row/column sparsity as follows.

Definition 2 (State-Feedback Mean-Square Stability): The system (1) with an initial condition $\mathbf{E}[x_0 x_0^\top] = \Sigma_0$ is called (mean-square) stabilizable if there exists a mean-square stabilizing state-feedback control of the form $u_t = K x_t$, where K is a constant matrix.

Definition 3 (Output-Feedback Mean-Square Stability): Given the output of the system (1), such that $y_t = C x_t$, the system (1) with an initial condition $\mathbf{E}[x_0 x_0^\top] = \Sigma_0$ is called (mean-square) stabilizable if there exists a mean-square stabilizing output-feedback control of the form $u_t = K y_t$, where K is a constant matrix.

Definition 4 (Row/Column Sparsity): A matrix $K \in \mathbb{R}^{n \times m}$ is called row-sparse (column-sparse) if there are rows (columns) where all elements are zero.

B. Stabilization With a Reduced Number of State-Feedback Controllers

Assume the system (1) is open-loop mean-square unstable and stabilizable via the state-feedback control, the goal of this section is to identify potential row-sparsity patterns of the closed-loop state-feedback control law in the form $u_t = K x_t$, such that the closed-loop system described by

$$\begin{aligned} dx_t &= (A_0 + B_0 K) x_t dt + \sum_{i=1}^k \sigma_i A_i x_t d\beta_{it} \\ &\quad + \sum_{j=1}^{\ell} \rho_j B_j K x_t d\delta_{jt} \end{aligned} \quad (2)$$

is mean-square stable. To guarantee the mean-square stability of the closed-loop system (2), the static state-feedback control gain matrix K exists if and only if there exists a matrix $X \in \mathbb{S}_{++}^n$ such that the following condition holds [22]:

$$\begin{aligned} (A_0 + B_0 K)^\top X + X(A_0 + B_0 K) &+ \sum_{i=1}^k \sigma_i^2 A_i^\top X A_i \\ &+ \sum_{j=1}^{\ell} \rho_j^2 K^\top B_j^\top X B_j K < 0. \end{aligned}$$

Pre- and postmultiplying the above inequality by $P = X^{-1}$ and introducing a new variable $Y = KP$, we arrive at the following condition:

$$\begin{aligned} A_0 P + P A_0^\top + B_0 Y + Y^\top B_0^\top &+ \sum_{i=1}^k \sigma_i^2 (A_i P)^\top P^{-1} A_i P \\ &+ \sum_{j=1}^{\ell} \rho_j^2 (B_j Y)^\top P^{-1} B_j Y < 0 \end{aligned} \quad (3)$$

where $P = P^\top \in \mathbb{S}_{++}^n$ and $Y \in \mathbb{R}^{m \times n}$ are the matrix variables. A stabilizing state-feedback controller can be reconstructed by $K = YP^{-1}$. Leveraging the Schur's Lemma, we transform the condition (3) together with $P \succ 0$ into a LMI

$$\begin{bmatrix} A_0P + PA_0^\top + B_0Y + Y^\top B_0^\top & Z \\ Z^\top & Z_P \end{bmatrix} \prec 0 \quad (4)$$

where $Z = [\sigma_1 PA_1^\top, \dots, \sigma_k PA_k^\top, \rho_1 Y^\top B_1^\top, \dots, \rho_\ell Y^\top B_\ell^\top]$ and

$$Z_P = \text{blkdiag}(\underbrace{-P, \dots, -P}_{k+\ell}).$$

If Y is row sparse, then the state-feedback law K is row sparse as well since postmultiplication preserves the zero-row structure. Hence, we promote row sparsity of Y through the following SDP:

$$\min_{Y, P} \|Y\|_r, \quad \text{s.t. (4) and } P \succ 0 \quad (5)$$

where $\|Y\|_r$ represents a generic row-sparsity induced function that can be chosen from row-norm [17], group LASSO [27] and sparse Group LASSO [27]. Note that the stabilizing state-feedback control matrix K_{rsp} with an identified row sparse pattern can be obtained from the solution P_{rsp}^* , Y_{rsp}^* of the above SDP problem, with the linear feedback control matrix calculated as $K_{\text{rsp}} = Y_{\text{rsp}}^* P_{\text{rsp}}^{*-1}$. Note that having additional additive noise in the system dynamic does not change the overall sparse control design process as long as the additive noise has zero mean and finite covariance. Under these assumptions, an additional term will appear in (3) as a function of the covariance of the additive noise. The modified LMI will be included in the proposed sparsity-promoting problems in the rest of the brief. We refer interested readers to [28] for more detailed information.

C. Stabilization With a Reduced Number of Output-Feedback Controllers via a Low-Dimensional Output

In this section, we present a stabilization solution to reduce the number of output-feedback controllers $u_t = Ky_t$ via a low-dimensional output $y_t = Cx_t$, where $y_t \in \mathbb{R}^n$ is the output vector and $C \in \mathbb{R}^{n_y \times n}$ is the output matrix. Assume the system (1) has the exact measurements of the full states and is stabilizable via an output-feedback control law K . Note that the potential stabilization solutions via output-feedback controllers highly depend on the C matrix. Here, we assume that there is at least one low-dimensional output that can enable the sparse control structure. The goal of this section is to obtain a low-dimensional system output matrix C and a column-row sparse output-feedback matrix K to stabilize (1). To have a low-dimensional output matrix C and a column sparse matrix K , we first change the row sparsity promoting function $\|Y\|_r$ in (5) to a generic column sparsity induced norm $\|Y\|_c$ and come to

$$\min_{Y, P} \|Y\|_c, \quad \text{s.t. (4) and } P \succ 0. \quad (6)$$

Similar to the row sparse state-feedback law from (5), the solution of (6) promotes the column sparsity on Y_c^* . Note that the feedback law is in the form

$$u_t = Y_{\text{csp}}^* P_{\text{csp}}^{*-1} x_t \quad (7)$$

where Y_{csp}^* and P_{csp}^* are the solution of (6). Interestingly, the sparsity pattern of the output-feedback $u_t = K_{\text{csp}} y_t$ can be attained by mapping the matrix multiplication of the term $Y_{\text{csp}}^* P_{\text{csp}}^{*-1}$ in (7) and the term $K_{\text{csp}} C_{\text{rsp}}$ in

$$u_t = K_{\text{csp}} y_t = K_{\text{csp}} C_{\text{csp}} x_t.$$

The output-feedback law K_{csp} consists of nonzero-columns of Y_{csp}^* and the output matrix C_{rsp} takes the rows of P_{csp}^{*-1} with same indices. In this way, we reduce the number of outputs in $y_t = C_{\text{csp}} x_t$ while stabilizing the system by a column sparse output-feedback $u_t = K_{\text{csp}} y_t$.

