AFFINE SPRINGER FIBERS, PROCESI BUNDLES, AND CHEREDNIK ALGEBRAS

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Abstract

Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{t}$ be its Cartan subalgebra, and let W be the Weyl group. The goal of this paper is to prove an isomorphism between suitable completions of the equivariant Borel–Moore homology of certain affine Springer fibers for $\mathfrak{g}$ and the global sections of a bundle related to a Procesi bundle on the smooth locus of a partial resolution of $(\mathfrak{t} \oplus \mathfrak{t}^*)/W$. We deduce some applications of our isomorphism including a conditional application to the center of the small quantum group. Our main method is to compare certain bimodules over rational and trigonometric Cherednik algebras.

Contents

1. Introduction

Let G be a connected reductive algebraic group over $\mathbb{C}$, let $\mathfrak{g}$ be its Lie algebra, let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra, let T be the corresponding maximal torus, and let W be the Weyl group. Pick a nonnegative integer d.

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The goal of this paper is to relate two different geometric objects, "coherent" and "constructible," constructed from these data.

First, we describe the "coherent" object. Consider the Poisson variety $Y := (\mathfrak{t} \oplus \mathfrak{t}^*)/W$. We will choose a suitable partial Poisson resolution X of Y (Section 2.1). For example, in the case of $G = \mathrm{GL}_n$, the variety X is going to be the Hilbert scheme of points in $\mathbb{C}^2$. When $\mathfrak{g}$ is simply laced, X is going to be the so-called *$\mathbb{Q}$-factorial terminalization of Y* (see [9] for the general construction or [38, Section 2.2] for a discussion in the present settings). In types B/C, F_4, and G_2 we get some intermediate partial resolution. See Section 2.1 for details. In all cases, we are going to have $\mathrm{codim}_X X^{\mathrm{sing}} \geq 4$.

The smooth locus X^{reg} comes with several important vector bundles. There is a "Procesi bundle" $\mathcal{P}^{\mathrm{reg}}$ that will be constructed in Section 2.2 based on results from [38]. One important property of $\mathcal{P}^{\mathrm{reg}}$ we need right now is that its endomorphism algebra is

$$H := \mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]\#W. \tag{1.1}$$

In the case when $G = \mathrm{GL}_n(\mathbb{C})$, we recover Haiman's Procesi bundle on the Hilbert scheme (see [28]). When $\mathfrak{g}$ is simply laced and hence X is a $\mathbb{Q}$-factorial terminalization of Y, $\mathcal{P}^{\mathrm{reg}}$ is the restriction of the Procesi sheaf $\mathcal{P}$ on X (see [38, Section 4]) to X^{reg}. In types B/C, F_4, and G_2, we consider the sheaf $\mathcal{P}$ obtained by the pushforward of the Procesi sheaf from the $\mathbb{Q}$-factorial terminalization to X and then restrict it to X^{reg}. The sign invariants in $\mathcal{P}^{\mathrm{reg}}$ is a line bundle to be denoted by $\mathcal{O}^{\mathrm{reg}}(1)$. Its dth tensor power will be denoted by $\mathcal{O}^{\mathrm{reg}}(d)$.

So, for $d \in \mathbb{Z}_{>0}$, we can consider the H-bimodule

$$B_d := \Gamma\big(X^{\mathrm{reg}}, \mathcal{P}^{\mathrm{reg},*} \otimes \mathcal{O}^{\mathrm{reg}}(d) \otimes \mathcal{P}^{\mathrm{reg}}\big). \tag{1.2}$$

This is the first of the two objects we are interested in.

There are a number of reasons to be interested in the bimodule B_d. First, consider the case when $G = \mathrm{GL}_n$. The bimodule B_d is closely related to the dth power of the so-called *∇-operator on symmetric polynomials* (cf. [12]). In more detail, the algebra H and the variety X come with natural actions of $(\mathbb{C}^\times)^2$. The functor $R\,\mathrm{Hom}(\mathcal{P}, \bullet)$ gives an equivalence from the derived category of $(\mathbb{C}^\times)^2$-equivariant coherent sheaves on X to the derived category of bigraded H-modules (see, e.g., [29, Theorem 5.3.2]). The K_0-groups of both categories are identified with the degree n symmetric polynomials with coefficients in $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ in such a way that the irreducible H-modules in bi-degree $(0, 0)$ are sent to the Schur polynomials. The operator of tensoring with B_d is the ∇-operator of Bergeron and Garcia (in the Hilbert scheme intepretation this operator is just the twist by $\mathcal{O}(1)$). It would be interesting to see whether this observation can be generalized to the case of general G.

Another reason to care about B_d is that these bimodules (or their variants) are expected to appear in a variety of other contexts. The subject of this paper is their connection to the affine Springer theory. Another prospective appearance is the study of character sheaves on semisimple Lie algebras and the usual Springer theory: the bimodules B_d are expected to be related to the central elements $T_{w_0}^{2d}$ in the Hecke category. A related appearance should be in the study of invariants of torus knots (see [27]).

The second object we care about, the "constructible" one, is the equivariant Borel–Moore homology of a suitable affine Springer fiber for the group G.

Fix a regular element $s \in \mathfrak{t}$. Let t be an indeterminate so that we can form the loop algebra $\mathfrak{g}((t))$. Consider the element $e_d := st^d \in \mathfrak{g}((t))$. This element gives rise to the affine Springer fiber $\mathcal{F}l_{e_d}$ in the affine flag variety $\mathcal{F}l$ for G; sometimes it is called an *equivalued unramified* affine Springer fiber. The maximal torus T, the centralizer of s, acts on $\mathcal{F}l_{e_d}$. So we can consider the equivariant Borel–Moore homology $H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d})$.

It turns out that $H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d})$ also carries a bimodule structure but for a somewhat different algebra. Namely, let $T^\vee$ denote the Langlands dual torus. Consider the algebra $H^\times := \mathbb{C}[T^*T^\vee]\#W$. The algebra $H^\times$ acts on $H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d})$ from the left by what we call the *CS (Chern–Springer) action*; such an action exists for any homogenous affine Springer fiber, as the construction in Section A.1 shows. For our particular choices of e_d we also have a commuting $H^\times$-action that we call the *ECM (equivariant-centralizer-monodromy) action*. The action of the Weyl group W in the ECM action can be related in type A to the S_n-action introduced in [49] on the cohomology of Hessenberg varieties. This action in the Hessenberg context is further studied in [48].

Now we explain a relation between H and $H^\times$. For $G = \mathrm{GL}_n$, the algebra $H^\times$ is a localization of H. In the general case, the algebras H and $H^\times$ share a common "completion." Namely, set

$$H^\wedge := H \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}[\mathfrak{t}]^{\wedge_0}, \tag{1.3}$$

where $\mathbb{C}[\mathfrak{t}]^{\wedge_0}$ is the completion of $\mathbb{C}[\mathfrak{t}]$ at zero. The same algebra $H^\wedge$ arises as $H^\times \otimes_{\mathbb{C}[T]} \mathbb{C}[T]^{\wedge_1}$, where we identify $\mathbb{C}[\mathfrak{t}]^{\wedge_0}$ with $\mathbb{C}[T]^{\wedge_1}$ by means of $\exp : \mathfrak{t} \to T$. Then we consider

$$B_d^\wedge := B_d \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}[\mathfrak{t}]^{\wedge_0},$$

an $H^\wedge$-bimodule, as well as

$$H_{\mathrm{BM}}^T(\mathcal{F}l_{e_d})^\wedge := H_{\mathrm{BM}}^T(\mathcal{F}l_{e_d}) \otimes_{\mathbb{C}[T]} \mathbb{C}[T]^{\wedge_1},$$

also an $H^\wedge$-bimodule.

THEOREM 1.1
There is an $H^\wedge$-bilinear isomorphism $B_d^\wedge \xrightarrow{\sim} H_T^{BM}(\mathcal{F}l_{e_d})^\wedge$.

Note that both sides are graded: $H_T^{BM}(\mathcal{F}l_{e_d})^\wedge$ is graded by the homological degree, and $B_d^\wedge$ from a $\mathbb{C}^\times$-equivariant structure on $\mathcal{P}^{\text{reg},*} \otimes \mathcal{O}^{\text{reg}}(d) \otimes \mathcal{P}^{\text{reg}}$ that will be explained in Section 3.3. We will see below that one can achieve that the isomorphism in Theorem 1.1 is grading preserving.

Now we explain how Theorem 1.1 relates to the previous work. In [32], Kivinen studied the spherical version of $\mathcal{F}l_{e_d}$ and proved a spherical version of Theorem 1.1 in the case of $G = \text{GL}_n$. "Spherical" means that B_d is replaced with ϵB_d for the trivial idempotent ϵ in $\mathbb{C}W = \mathbb{C}S_n$. On the level of Springer fibers, this means that we take the Springer fiber in the affine Grassmannian instead of the affine flag variety. Also note that Kivinen works with localizations, which is only possible for $G = \text{GL}_n$. Even stronger, one can prove an analog of Theorem 1.1 for localizations using the methods of this paper, but we are not going to discuss this. In fact, one should be able to prove a version of Theorem 1.1 for B_d itself and a suitable modification of $\mathcal{F}l_{e_d}$, but this will be addressed elsewhere.

The bimodule B_1 for $G = \text{GL}_n$ also appears in the recent paper of Carlsson and Mellit (see [12, Conjecture 3.7]). Note that a statement similar to Theorem 1.1 (in the GL_n-case) is conjectured in [12, Section 3.3]. We will deduce (see [12, Conjecture 3.7]) from Theorem 1.1 combined with other statements that are used in its proof in Section 7.3. A motivation for [12] was to get some geometric understanding of the ∇-conjecture (saying that the image of every Schur polynomial under the ∇-operator is Schur positive up to a sign). As we have mentioned above, the ∇-operator is directly related to the bimodule B_1. On the other hand, [13] contains an interpretation of a stronger version of the ∇-conjecture in terms of the geometry of spaces related to the affine Springer fibers. This serves as a motivation for having results like Theorem 1.1.

Here is another important application of Theorem 1.1. Let $\mathbb{C}_{\text{triv}}$ denote the 1-dimensional irreducible representation of $H^\wedge$, where $\mathfrak{t}$ and $\mathfrak{t}^*$ act by 0, and W acts via the trivial representation.

THEOREM 1.2
We have

$$\dim B_d \otimes_H \mathbb{C}_{\text{triv}} = \dim H_T^{BM}(\mathcal{F}l_{e_d}) \otimes_{H^\times} \mathbb{C}_{\text{triv}} = (dh+1)^{\dim \mathfrak{t}},$$

where h denotes the Coxeter number of W. Moreover, as a W-module, $B_d \otimes_H \mathbb{C}_{\text{triv}}$ is isomorphic to $\mathbb{C}(\Lambda_0/(dh+1)\Lambda_0)$, where we write Λ_0 for the coroot lattice.

In fact, we show that the first dimension is $\geq (dh+1)^{\dim \mathfrak{t}}$, while the second dimension is $\leq (dh+1)^{\dim \mathfrak{t}}$. The latter is done by using an argument similar to one in [5, Proposition 2.9].

Now we explain a reason to be interested in $H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d})$. It is expected that for $d=1$, this bimodule is closely related to the center of the principal block of the small quantum group $\mathfrak{u}_\epsilon(\mathfrak{g}^\vee)$, where ϵ is an odd root of unity (see [5, Theorem 4.12]). We remark that

$$\left(H^T_{\mathrm{BM}}(\mathcal{F}l_{e_1})^\wedge \otimes_{\mathbb{C}[T^*T^\vee]^\wedge} \mathbb{C}_{\mathrm{triv}}\right)^* = H^*(\mathcal{F}l_{e_1})^\Lambda,$$

where Λ stands for the character lattice of T. Let $G^\vee$ denote the Langlands dual group, and let $T^\vee$ be its maximal torus. Let Z denote the center of the principal block of $\mathfrak{u}_\epsilon(\mathfrak{g}^\vee)$. The group $G^\vee$ acts on Z by algebra automorphisms. The main conjecture of [5, Conjecture A] relates the subalgebra $Z^{T^\vee}$ of Z to the cohomology of $\mathcal{F}l_{e_1}$ (there are also connections of the equivariant cohomology to the center, but we are not going to discuss that). Namely, it is conjectured in [5, Conjecture A] that $Z^{T^\vee}$ is isomorphic to $H^*(\mathcal{F}l_e)^\Lambda$. Modulo the conjecture from [5, Conjecture A], Theorem 1.2 shows that the dimension of the W-invariant part in $Z^{T^\vee}$ has dimension $(h+1)^{\dim \mathfrak{t}}$.

For $G=\mathrm{SL}_n$, we can say more. Using Theorem 1.1 combined with Haiman's $n!$ theorem [28], one can show that, modulo the conjecture from [5, Conjecture A], W acts trivially on $Z^{T^\vee}$. This implies that $G^\vee$ acts trivially on Z, so $\dim Z = (n+1)^{n-1}$. This will confirm a conjecture from [34]. See Section 7.4 for details.

Now we explain two key ideas of the proof of Theorem 1.1. First, unsurprisingly, we use the induction on d. Our second, and main, idea is to use a one-parameter deformation: it turns out that we can deform both B_d and $H^T_{\mathrm{BM}}(\mathcal{F}l_{e_d})$. For a complex number c, we can consider the rational Cherednik algebra $H_{\hbar,c}$ over $\mathbb{C}[\hbar]$ (see Section 2.3), deforming H, and the trigonometric Cherednik algebra $H^\times_{\hbar,c}$ (see Section 2.4), deforming $H^\times$. The H-bimodule B_d deforms to a bimodule over $H_{\hbar,d}$ (acting on the left) and $H_{\hbar,0}$ (acting on the right). This is achieved by quantizing the Procesi bundle $\mathcal{P}^{\mathrm{reg}}$ and the line bundle $\mathcal{O}^{\mathrm{reg}}(1)$. The bimodule $H^T_{\mathrm{BM}}(\mathcal{F}l_{e_d})$ deforms to a bimodule over $H^\times_{\hbar,d}$ and $H^\times_{\hbar,0}$. The deformation in this case is done by considering the equivariant BM homology for $T\times\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts by the loop rotation. Note that, for each $c\in\mathbb{C}$, the algebras $H_{\hbar,c}$ and $H^\times_{\hbar,c}$ share common partial completions (at 0 and 1, resp.). We will see that we have a deformed version of the isomorphism from Theorem 1.1, which turns out to be easier to establish.

In fact, the representations of rational Cherednik algebras appeared in the context of affine Springer theory previously (see [47]). In particular, for a suitable "elliptic" element e'_d (different from e_d), it was shown that $H^{\mathrm{BM}}_{\mathbb{C}^\times}(\mathcal{F}l_{e'_d})$ admits a filtration (the so-called *perverse filtration*) with an action of $H_{\hbar,d+1/h}$ on the associated graded space turning $\operatorname{gr} H^{\mathrm{BM}}_{\mathbb{C}^\times}(\mathcal{F}l_{e'_d})$ into a deformation of the unique irreducible finite dimensional module of the quotient $H_{d+1/h} := H_{\hbar,d+1/h}/(\hbar-1)$. Some techniques we use are the same as in [47]: both our action of $H^\times_{d,\hbar}$ and their action of $H^\times_{d,\hbar}$ on the equivariant cohomology come from the Springer–Chern construction. But this is where the similarities essentially end; for example, there is no bimodule structure in their construction, and our techniques of identifying $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ with a Cherednik algebra bimodule are very different. Note also that there is no connection between $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ and $H^{\mathrm{BM}}_{\mathbb{C}^\times}(\mathcal{F}l_{e'_d})$. However, there is a connection between $H^{\mathrm{BM}}(\mathcal{F}l_{e_d})$ and $H^{\mathrm{BM}}(\mathcal{F}l_{e'_d})$ (see [5, Corollary 2.14]).

Note that the bimodule B_d admits a bigrading, while $H^{\mathrm{BM}}_T(\mathcal{F}l_{e_d})$ a priori only has one grading. A question that remains open is to understand the second grading of B_d under the isomorphism in Theorem 1.1 on $H^{\mathrm{BM}}_T(\mathcal{F}l_{e_d})$. In the result of [47], this second grading is understood as coming from the perverse filtration, but this filtration is easy to construct for elliptic elements of $\mathfrak{g}((t))$ and is more subtle in the case of nonelliptic elements such as e_d. The combinatorics of the bigraded module $H^{\mathrm{BM}}(\mathcal{F}l_{e_{d'}})$ has been studied in [14], [26], and [30] in the case of elliptic elements such as those studied in [47].

We finish the introduction by describing the content of the paper. In Section 2 we discuss generalities on partial Poisson resolutions of $Y = (\mathfrak{t}\oplus\mathfrak{t}^*)/W$, Procesi sheaves on them, and rational and trigonometric Cherednik algebras. This section mostly contains known results and their easy modifications.

In Section 3 we construct a deformation of B_d. A key result used in the construction is that the pushforward from X^{reg} to X of the vector bundle $\mathcal{P}^{\mathrm{reg},*}\otimes\mathcal{O}^{\mathrm{reg}}(d)\otimes\mathcal{P}^{\mathrm{reg}}$ is a Cohen–Macaulay sheaf without higher cohomology. Two key ingredients for this result are the construction of the Procesi sheaves via quantizations in characteristic p and the following claim of independent interests: the pushforward to X of a line bundle on X^{reg} is Cohen–Macaulay.

In Section 4 we provide some background on the equivariant Borel–Moore homology and equivalued unramified affine Springer fibers. This section does not contain any new results.

In Section 5 we construct actions of $H^\times_{\hbar,d}$, $H^\times_{\hbar,0}$ on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ and establish some properties of the resulting bimodule. A key technique is the localization theorem for equivariant BM homology. This section relies on an Appendix by the authors and Kivinen to check the relations for the action of $H^\times_{\hbar,d}$.

In Section 6 we prove Theorems 1.1 and 1.2. And then in Section 7 we discuss applications of our own main results to conjectures of Carlsson and Mellit and to the center of the small quantum group.

The bimodules B_d and related objects were previously studied mostly in type A. The proof of Theorem 1.1 does simplify in this case as many prerequisite constructions are easier. The reader interested in type A only can essentially skip Section 2.1 and parts of Section 3: the claim of Proposition 3.2 is vacuous in type A, and (2) of Proposition 3.1 follows directly from (1) of that proposition.

2. Procesi sheaves and Cherednik algebras

In this section we recall various generalities related to the algebras $H = \mathbb{C}[T^*\mathfrak{t}^*]\#W$, $H^\times = \mathbb{C}[T^*T^\vee]\#W$, their deformations—the rational and trigonometric Cherednik algebras, and the bimodule B_d. In particular, we discuss a partial resolution X of Y, and Procesi sheaves on X.

2.1. Partial Poisson resolutions of Y

Let $Y = (\mathfrak{t} \oplus \mathfrak{t}^*)/W$. The goal of this section is to construct a partial Poisson resolution X of Y mentioned in the introduction.

The variety Y is a conical symplectic singularity. As such, it admits a $\mathbb{Q}$-factorial terminalization, to be denoted by $\tilde{X}$ (see [9] or [38, Section 2.2]). This is another, generally, singular symplectic variety together with a projective birational morphism $\rho : \tilde{X} \to Y$. The variety $\tilde{X}$ is $\mathbb{Q}$-factorial and has terminal singularities, in particular, $\operatorname{codim}_{\tilde{X}} \tilde{X}^{\text{sing}} \geq 4$ (see [46]). We remark that $\tilde{X}$ is not unique. Below we will need a special choice of $\tilde{X}$ for some W.

Note that Y carries a natural action of $(\mathbb{C}^\times)^2$, by dilations of $\mathfrak{t}$ and of $\mathfrak{t}^*$. This action lifts to $\tilde{X}$, making ρ equivariant (cf. [43, Proposition A.7]). We will also consider the contracting torus $\{(t,t) \mid t \in \mathbb{C}^\times\} \subset (\mathbb{C}^\times)^2$. The Poisson bracket on $\mathcal{O}_{\tilde{X}}$ has weight -2 with respect to the action of this torus.

We will need to understand the structure of the exceptional divisor D of $\tilde{X} \to Y$. For each irreducible component of this divisor, its image in Y is the closure of a codimension 2 leaf (see the proof of [40, Proposition 2.14]). Such leaves are in bijection with conjugacy classes of reflections in W. All formal slices to these leaves in Y are of type A_1. Therefore the preimage of the closure of such a leaf is irreducible. So we get a bijection between the conjugacy classes of reflections in W and the irreducible components of the exceptional divisor. For a reflection s we write D_s for the corresponding component. So in the class group we have $D = \sum D_s$, where the sum is taken over the representatives of conjugacy classes.

We proceed to defining a partial resolution X of Y.

When $\mathfrak{g}$ is simple and simply laced, we set $X := \tilde{X}$. For example, for $\mathfrak{g} = \mathfrak{sl}_n$, we get a slight modification of $\operatorname{Hilb}_n(\mathbb{C}^2)$, the Hilbert scheme of n points in $\mathbb{C}^2$. Namely, this variety maps to $(\mathbb{C}^n \oplus \mathbb{C}^{n*})/S_n$ and our X is the preimage of $(\mathfrak{t} \oplus \mathfrak{t}^*)/S_n$. Note that in this case X is smooth (and symplectic).

Assume again that $\mathfrak{g}$ is simple and simply laced. Note that since $\tilde{X}$ is $\mathbb{Q}$-factorial, there is $\ell > 0$ such that the line bundle $\mathcal{O}(\ell D)$ on $\tilde{X}^{\mathrm{reg}}$ extends to a line bundle on $\tilde{X}$. The extension, also denoted by $\mathcal{O}(\ell D)$, is ample.

Now consider the case when W is of type B_n, F_4, or G_2. In this case, there are two codimension 2 symplectic leaves in Y, corresponding to the two conjugacy classes of reflections. We consider the bundle $\mathcal{O}(D)$ on $\tilde{X}^{\mathrm{reg}}$ associated to the divisor D. Again, we can find ℓ such that $\mathcal{O}(\ell D)$ extends to $\tilde{X}$. But now $\mathcal{O}(\ell D)$ may fail to be ample. For example, this is the case in type B_n for $n > 1$. According to [45], possible $\mathbb{Q}$-factorial terminalizations of Y are in bijection with chambers of a suitable hyperplane arrangement in $\mathrm{Pic}(\tilde{X}') \otimes_{\mathbb{Z}} \mathbb{R}$ (where $\tilde{X}'$ is any fixed $\mathbb{Q}$-factorial terminalization; these spaces are identified for different $\tilde{X}'$) modulo the action of the Namikawa–Weyl group from [44]: we send $\tilde{X}$ to its ample cone. We choose $\tilde{X}$ so that $\mathcal{O}(\ell D)$ lies in the closure of the ample cone of $\tilde{X}$.

PROPOSITION 2.1

There is an irreducible singular symplectic variety X with projective birational morphisms $\bar{\rho} : \tilde{X} \to X$ and $\underline{\rho} : X \to Y$ such that

(i) $\operatorname{codim}_X X^{\mathrm{sing}} \geq 4$,

(ii) *for some $\ell > 0$, the bundle $\mathcal{O}(\ell D)$ is lifted from an ample line bundle on X.*

Proof

Intermediate partial resolutions X between $\tilde{X}$ and Y (that are normal, hence singular symplectic) are classified by faces of the ample cone of $\tilde{X}$ in such a way that for a given face, C_0, for any rational point, $\chi \in C_0$, a positive rational multiple of χ is pulled back from an ample line bundle on the corresponding partial resolution. This follows, for example, from [31, Theorem 3-2-1]. In more detail, X in that theorem is our $\tilde{X}$, and S there is our Y. The nef cone $\overline{NE}(\tilde{X}/Y)$, by definition, is spanned by the numerical equivalence classes of curves in fibers of ρ. A Cartier divisor H there is a positive multiple of χ. It is ρ-nef in the terminology of [31, Theorem 3-2-1] because of the theorem of Kleiman [31, Theorem 0-1-2] that states that the nef cone is dual to the ample cone. To apply [31, Theorem 3-2-1], we take Δ there as in the proof of [45, Lemma 1], so that $(\tilde{X}, \Delta)$ is klt.

We provide details on the latter claim for readers' convenience. Note that in our case, Y is $\mathbb{Q}$-factorial. Indeed, let V^0 denote the locus of points in $\mathfrak{t} \oplus \mathfrak{t}^*$ without stabilizers in W. The complement to this locus has codimension 2. Therefore, we have isomorphisms

$$\mathrm{Cl}(Y) \cong \mathrm{Pic}(V^0/W) \cong \mathrm{Pic}^W(V^0) \cong \mathrm{Pic}^W(V) \cong \mathrm{Hom}_{\mathrm{Groups}}(W, \mathbb{C}^\times).$$

The latter group is finite; hence, Y is $\mathbb{Q}$-factorial. So, we can use the construction explained in [33] to get an effective divisor D' on $\tilde{X}$ supported on the exceptional locus of ρ that pairs negatively with every nonzero class in $\overline{NE}(\tilde{X}/Y)$. We take $\Delta = \epsilon D'$ for a very small positive rational number ϵ. The claim that $(\tilde{X}, \Delta)$ is klt follows from the definition of terminal singularities (see [31, Definition 0-2-6]).

So, the conditions of [31, Theorem 3-2-1] are satisfied. Then Y in that theorem is our X.

Explicitly, let $\mathcal{L}$ be a line bundle on $\tilde{X}$ that is a positive multiple of χ. Then X is the image of $\tilde{X}$ in $Y \times \mathbb{P}(V)$ for a sufficiently large integer d, where V is a finite dimensional generating subspace of the $\mathbb{C}[Y]$-module $\Gamma(\mathcal{L}^d)$.

In particular, we get a unique partial resolution X satisfying (ii). We need to show that it satisfies (i) as well. Assume the contrary: $\operatorname{codim}_X X^{\text{sing}} = 2$. Since $\operatorname{codim}_{\tilde{X}} \tilde{X}^{\text{sing}} \geq 4$ (see [46]), an irreducible component of $\bar{\rho}^{-1}(X^{\text{sing}})$ is a divisor. On the other hand, as argued in the second paragraph of the proof of [40, Proposition 2.14], the image in Y of an irreducible divisor under ρ either intersects Y^{reg} or coincides with the closure of a codimension 2 leaf. It follows that a codimension 2 leaf in X maps to a codimension 2 leaf in Y. This contradicts the claim that some multiple of D corresponds to an ample line bundle on X. So X satisfies (i) as well, which finishes the proof. □

We note that, by the construction, $(\mathbb{C}^\times)^2$ acts on X and the morphisms $\bar{\rho}$ and $\underline{\rho}$ are equivariant.

Remark 2.2
When W is of type B_n, the varieties $\tilde{X}$ (which is actually smooth), X, and Y can be realized as Nakajima quiver varieties for the affine quiver of type $\tilde{A}_2$ with dimension vector $n\delta$ and unit framing at the extending vertex 0. The Nakajima quiver varieties (see [42] for generalities) with these data depend on the choice of a pair of integers (θ_0, θ_1), and we write $\mathcal{M}^{(\theta_0,\theta_1)}$ for the corresponding quiver variety. We take $\tilde{X} = \mathcal{M}^{(1,1)}$, $X := \mathcal{M}^{(0,1)}$, and $Y := \mathcal{M}^{(0,0)}$.

We now sketch the argument. Recall that $\mathcal{M}^{(\theta_0,\theta_1)}$ is defined as the GIT quotient for an action of the group $\mathrm{GL}_n \times \mathrm{GL}_n$ on an affine variety with respect to the character $\theta : (g_0, g_1) \mapsto \det(g_0)^{\theta_0} \det(g_0)^{\theta_1}$. Let $\mathcal{L}$ denote the line bundle on $\tilde{X}$ corresponding to the character $(g_0, g_1) \mapsto \det(g_1)$. The slices to codimension 2 singularities in Y look like $\mathbb{C}^2/\{\pm 1\}$ (meaning that the formal slice is the formal neighborhood of 0 in $\mathbb{C}^2/\{\pm 1\}$). Over these slices $\tilde{X} \to Y$ looks like $T^*\mathbb{P}^1 \to \mathbb{C}^2/\{\pm 1\}$. To check that $X = \mathcal{M}^{(0,1)}$, we need to show that $\mathcal{L}$ restricts to the same line bundle on both $\mathbb{P}^1$s. Recall (see [42, Section 6] or [8, Section 2.1.6]) that the slices in $\mathcal{M}^{(\theta_0,\theta_1)}$ can also be realized as Nakajima quiver varieties but for smaller quivers. The groups we need to quotient out are realized as subgroups in $\mathrm{GL}_n \times \mathrm{GL}_n$ and the characters used to take the GIT quotients are obtained by restricting θ to these subgroups. The relevant computations were performed in [35, Section 6.5]. Our conclusion is that the restriction of the line bundle on $\tilde{X}$ corresponding to θ to the two $\mathbb{P}^1$ is given by the same formulas as in [35, Theorem 6.2.1] (up to the sign—the sign convention in that paper is opposite, and in the formulas we need to take $h = 0$): if the restrictions are $\mathcal{O}(k)$ and $\mathcal{O}(c)$ for $k, c \in \mathbb{Z}$, then $\theta_0 = (c-k)/2$ and $\theta_1 = c/2$.

The reason why we need to use X (instead of $\tilde{X}$) is that the Serre vanishing theorem holds: since $\mathcal{O}(\ell d)$ is ample on X, for every coherent sheaf $\mathcal{F}$ on X we have $H^i(X, \mathcal{F} \otimes \mathcal{O}(\ell D)^d) = 0$ for all $i > 0$ and d is sufficiently large (depending on $\mathcal{F}$). This plays a crucial role in the proof of Proposition 3.1 below.

2.2. Procesi sheaves

The goal of this section is to produce a Procesi sheaf on X. The case of Procesi sheaves on $\tilde{X}$ was handled in [38, Section 4].

Let us recall the construction of the latter. We can reduce $\tilde{X}$ mod p for $p \gg 0$. Namely, set $\mathbb{F} := \overline{\mathbb{F}}_p$. Then we can define the reduction $\tilde{X}_{\mathbb{F}}$ to $\mathbb{F}$. Since p is sufficiently large, $\tilde{X}_{\mathbb{F}}$ is a singular symplectic variety with $\operatorname{codim}_{\tilde{X}_{\mathbb{F}}} \tilde{X}_{\mathbb{F}}^{\mathrm{sing}} \geq 4$ and vanishing higher cohomology of the structure sheaf. In [38, Section 4.2], the second named author constructed a filtered quantization $\mathcal{D}_{\mathbb{F}}$ of the structure sheaf $\mathcal{O}_{\tilde{X}_{\mathbb{F}}}$, whose global sections are $\mathbb{A}(\mathfrak{t}_{\mathbb{F}} \oplus \mathfrak{t}_{\mathbb{F}}^*)^W$, where $\mathbb{A}$ stands for the Weyl algebra of a symplectic vector space. Consider the Frobenius morphism $\mathrm{Fr} : \tilde{X}_{\mathbb{F}} \to \tilde{X}_{\mathbb{F}}^{(1)}$ and the pushforward $\mathrm{Fr}_* \mathcal{D}_{\mathbb{F}}$. The restriction of this sheaf of algebras to the regular locus is an Azumaya algebra [38, Lemma 4.3]. Consider the completion $\mathbb{F}[Y^{(1)}]^{\wedge_0}$ of $\mathbb{F}[Y^{(1)}]$ at 0. We denote its spectrum by $Y_{\mathbb{F}}^{(1)\wedge}$. Consider the scheme

$$\tilde{X}_{\mathbb{F}}^{(1),\wedge} := Y_{\mathbb{F}}^{(1)\wedge} \times_{Y_{\mathbb{F}}^{(1)}} \tilde{X}_{\mathbb{F}}^{(1)}. \tag{2.1}$$

It was shown in [38, Section 4.3] that the restriction of $\mathrm{Fr}_* \mathcal{D}_{\mathbb{F}}$ to the regular locus in $\tilde{X}_{\mathbb{F}}^{(1),\wedge}$ splits. Moreover, it was shown there that we can find a Morita equivalent sheaf of algebras $\mathcal{A}_{\mathbb{F}}$ on $\tilde{X}_{\mathbb{F}}^{(1),\wedge}$ whose global sections are $\mathbb{F}[\mathfrak{t}^{(1)} \oplus \mathfrak{t}^{(1)*}]^{\wedge_0} \# W$. Let ϵ denote the averaging idempotent in $\mathbb{F}W$. Set $\tilde{\mathcal{P}}_{\mathbb{F}}^{\wedge} := \mathcal{A}_{\mathbb{F}}\epsilon$. Then the restriction of $\tilde{\mathcal{P}}_{\mathbb{F}}^{\wedge}$ to $\tilde{X}_{\mathbb{F}}^{(1),\wedge,\mathrm{reg}}$ is a splitting bundle for the Azumaya algebra

$$\mathcal{A}_{\mathbb{F}}|_{\tilde{X}_{\mathbb{F}}^{(1),\wedge,\mathrm{reg}}}.$$

Also note that $\mathcal{A}_{\mathbb{F}}$ is a maximal Cohen–Macaulay sheaf that coincides with the endomorphism sheaf of $\tilde{\mathcal{P}}_{\mathbb{F}}^{\wedge}$. Note that, by the construction, we have

$$\epsilon \tilde{\mathcal{P}}_{\mathbb{F}}^{\wedge} = \mathcal{O}_{\tilde{X}_{\mathbb{F}}^{(1),\wedge}}.$$

Consider the contracting $\mathbb{F}^{\times}$-action on $X_{\mathbb{F}}^{(1)\wedge}$. Then $\tilde{\mathcal{P}}_{\mathbb{F}}^{\wedge}$ can be shown to admit an $\mathbb{F}^{\times}$-equivariant structure. Using this, we can extend $\tilde{\mathcal{P}}_{\mathbb{F}}^{\wedge}$ to an $\mathbb{F}^{\times}$-equivariant maximal

Cohen–Macaulay sheaf on $\tilde{X}^{(1)}_{\mathbb{F}}$ to be denoted by $\tilde{\mathcal{P}}_{\mathbb{F}}$ (see [38, Lemma 4.6]). By the same lemma, we can modify the $\mathbb{F}^{\times}$-equivariant structure on $\tilde{\mathcal{P}}_{\mathbb{F}}$ so that we get a graded algebra isomorphism $\mathrm{End}(\tilde{\mathcal{P}}_{\mathbb{F}}) \xrightarrow{\sim} \mathbb{F}[\mathfrak{t}^{(1)} \oplus \mathfrak{t}^{(1)*}]\#W$.

Finally, we can lift $\tilde{\mathcal{P}}_{\mathbb{F}}$ to characteristic 0 (see [38, Section 4.4]). We get a maximal Cohen–Macaulay sheaf $\tilde{\mathcal{P}}$ on $\tilde{X}$ with the following properties:

(i) we have a graded algebra isomorphism $\mathrm{End}(\tilde{\mathcal{P}}) \xrightarrow{\sim} H$,

(ii) $\mathcal{E}nd(\tilde{\mathcal{P}})$ is a maximal Cohen–Macaulay module,

(iii) $H^i(\tilde{X}, \mathcal{E}nd(\tilde{\mathcal{P}})) = 0$ for $i > 0$,

(iv) $\epsilon\tilde{\mathcal{P}} \xrightarrow{\sim} \mathcal{O}_{\tilde{X}}$, a $\mathbb{C}^{\times}$-equivariant isomorphism.

Sheaves $\tilde{\mathcal{P}}$ satisfying (i)–(iv) are called *Procesi sheaves* on $\tilde{X}$.

We note that, for the same reason as in [38, Lemma 4.6], $\tilde{\mathcal{P}}$ can also be made equivariant with respect to $(\mathbb{C}^{\times})^2$, and the isomorphisms in (i) and (iv) can be assumed to be $(\mathbb{C}^{\times})^2$-equivariant. As remarked in [38, Remark 4.8], the argument in [36] classifying the Procesi bundles in the smooth case carries over to the singular case. So the bundles $\tilde{\mathcal{P}}$ on $\tilde{X}$ satisfying (i)–(iv) are classified by the elements of the Namikawa–Weyl group of Y introduced in [44]. We will denote this group by W_Y. This group is $\prod_s(\mathbb{Z}/2\mathbb{Z})$, where s runs over representatives of conjugacy classes of reflections in W. Below, in Section 2.3, we will recall how the classification of Procesi sheaves works.

To finish the section, we discuss Procesi sheaves on X. Recall the birational projective morphism $\bar{\rho}: \tilde{X} \to X$ from Proposition 2.1. Set

$$\mathcal{P} := \bar{\rho}_* \tilde{\mathcal{P}}.$$

LEMMA 2.3

The sheaf $\mathcal{P}$ on X has properties completely analogous to (i)–(iv).

Proof

First of all, note that $R\overline{\rho}_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. This is because $\tilde{X}$ and X are singular symplectic, and $\overline{\rho}$ is birational and projective. Indeed, by [1, Proposition 1.3], singular symplectic varieties have rational singularties. So, for any resolution of singularities $\pi: Z \to \tilde{X}$, we have $R\pi_*\mathcal{O}_Z = \mathcal{O}_{\tilde{X}}$ and $R(\overline{\rho} \circ \pi)_*\mathcal{O}_Z = \mathcal{O}_X$, implying $R\overline{\rho}_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.

For similar reasons, $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. So the same is true over $\mathbb{F}$ (assuming, as always, that $p \gg 0$). Therefore, $R^i\overline{\rho}_*\mathcal{D} = 0$ for $i > 0$, and the sheaf $\overline{\rho}_*\mathcal{D}$ is a filtered quantization of $\mathcal{O}_{X_{\mathbb{F}}}$, and has no higher cohomology. From here we deduce that $\overline{\rho}_*\mathcal{A}_{\mathbb{F}}$ is a maximal Cohen–Macaulay sheaf without higher cohomology. Moreover, $\overline{\rho}_*\tilde{\mathcal{P}}^{\wedge}_{\mathbb{F}} = \epsilon\overline{\rho}_*\mathcal{A}_{\mathbb{F}}$. Now note that $\mathcal{P}$ is obtained from $\overline{\rho}_*\tilde{\mathcal{P}}^{\wedge}_{\mathbb{F}}$ in the same way as $\tilde{\mathcal{P}}$ is obtained from $\tilde{\mathcal{P}}^{\wedge}_{\mathbb{F}}$. It follows that the natural homomorphism $\mathrm{End}(\tilde{\mathcal{P}}) \to \mathrm{End}(\mathcal{P})$ is an isomorphism yielding (i). Conditions (ii) and (iii) also follow, while (iv) is immediate from the construction of $\mathcal{P}$. ☐

2.3. Rational Cherednik algebras

Let us write S for the set of reflections in W. Let $c : S \to \mathbb{C}$ be a W-invariant function. Let $\hbar$ be an independent variable. Then we can define the rational Cherednik algebra $H_{\hbar,c}$ as the quotient of $T(\mathfrak{t} \oplus \mathfrak{t}^*)[\hbar]\#W$ by the following relations:

$$[x, x'] = [y, y'] = 0, \qquad [y, x] = \hbar\Big(\langle y, x\rangle - \sum_{s \in S} c(s)\langle y, \alpha_s^\vee\rangle\langle x, \alpha_s\rangle s\Big).$$

Here $x, x' \in \mathfrak{t}$, y, $y' \in \mathfrak{t}^*$, and α_s, $\alpha_s^\vee$ denote the positive root and the positive coroot corresponding to a reflection s. For example, $H_{\hbar,0} = D_\hbar(\mathfrak{t}^*)\#W$, where we write $D_\hbar(\mathfrak{t}^*)$ for the algebra of homogenized differential operators on $\mathfrak{t}^*$.

We will write H_c for the specialization of $H_{\hbar,c}$ to $\hbar = 1$.

Now we will discuss a connection between the rational Cherednik algebras and Procesi sheaves. We start with the Procesi sheaves on $\tilde{X}$, the case treated in [38, Section 5.1].

The formal quantizations of $\tilde{X}^{\text{reg}}$ with a compatible action of the contracting torus are classified by the points of $H^2(\tilde{X}^{\text{reg}}, \mathbb{C})$ (see [35, Section 2.3]). We note that the first Chern class map induces an isomorphism $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Pic}(\tilde{X}^{\text{reg}}, \mathbb{C}) \xrightarrow{\sim} H^2(\tilde{X}^{\text{reg}}, \mathbb{C})$, both spaces have dimensions equal to the number of conjugacy classes of reflections in W. The quantizations of $\tilde{X}$ are in a natural bijection with those of $\tilde{X}^{\text{reg}}$ via push-forward and pullback (see [10, Proposition 3.4]). Let us write $\tilde{\mathcal{D}}_{\hbar,\lambda}$ for the formal quantization of $\tilde{X}$ corresponding to λ. Note that $\tilde{\mathcal{D}}_{\hbar,\lambda}$ also has an action of the torus $(\mathbb{C}^\times)^2$, and the action of the Hamiltonian subtorus $\{(t, t^{-1}) \mid t \in \mathbb{C}^\times\}$ is still Hamiltonian.

The algebra of global sections $\Gamma(\tilde{\mathcal{D}}_{\hbar,\lambda})$ is related to the rational Cherednik algebra $H_{\hbar,c}$ as follows. Consider the spherical subalgebra $\epsilon H_{\hbar,c}\epsilon$, a graded quantization of $\mathbb{C}[Y]$. We can consider the subalgebra $\Gamma(\tilde{\mathcal{D}}_{\hbar,\lambda})^{\text{fin}}$ of $\mathbb{C}^\times$-locally finite elements in $\Gamma(\tilde{\mathcal{D}}_{\hbar,\lambda})$ with respect to the contracting $\mathbb{C}^\times$-action. Then we have

$$\Gamma(\tilde{\mathcal{D}}_{\hbar,\lambda})^{\text{fin}} \cong \epsilon H_{\hbar,c_\lambda}\epsilon, \tag{2.2}$$

where c_λ is computed as follows. The Chern classes of the line bundles $\mathcal{O}(D_s)$ form a basis in $H^2(\tilde{X}^{\text{reg}}, \mathbb{C})$. Let λ_s be the coefficient of the basis element corresponding to s in λ.

Definition 2.4

By definition, c_λ sends $s \in S$ to $\lambda_s - \frac{1}{2}$.

Isomorphism (2.2) follows from [40, Proposition 3.17].

Note that the Namikawa–Weyl group W_Y acts on $H^2(\tilde{X}^{\text{reg}}, \mathbb{C})$ by changing signs of the coordinates λ_s; this follows, for example, from [40, Section 3.6]. In particular, we get a W_Y-action on the affine space of parameters c. Two W_Y-conjugate parameters give rise to the same algebra $\Gamma(\mathcal{D}_{\hbar,\lambda})^{\mathrm{fin}}$ (see [10, Proposition 3.10]).

Now we discuss a connection of rational Cherednik algebras with Procesi sheaves, established in [35] in a special case and in [38] in the general case. See, in particular, [38, Section 5.1]. Let $\tilde{\mathcal{P}}^{\mathrm{reg}}$ denote the restriction of $\tilde{\mathcal{P}}$ to $\tilde{X}^{\mathrm{reg}}$; this is a vector bundle. Let $\tilde{\mathcal{D}}^{\mathrm{reg}}_{\hbar,\lambda}$ be the restriction of $\tilde{\mathcal{D}}_{\hbar,\lambda}$. Since $\mathcal{E}nd(\tilde{\mathcal{P}})$ is a maximal Cohen–Macaulay module that has no higher cohomology (see Section 2.2), the condition $\operatorname{codim}_{\tilde{X}} \tilde{X}^{\mathrm{sing}} \geq 4$ implies that $\Gamma(\mathcal{E}nd(\tilde{\mathcal{P}}^{\mathrm{reg}})) = H$ and $H^i(\mathcal{E}nd(\tilde{\mathcal{P}}^{\mathrm{reg}})) = 0$ for $i = 1, 2$. So we have a unique quantization of $\tilde{\mathcal{P}}^{\mathrm{reg}}$ to a left $\tilde{\mathcal{D}}^{\mathrm{reg}}_{\hbar,\lambda}$-module to be denoted by $\tilde{\mathcal{P}}^{\mathrm{reg}}_{\hbar,\lambda}$. This quantization is $(\mathbb{C}^\times)^2$-equivariant. Set

$$\tilde{\mathcal{E}}^{\mathrm{reg}}_{\hbar,\lambda} := \mathcal{E}nd_{\tilde{\mathcal{D}}^{\mathrm{reg}}_{\hbar,\lambda}}(\tilde{\mathcal{P}}^{\mathrm{reg}}_{\hbar,\lambda})^{\mathrm{opp}}. \tag{2.3}$$

This is a sheaf of $\mathbb{C}[[\hbar]]$-algebras on $\tilde{X}^{\mathrm{reg}}$ deforming $\mathcal{E}nd(\tilde{\mathcal{P}}^{\mathrm{reg}})^{\mathrm{opp}}$.

As was argued in [38, Section 5.1],

$$\Gamma(\tilde{\mathcal{E}}^{\mathrm{reg}}_{\hbar,\lambda})^{\mathrm{fin}} \xrightarrow{\sim} H_{\hbar,c'(\lambda)} \tag{2.4}$$

for an affine map $\lambda \mapsto c'(\lambda)$. By multiplying the source and the target of (2.4) with ϵ on the left and on the right, we get $\Gamma(\tilde{\mathcal{D}}^{\mathrm{reg}}_{\hbar,\lambda})^{\mathrm{fin}} \xrightarrow{\sim} \epsilon H_{\hbar,c'(\lambda)}\epsilon$ that gives the identity endomorphism of $\mathbb{C}[Y]$ after taking the quotient by $\hbar = 0$. So, as argued in [38, Section 5.1], we have an element $w \in W_Y$ such that $c'(\lambda) = wc_\lambda$. This element depends on the choice of $\tilde{\mathcal{P}}$. This defines a bijection between the set of possible Procesi bundles and W_Y, already mentioned in Section 2.2. This was proved in [36, Theorem 1.1] in the case when $\tilde{X}$ is smooth and carries over to the general case verbatim.

We will always choose $\tilde{\mathcal{P}}$ corresponding to the unit element in W_Y.

Let ϵ_- denote the sign idempotent in $\mathbb{C}W$. Using the previous discussion, we can describe $\tilde{\mathcal{P}}^{\mathrm{reg}}\epsilon_-$, the sign component of $\tilde{\mathcal{P}}^{\mathrm{reg}}$.

LEMMA 2.5
We have $c_1(\tilde{\mathcal{P}}^{\mathrm{reg}}\epsilon_-) = \frac{1}{2}c_1(\mathcal{O}(D))$.

Proof
Let s be a reflection in W. Pick a point $y \in Y$ lying in the symplectic leaf corresponding to s. We set $Y^{\wedge y} := \operatorname{Spec}(\mathbb{C}[Y]^{\wedge y})$ and $X^{\wedge y} := Y^\wedge_y \times_Y \tilde{X}$.

Pick $\lambda \in H^2(\tilde{X}^{\mathrm{reg}}, \mathbb{C})$ and set $c := c_\lambda$. We can also consider the completion $H^{\wedge y}_{\hbar,c}$. As was checked in [6, Section 3.3], this is a matrix algebra of size $|W|/2$ over $\underline{H}^{\wedge 0}_{\hbar,c(s)}$, the completed rational Cherednik algebra for $(\mathfrak{t}, \langle s\rangle)$ with parameter $c(s)$. On the other hand, analogously to [36, Proposition 4.1], $\tilde{\mathcal{P}}^{\wedge y} := \tilde{\mathcal{P}}|_{\tilde{X}^{\wedge y}}$ coincides with $\operatorname{Hom}_{\mathbb{C}\{1,s\}}(\mathbb{C}W, \underline{\mathcal{P}}^{\wedge 0})$, where $\underline{\mathcal{P}}$ is the Procesi bundle over $(\mathfrak{t} \oplus \mathfrak{t}^*)^s \times T^*\mathbb{P}^1$. Let $i : X^{\wedge y} \to \tilde{X}^{\mathrm{reg}}$ be the embedding. Then we have the pullback map

$$i^*: H^2(\tilde{X}^{\mathrm{reg}}, \mathbb{C}) = H^2_{DR}(\tilde{X}^{\mathrm{reg}}) \to H^2_{DR}(X^{\wedge y}) \cong H^2\big((\mathfrak{t}\oplus\mathfrak{t}^*)^s \times T^*\mathbb{P}^1, \mathbb{C}\big) = \mathbb{C}.$$

The isomorphism $\mathrm{End}(\mathcal{P}_{\hbar,\lambda}) \xrightarrow{\sim} H^{\wedge_\hbar}_{\hbar,\lambda}$ gives rise to an isomorphism $\mathrm{End}(\underline{\mathcal{P}}^{\wedge 0}_{\hbar,i^*(\lambda)}) \xrightarrow{\sim} \underline{H}^{\wedge 0}_{\hbar,c(s)}$. By Definition 2.4, the isomorphism of parameter spaces corresponding to $\underline{\mathcal{P}}$ sends $i^*(\lambda)$ to $i^*(\lambda) - \frac{1}{2}$. The two possibilities for $\underline{\mathcal{P}}$ are $\mathcal{O}\oplus\mathcal{O}(1)$ and $\mathcal{O}\oplus\mathcal{O}(-1)$. The map between the parameter spaces we have is realized by the former. This is an easy special case of [36, Section 4.2], for example.

In particular, using the direct analog of [36, