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RESEARCH ARTICLE

On the atomic structure of torsion-free monoids

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Abstract
Let M be a cancellative and commutative (additive) monoid. The monoid M is atomic if every non-invertible element can be written as a sum of irreducible elements, which are also called atoms. Also, M satisfies the ascending chain condition on principal ideals (ACCP) if every increasing sequence of principal ideals (under inclusion) becomes constant from one point on. In the first part of this paper, we characterize torsion-free monoids that satisfy the ACCP as those torsion-free monoids whose submonoids are all atomic. A submonoid of the nonnegative cone of a totally ordered abelian group is often called a positive monoid. Every positive monoid is clearly torsion-free. In the second part of this paper, we study the atomic structure of certain classes of positive monoids.

Keywords Totally ordered group · Positive monoid · ACCP · Atomic monoid · Ordered abelian group

1 Introduction

A cancellative and commutative additive monoid is called atomic if every non-invertible element can be expressed as a sum of irreducibles, also called atoms. Motivated by the celebrated paper [1] by Anderson, Anderson, and Zafrullah, the property of being atomic has received a great deal of attention in the literature during the last three decades. Although in [1] the authors only considered atomicity in the context of integral domains, the same notion has been significantly extended to and explored in a variety of different contexts. Perhaps the first influential generalization is

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$$\textbf{UFM} \Longrightarrow \textbf{FFM} \Longrightarrow \textbf{BFM} \Longrightarrow \textbf{ACCP}$$

Fig. 1 A monoid adaptation of a fragment of the diagram introduced in 1990 by Anderson, Anderson, and Zafrullah to study factorizations in the context of integral domains

due to Halter-Koch, who generalized in [34] to the context of cancellative and commutative monoids some atomic notions introduced in [1] for integral domains. Although it goes outside the scope of this paper, it is worth emphasizing that atomicity has also been studied in non-cancellative and non-commutative algebraic structures, including commutative rings with nonzero zero-divisors [20, 35], non-commutative monoids [4, 5], and even more general scenarios [14, 15].

Even before the nineties, atomicity was sporadically studied in the context of integral domains, mainly in connection to the ACCP. A cancellative and commutative monoid (or an integral domain) satisfies the ascending chain condition on principal ideals (ACCP) if every increasing sequence of principal ideals (under inclusion) becomes constant from one point on. The first example of an atomic domain not satisfying the ACCP was constructed by Grams in [33], correcting Cohn's misled assertion that being atomic and satisfying the ACCP were equivalent conditions in the context of integral domains. Zaks later constructed in [39] two more examples of atomic domains that do not satisfy the ACCP (one of them was suggested by Cohn). Further examples of atomic domains that do not satisfy the ACCP have been constructed since then (see, for instance, [6, 30–32, 38]), although none of these examples has a trivial construction.

A cancellative and commutative monoid M is called hereditarily atomic if every submonoid of M is atomic. Hereditary atomicity in the context of integral domains was recently studied by Coykendall, Hasenauer, and the first author in [17]. In this paper, we use hereditary atomicity to characterize the ACCP property in the context of reduced torsion-free monoids (a monoid is called reduced if its group of invertible elements is trivial, while it is called torsion-free if its difference group is torsion-free). Indeed, the fundamental result we establish in this paper is Theorem 3.1, which states the following.

Theorem *A reduced torsion-free monoid satisfies the ACCP if and only if it is hereditarily atomic.*

It is unknown to the authors whether the two conditions in the statement of the theorem are equivalent after dropping the torsion-free condition. To motivate further research we leave this as an open question at the end of Sect. 3.

In Sect. 4, we exhibit several examples to illustrate the variation of atomic behavior in the class of positive monoids. A positive monoid is a submonoid of the nonnegative cone of a totally ordered abelian group. Positive monoids form a special class of torsion-free monoids. In Sect. 4, we consider properties stronger than atomicity, in the direction of the taxonomic classes of atomicity proposed and investigated by Anderson, Anderson, and Zafrullah in [1]. The considered properties include the bounded and the finite factorization properties, which were both introduced in [1] to better understand the atomicity of integral domains in the context of the methodological diagram illustrated in Fig. 1, where UFM (resp., FFM and BFM) stands for unique factorization monoid (resp., finite factorization monoid and bounded factorization monoid).

ATM $\Longrightarrow$ **NAM** $\Longrightarrow$ **AAM** $\Longrightarrow$ **QAM**

Fig. 2 A chain of implications extending to the right the one in Fig. 1 and consisting of properties weaker than being atomic

In the same section, we also consider properties weaker than atomicity, including the property of being almost atomic and that of being quasi-atomic, both introduced by Boynton and Coykendall in [7] to study divisibility in integral domains, as well as the property of being nearly atomic, recently introduced by Lebowitz-Lockard in [36]. These properties also form a nested diagram of atomic classes: this is illustrated in Fig. 2, where the non-standard acronym ATM (resp., NAM, AAM, QAM) stands for atomic monoid (resp., nearly atomic monoid, almost atomic monoid, quasi-atomic monoid).

Our primary purpose in Sect. 5 is to consider the atomic structure of certain classes of positive monoids that are often useful to construct needed (counter)examples in factorization theory: they are monoids of the form $\{0\} \cup G_{\geq a}$, where G is a totally ordered group and a is an element in the nonnegative cone of G. We call these monoids conductive positive monoids. Examples of commutative rings in factorization theory based on conductive positive monoids can be found in [1, Example 2.7(a)] and [28, Theorem 4.4].

2 Background

Following usual conventions, we let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of integers, rational numbers, and real numbers, respectively. We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive and nonnegative integers, respectively. In addition, we let $\mathbb{P}$ denote the set of primes. For $b, c \in \mathbb{Z}$ with $b \leq c$, we let $[\![b, c]\!]$ denote the set of integers between b and c; that is, $[\![b, c]\!] := \{m \in \mathbb{Z} \mid b \leq m \leq c\}$. In addition, for $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we set $S_{\geq r} = \{s \in S \mid s \geq r\}$ and $S_{>r} = \{s \in S \mid s > r\}$. For $q \in \mathbb{Q}_{>0}$, the relatively prime positive integers n and d such that $q = \frac{n}{d}$ are denoted here by $\mathsf{n}(q)$ and $\mathsf{d}(q)$, respectively. Finally, for any $r \in \mathbb{R}$, we let $\mathbb{Z}[r]$ denote the subring $\{f(r) \mid f(x) \in \mathbb{Z}[x]\}$ of $\mathbb{R}$.

2.1 Atomic notions in monoids

A *monoid* is a semigroup with an identity element. However, in the context of this paper we will tacitly assume that monoids are both cancellative and commutative. Let M be a monoid written additively. We set $M^\bullet = M \setminus \{0\}$, and we say that M is *trivial* if $M^\bullet$ is empty. The invertible elements of M form a group, which we denote by $\mathcal{U}(M)$, and M is called *reduced* if $\mathcal{U}(M)$ is the trivial group. The *difference group* $\text{gp}(M)$ of M is the unique abelian group $\text{gp}(M)$ up to isomorphism such that any abelian group containing an isomorphic image of M also contains an isomorphic image of $\text{gp}(M)$. The *rank* of M is, by definition, the rank of $\text{gp}(M)$ as a $\mathbb{Z}$-module or, equivalently, the dimension of the vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$. The rank of M is denoted by $\text{rank}\, M$. The *reduced monoid* of M is the quotient $M/\mathcal{U}(M)$, which is denoted by M_{red}. For

$b, c \in M$, we say that c *divides* b *in* M if there exists $d \in M$ such that $b = c + d$; in this case, we write $c \mid_M b$ (we reserve the standard notation $m \mid n$, with no subscripts, to express that m divides n in the multiplicative monoid $\mathbb{Z}\setminus\{0\}$). A submonoid M' of M is called a *divisor-closed submonoid* if every element of M dividing an element of M' in M belongs to M'. If S is a subset of M, then we let $\langle S \rangle$ denote the smallest submonoid of M containing S, in which case, we say that S is a *generating set* of $\langle S \rangle$. The monoid M is called *finitely generated* provided that $M = \langle S \rangle$ for some finite subset S of M.

A non-invertible element $a \in M$ is called an *atom* of M if whenever $a = b + c$ for some $b, c \in M$, either $b \in \mathcal{U}(M)$ or $c \in \mathcal{U}(M)$. We let $\mathcal{A}(M)$ denote the set consisting of all the atoms of M. If $\mathcal{A}(M)$ is empty, M is called *antimatter*. Observe that if M is a reduced monoid, then $\mathcal{A}(M)$ is contained in every generating set of M. An element of M is called *atomic* if it is invertible or it can be written as a sum of finitely many atoms. The monoid M is called *atomic* if every element of M is atomic. In addition, the monoid M is called *hereditarily atomic* provided that every submonoid of M is atomic. Following [1], we say that M is *strongly atomic* if for any elements $b, c \in M$, there exists an atomic element $d \in M$ that is a common divisor of b and c such that the only common divisors of $b - d$ and $c - d$ are invertible elements. It follows from the definitions that every strongly atomic monoid is atomic. The converse does not hold in general and we exhibit a counterexample in Example 4.3 (the first of such examples seems to be given by Roitman in [38, Example 5.2]). It is well known that every monoid satisfying the ACCP is strongly atomic (see [1, Theorem 1.3]).

We proceed to introduce some properties that are weaker than that of being atomic. An element $c \in M$ is called *quasi-atomic* (resp., *almost atomic*) provided that there exists an element (resp., an atomic element) $b \in M$ such that $b+c$ is atomic. Following [7], we say that M is *quasi-atomic* (resp., *almost atomic*) if every non-invertible element of M is quasi-atomic (resp., almost atomic). It follows directly from the definitions that every almost atomic monoid is quasi-atomic. Following [36], we say that M is *nearly atomic* if there exists $b \in M$ such that for each non-invertible element $c \in M$, the element $b + c$ is atomic. It follows directly from the definitions that every atomic monoid is nearly atomic, and one can prove that every nearly atomic monoid is almost atomic by mimicking the proof of [36, Lemma 5]. Therefore the properties of being nearly atomic, almost atomic, and quasi-atomic are nested weaker notions of atomicity.

A subset I of M is called an *ideal* of M if the set $I + M := \{b + c \mid b \in I \text{ and } c \in M\} \subseteq I$ or, equivalently, if $I + M = I$. If I is an ideal of M such that $I = b+M := \{b+c \mid c \in M\}$ for some $b \in M$, then I is called *principal*. The monoid M satisfies the *ascending chain condition on principal ideals* (ACCP) provided that every ascending chain $(b_n + M)_{n\in\mathbb{N}}$ of principal ideals of M stabilizes; that is, there exists $n_0 \in \mathbb{N}$ such that $b_n + M = b_{n+1} + M$ for every $n \geq n_0$. It is not hard to check that if M satisfies the ACCP, then M is atomic (see [23, Proposition 1.1.4]). As we have mentioned in the introduction, the converse of this statement does not hold.

The *free commutative monoid* on a set S is the commutative monoid whose elements are formal linear combinations of elements of S. The free commutative monoid on $\mathcal{A}(M_{\text{red}})$ is denoted by $\mathsf{Z}(M)$. Let $\pi : \mathsf{Z}(M) \to M_{\text{red}}$ be the unique monoid homomorphism fixing every element of $\mathcal{A}(M_{\text{red}})$. If $z := a_1 \cdots a_\ell \in \mathsf{Z}(M)$ for some

$a_1, \ldots, a_\ell \in \mathcal{A}(M_{\text{red}})$, then ℓ is called the *length* of z and is denoted by $|z|$. For every $b \in M$, we set

$$\mathsf{Z}(b) := \mathsf{Z}_M(b) := \pi^{-1}(b + \mathcal{U}(M)),$$

and the elements of $\mathsf{Z}(b)$ are called *factorizations* of b. A recent survey on factorization theory in commutative monoids by Geroldinger and Zhong can be found in [25]. If $|\mathsf{Z}(b)| = 1$ for every $b \in M$, then M is called a *unique factorization monoid* (UFM). Following Zaks [40], we say that M is a *half-factorial monoid* (HFM) if M is atomic and any two factorizations of the same element have the same length. On the other hand, if M is atomic and $|\mathsf{Z}(b)| < \infty$ for every $b \in M$, then M is called a *finite factorization monoid* (FFM). It follows directly from the definitions that every UFM is both an HFM and an FFM. None of the notions of HFM and FFM implies the other, even in the context of positive monoids (see, for instance, Examples 4.9 and 5.12). In addition, it follows from [23, Proposition 2.7.8] that every finitely generated monoid is an FFM. Now, for every $b \in M$, we set

$$\mathsf{L}(b) := \mathsf{L}_M(b) := \{|z| \mid z \in \mathsf{Z}(b)\}.$$

If M is atomic and $|\mathsf{L}(b)| < \infty$ for every $b \in M$, then M is called a *bounded factorization monoid* (BFM). According to [34, Theorem 1], the monoid M is a BFM if it admits a *length function*, that is, a function $\ell \colon M \to \mathbb{N}_0$ satisfying the following two properties:

(i) $\ell(u) = 0$ if and only if $u \in \mathcal{U}(M)$;
(ii) $\ell(b + c) \geq \ell(b) + \ell(c)$ for all $b, c \in M$.

It follows directly from the definitions that every monoid that is either an HFM or an FFM is also a BFM. In addition, it is not hard to argue that every BFM must satisfy the ACCP (see [23, Corollary 1.4.4]). A recent survey on the bounded factorization and finite factorization properties by Anderson and the first author can be found in [2]. The monoid M is called a *length-factorial monoid* (LFM) provided that M is atomic and also that any two distinct factorizations of the same element have different lengths. The notion of length-factoriality was introduced by Coykendall and Smith in [18] under the term 'other-half-factoriality'. The same notion has been considered recently by Chapman et al. in [10] and by Geroldinger and Zhong in [24]. It follows directly from the definitions that every UFM is an LFM. Moreover, observe that the notion of being an LFM complements the notion of being an HFM in the sense that a monoid is a UFM if and only if it is both an HFM and an LFM. Finally, it follows from [8, Proposition 3.1] that every LFM is an FFM.

The classes of monoids we have introduced in this subsection are represented in the chain of implications illustrated in Fig. 3, which consists of nested classes of monoids and extends simultaneously the diagrams previously shown in Figs. 1 and 2. For the sake of consistency, in the diagram shown in Fig. 3, we let the (nonstandard) acronym SAM (resp., ATM, NAM, AAM, and QAM) stand for strongly atomic monoid (resp., atomic monoid, nearly atomic monoid, almost atomic monoid, and quasi-atomic monoid).

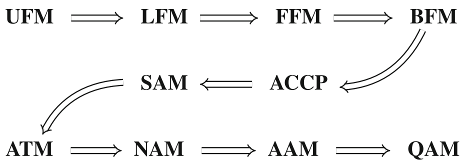

Fig. 3 A chain of implications connecting those in Figs. 1 and 2 through the property of being strongly atomic

2.2 Totally ordered groups

We proceed to review some basic notions related to totally ordered abelian groups that we will be using later. A *totally ordered abelian group* is a pair (G, G^+), where G is an abelian (additive) group and G^+ is a subset of G containing 0 and satisfying the following two conditions:

(1) for all $g, h \in G^+$, $g + h \in G^+$, and
(2) for each $g \in G \backslash \{0\}$, exactly one of the conditions $g \in G^+$ and $-g \in G^+$ holds.

In this case, G^+ is called a *nonnegative cone* of G, and G^+ induces a total order on G, namely, $g \le h$ in G whenever $h - g \in G^+$. On the other hand, if G is an abelian group and $\le$ is a total order on G compatible with the operation of G, then $\{g \in G \mid g \ge 0\}$ is a nonnegative cone of G inducing the order $\le$. We can see that every totally ordered group is torsion-free. When there seems to be no risk of ambiguity, we will often abuse notation, writing G instead of the more cumbersome notation (G, G^+).

Let G be a totally ordered abelian group. For each $g \in G$ set $|g| = g$ if $g \in G^+$ and $|g| = -g$ otherwise. For $g, h \in G$, we write $g = \mathbf{O}(h)$ if $|g| \le n|h|$ for some $n \in \mathbb{N}$, and $g \sim h$ if both $g = \mathbf{O}(h)$ and $h = \mathbf{O}(g)$ hold. Also, for any $g, h \in G$ with $g = \mathbf{O}(h)$ and $g \nsim h$, we write $g \ll h$ or, equivalently, $h \gg g$. It is clear that $\sim$ defines an equivalence relation on G. Let

$$v \colon G \setminus \{0\} \to \Gamma_G := (G \setminus \{0\})/\sim$$

be the quotient map. Setting $v(g) \preceq v(h)$ for any $g, h \in G \backslash \{0\}$ with $h = \mathbf{O}(g)$, one finds that $(\Gamma_G, \preceq)$ is a well-defined totally ordered set, which is called the *value set* of G. One can verify that for all $g, h \in G$ such that $0 \notin \{g, h, g + h\}$,

$$v(g + h) \ge \min\{v(g), v(h)\},$$

where the equality holds provided that $v(g) \ne v(h)$ or $g, h \in G^+$. The elements of Γ_G are called *Archimedean classes* of G, and the quotient map v is called the *Archimedean valuation* on G. The totally ordered group G is called *Archimedean* if and only if its value set Γ_G is a singleton.

A *positive monoid of a totally ordered group* G is a submonoid of G contained in G^+. A monoid M is called a *positive monoid* if it is isomorphic to a positive monoid of a totally ordered group G. According to Hölder's theorem, a totally ordered abelian group is Archimedean if and only if it is order-isomorphic to a subgroup of the additive group $\mathbb{R}$. Thus, additive submonoids of $\mathbb{R}_{\ge 0}$ account, up to isomorphism, for all positive monoids of Archimedean groups. In addition, nontrivial additive submonoids of $\mathbb{Q}_{\ge 0}$, also known as *Puiseux monoids*, account up to isomorphism for all the rank-one positive monoids (see [22, Theorem 3.12] and [21, Section 24]). Cofinite submonoids of the additive monoid $\mathbb{N}_0$ are called *numerical monoids*. Every numerical monoid is then a Puiseux monoid, and it is well known that a Puiseux monoid is finitely generated if and only if it is isomorphic to a numerical monoid. The atomic structure and arithmetic of factorizations of Puiseux monoids have been systematically studied during the last half-decade (see [22] and references therein).

3 Hereditary atomicity and the ACCP

It turns out that in the class consisting of reduced torsion-free monoids, being hereditarily atomic and satisfying the ACCP are equivalent conditions. The condition that the monoid is reduced is not necessary for one of the implications, namely, that being hereditarily atomic implies satisfying the ACCP. We will argue this last observation in Proposition 3.3 at the end of this section.

Theorem 3.1 *For a reduced torsion-free monoid M, the following conditions are equivalent.*

(a) *M satisfies the ACCP.*
(b) *M is hereditarily atomic.*

Proof (a) $\Rightarrow$ (b): Since M is a reduced monoid, the fact that M satisfies the ACCP immediately implies that every submonoid of M satisfies the ACCP. Now the implication follows from the fact that every monoid satisfying the ACCP is atomic.

(b) $\Rightarrow$ (a): Let G be the difference group of M. Since M is torsion-free, G is a torsion-free abelian group. Then it follows from [26, Theorem 3.2] that G can be turned into a totally ordered group $(G, \le)$ in such a way that M is a submonoid of the nonnegative cone G^+ of G. Because G is totally ordered, it follows from the Hahn embedding theorem that G can be embedded as an ordered group into the totally ordered group $\mathfrak{F}(\Gamma_G, \mathbb{R})$ consisting of all functions from Γ_G to $\mathbb{R}$ that vanish outside a well-ordered set, where Γ_G is the value set of G. In particular, G^+ is a submonoid of $\mathfrak{F}(\Gamma_G, \mathbb{R})^+$. Since $\mathfrak{F}(\Gamma_G, \mathbb{R})$ is a divisible group, after replacing G by $\mathfrak{F}(\Gamma_G, \mathbb{R})$, we can assume that G is a totally ordered divisible abelian group and, in particular, g and g/n are in the same Archimedean class of G for all $g \in G$ and $n \in \mathbb{N}$.

Suppose, by way of contradiction, that M does not satisfy the ACCP. Then there is a sequence of principal ideals $(q_n + M)_{n \in \mathbb{N}_0}$ such that $a_{n+1} := q_n - q_{n+1} \in M^\bullet$ for every $n \in \mathbb{N}_0$. We split the rest of the proof into two cases.

Case 1: There exists an Archimedean class of G containing infinitely many terms of the sequence $(a_n)_{n \in \mathbb{N}}$. In this case, there is a subsequence $(b_n)_{n \in \mathbb{N}}$ of $(a_n)_{n \in \mathbb{N}}$ whose terms are in the same Archimedean class of G. Now set $q'_n := q_0 - \sum_{i=1}^n b_i$ for each $n \in \mathbb{N}$. The fact that $(q_n + M)_{n \in \mathbb{N}}$ is a non-stabilizing ascending chain of principal ideals of M implies that $(q'_n + M)_{n \in \mathbb{N}}$ is also a non-stabilizing ascending chain of principal ideals of M. Therefore, after replacing $(q_n + M)_{n \in \mathbb{N}}$ by $(q'_n + M)_{n \in \mathbb{N}}$ if necessary, we can assume that all the terms of the sequence $(a_n)_{n \in \mathbb{N}}$ belong to the same Archimedean class of G. Because M is hereditarily atomic, its submonoid

$N := \langle a_n, q_n \mid n \in \mathbb{N} \rangle$ is atomic. Now for each $n \in \mathbb{N}_0$ the equality $q_n = q_{n+1} + a_{n+1}$ ensures that $q_n \notin \mathcal{A}(N)$. This, along with the fact that $q_0 = q_1 + a_1 \in N$, implies that $q_0 \in \langle a_n \mid n \in \mathbb{N} \rangle$. Hence $a_n \sim q_0$ for every $n \in \mathbb{N}$.

Now observe that, for any $k \in \mathbb{N}$, the term a_k cannot be a lower bound of the set $\{a_n \mid n > k\}$ as, otherwise, we could take $n \in \mathbb{N}$ such that $na_k > q_0$ to obtain that $\sum_{i=1}^{k+n} a_i > \sum_{i=k+1}^{k+n} a_i \geq na_k > q_0$, which is not possible. Therefore, after replacing $(a_n)_{n\in\mathbb{N}}$ by a suitable subsequence and redefining $(q_n)_{n\in\mathbb{N}}$ as we did in the previous paragraph, we can assume that $(a_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence. In the same vein, we observe that for any $k \in \mathbb{N}$, there must exist $\ell \in \mathbb{N}_{>k}$ such that $a_k - a_\ell \sim q_0$ as otherwise (i.e., $a_k - a_{k+i} \ll q_0$ for every $i \in \mathbb{N}$), we could take $n \in \mathbb{N}$ such that $q_0 < na_k$ to obtain that $q_0 - \sum_{i=k}^{k+n} a_i < -a_k + \sum_{i=k+1}^{k+n} (a_k - a_i) < 0$, which is not possible. Thus, after replacing $(a_n)_{n\in\mathbb{N}}$ by a suitable subsequence and redefining $(q_n)_{n\in\mathbb{N}}$ accordingly, we can assume that $a_n - a_{n+1} \sim q_0$ for every $n \in \mathbb{N}$. Set $s_n := \sum_{i=1}^{n} a_i$ for each $n \in \mathbb{N}$.

Our current goal is to construct a sequence $(a'_n)_{n\in\mathbb{N}}$ with its terms in M such that, for every $n \in \mathbb{N}$, the following conditions hold:

(1) $q_0 \notin \langle a'_1, \ldots, a'_n \rangle$,
(2) $q_0 \sim a'_i$ for every $i \in [\![1, n]\!]$, and
(3) $s'_n := \sum_{i=1}^{n} a'_i$ divides s_m in M for some $m \in \mathbb{N}$.

We proceed inductively. Since $q_0 = \mathbf{O}(a_1)$, the set $G^+ \cap \left\{\frac{q_0}{n} - a_1 \mid n \in \mathbb{N}\right\}$ is finite. Now the fact that the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing ensures the existence of $i \in \mathbb{N}$ such that $a_i \notin \left\{\frac{q_0}{n} - a_1 \mid n \in \mathbb{N}\right\}$, whence $q_0 \notin \langle a_1 + a_i \rangle$. Setting $a'_1 := a_1 + a_i$, we can see that conditions (1)–(3) above hold for $n = 1$. Now suppose that we have already found $a'_1, \ldots, a'_n \in M$ such that conditions (1)–(3) hold. Fix $m \in \mathbb{N}$ such that s'_n divides s_m in M. Set $Q := q_0 - \langle a'_1, \ldots, a'_n \rangle$. From the fact that $q_0 \sim a'_i$ for every $i \in [\![1, n]\!]$, one can infer that the set $G^+ \cap Q$ is finite. Also, $q = \mathbf{O}(a_{m+1})$ for each $q \in G^+ \cap Q$. Therefore the set $G^+ \cap \left\{\frac{q}{n} - a_{m+1} \mid q \in Q \text{ and } n \in \mathbb{N}\right\}$ is also finite. As $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing, there is a $j \in \mathbb{N}$ with $j > m + 1$ such that $a_j \notin \left\{\frac{q}{n} - a_{m+1} \mid q \in Q \text{ and } n \in \mathbb{N}\right\}$. Thus, Q is disjoint from $\langle a_{m+1} + a_j \rangle$. So after setting $a'_{n+1} = a_{m+1} + a_j$, we see that $q_0 \notin \langle a'_1, \ldots, a'_{n+1} \rangle$ and also that $q_0 \sim a_{m+1} + a_j \sim a'_{n+1}$. In addition, observe that $s'_{n+1} := \sum_{i=1}^{n+1} a'_i = s'_n + a_{m+1} + a_j$ divides $s_m + a_{m+1} + a_j$ in M, which implies that s'_{n+1} divides s_j in M. Hence we can assume the existence of a sequence $(a'_n)_{n\in\mathbb{N}}$ satisfying conditions (1)–(3) above.

Finally, let $(r_n)_{n\in\mathbb{N}}$ be the sequence defined as follows: take $r_0 = q_0$ and take $r_n = q_0 - s'_n$ for every $n \in \mathbb{N}$. Because $r_n = q_0 - s'_n = q_0 - s'_{n+1} + (s'_{n+1} - s'_n) = r_{n+1} + a'_{n+1}$, we see that $(r_n + M)_{n\in\mathbb{N}}$ is a non-stabilizing ascending chain of principal ideals of M. Now consider the submonoid $N' := \langle a'_n, r_n \mid n \in \mathbb{N} \rangle$ of M. Since M is hereditarily atomic, N' is atomic. In addition, the fact that $(r_n + M)_{n\in\mathbb{N}}$ is a non-stabilizing ascending chain of principal ideals implies that $\mathcal{A}(N') \subseteq \{a'_n \mid n \in \mathbb{N}\}$. This, along with the fact that $q_0 = r_1 + a'_1 \in N'$, guarantees the existence of $c_1, \ldots, c_n \in \mathbb{N}_0$ such that $q_0 = \sum_{i=1}^{n} c_i a'_i$. However, this contradicts that $q_0 \notin \langle a'_1, \ldots, a'_n \rangle$.

Case 2: Each Archimedean class of G contains only finitely many terms of the sequence $(a_n)_{n\in\mathbb{N}}$. Then there exists a subsequence $(b_n)_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that each Archimedean class of G contains at most one term of $(b_n)_{n\in\mathbb{N}}$ and, after replacing

$(q_n)_{n\in\mathbb{N}}$ by $(q_0 - \sum_{i=1}^n b_i)_{n\in\mathbb{N}}$, one can assume that each Archimedean class of F contains at most one term of the sequence $(a_n)_{n\in\mathbb{N}}$.

Because $q_0 > \sum_{i=1}^n a_i$ for every $n \in \mathbb{N}$, we see that $a_n = \mathbf{O}(q_0)$ for every $n \in \mathbb{N}$. On the other hand, the fact that $(q_n + M)_{n\in\mathbb{N}}$ is a non-stabilizing ascending chain of principal ideals, together with the fact that $\langle a_n, q_n \mid n \in \mathbb{N}\rangle$ is an atomic monoid, guarantees that $q_0 \in \langle a_n \mid n \in \mathbb{N}\rangle$, which in turns implies that $q_0 \sim a_{k_1}$ for some $k_1 \in \mathbb{N}$. Observe that $a_{k_1} \gg a_n$ for any $n > k_1$. Now suppose that we have found $k_1, \dots, k_m \in \mathbb{N}$ with $k_1 < \cdots < k_m$ such that $q_0 \sim a_{k_1} \gg \cdots \gg a_{k_m} \gg a_n$ for all $n > k_m$. Then set $q_0' := q_0 - \sum_{i=1}^m a_{k_i}$ and, for each $n \in \mathbb{N}$, set $a_n' := \sum_{i=1}^n a_{k_m+i}$ and $q_n' = q_0' - a_n'$. Observe that $(q_n' + M)_{n\in\mathbb{N}}$ is a non-stabilizing ascending chain of principal ideals of M. Proceeding as we did before, we find that $q_0' \sim a_{k_m+j}$ for some $j \in \mathbb{N}$. Since $q_0' > \sum_{i=1}^n a_{k_m+i}$ for all $n \in \mathbb{N}$, it follows that $a_{k_m+j} \gg a_n$ for every $n > k_m + j$. Then after setting $k_{m+1} := k_m + j$, we obtain $q_0 \sim a_{k_1} \gg \cdots \gg a_{k_{m+1}} \gg a_n$ for all $n > k_{m+1}$. Hence, after replacing $(q_n)_{n\in\mathbb{N}}$ by $(q_0 - \sum_{i=1}^n a_{k_i})_{n\in\mathbb{N}}$, one can assume that $q_0 \sim a_1$ and also that $a_n \gg a_{n+1}$ for every $n \in \mathbb{N}$. Now fix any $\ell \in \mathbb{N}$. Since $q_\ell = q_{\ell+1} + a_{\ell+1}$, we see that $a_{\ell+1} = \mathbf{O}(q_\ell)$. On the other hand, the atomicity of $\langle a_n, q_n \mid n \ge \ell + 1\rangle$ ensures that $q_\ell \in \langle a_n \mid n \ge \ell + 1\rangle$, and so $a_{\ell+1} \gg a_{\ell+n}$ for every $n \ge 2$ guarantees that $q_\ell \sim a_{\ell+1}$.

Finally, write $q_0 = \sum_{i=1}^k c_i a_i$ for some $c_1, \dots, c_k \in \mathbb{N}_0$. Note that $c_1 \ge 1$ because $q_0 \sim a_1 \gg a_n$ for every $n \ge 2$. Since $a_1 \gg a_2$, the fact that $(c_1 - 1)a_1 + \sum_{i=2}^k c_i a_i = q_0 - a_1 = q_1 \sim a_2$ implies that $c_1 = 1$. Set $j := \max\{i \in [\![1, k]\!] \mid c_i = 1\}$, and observe that $j < k$ as $q_0 - \sum_{i=1}^k a_i = q_k \neq 0$. As $a_{j+1} \gg a_n$ for every $n \ge j + 2$, the fact that $\sum_{i=j+1}^k c_i a_i = q_0 - \sum_{i=1}^j a_i = q_j \sim a_{j+1}$ guarantees the inequality $c_{j+1} \ge 1$. Finally, since $a_{j+1} \gg a_{j+2}$, the fact that $(c_{j+1} - 1)a_{j+1} + \sum_{i=j+2}^k c_i a_i = q_0 - \sum_{i=1}^{j+1} a_i = q_{j+1} \sim a_{j+2}$ implies that $c_{j+1} = 1$. However, this contradicts the maximality of j. □

With notation as in Theorem 3.1, the condition that the monoid is reduced is not superfluous. This is illustrated in the following example.

Example 3.2 The additive abelian group $\mathbb{Z}^2$ trivially satisfies the ACCP as a monoid. To see that $\mathbb{Z}^2$ is not hereditarily atomic, let M be the nonnegative cone of $\mathbb{Z}^2$ with respect to the lexicographical order with priority in the second coordinate. It is clear that M is a submonoid of $\mathbb{Z}^2$, and it is not hard to verify that $\mathcal{A}(M) = \{(1, 0)\}$. Thus, M is not atomic, and so $\mathbb{Z}^2$ is not a hereditarily atomic monoid.

On the other hand, the condition of being reduced is not needed for the other implication of Theorem 3.1, as the following proposition shows.

Proposition 3.3[1] *If a torsion-free monoid is hereditarily atomic, then it satisfies the ACCP.*

Proof Let M be a hereditarily atomic torsion-free monoid and assume, towards a contradiction, that M does not satisfy the ACCP. Let $(b_n + M)_{n\in\mathbb{N}}$ be an ascending chain of principal ideals of M that does not stabilize. Set $c_n := b_n - b_{n+1}$ for every

[1] Proposition 3.3 was kindly observed by Ben Li, who also suggested the given proof.

$n \in \mathbb{N}$. After replacing $(b_n + M)_{n\in\mathbb{N}}$ by a suitable subsequence, we can assume that $c_n \notin \mathcal{U}(M)$ for any $n \in \mathbb{N}$. Now consider the submonoid $M' := \langle b_n, c_n \mid n \in \mathbb{N}\rangle$ of M. Since M is a hereditarily atomic torsion-free monoid, so is M'. Observe that $(b_n + M')_{n\in\mathbb{N}}$ is an ascending chain of principal ideals of M', which does not stabilize in M' because $(b_n + M)_{n\in\mathbb{N}}$ does not stabilize in M. Thus, M' does not satisfy the ACCP and, therefore M' is not reduced by virtue of Theorem 3.1. As a result, some element in the defining generating set of M' must be invertible. Since $c_n \notin \mathcal{U}(M')$ for any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $b_m \in \mathcal{U}(M') \subseteq \mathcal{U}(M)$. This implies that $b_n + M = M$ for every $n \geq m$, which contradicts that the chain of principal ideals $(b_n + M)_{n\in\mathbb{N}}$ does not stabilize in M. □

According to Proposition 3.3, when a torsion-free monoid is atomic but does not satisfy the ACCP, then it contains a submonoid that is not atomic. As the following example illustrates, sometimes we can identify such submonoids.

Example 3.4 (1) Let $(p_n)_{n\geq 0}$ be the strictly increasing sequence whose underlying set is $\mathbb{P} \setminus \{2\}$. Now consider the Puiseux monoid

$$M := \Big\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N}_0 \Big\rangle.$$

The monoid M is often called the Grams monoid as it is the main ingredient in Grams' construction of the first atomic domain that does not satisfy the ACCP, namely, the localization of the monoid algebra $\mathbb{Q}[M]$ at the multiplicative subset $\{f \in \mathbb{Q}[M] \mid f(0) \neq 0\}$ (see [33, Theorem 1.3] for more details). One can readily show that the monoid M is atomic with $\mathcal{A}(M) = \big\{\frac{1}{2^n p_n} \mid n \in \mathbb{N}_0\big\}$. However, M does not satisfy the ACCP because the sequence $(\frac{1}{2^n} + M)_{n\geq 0}$ is an ascending chain of principal ideals of M that does not stabilize. Since M is torsion-free, in light of Proposition 3.3 there is a submonoid of M that is not atomic. Indeed, $\big\langle \frac{1}{2^n} \mid n \in \mathbb{N}_0 \big\rangle$ is one of the non-atomic submonoids of M.

(2) Take $q \in \mathbb{Q} \cap (0, 1)$ such that $q^{-1} \notin \mathbb{N}$, and consider the Puiseux monoid $M_q := \langle q^n \mid n \in \mathbb{N}_0\rangle$. It follows from [29, Theorem 6.2] that M_q is an atomic monoid with $\mathcal{A}(M_q) = \{q^n \mid n \in \mathbb{N}_0\}$. On the other hand, observe that $\mathsf{n}(q)q^n = \mathsf{n}(q)q^{n+1} + (\mathsf{d}(q) - \mathsf{n}(q))q^{n+1}$ for every $n \in \mathbb{N}_0$, which implies that $(\mathsf{n}(q)q^n + M_q)_{n\geq 0}$ is an ascending chain of principal ideals of M_q that does not stabilize. Hence M_q does not satisfy the ACCP and, by virtue of Proposition 3.3, it must contain a submonoid that is not atomic. Let us explicitly find a non-atomic submonoid of M_q. Since $M_{q^k} := \langle q^{kn} \mid n \in \mathbb{N}_0\rangle$ is a submonoid of M_q for every $k \in \mathbb{N}$, after replacing M_q by M_{q^k} for some k sufficiently large, we can further assume that $q \leq \frac{1}{2}$. In particular, $\mathsf{d}(q) - \mathsf{n}(q) \geq 2$. We claim that the submonoid

$$M := \big\langle \mathsf{n}(q)q^n, (\mathsf{d}(q) - \mathsf{n}(q))q^n \mid n \in \mathbb{N}_0 \big\rangle$$

of M_q is not atomic. It is enough to argue that $\mathsf{n}(q)$ cannot be written as a sum of atoms in M. We first observe that the inclusion $\mathcal{A}(M) \subseteq \{\mathsf{n}(q)q^n, (\mathsf{d}(q) - \mathsf{n}(q))q^n \mid n \in \mathbb{N}_0\}$ holds. Since $\mathsf{n}(q)q^n = \mathsf{n}(q)q^{n+1} + (\mathsf{d}(q) - \mathsf{n}(q))q^{n+1}$ for every $n \in \mathbb{N}_0$, we see that $\mathsf{n}(q)q^n \notin \mathcal{A}(M)$ for any $n \in \mathbb{N}_0$. In particular, $\mathsf{n}(q)$ is not an atom of M. Suppose, by way of contradiction, that $\mathsf{Z}_M(\mathsf{n}(q))$ is nonempty, and write

$$\mathsf{n}(q) = \sum_{i=0}^{\ell} c_i(\mathsf{d}(q) - \mathsf{n}(q))q^i$$

for some $c_0, \dots, c_\ell \in \mathbb{N}_0$. Hence $\mathsf{d}(q)^\ell \mathsf{n}(q) = \sum_{i=0}^{\ell} c_i(\mathsf{d}(q) - \mathsf{n}(q))\mathsf{n}(q)^i \mathsf{d}(q)^{\ell - i}$, from which we obtain that $\mathsf{d}(q) - \mathsf{n}(q) \mid \mathsf{d}(q)^\ell \mathsf{n}(q)$. Since $\mathsf{d}(q) - \mathsf{n}(q)$ is relatively prime with both $\mathsf{n}(q)$ and $\mathsf{d}(q)$, it follows that $\mathsf{d}(q) - \mathsf{n}(q) = 1$. However, this contradicts the inequality $\mathsf{d}(q) - \mathsf{n}(q) \ge 2$.

Although we could not answer the question of whether the torsion-free condition is superfluous in Proposition 3.3, we are inclined to believe that this is the case.

Conjecture 3.5 *Every hereditarily atomic monoid satisfies the ACCP.*

4 Atomicity in positive monoids through examples

Let us begin by considering positive monoids whose sets of nonzero elements are not in a neighborhood of 0. In terms of atomicity, this condition is quite strong for a positive monoid of the additive group of an Archimedean field: indeed, if M is such a monoid and 0 is not a limit point of $M^\bullet$ in the order topology, then M is a BFM [27, Proposition 4.5]. The same result can be easily extended to any positive monoid.

Proposition 4.1 *Let M be a positive monoid of an Archimedean ordered group G. If* 0 *is not a limit point of $M^\bullet$ in G, then M is a BFM.*

Proof By virtue of Hölder's theorem, we can identify G with an additive subgroup of $\mathbb{R}$, in which case, M is an additive submonoid of $\mathbb{R}$. Hence it follows from [27, Proposition 4.5] that M is a BFM. □

When M is a positive monoid of a totally ordered group that is not necessarily Archimedean, the fact that 0 is not a limit point of $M^\bullet$ is not a strong condition from the factorization-theoretical point of view. The following examples shed some light upon this observation.

Example 4.2 Consider the abelian group $G := \mathbb{Q} \times \mathbb{Q}$ with the lexicographical order with priority in the first component; that is, $(x, y) \le (x', y')$ in G provided that either $x < x'$ or $x = x'$ and $y \le y'$. Observe that $G^+ = (\{0\} \times \mathbb{Q}_{\ge 0}) \cup (\mathbb{Q}_{>0} \times \mathbb{Q})$.

(1) *A positive monoid of G that is antimatter.* The monoid $M := \{(0, 0)\} \cup (\mathbb{Q}_{>0} \times \mathbb{Q})$ of G is a positive monoid of G, and $(0, 0)$ is not a limit point of $M^\bullet$ with respect to the order topology of G. In addition, for each $b \in M^\bullet$, the element $\frac{1}{2}b$ also belongs to $M^\bullet$. As a consequence, M is antimatter. Our conclusion here implies that M is not Archimedean as, otherwise, the fact that $(0, 0)$ is not a limit point of $M^\bullet$ would imply that M is a BFM and, therefore, atomic.

(2) *A positive monoid of G that is neither antimatter nor atomic.* Now consider the positive monoid $M := G^+ \cap (\mathbb{Z} \times \mathbb{Z}) = (\{0\} \times \mathbb{N}_0) \cup (\mathbb{N} \times \mathbb{Z})$ of G. As in the previous example, $(0, 0)$ is not a limit point of $M^\bullet$ with respect to the order topology of G. Since $a := (0, 1)$ is the minimum nonzero element of M, it must be an atom. On the other hand, each nonzero element of M is divisible by a and, therefore, $\mathcal{A}(M) = \{a\}$. However, $\langle a \rangle = \{0\} \times \mathbb{N}_0$ is a proper submonoid of M, which implies that M is not atomic.

We proceed to construct an atomic positive monoid that is not strongly atomic.

Example 4.3 Fix $q \in \mathbb{Q}$ such that $q \in (0, 1)$ and $q^{-1} \notin \mathbb{N}$, and then set $M_q := \langle q^n \mid n \in \mathbb{N}_0 \rangle$. As mentioned in Example 3.4(2), the monoid M_q is atomic with $\mathcal{A}(M_q) = \{q^n \mid n \in \mathbb{N}_0\}$. Take distinct irrational numbers $\alpha, \beta \in \mathbb{R}_{>0}$ with $1 < \alpha < \beta$ such that the set $\{1, \alpha, \beta\}$ is linearly independent over $\mathbb{Q}$. Now set $S := \{s \in M_q \mid s < \alpha\}$. As $1 < \alpha$, the inclusion $\mathcal{A}(M_q) \subseteq S$ holds. Observe that S is a countable set. Let $\varphi \colon S \to \mathbb{P}$ be an injective function. Now consider the monoid

$$M_{\alpha,\beta} = \left\langle s, \frac{\alpha - s}{\varphi(s)}, \frac{\beta - s}{\varphi(s)} \;\middle|\; s \in S \right\rangle.$$

Because $\sup S = \alpha$ and $\alpha < \beta$, we see that $M_{\alpha,\beta}$ is a positive monoid of $\mathbb{R}$. Note that $\alpha, \beta \in M_{\alpha,\beta}$ and also that M_q is a submonoid of $M_{\alpha,\beta}$. Using the fact that the set $\{1, \alpha, \beta\}$ is linearly independent over $\mathbb{Q}$, we can verify that no element of S is divisible by either $\frac{\alpha-s}{\varphi(s)}$ or $\frac{\beta-s}{\varphi(s)}$ in $M_{\alpha,\beta}$ for any $s \in S$. From this, we can deduce that every atom of M_q is also an atom of $M_{\alpha,\beta}$; that is, $\{q^n \mid n \in \mathbb{N}_0\} \subseteq \mathcal{A}(M_{\alpha,\beta})$. Therefore $\mathcal{A}(M_{\alpha,\beta}) \cap S = \{q^n \mid n \in \mathbb{N}_0\}$. Once again we can use that $\{1, \alpha, \beta\}$ is linearly independent over $\mathbb{Q}$ to argue that $\frac{\alpha-s}{\varphi(s)}$ and $\frac{\beta-s}{\varphi(s)}$ are atoms of $M_{\alpha,\beta}$ for every $s \in S$. As a result,

$$\mathcal{A}(M_{\alpha,\beta}) = \left\{q^n \mid n \in \mathbb{N}_0\right\} \bigcup \left\{ \frac{\alpha - s}{\varphi(s)}, \frac{\beta - s}{\varphi(s)} \;\middle|\; s \in S \right\}.$$

It is clear now that every element of $M_{\alpha,\beta}$ can be expressed as a sum of atoms, which means that $M_{\alpha,\beta}$ is an atomic monoid.

Let us proceed to argue that $M_{\alpha,\beta}$ is not strongly atomic. To do so, suppose that $d \in M$ is a common divisor of α and β in $M_{\alpha,\beta}$. It follows from the linear independence of $\{1, \alpha, \beta\}$ that $\frac{\beta-s}{\varphi(s)} \nmid_{M_{\alpha,\beta}} \alpha$ for any $s \in S$. Similarly, $\frac{\alpha-s}{\varphi(s)} \nmid_{M_{\alpha,\beta}} \beta$ for any $s \in S$. Therefore $d \in M_q$, and so the inequality $d < \alpha$ ensures that $d \in S$. Now, after taking $k \in \mathbb{N}$ sufficiently large so that $d + q^k \in S$, we see that

$$\alpha - d = \varphi(d + q^k)\frac{\alpha - (d + q^k)}{\varphi(d + q^k)} + q^k \quad \text{and} \quad \beta - d = \varphi(d + q^k)\frac{\beta - (d + q^k)}{\varphi(d + q^k)} + q^k,$$

which implies that q^k is a nonzero common divisor of both $\alpha - d$ and $\beta - d$ in $M_{\alpha,\beta}$. Therefore there is no common divisor d of α and β in $M_{\alpha,\beta}$ such that the only common divisor of $\alpha - d$ and $\beta - d$ is 0. Hence $M_{\alpha,\beta}$ is not strongly atomic.

Now we consider the monoid $M := M_{\alpha,\beta} \times \mathbb{N}_0$. Observe that M is a positive monoid of the abelian group $G := \mathbb{R} \times \mathbb{Z}$ endowed with the lexicographical order with priority in the first component. Also, we note that $(0, 0)$ is not a limit point of $M^\bullet$ with respect to the order topology of G. It is routine to check that the direct product of two atomic monoids is also an atomic monoid. As a consequence, M is atomic. Since M has a divisor-closed submonoid that is isomorphic to $M_{\alpha,\beta}$, namely $M_{\alpha,\beta} \times \{0\}$, we conclude that M is not strongly atomic.

Observe that the atomic positive monoid M that is not strongly atomic in Example 4.3 has rank 4, while the monoid $M_{\alpha,\beta}$ in the same example is a rank-3 atomic monoid that is not strongly atomic. As of now, we still do not know of any atomic monoid of rank at most 2 that is not strongly atomic. The purpose of the next question is to motivate further research in this direction. Recall that a positive monoid has rank-1 if and only if it is a Puiseux monoid up to isomorphism.

Question 4.4 *Can we construct an atomic Puiseux monoid that is not strongly atomic?*

Now we exhibit a positive monoid that is strongly atomic but does not satisfy the ACCP.

Example 4.5 As in the previous example, for $q \in \mathbb{Q} \cap (0, 1)$ with $q^{-1} \notin \mathbb{N}$, consider the Puiseux monoid $M_q := \langle q^n \mid n \in \mathbb{N}_0 \rangle$ whose set of atoms is $\mathcal{A}(M_q) = \{q^n \mid n \in \mathbb{N}_0\}$. Indeed, it follows from [32, Example 3.8] and [32, Proposition 3.10(2)] that M_q is a strongly atomic monoid. Now observe that $\mathsf{d}(q)q^n = (\mathsf{d}(q) - \mathsf{n}(q))q^n + \mathsf{d}(q)q^{n+1}$ for every $n \in \mathbb{N}$, which implies that $\big(\mathsf{d}(q)q^n + M_q\big)_{n \ge 0}$ is an ascending chain of principal ideals of M_q that does not stabilize. As a result, M_q does not satisfy the ACCP. Following the lines of the last paragraph of Example 4.3, we can verify that $M := M_q \times \mathbb{N}_0$ is a positive monoid of the totally ordered abelian group $\mathbb{Q} \times \mathbb{Z}$ (under the lexicographical order with priority in the first component) such that M is strongly atomic, M does not satisfy the ACCP, and $(0, 0)$ is not a limit point of $M^\bullet$ with respect to the order topology of $\mathbb{Q} \times \mathbb{Z}$.

Besides the monoids exhibited in Examples 4.3 and 4.5, other atomic positive monoids that do not satisfy the ACCP have appeared in recent literature as ingredients to construct monoid algebras with certain factorization properties. For instance, see [16, Proposition 5.1], [30, Proposition 4.1], and [32, Section 3].

Lastly, we provide an example of a positive monoid that satisfies the ACCP but is not a BFM.

Example 4.6 Consider the Puiseux monoid $M_0 := \big\langle \frac{1}{p} \mid p \in \mathbb{P} \big\rangle$ of $\mathbb{Q}$. It is known that M_0 satisfies the ACCP (see [2, Example 3.3]). On the other hand, one can easily check that $\mathcal{A}(M_0) = \big\{ \frac{1}{p} \mid p \in \mathbb{P} \big\}$. This implies that $\mathbb{P} \subseteq \mathsf{L}_{M_0}(1)$ and, therefore, M_0 is not a BFM. Proceeding as in the previous two examples, we can see that $M := M_0 \times \mathbb{N}_0$ is a positive monoid of the totally ordered group $\mathbb{Q} \times \mathbb{Z}$ (under the lexicographical order with priority in the first component) such that $(0, 0)$ is not a limit point of $M^\bullet$ with respect to the order topology. Finally, from the fact that M_0 satisfies the ACCP but is not a BFM, we can deduce that M satisfies the ACCP but is not a BFM.

Here is an example of a positive monoid that is a BFM but not an FFM.

Example 4.7 Consider the positive monoid $M := \{0\} \cup \mathbb{R}_{\ge 1}$. Since $(\mathbb{R}, +)$ is an Archimedean ordered group and 0 is not a limit point of $M^\bullet$, it follows from Proposition 4.1 that M is a BFM. One can readily see that $\mathcal{A}(M) = [1, 2)$. For each $n \in \mathbb{N}$ with $n \ge 3$, we can observe that the formal decomposition $3 = \left(\frac{3}{2} - \frac{1}{n}\right) + \left(\frac{3}{2} + \frac{1}{n}\right)$ yields a factorization of 3 in M. Hence $|\mathsf{Z}(3)| = \infty$, and so M is not an FFM.

Before providing an example of a positive monoid that is an FFM but not a UFM, we identify a class of positive monoids that are FFMs. We say that a positive monoid is *increasing* if it can be generated by an increasing sequence. It is clear that if M is an increasing monoid, then there is a neighborhood of 0 (with respect to the order topology) that is disjoint from $M^\bullet$. Increasing positive monoids of the additive group of an ordered field are FFMs (see [27, Theorem 5.6]). The same statement holds for any increasing positive monoid.

Theorem 4.8 *Each increasing positive monoid of a totally ordered group is an FFM.*

Proof The proof of [27, Theorem 5.6] can be mimicked as it does not use the multiplicative structure of the ordered field. □

For the sake of completeness, we provide some simple examples of positive monoids that are FFMs but not UFMs.

Example 4.9 (1) Every numerical monoid is increasing and so an FFM. Thus, every numerical monoid different from $\mathbb{N}_0$ is an FFM that is not a UFM.

(2) Take $q \in \mathbb{Q}_{>1}$ such that $q \notin \mathbb{N}$, and consider the Puiseux monoid $M_q := \langle q^n \mid n \in \mathbb{N}_0 \rangle$ (the class of Puiseux monoids similarly defined but parameterized by $q \in \mathbb{Q} \cap (0, 1)$ with $q^{-1} \notin \mathbb{N}$ was already considered in Example 3.4(2) and in Examples 4.3 and 4.5). It is clear that M_q is an increasing positive monoid, so M_q is an FFM. It follows from [29, Theorem 6.2] that $\mathcal{A}(M_q) = \{q^n \mid n \in \mathbb{N}_0\}$. Since $\mathsf{n}(q) = \mathsf{n}(q) \cdot 1 = \mathsf{d}(q) \cdot q$, we see that M_q is not even an HFM.

5 Conductive positive monoids

As mentioned in the introduction, monoids similar to the one in Example 4.7 have been considered in the literature before, mainly to construct examples of commutative monoid algebras satisfying certain desired properties. With Example 4.7 in mind, we make the following more general definition.

Definition 5.1 Let G be a totally ordered group G, and take a nonzero $a \in G^+$. Then we call $M_a := \{0\} \cup G_{\ge a}$ the positive monoid of G *conducted* by a or simply a *conductive positive monoid*.

A version of conductive positive monoids was considered in [4] and a special class of them was more recently considered in [3]. Throughout this section, we restrict our attention to the atomicity of conductive positive monoids.

5.1 Bounded factorization conductive positive monoids

It immediately follows from Proposition 4.1 that when G is Archimedean, M_a is a BFM for each nonzero $a \in G^+$. However, when G is not Archimedean, M_a may not be even atomic for some nonzero $a \in G^+$. In the next proposition, for any nontrivial totally ordered group G, we provide conditions equivalent to that of M_a being atomic. First, let us take a look at the following motivating example.

Example 5.2 Consider the totally ordered group $G = \mathbb{Z} \times \mathbb{Z}$ with the lexicographical order with priority in the first component, in which case, $G^+ = (\{0\} \times \mathbb{N}_0) \cup (\mathbb{N} \times \mathbb{Z})$. Observe that $G \setminus \{0\}$ consists of two Archimedean classes, whose intersections with G^+ are $C_1 := \{0\} \times \mathbb{N}$ and $C_2 := \mathbb{N} \times \mathbb{Z}$. Fix a nonzero $a \in G^+$. If $a \in C_1$, then $v(a) \succ \min \Gamma_G$ and M_a is not atomic (cf. Proposition 5.3): indeed, $\langle [a, 2a) \rangle^\bullet \subset C_1$ and $M_a = \langle [a, 2a) \rangle \cup C_2$. If $a \in C_2$, then $v(a) = \min \Gamma_G$ and M_a is a BFM (cf. Proposition 5.3): indeed, in this case, the restriction of the function $M_a \to \mathbb{N}_0$ given by $(m, n) \mapsto m$ is clearly a length function of M_a (see [2, Proposition 3.1]).

With Example 5.2 in mind, we can characterize the conductive positive monoids that are atomic in several ways.

Proposition 5.3 *Let G be a nontrivial totally ordered group, and fix a nonzero $a \in G^+$. Then $\mathcal{A}(M_a) = [a, 2a)$, and the following conditions are equivalent.*

(a) *M_a is a BFM.*
(b) *M_a satisfies the ACCP.*
(c) *M_a is strongly atomic.*
(d) *M_a is atomic.*
(e) *M_a is nearly atomic.*
(f) *M_a is almost atomic.*
(g) *M_a is quasi-atomic.*
(h) *$M_a^\bullet \subset v(a)$.*
(i) *$v(a) = \min \Gamma_G$.*

Proof For each $x \in M_a$ with $x \geq 2a$, we see that $x - a \in M_a^\bullet$ and, therefore, $\mathcal{A}(M_a) \subseteq [a, 2a)$. On the other hand, $2a$ is a lower bound for the set $G_{\geq a} + G_{\geq a}$, which implies that $\mathcal{A}(M_a) \subseteq [a, 2a)$. Thus, $\mathcal{A}(M_a) = [a, 2a)$. Let us show now that the conditions (a)–(i) above are equivalent.

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e): This is clear.

(e) $\Rightarrow$ (f): Suppose that M is nearly atomic, and then take $b \in M$ such that $b + c$ is atomic for every nonzero $c \in M$. If $b = 0$, then M_a is atomic and, therefore, almost atomic. Suppose, on the other hand, that $b > 0$. In this case, $2b$ is an atomic element of M. In addition, for each $c \in M$, the element $2b + c = b + (b + c)$ is an atomic element of M. As a consequence, M is almost atomic.

(f) $\Rightarrow$ (g): This is clear.

(g) $\Rightarrow$ (h): Assume that M_a is quasi-atomic. Take $b \in M_a^\bullet$. As M_a is quasi-atomic, we can pick $c \in M_a$ such that $b + c = a_1 + \cdots + a_n$ for some $a_1, \dots, a_n \in [a, 2a)$. Hence $b \leq b + c = a_1 + \cdots + a_n < 2na$. This implies that $b = \mathbf{O}(a)$, and it is clear that $a = \mathbf{O}(b)$. Thus, $b \sim a$ for all nonzero $b \in M_a$, whence the inclusion $M_a^\bullet \subset v(a)$ holds.

(h) $\Rightarrow$ (i): Suppose that the inclusion $M_a^\bullet \subset v(a)$ holds. Fix a nonzero $g \in G$, and let us verify that $v(g) \succeq v(a)$. After replacing g by $-g$ if necessary, we can assume that $g \in G^+$. If $g < a$, then $v(g) \succeq v(a)$ and, otherwise, $g \in M_a \subset v(a)$, which implies that $v(g) = v(a)$.

(i) $\Rightarrow$ (a): Assume that $v(a) \preceq v(g)$ for all nonzero $g \in G$. Fix $b \in M_a^\bullet$. The fact that $v(a) \preceq v(b)$ implies that $b = \mathbf{O}(a)$ and, therefore, $b \le Na$ for some $N \in \mathbb{N}$. Now suppose that $b = a_1 + \cdots + a_\ell$ for some $a_1, \ldots, a_\ell \in M_a^\bullet$. Then $\ell a \le a_1 + \cdots + a_\ell = b$ and so $\ell a \le b \le Na$. Thus, b cannot be written as a sum of more than N elements of $M_a^\bullet$. Now assuming that ℓ is as large as it could possibly be, we obtain that the summands in the right-hand side of $b = a_1 + \cdots + a_\ell$ are atoms of $M_a^\bullet$, which implies not only that b is an atomic element of M_a but also that $\mathsf{L}_{M_a}(b)$ is bounded. Hence M_a is a BFM. $\square$

No two of the conditions (a)–(d) in Proposition 5.3 are equivalent in the class of positive monoids. Indeed, we have seen that the positive monoid M_0 of Example 4.6 satisfies the ACCP but is not a BFM, the positive monoid M_q of Example 4.5 is strongly atomic but does not satisfy the ACCP, and the positive monoid $M_{\alpha,\beta}$ of Example 4.3 is atomic but not strongly atomic. Now we proceed to provide examples to illustrate that, in the class of positive monoids, no two of the conditions (d)–(g) are equivalent. Using the idea in Example 4.3, we begin with an example of a positive monoid that is nearly atomic but not atomic.

Example 5.4 Let α be a positive irrational number, and let $\varphi \colon \mathbb{Q}_{\ge 0} \to \mathbb{P}$ be an injective function. Then consider the positive monoid M of the totally ordered abelian group $\mathbb{R}$ defined as follows:

$$M := \left\langle q, \frac{\alpha + q}{\varphi(q)} \;\middle|\; q \in \mathbb{Q}_{\ge 0} \right\rangle.$$

From the irrationality of α, one can readily argue that $\frac{\alpha+q}{\varphi(q)} \in \mathcal{A}(M)$ for all $q \in \mathbb{Q}_{\ge 0}$. We proceed to show that M is nearly atomic. In order to do so, we first observe that $\alpha = \varphi(0)\frac{\alpha}{\varphi(0)} \in M$, and then we claim that $\alpha + r$ is an atomic element of M for each $r \in M$. Take $r \in M$, and observe that we can write $r = q_0 + a_1 + \cdots + a_n$ for some $q_0 \in \mathbb{Q}_{\ge 0}$ and $a_1, \ldots, a_n \in \mathcal{A}(M)$. Then

$$\alpha + r = (\alpha + q_0) + \sum_{k=1}^{n} a_k = \varphi(q_0)\frac{\alpha + q_0}{\varphi(q_0)} + \sum_{k=1}^{n} a_k,$$

which illustrates that $\alpha + r$ is an atomic element of M. As a consequence, M is nearly atomic. Finally, the irrationality of α guarantees that no element in $M \cap \mathbb{Q}_{>0}$ can be written as a sum of atoms in M, whence M is not atomic.

Let us now construct an almost atomic positive monoid that is not nearly atomic.

Example 5.5 Let M_0 be the monoid generated by the reciprocals of the prime numbers; that is, $M_0 = \left\langle \frac{1}{p} \mid p \in \mathbb{P} \right\rangle$, and then let G be the difference group of M_0. Now consider the Puiseux monoid $M := M_0 \cup G_{\geq 1}$. We mentioned in Example 4.6 that $\mathcal{A}(M_0) = \left\{ \frac{1}{p} \mid p \in \mathbb{P} \right\}$. Therefore the fact that $\max \mathcal{A}(M_0) < 1$ ensures that no element of $G_{\geq 1}$ divides any atom of M_0 in M. As a consequence, $\{\frac{1}{p} \mid p \in \mathbb{P}\} \subseteq \mathcal{A}(M)$. Moreover, because each element in $M_{>1}$ is divisible by an atom $\frac{1}{p}$ in M for a sufficiently large $p \in \mathbb{P}$, it follows that $\mathcal{A}(M) = \{\frac{1}{p} \mid p \in \mathbb{P}\}$. Since any element of M can be written as a difference between two elements of M_0, any element of M can be written as a difference between two sums of atoms of M. Thus, M is almost atomic.

On the other hand, we claim that M is not nearly atomic. Suppose, for the sake of a contradiction, that there exists $q \in M$ such that $q + r$ is an atomic element in M for all $r \in M^\bullet$. Let P_q be the set of all primes dividing $\mathsf{d}(q)$. Since P_q is a finite set, and the series $\sum_{n \in \mathbb{N}} \frac{1}{p_n}$ is divergent (here p_n is the n-th prime number), we can find a nonempty finite subset S of $\mathbb{P}$ such that $S \cap P_q$ is empty and $\sum_{p \in S} \frac{1}{p} > q + 2$. Now consider the element $r := 1 + \prod_{p \in S} \frac{1}{p}$. Observe that $r \in M^\bullet$ and $r < 2$. Because $q + r$ is atomic in M, there exist $p'_1, \dots, p'_k \in \mathbb{P}$ not necessarily distinct such that $q + r = \sum_{j=1}^k \frac{1}{p'_j}$. Now the fact that $S \cap P_q$ is empty implies that $S \subseteq \{p'_1, \dots, p'_k\}$, and so

$$q + r = \sum_{j=1}^{k} \frac{1}{p'_j} \geq \sum_{p \in S} \frac{1}{p} > q + 2,$$

which contradicts that $r < 2$. As a consequence, we conclude that M is an almost atomic monoid that is not nearly atomic.

Lastly, we construct a quasi-atomic positive monoid that is not almost atomic.

Example 5.6 Let M be the additive submonoid of $\mathbb{Q}_{\geq 0}$ generated by the set $S := \mathbb{Z}[\frac{1}{2}]_{\geq 0} \cup \mathbb{Z}[\frac{1}{3}]_{\geq 4/3}$, which is a positive monoid of $\mathbb{Q}$. It is clear that $\frac{4}{3} \in \mathcal{A}(M)$. Hence each element in $4\mathbb{N}$ is an atomic element in M. Now for each $q \in M^\bullet$, we can take the element $b := (4\mathsf{d}(q) - 1)q$, and conclude that $b + q = 4\mathsf{n}(q)$ is an atomic element in M. Hence M is quasi-atomic.

To argue that M is not almost atomic, we first verify that $\mathsf{d}(q)$ is a power of 3 for every $q \in \mathcal{A}(M)$. To do so, let q be an atom of M. Observe that $6 \nmid \mathsf{d}(q)$ as, if this were not the case, an element in the defining generating set S of M would have its denominator divisible by 6. In addition, observe that $\mathsf{d}(q)$ cannot be a power of 2 as, otherwise, $\frac{1}{2}q$ would divide q in M. Hence the denominator of each atom of M is a power of 3. As a consequence, $\mathcal{A}(M) \subseteq \mathbb{Z}[\frac{1}{3}]_{\geq 4/3}$, which implies that the difference group G of $\langle \mathcal{A}(M) \rangle$ is contained in $\mathbb{Z}[\frac{1}{3}]$. Therefore the fact that $\frac{1}{2} \in M \setminus G$ guarantees that M is not almost atomic.

5.2 Finite factorization conductive positive monoids

Our primary task in this subsection is to characterize the conductive positive monoids that are FFMs. It turns out that the characterization that we proceed to establish is based solely on the defining groups.

Proposition 5.7 *Let G be a nontrivial totally ordered group, and fix a nonzero $a \in G^+$. Then the following conditions are equivalent.*

(a) *M_a is an FFM.*
(b) *G is a cyclic group.*

Proof (a) ⇒ (b): Suppose first that M_a is an FFM. Then M_a is atomic, and so it follows from Proposition 5.3 that $\mathcal{A}(M_a) = [a, 2a)$ and $v(a) = \min \Gamma_G$. Now observe that for each $b \in [a, 2a)$, the element $3a - b$ belongs to $[a, 2a)$ and, therefore, $b + (3a - b)$ is a factorization of $3a$ in M_a. As a consequence, the fact that $|\mathsf{Z}_{M_a}(3a)| < \infty$ guarantees that $|[a, 2a)| < \infty$. Now observe that if there existed $g \in G \backslash \{0\}$ such that $v(a) \prec v(g)$, then $a + (G^+ \cap v(g)) \subseteq (a, 2a)$, which is not possible because $(a, 2a)$ is finite and $G^+ \cap v(g)$ contains the infinite set $\mathbb{N}|g|$. Hence $|\Gamma_G| = 1$, which means that G is Archimedean. By virtue of Hölder's theorem, we can assume that G is a nontrivial additive subgroup of $\mathbb{R}$.

For the second part of the proof, we need the following claim.
Claim: G is dense in $\mathbb{R}$ unless G is cyclic.
Proof of Claim: Suppose that $\operatorname{rank} G \geq 2$, and take integrally independent elements $g, h \in G$; that is, the elements g and h are linearly independent over $\mathbb{Q}$. Then $h \neq 0$ and $\frac{g}{h} \notin \mathbb{Q}$. It is well known and not hard to verify that the set $\{m + n\frac{g}{h} \mid m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Hence G is dense in $\mathbb{R}$. On the other hand, suppose that $\operatorname{rank} G = 1$. Fix a nonzero $g_0 \in G$, and then set $G_0 := \frac{1}{g_0}G$. It is clear that G_0 is a copy of G inside $\mathbb{Q}$. It is well known that every subgroup of $\mathbb{Q}$ is the union of a (possibly infinite) ascending sequence of cyclic groups (see [26, Corollary 2.8]). Therefore G_0 is either an infinite cyclic group or 0 is a limit point of $G_0 \backslash \{0\}$, which implies that G_0 is dense in $\mathbb{Q}$. As a consequence, G is either an infinite cyclic group or a dense subset of $\mathbb{R}$, and the claim follows.

Now assume, for the sake of a contradiction, that G is not cyclic. Then it follows from the established claim that G is dense in $\mathbb{R}$. However, this is a contradiction as the interval $[a, 2a)$ of G is finite. Hence G must be a cyclic subgroup of $\mathbb{R}$.

(b) ⇒ (a): Finally, suppose that $G \cong \mathbb{Z}$. Since M_a is a submonoid of G^+ and $G^+ \cong \mathbb{N}_0$, it follows that M_a is isomorphic to a numerical monoid and, therefore, it is finitely generated. Hence M_a is an FFM. □

We can use Propositions 5.3 and 5.7 in tandem to construct further examples of BFMs that are not FFMs (see Example 4.7). To illustrate this, we revisit Example 5.2.

Example 5.8 For the totally ordered group $G = \mathbb{Z} \times \mathbb{Z}$ with the lexicographical order with priority in the first component, we have seen in Example 5.2 that $G \setminus \{0\}$ consists of two Archimedean classes and that M_a is a BFM provided that $a \in G^+$ and $v(a) = \min \Gamma_G$. In addition, in light of Proposition 5.7, the fact that $|\Gamma_G| = 2$ ensures that M_a is not an FFM.

5.3 Length-factorial and half-factorial conductive positive monoids

In this final subsection, our main purpose is to characterize the conductive positive monoids that are LFMs as well as the conductive positive monoids that are HFMs. Let us begin with the conductive positive monoids that are LFMs.

Proposition 5.9 *Let G be a nontrivial totally ordered group, and fix a nonzero $a \in G^+$. Then the following conditions are equivalent.*

(a) *M_a is an LFM.*
(b) *$G = \mathbb{Z}b$ for some nonzero $b \in G^+$, and $a \in \{b, 2b\}$.*

Proof (a) $\Rightarrow$ (b): Suppose that M_a is an LFM. Then M_a is also an FFM and, therefore, the group G is cyclic by virtue of Proposition 5.7. Take a nonzero $b \in G$ such that $G = \mathbb{Z}b$. After replacing b by $-b$ if necessary, we can assume that $b \in G^+$. Assume, by way of contradiction, that $a \geq 3b$. It follows from Proposition 5.3 that $\mathcal{A}(M_a) = [\![a, 2a - b]\!]$, and the fact that $a \geq 3b$ guarantees that $|\mathcal{A}(M_a)| \geq 3$. Take $c_1, c_2, c_3 \in \mathbb{N}_{\geq 3}$ such that $c_1 b$, $c_2 b$, and $c_3 b$ are distinct atoms of M_a and assume, without loss of generality, that $c_1 < c_2 < c_3$. Now take $m, n \in \mathbb{N}$ such that $m(c_2 - c_1) = n(c_3 - c_2)$, and observe that $(m+n)c_2 b$ and $mc_1 b + nc_3 b$ yield two distinct factorizations of the same element of M_a having the same length. This contradicts, however, that M_a is an LFM. Hence $a \leq 2b$ and, as a is a nonzero element in the nonnegative cone of $\mathbb{Z}b$, we can conclude that $a \in \{b, 2b\}$.

(b) $\Rightarrow$ (a): Suppose now that $G = \mathbb{Z}b$ for some nonzero $b \in G^+$, and $a \in \{b, 2b\}$. Since G is the infinite cyclic group, M_a is isomorphic to a submonoid of the additive monoid $\mathbb{N}_0$. As M_a is generated by at most two elements, it follows from [18, Example 2.13] that M_a is an LFM. □

We conclude discussing the half-factorial property in the context of conductive positive monoids. Recall that an atomic monoid M is half-factorial (or an HFM for short) if any two factorizations of the same element have the same length. Half-factoriality has been systematically studied for almost four decades (see the recent paper [37] and references therein): a survey on the advances on half-factoriality until 2000 was given by Chapman and Coykendall in [9]. It turns out that for conductive positive monoids the condition of being a UFM and that of being an HFM are equivalent. In fact, we can characterize both properties as follows.

Proposition 5.10 *Let G be a nontrivial totally ordered group, and fix a nonzero $a \in G^+$. Then the following conditions are equivalent.*

(a) *M_a is a UFM.*
(b) *M_a is an HFM.*
(c) *G is cyclic, and $M_a = G^+$.*

Proof (a) $\Rightarrow$ (b): This follows from the definitions.

(b) $\Rightarrow$(c): Assume that M_a is an HFM. We claim that the open interval $(a, 2a)$ is empty. Suppose, by way of contradiction, that this is not the case, and take $b \in (a, 2a)$. Since $3a - b \in (a, 2a)$, it follows from Proposition 5.3 that $3a - b \in \mathcal{A}(M_a)$. Since

$$[\,\textbf{UFM} \iff \textbf{HFM}\,] \implies \textbf{LFM} \implies \textbf{FFM} \implies [\,\textbf{BFM} \iff \textbf{QAM}\,]$$

Fig. 4 Adaptation of the chain of implications shown in Fig. 3 for the class of conductive positive monoids. None of the three implication arrows is reversible in this class. The atomic properties between being a BFM and being a quasi-atomic monoid in Proposition 5.3 have been omitted for simplicity

$a, b \in \mathcal{A}(M_a)$ by the same proposition, the equality $a+a+a = b+(3a-b)$ yields two factorizations of the same element with different lengths, which contradicts that M_a is an HFM. Thus, $(a, 2a)$ is empty, as we claimed. Then it follows from Proposition 5.3 that $\mathcal{A}(M_a) = \{a\}$. Now if $c \in G^+ \cap [0, a)$, then the fact that $2a - c \in (a, 2a]$ ensures that $c = 0$. Therefore a must be the smallest nonzero element of G^+, which implies that $M_a = G^+$. Moreover, since M_a is atomic, every element of M_a must be the sum of copies of a, which is the only atom of M_a. As a consequence, $G^+ = M_a = \mathbb{N}_0 a$, and so $G = \mathbb{Z}a$ is the cyclic group.

(c) $\Rightarrow$ (a): If G is cyclic and $M_a = G^+$, then $M_a \cong \mathbb{N}_0$, whence M_a is a UFM. □

In the direction of Proposition 5.10, it is worth mentioning that the property of being a UFM and that of being an HFM are also equivalent in the class consisting of all rank-one positive monoids. Indeed, every positive monoid is torsion-free, and it follows from [22, Theorem 3.12] that every rank-one torsion-free monoid is either a group or a Puiseux monoid (up to isomorphism). In addition, it was proved in [28, Proposition 4.3] that a Puiseux monoid is an HFM if and only if it is a UFM. Therefore we obtain the following remark.

Remark 5.11 A rank-one positive monoid is a UFM if and only if it is an HFM.

For higher ranks, however, there are positive monoids that are HFM but not UFM. Indeed, in the following example we exhibit a rank-two positive monoid that is an HFM but is not even an FFM.

Example 5.12 Consider the totally ordered group $G = \mathbb{Z} \times \mathbb{Z}$ with the lexicographical order with priority in the first component, in which case, $G^+ = (\{0\} \times \mathbb{N}_0) \cup (\mathbb{N} \times \mathbb{Z})$. Now consider the monoid $M := \{(0, 0)\} \cup (\mathbb{N} \times \mathbb{Z})$, which is a positive monoid of G. We first observe that $\mathcal{A}(M) = \{(1, n) \mid n \in \mathbb{Z}\}$ and, therefore, M is atomic. In addition, we can see that each factorization of an element $(m, n) \in M^\bullet$ consists of precisely m atoms (counting repetitions). Hence M is an HFM. On the other hand, since $(2, 0) = (1, -n) + (1, n)$ for every $n \in \mathbb{N}$, the element $(2, 0)$ has infinitely many factorizations in M (indeed, one can similarly see that every nonzero element of M that is not an atom has infinitely many factorizations). As a consequence, M is not an FFM.

We conclude this paper considering the atomic diagram in Fig. 3 restricted to the class of conductive positive monoids. The implications in the diagram shown in Fig. 4 summarize the main results we have established in this section.

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