After we identify the column sparsity pattern of the output-feedback law K_{csp} , we continue to explore the potential row sparsity to reduce the number of the output-feedback controllers. Having the knowledge of potential column sparsity of variable Y by solving (6), we integrate the identified zero-column pattern of the solution Y_{csp}^* as additional constraints into (5) and come to

$$\min_{Y, P} \|Y\|_r \quad (8a)$$

$$\text{s.t. } Y_{c,i} = 0 \quad \forall i \in \mathcal{C} \quad (8b)$$

$$P \succ 0 \text{ and (4)} \quad (8c)$$

where $Y_{c,i}$ indicates the i th column of the variable Y and the set $\mathcal{C}$ collects the indices of all zero columns of Y_{csp}^* . The solution of (8) are Y_{sp}^* and P_{sp}^* . We adopt the output-feedback law K_{sp} as the same row-column sparsity of Y_{sp}^* and the output matrix C_{sp} consists of the zero rows of P_{sp}^{*-1} . The feedback control structure $u_t = K_{\text{sp}} y_t$ has the row-sparsity such that the corresponding controllers can be removed with a low-dimensional output $y_t = C_{\text{sp}} x_t$. By solving these two sparsity-promoting problems (6) and (8) in sequence, we can identify the potential sparsity structure to design low-dimensional outputs and remove unnecessary output-feedback controllers for stabilizing. Clearly, the sparse control structure with a subset of outputs and controllers will reduce system performance. We will discuss the performance degradation and sparsity tradeoffs in the rest of this brief.

Remark 1 (Sparsity-Promoting Norms): Given a matrix $Y \in \mathbb{R}^{m \times n}$, the row and column sparsity can be induced by various sparsity-promoting matrix norms, respectively, defined as [17]

$$\|Y\|_{\text{row}} = \sum_{i=1}^m \|Y_{r,i}\|_\infty, \quad \|Y\|_{\text{col}} = \sum_{j=1}^n \|Y_{c,j}\|_\infty$$

where $\|Y_{r,i}\|_\infty$ and $\|Y_{c,j}\|_\infty$ are the maximum absolute values of the i th row and j th column of matrix Y , respectively. Row and column sparsity can also be induced by the row and column group LASSO [27]

$$\|Y\|_{\text{rGL}} = \sum_{i=1}^m \|Y_{r,i}\|_2, \quad \|Y\|_{\text{cGL}} = \sum_{j=1}^n \|Y_{c,j}\|_2$$

where $\|Y_{r,i}\|_2$ and $\|Y_{c,j}\|_2$ are the vector ℓ_2 -norms of the i th row and j th column of matrix Y , respectively. The row and column sparse group LASSO [27] can also promote the

sparsity pattern

$$\begin{aligned}\|Y\|_{\text{rSGL},\mu} &= \sum_{i=1}^m (1-\mu)\|Y_{r,i}\|_1 + \mu\|Y_{r,i}\|_2 \\ \|Y_c\|_{\text{cSGL},\mu} &= \sum_{j=1}^n (1-\mu)\|Y_{c,j}\|_1 + \mu\|Y_{c,j}\|_2\end{aligned}$$

where $\|Y_{r,i}\|_1$ and $\|Y_{c,j}\|_1$ are the vector ℓ_1 -norms of the i th row and j th column of matrix Y , respectively. The prescribed constant $\mu \in [0, 1]$ quantifies the weight on two combined norms. In the rest of this brief, we refer to $\|Y\|_{\text{reg}}$ as a generic sparsity-promoting regularizer in the following optimal design formulation.

It is worth emphasizing that the above sparse structure design problems are also applicable to the systems with open-loop mean-square stability for allowing performance tradeoffs, which will be discussed in the rest of this brief. Note that the proposed sparsity-promoting problems (6) and (8) are SDP problems, which can be solved by many off-the-shelf solvers, e.g., MOSEK.

D. Tradeoff Between Performance and Degree of Sparsity

We now consider an application of our approach to the LQRm noise for the system (1) given an initial condition $\mathbf{E}[x_0 x_0^\top] = \Sigma_0$

$$\min_{u_t(\cdot)} J(\Sigma_0, u(\cdot)) = \mathbf{E} \int_0^\infty (x_t^\top Q x_t + u_t^\top R u_t) dt \quad (9a)$$

$$\begin{aligned}\text{s.t. } dx_t &= (A_0 x_t + B_0 u_t) dt + \sum_{i=1}^k \sigma_i A_i x_t d\beta_{it} \\ &+ \sum_j \rho_j B_j u_t d\gamma_{jt}\end{aligned} \quad (9b)$$

where Q and R are given positive definite. The objective is to determine an optimal linear state-feedback (output-feedback) control law to tradeoff the LQRm closed-loop performance and the sparsity of the linear feedback control gain K .² For state-feedback control $u_t = K x_t$, we are ultimately interested in a regularized LQRm problem with an alternative objective

$$\min_K J(\Sigma_0, u_t(K)) + \gamma J_{\text{reg}}(K)$$

where $J_{\text{reg}}(K)$ is a sparsity-promoting function of K and γ specifies the importance of its sparsity. Together with the stability constraint (4), the optimal control problem of determining stabilizing closed-loop state-feedback K that minimizes the LQRm cost and detecting the potential sparsity structure of K can be reformulated as the following optimization problem [28]:

$$\min_{Y,P} \text{Tr}(\Sigma_0 P^{-1}) + \gamma \|Y\|_{\text{reg}} \quad (10a)$$

$$\text{s.t. } P \succ 0 \quad (10b)$$

$$\begin{aligned}A_0 P + P A_0^\top + B_0 Y + Y B_0^\top + \sum_{i=1}^k \sigma_i P A_i^\top P^{-1} A_i P \\ + \sum_{j=1}^l \beta_j Y^\top B_j^\top P^{-1} B_j Y + Y^\top R Y + P Q P \prec 0.\end{aligned} \quad (10c)$$

²We here omit the regularized LQRm problem with static output-feedback control. The corresponding reformulation closely follows the line of the steps as state-feedback control but has the output-feedback control law $u_t = K y_t = K C x_t$ instead.

Note that (10) is intractable due to the matrix inverse of variable P^{-1} in the objective and the nonlinear multiplication in (10c). We then introduce new (slack) variables $\Pi \in \mathbb{R}^{n \times n}$ and $\kappa \in \mathbb{R}$, while minimizing the upper bound of the LQRm cost, i.e., $J = \text{Tr}(\Sigma_0 P^{-1}) \leq \kappa$. By leveraging the Schur's Lemma [28], [29], we come to the following SDP problem:

$$\min_{Y,P,\Pi,\kappa} \kappa + \gamma \|Y\|_{\text{reg}} \quad (11a)$$

$$\text{s.t. } \text{Tr}(\Pi) \leq \kappa, \begin{bmatrix} \Pi & \Sigma_0^{\frac{1}{2}} \\ \Sigma_0^{\frac{1}{2}} & P \end{bmatrix} \succeq 0, \quad P \succ 0 \quad (11b)$$

$$\begin{bmatrix} A_0 P + P A_0^\top + B_0 Y + Y^\top B_0^\top & Z & Y^\top & P \\ Z^\top & Z_P & 0 & 0 \\ Y & 0 & -R^{-1} & 0 \\ P & 0 & 0 & -Q^{-1} \end{bmatrix} \prec 0. \quad (11c)$$

The solution Y^* , P^* , Π^* and κ^* defines an sparse stabilizing law $K_{\text{sp}} = Y^* P^{*-1}$ and the upper bound of the LQRm cost $J^*(K) = \text{Tr}(\Sigma_0 P^{*-1}) \leq \kappa^*$. Note that (11) is convex and can be solved by several academic and commercial SDP solvers. Our proposed method may identify a quite sparse control structure with only a small loss in performance, as demonstrated by the numerical studies in Section III.

III. APPLICATION TO POWER GRIDS

We apply our proposed methodology to devise an optimal wide-area frequency control scheme for a power transmission system. The objective is to design a sparse linear feedback frequency control architecture, which stabilizes the frequency dynamics with modeling errors as multiplicative noise. Consider a lossless transmission system [30] modeled as a graph $G = (\mathcal{N}, \mathcal{E})$ with nodes (or buses) $\mathcal{N} = \{1, \dots, N\}$ and edges (or lines) $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$. The topology of grid is represented by the Laplacian matrix $L \in \mathbb{R}^{N \times N}$ induced by the line susceptances b_{ij} for all $(i, j) \in \mathcal{E}$ (see [31]). We partition all buses into the buses with generators $\mathcal{G}$ (i.e., synchronous machines and inverter-based generators) and the buses with frequency-sensitive loads $\mathcal{L}$, where $\mathcal{N} = \mathcal{G} \cup \mathcal{L}$. The states of the network for all $i \in \mathcal{N}$ are the angle θ_i and frequency ω_i of the sinusoidal voltage signals. The associated system dynamics [32], [33] derived from linearized swing equations are given by

$$\dot{\theta}_i = \omega_i \quad \forall i \in \mathcal{N} \quad (12a)$$

$$M_i \ddot{\theta}_i + D_{g,i} \dot{\theta}_i = - \sum_{(i,j) \in \mathcal{E}} b_{ij} (\theta_i - \theta_j) + u_i \quad \forall i \in \mathcal{G} \quad (12b)$$

$$D_{l,i} \dot{\theta}_i = - \sum_{(i,j) \in \mathcal{E}} b_{ij} (\theta_i - \theta_j) + u_i \quad \forall i \in \mathcal{L} \quad (12c)$$

where $u_i \in \mathbb{R}$ is a controllable generation or load for all $i \in \mathcal{N}$. A generator bus $i \in \mathcal{G}$ is characterized by its inertia $M_i \in \mathbb{R}_{++}$ (rotational or virtual inertia) and its droop coefficient $D_{g,i} \in \mathbb{R}_{++}$. A frequency-sensitive load bus $i \in \mathcal{L}$ is characterized by its sensitivity coefficient $D_{l,i} \in \mathbb{R}_{++}$.

Here, we consider inertia variations caused by inverter-based generation or modeling errors, which are modeled by treating the inertia parameters M_i as multiplicative noise rather than simply a constant [31]. The inertia parameters for all $i \in \mathcal{G}$, M_i can be modeled as random parameters with the mean

value $\bar{M}_i$ and the variance $\bar{\sigma}_i^2$ [31]. In general, the modeling errors of the network topology L and frequency-sensitive coefficients $D_{\ell,i}/D_{g,i}$ can also be treated as multiplicative noises in (12). For simplicity, only the inertia parameter randomness is considered. The above dynamics can be formulated in a generalized form with outputs as a multiinput multioutput stochastic linear system with multiplicative noise

$$\begin{aligned} dx_t &= A_0 x_t dt + \sum_{i=1}^{|\mathcal{G}|} \sigma_i (A_i x_t + B_i u_t) d\beta_{it} \\ y_t &= C x_t, \quad x_t = [\theta_{g,t}^\top, \omega_{g,t}^\top, \theta_{l,t}^\top]^\top \end{aligned} \quad (13)$$

where

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & I & 0 \\ -M^{-1}L_{gg} & -M^{-1}D_g & -M^{-1}L_{gl} \\ -D_l^{-1}L_{lg} & 0 & -D_l^{-1}L_{ll} \end{bmatrix} \\ B_0 &= \begin{bmatrix} 0 \\ M^{-1} \\ 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 0 & 0 \\ R_i L_{gg} & R_i D_g & R_i L_{gl} \\ 0 & 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ R_i \\ 0 \end{bmatrix} \end{aligned}$$

and the vectors $\theta_{g,t} \in \mathbb{R}^{|\mathcal{G}|}/\theta_{l,t} \in \mathbb{R}^{|\mathcal{L}|}$ and $\omega_{g,t} \in \mathbb{R}^{|\mathcal{G}|}$ collect the angle states of generation/load buses and the frequency states of generation buses, respectively. Note that the inertia parameter for every generator M_i , $i \in \mathcal{G}$ appears in the state compact form (13) as the inverse distribution of M_i with the mean value $\bar{M}_i^{-1}$ and the variance σ_i^2 . The matrix $M^{-1} := \text{diag}(\bar{M}_i^{-1}) \in \mathbb{R}^{|\mathcal{G}| \times |\mathcal{G}|}$ collects the mean values of the inverse distribution of M_i . The modeling error of the inverse of inertia parameter $d\beta_i$ for all $i \in \mathcal{G}$ is considered as an independent Wiener process normalized by σ_i . The diagonal matrix $D_g \in \mathbb{R}^{|\mathcal{G}| \times |\mathcal{G}|}$ collects the droop coefficients $D_{g,i}$ at all buses $i \in \mathcal{G}$ and the diagonal matrix $D_l \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{L}|}$ collects the sensitivity coefficients $D_{l,i}$ at all buses $i \in \mathcal{L}$. The matrices $L_{gg} \in \mathbb{R}^{|\mathcal{G}| \times |\mathcal{G}|}$, $L_{lg} \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{G}|}$ and $L_{gl} \in \mathbb{R}^{|\mathcal{G}| \times |\mathcal{L}|}$, $L_{ll} \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{L}|}$ are derived from the original Laplacian matrix L , which represent the weighted connections between different types of buses (i.e., load and generation). We define an *inertia disturbance allocation* matrix $R_i \in \mathbb{R}^{N \times N}$ in A_i and B_i associated with each bus $i \in \mathcal{G}$. The elements in R_i are all zeros except for one diagonal element $r_{ii} = 1$, which maps the corresponding inertia disturbance $d\beta_i$ onto bus i . If the inertia variation at bus i is insignificant, we set $R_i = 0$ to remove the inertia disturbance on i th bus. In the rest of this brief, we conduct numerical experiments on two power networks. A small power system is used to numerically show the design results and a large-scale power system is utilized to demonstrate the computational-affordability of the proposed LMI approach.

We first test our approach on a toy example, i.e., a four-area interconnected power system. The grid topology and line parameters can be found in [31]. The damping/frequency sensitivity coefficients are set to $D_i = 10$ for all buses $i \in \mathcal{N}$. The mean value of inertia is $\bar{M}_i = 10$ for all $i \in \mathcal{G}$ and the standard deviation of the inverses distribution of M_i^{-1} is 10% of the inverse mean value, i.e., $\sigma_i = 10\% \bar{M}_i^{-1}$ for all $i \in \mathcal{G}$. Note that the open-loop dynamics (12) is not mean-square stable if the multiplicative noise variance is significant. More details of modeling and stability analysis

related to this example can be found in our previous work [31]. In this brief, we mainly focus on finding the sparse structure of the closed-loop feedback control for generators that stabilize the frequency dynamics with inertia disturbances. The initial condition of states is $\Sigma_0 = 0.1I$ and the coefficients of the LQRm cost are $Q = I$ and $R = I$.

A. Reducing the Number of State-Feedback Controllers

We first check if the stochastic linear system (13) is mean-square stabilizable (no sparsity induced) via a closed-loop state-feedback controller $u_t = Kx_t$ by solving (11) with $\gamma = 0$. The obtained solution is

$$\begin{aligned} K_0 &= \begin{bmatrix} -0.2329 & -0.0939 & -0.0236 & -0.2740 & -0.0879 & -0.0216 \\ -0.0941 & -0.3008 & -0.0823 & -0.0878 & -0.3358 & -0.0770 \\ -0.0236 & -0.0822 & -0.2075 & -0.0216 & -0.0771 & -0.2503 \end{bmatrix}. \end{aligned}$$

Having bus 4 grounded (as an infinite bus) for model reduction [31], the system has six states, three inputs and three independent multiplicative noises, and all three generators participate in stabilizing the grid. The state-feedback law K is fully populated and the LQRm cost is $\kappa_0^* = 1.7915$. To obtain a row sparse solution K for the reduced number of state-feedback controllers, we again solve the regularized LQRm (11) using the row-norm $\|Y\|_{\text{row}}$ with $\gamma = 4$, which results at a sparse structure

$$\begin{aligned} K_{\text{rsp},4} &= \begin{bmatrix} -0.0114 & -0.0153 & -0.0103 & -0.0110 & -0.0147 & -0.0099 \\ -0.0718 & -0.0960 & -0.0645 & -0.0692 & -0.0925 & -0.0622 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

This leads to a slight increase of LQRm cost $\kappa_4^* = 1.8757$. The row sparse state-feedback control law $K_{\text{rsp},4}$ indicates that the generator 3 is not necessarily required to stabilize the system but at the expense of 4.67% decrease of the closed-loop performance.

B. Reducing the Number of Output-Feedback Controllers via a Low-Dimensional Output

To obtain a sparse structure of the output-feedback control $u_t = Ky_t = KCx_t$, we first solve (11) by using the column-norm $\|Y\|_{\text{col}}$ as the sparsity regularizer with $\gamma = 0.5$. We attain the solution $Y_{\text{csp},0.5}^*$ and $P_{\text{csp},0.5}^*$, specifically

$$Y_{\text{csp},0.5}^* = \begin{bmatrix} -0.0177 & -0.0400 & 0.0078 & 0 & 0 & 0 \\ -0.0177 & -0.0439 & -0.0080 & 0 & 0 & 0 \\ 0.0074 & -0.0387 & -0.0080 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we can start to shape two matrices K_{csp} and C_{rsp} . Since we have $K = YP^{-1}$, the sparse output-feedback controller K_{csp} includes the nonzero columns of $Y_{\text{csp},0.5}^*$, resulting in to $Y_{\text{csp},0.5}^*$

$$K_{\text{csp},0.5} = \begin{bmatrix} -0.0177 & -0.0400 & 0.0078 \\ -0.0177 & -0.0439 & -0.0080 \\ 0.0074 & -0.0387 & -0.0080 \end{bmatrix}.$$

The associated 3-D output matrix consists of the first three rows of the solution $P_{\text{csp},0.5}^{*-1}$, i.e.,

$$C_{\text{rsp},0.5} = \begin{bmatrix} 2.9985 & 0.9354 & 0.2549 & 2.3444 & 0.9222 & 0.2352 \\ 0.9354 & 3.6956 & 0.8172 & 0.9155 & 3.0168 & 0.8046 \\ 0.2549 & 0.8172 & 2.7508 & 0.2349 & 0.8127 & 2.0984 \end{bmatrix}.$$

We now move further to reduce the number of output-feedback controllers by exploring the row sparsity of K_{csp} . Forcing the last three columns of Y to equal zero as additional constraints, we then solve (11) again use the row-norm regularizer and $\gamma = 2.6$. The solution $Y_{\text{sp},2.6}^*$ has a row-column sparse pattern

$$Y_{\text{sp},2.6}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0130 & -0.0130 & -0.0130 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we adopt

$$K_{\text{sp},2.6} = \begin{bmatrix} 0 & 0 & 0 \\ -0.0130 & -0.0130 & -0.0130 \\ 0 & 0 & 0 \end{bmatrix}$$

as a row-column sparse output-feedback law. The output matrix $C_{\text{sp},2.6}$ is composed of the first three rows of $P_{\text{sp},2.6}^{*-1}$

$$C_{\text{sp},2.6} = \begin{bmatrix} 3.0963 & 1.0656 & 0.3010 & 2.4546 & 1.0592 & 0.2839 \\ 1.0656 & 3.8739 & 0.9181 & 1.0653 & 3.2085 & 0.9221 \\ 0.3010 & 0.9181 & 2.8063 & 0.2843 & 0.9176 & 2.1612 \end{bmatrix}.$$

At the end, the designed output-feedback controller uses one generator with a 3-D output feedback to stabilize the system at the expense of 5.5% LQRm cost increase compared to $\kappa_0^* = 1.7915$. Overall, the design procedure of a sparse output feedback controller via a low-dimensional output can be summarized as follows.

- 1) solve the relaxed LQR problem (11) to detect a column-sparse pattern of matrix $Y_{\text{csp},\gamma}^*$ and the associated solution $P_{\text{csp},\gamma}^*$;
- 2) build the column sparse control law matrix $K_{\text{csp},\gamma}$ from $Y_{\text{csp},\gamma}^*$ and the associated low-dimensional output matrix $C_{\text{rsp},\gamma}$ from $P_{\text{csp},\gamma}^*$;
- 3) resolve the relaxed LQR problem (11) again by forcing the zero columns of Y as additional constraints;
- 4) build the row-column sparse feedback control law $K_{\text{sp},\gamma}$ from the solution $Y_{\text{sp},\gamma}$ and update the associated output matrix $C_{\text{sp},\gamma}$.

C. Tradeoff the System Performance and the Degree of Sparsity

To discuss the computational cost of our approach on a large-scale system, we use the IEEE 39-bus New England transmission system to visualize the tradeoffs between the sparsity degree of the structure and the LQRm cost under various multiplicative noise settings. This model consists of 39 buses and ten generators, where the generator 10 is an equivalent aggregated model [34]. We use the MOSEK SDP solver [35] via the MATLAB interface CVX [36] on a laptop with 16 GB memory and 2.3 GHz Intel Core i7-10510U CPU. It took 121.7 s to solve (11) with ten inputs, 47 states, and ten independent multiplicative noises. The inertia mean value is $M_i = 10$ for all generators $i \in \mathcal{G}$ and the damping/frequency sensitivity coefficient is set to $D_i = 10$ for all buses $i \in \mathcal{N}$. We vary the sparsity importance γ to tradeoff the sparsity degree of the state-feedback control law K and the LQRm performance. The LQRm cost coefficients are $Q = I$ and $R = I$. The initial state condition is $\Sigma_0 = 0.1I$.

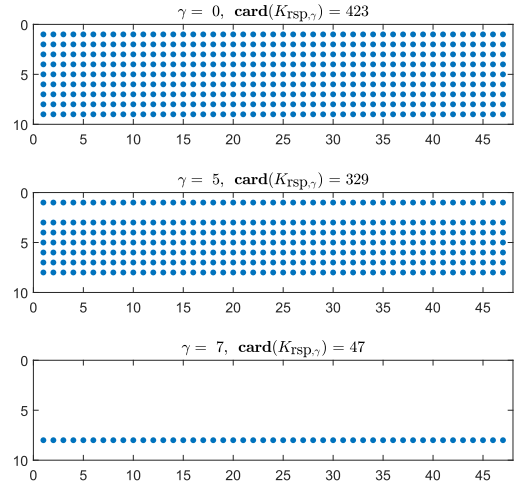


Fig. 1. Sparsity patterns of the closed-loop state-feedback K resulting from the row-norm regularizer under various γ . The white elements represent (nearly) zero entries.

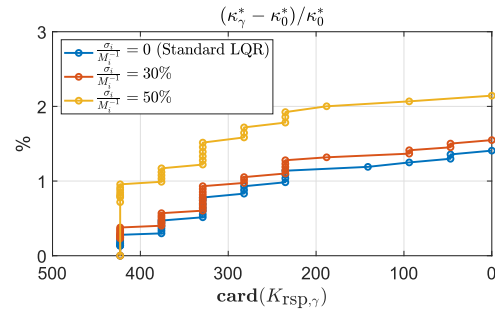


Fig. 2. Sparsity comparison under different row-sparsity regularizers.

A few of sparsity patterns under different sparsity importance scenarios (i.e., $\gamma = 0, 5$, and 7) are presented in Fig. 1. For $\gamma = 0$, the optimal feedback gain K_0 is fully populated, thereby requiring all ten generators contributing to a mean-square stabilizing solution. As γ increases, the rows of the state-feedback matrix K_γ become significantly sparse whereas the relative cost objective $(\kappa_\gamma^* - \kappa_0^*)/\kappa_0^*$ increase only slightly, see Fig. 2. In particular, for $\gamma = 5$, the identified control architecture indicates that the controllers of generators 2, 9, and 10 are not necessarily required to stabilize the system. As γ increases to seven, most of the stabilization burden is on generator 8 but only with 1.4% LQRm cost increase.

We next compare the LQRm cost with various levels of multiplicative noise shown in Fig. 2. As the standard deviation σ of the multiplicative noise increases, the LQRm cost becomes larger since more control effort is required for stabilizing the system-level disturbances. In addition, two sparsity-promoting structures of the noise-aware ($\sigma_i/\bar{M}_i^{-1} = 50\%$) and noise-unaware ($\sigma = 0$) state-feedback controllers $K_{\text{rsp},\gamma}$ are given in Fig. 3. We observe that more generators need to be included for stabilizing due to the significant system-level disturbance. This also implies that a noise-unaware state-feedback controller may fail to stabilize a stochastic linear system with substantial multiplicative noise in a mean-square sense. In practice, the multiplicative noise inherently comes with the linearized system models and noisy control channels.

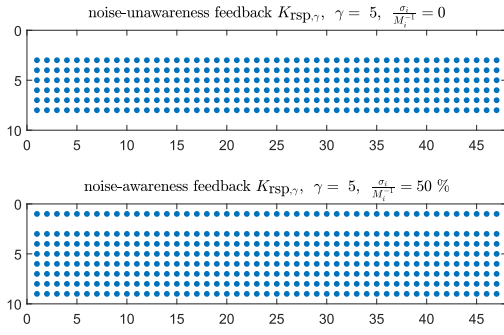


Fig. 3. Sparsity patterns of the noise-unaware and noise-aware state-feedback control law $K_{\text{rsp},\gamma}$.

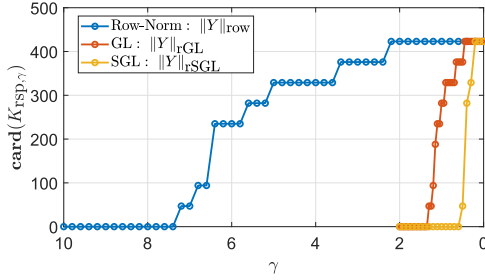


Fig. 4. Sparsity comparison under different row-sparsity regularizers with $\sigma_i/M_i = 10\%$.

This emphasizes the importance and necessity of having a noise-aware sparse architecture design to improve robustness to system-level disturbances for a mean-square stabilizing solution. In the end, we validate our approaches with various row sparsity-induced norms, such as the row-norm, the group LASSO, and the sparse group LASSO, see Fig. 4. All of three sparsity-promoting norms successfully induce various sparsity patterns under different γ . The sparse group LASSO (with $\mu = 0.5$) and the group LASSO lead to more aggressive sparsity patterns than the row-norm regularize.

Overall, we conclude that our approach successfully provides a sparse control solution, which reduces the number of controllers (with low-dimensional outputs) only at the expense of a small decrease of system performance. This proposed LMI approach is convex and computationally affordable for a large-scale dynamic system.

IV. CONCLUSION

This brief proposed a sparse feedback control architecture design for stochastic linear systems with multiplicative noise. We minimize the sparsity-promoting matrix norms subject to a mean-square stability LMI condition as an SDP problem to approximate the complicated and nonconvex combinatorial problem. For a large-scale dynamic system with network-wide disturbances, the designed sparse stabilizing solution successfully reduces the number of controllers, limits the unnecessary output information exchanges, and slightly trade-offs the LQRm cost. However, there remain several lines of future work that can extend the present design to various applications and understand the benefits and limitations to trade off system performance and sparsity of the control schemes. Future work could involve.

- 1) Applying the proposed approach to devise the sparse control architecture for different applications, such as water networks, traffic control, and others.
- 2) Conducting sensitivity analysis to reveal insights into the relative importance of actuators and sensors.

APPENDIX RESULTS FOR DISCRETE-TIME STOCHASTIC LINEAR SYSTEMS

In this appendix, we present the mean-square stability condition and the reformulation of the regularized LQRm problem for a discrete-time linear system with state- and input-dependent multiplicative noises. Consider a discrete-time stochastic linear system

$$x_{t+1} = \bar{A}_0 x_t + \bar{B}_0 u_t + \sum_{i=1}^k \bar{\sigma}_i \bar{A}_i x_t w_{it} + \sum_{j=1}^{\ell} \bar{\rho}_j \bar{B}_j u_t p_{jt} \quad (14)$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ denotes the control input vector, and $w_{it} (i = 1, \dots, k)$ and $p_{jt} (j = 1, \dots, \ell)$ denote the independent, identically random variables with

$$\begin{aligned} \mathbf{E}[w_{it}] &= 0, \quad \mathbf{E}[p_{jt}] = 0, \quad \mathbf{E}[w_{it}^2] = \bar{\sigma}_i^2, \quad \mathbf{E}[p_{jt}^2] = \bar{\rho}_j^2 \\ \mathbf{E}[w_{i_1,t} w_{i_2,t}] &= 0 \quad (i_1 \neq i_2), \quad \mathbf{E}[p_{j_1,t} p_{j_2,t}] = 0 \quad (j_1 \neq j_2) \end{aligned}$$

and

$$\mathbf{E}[w_{it} p_{jt}] = 0$$

$$\forall i, i_1, i_2 = 1, \dots, k, \text{ and } \forall j, j_1, j_2 = 1, \dots, \ell.$$

The scale factors $\bar{\sigma}_i$ and $\bar{\rho}_j$ indicate the standard deviation which normalize w_{it} and p_{jt} with the unit variance. The constant system matrices are $\bar{A}_0 \in \mathbb{R}^{n \times n}$ and $\bar{B}_0 \in \mathbb{R}^{n \times m}$. The state-dependent noise is allocated by matrix $\bar{A}_i \in \mathbb{R}^{n \times n}$, and the input-dependent noise is allocated by matrix $\bar{B}_j \in \mathbb{R}^{n \times m}$. The system (14) is stabilizable via the state-feedback control $u_t = K x_t$ if and only if there exists a matrix $X \in \mathbb{S}_{++}^n$ such that the following condition holds:

$$\begin{aligned} (\bar{A}_0 + \bar{B}_0 K)^\top X (\bar{A}_0 + \bar{B}_0 K) - X + \sum_{i=1}^k \bar{\sigma}_i^2 \bar{A}_i^\top X \bar{A}_i \\ + \sum_{j=1}^{\ell} \bar{\rho}_j^2 K^\top \bar{B}_j^\top X \bar{B}_j K < 0. \end{aligned}$$

We pre- and postmultiply the above inequality by $P = X^{-1}$ and introduce a new variable $Y = K P$, which leads to

$$\begin{aligned} (\bar{A}_0 P + \bar{B}_0 Y)^\top P^{-1} (\bar{A}_0 P + \bar{B}_0 Y) - P \\ + \sum_{i=1}^k \bar{\sigma}_i^2 P^\top \bar{A}_i^\top P^{-1} \bar{A}_i P + \sum_{j=1}^{\ell} \bar{\rho}_j^2 Y^\top \bar{B}_j^\top P^{-1} \bar{B}_j Y < 0. \quad (15) \end{aligned}$$

We then apply the Schur's Lemma on (15) and come to a LMI

$$\begin{bmatrix} P & (\bar{A}_0 P + \bar{B}_0 Y) & \bar{Z} \\ (\bar{A}_0 P + \bar{B}_0 Y)^\top & P & 0 \\ \bar{Z}^\top & 0 & \bar{Z}_P \end{bmatrix} \succ 0 \quad (16)$$

where $\bar{Z} = [\bar{\sigma}_1 \bar{A}_1 P, \dots, \bar{\sigma}_k \bar{A}_k P, \bar{\rho}_1 \bar{B}_1 Y, \dots, \bar{\rho}_\ell \bar{B}_\ell Y]$ and

$$\bar{Z}_P = \text{blkdiag}(\underbrace{X, \dots, X}_{k+\ell}).$$

Equation (16) can replace the stability condition (4) in the row sparsity-promoting problem (5) and the column sparsity-promoting problem (6) when the system dynamic is given in the discrete-time domain. Similar to (11), we present a relaxation of the regularized LQRm formulation for the stochastic discrete linear system (14) to minimize the upper bound of LQRm cost

$$\min_{Y, P, \kappa, \Pi} \kappa + \gamma \|Y\|_{\text{reg}} \quad (17a)$$

$$\text{s.t. } \mathbf{Tr}(\Pi) \leq \kappa, \begin{bmatrix} \Pi & \Sigma_0^{\frac{1}{2}} \\ \Sigma_0^{\frac{1}{2}} & P \end{bmatrix} \geq 0, P \succ 0 \quad (17b)$$

$$\begin{bmatrix} P & (\bar{A}_0 P + \bar{B}_0 Y) & \bar{Z} & Y & P \\ (\bar{A}_0 P + \bar{B}_0 Y)^\top & P & 0 & 0 & 0 \\ \bar{Z}^\top & 0 & \bar{Z}_P & 0 & 0 \\ Y^\top & 0 & 0 & R^{-1} & 0 \\ P & 0 & 0 & 0 & Q^{-1} \end{bmatrix} \succ 0 \quad (17c)$$

where $\kappa \in \mathbb{R}$ and $\Pi \in \mathbb{R}^{n \times n}$ are the slack variables. The importance of sparsity is defined by γ . The initial state condition is $\mathbf{E}[x_0 x_0^\top] = \Sigma_0$ and the cost matrices Q and R are positive definite. The solution Y^* , P^* , Π^* , and κ^* defines an sparse stabilizing law $K_{\text{sp}} = Y^* P^{*-1}$ and the upper bound of the LQRm cost $J^*(K) = \mathbf{Tr}(\Sigma_0 P^{*-1}) \leq \kappa^*$.

REFERENCES

- [1] C. M. Harris and D. M. Wolpert, "Signal-dependent noise determines motor planning," *Nature*, vol. 394, no. 6695, pp. 780–784, Aug. 1998.
- [2] E. Todorov, "Stochastic optimal control and estimation methods adapted to the noise characteristics of the sensorimotor system," *Neural Comput.*, vol. 17, no. 5, pp. 1084–1108, May 2005.
- [3] E. Todorov and W. Li, "Optimal control methods suitable for biomechanical systems," in *Proc. 25th Annu. Int. Conf. IEEE Eng. Med. Biol. Soc.*, Feb. 2003, pp. 1758–1761.
- [4] E. Todorov and M. I. Jordan, "Optimal feedback control as a theory of motor coordination," *Nature Neurosci.*, vol. 5, no. 11, pp. 1226–1235, Nov. 2002.
- [5] M. Fardad, F. Lin, and M. R. Jovanovic, "Sparsity-promoting optimal control for a class of distributed systems," in *Proc. Amer. Control Conf.*, Jun. 2011, pp. 2050–2055.
- [6] F. Lin, M. Fardad, and M. R. Jovanovic, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2426–2431, Sep. 2013.
- [7] S. Schuler, P. Li, J. Lam, and F. Allgöwer, "Design of structured dynamic output-feedback controllers for interconnected systems," *Int. J. Control*, vol. 84, no. 12, pp. 2081–2091, Dec. 2011.
- [8] N. Matni, "Communication delay co-design in $\mathcal{H}_2$ -distributed control using atomic norm minimization," *IEEE Trans. Control Netw. Syst.*, vol. 4, no. 2, pp. 267–278, Jun. 2017.
- [9] M. Fardad and M. R. Jovanovic, "On the design of optimal structured and sparse feedback gains via sequential convex programming," in *Proc. Amer. Control Conf.*, Jun. 2014, pp. 2426–2431.
- [10] N. Matni, "Distributed optimal control of cyber-physical systems: Controller synthesis, architecture design and system identification," Ph.D. thesis, California Inst. Technol., Pasadena, CA, USA, 2016.
- [11] N. Matni and V. Chandrasekaran, "Regularization for design," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 3991–4006, Dec. 2016.
- [12] M. R. Jovanović and N. K. Dhingra, "Controller architectures: Tradeoffs between performance and structure," *Eur. J. Control*, vol. 30, pp. 76–91, Jul. 2016.
- [13] A. Zare, H. Mohammadi, N. K. Dhingra, T. T. Georgiou, and M. R. Jovanovic, "Proximal algorithms for large-scale statistical modeling and sensor/actuator selection," *IEEE Trans. Autom. Control*, vol. 65, no. 8, pp. 3441–3456, Aug. 2020.
- [14] N. K. Dhingra and M. R. Jovanovic, "Convex synthesis of symmetric modifications to linear systems," in *Proc. Amer. Control Conf. (ACC)*, Jul. 2015, pp. 3583–3588.
- [15] N. K. Dhingra, X. Wu, and M. R. Jovanović, "Sparsity-promoting optimal control of systems with invariances and symmetries," *IFAC-PapersOnLine*, vol. 49, no. 18, pp. 636–641, 2016.
- [16] N. K. Dhingra, M. R. Jovanovic, and Z. Luo, "An ADMM algorithm for optimal sensor and actuator selection," in *Proc. 53rd IEEE Conf. Decis. Control*, Dec. 2014, pp. 4039–4044.
- [17] B. Polyak, M. Khlebnikov, and P. Shcherbakov, "An LMI approach to structured sparse feedback design in linear control systems," in *Proc. Eur. Control Conf. (ECC)*, Jul. 2013, pp. 833–838.
- [18] S. H. Moghaddam and M. R. Jovanović, "Customized algorithms for growing connected resistive networks," *IFAC-PapersOnLine*, vol. 49, no. 18, pp. 968–973, 2016.
- [19] N. K. Dhingra and M. R. Jovanovic, "A method of multipliers algorithm for sparsity-promoting optimal control," in *Proc. Amer. Control Conf. (ACC)*, Jul. 2016, pp. 1942–1947.
- [20] H. J. Kushner, *Stochastic Stability and Control*, vol. 33. New York, NY, USA: Academic, 1967.
- [21] B. Gravel, P. M. Esfahani, and T. Summers, "Learning optimal controllers for linear systems with multiplicative noise via policy gradient," *IEEE Trans. Autom. Control*, vol. 66, no. 11, pp. 5283–5298, Nov. 2021.
- [22] L. El Ghaoui, "State-feedback control of systems with multiplicative noise via linear matrix inequalities," *Syst. Control Lett.*, vol. 24, no. 3, pp. 223–228, Feb. 1995.
- [23] J. Lu and R. E. Skelton, "Robust variance control for systems with finite-signal-to-noise uncertainty," *Automatica*, vol. 36, no. 4, pp. 511–525, Apr. 2000.
- [24] E. Gershon, U. Shaked, and I. Yaesh, " H_∞ control and filtering of discrete-time stochastic systems with multiplicative noise," *Automatica*, vol. 37, no. 3, pp. 409–417, Mar. 2001.
- [25] M. Rami, X. Chen, J. Moore, and X. Zhou, "Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls," *IEEE Trans. Autom. Control*, vol. 46, no. 3, pp. 428–440, Mar. 2001.
- [26] Y. Phillips, "Controller design of systems with multiplicative noise," *IEEE Trans. Autom. Control*, vol. AC-30, no. 10, pp. 1017–1019, Oct. 1985.
- [27] M. E. Ahsen, N. Challapalli, and M. Vidyasagar, "Two new approaches to compressed sensing exhibiting both robust sparse recovery and the grouping effect," *J. Mach. Learn. Res.*, vol. 18, no. 1, pp. 1745–1768, 2017.
- [28] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: SIAM, 1994.
- [29] M. A. Rami and X. Yu Zhou, "Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls," *IEEE Trans. Autom. Control*, vol. 45, no. 6, pp. 1131–1143, Jun. 2000.
- [30] K. Purchala, L. Meeus, D. Van Dommelen, and R. Belmans, "Usefulness of DC power flow for active power flow analysis," in *Proc. IEEE Power Eng. Soc. Gen. Meeting*, Oct. 2005, pp. 454–459.
- [31] Y. Guo and T. H. Summers, "A performance and stability analysis of low-inertia power grids with stochastic system inertia," in *Proc. Amer. Control Conf. (ACC)*, Jul. 2019, pp. 1965–1970.
- [32] D. J. Hill and I. M. Y. Mareels, "Stability theory for differential/algebraic systems with application to power systems," *IEEE Trans. Circuits Syst.*, vol. 37, no. 11, pp. 1416–1423, Nov. 1990.
- [33] J. Machowski, Z. Lubosny, J. W. Bialek, and J. R. Bumby, *Power System Dynamics: Stability and Control*. Hoboken, NJ, USA: Wiley, 2020.
- [34] R. Zimmerman, C. E. Murillo-Sanchez, and R. J. Thomas, "MATPOWER: Steady-state operations, planning, and analysis tools for power systems research and education," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12–19, Feb. 2010.
- [35] MOSEK Aps, "MOSEK optimization toolbox for MATLAB," User's Guide and Reference Manual, Version 4.1, 2019.
- [36] M. Grant and S. Boyd, "CVX: MATLAB software for disciplined convex programming, version 2.1," 2014.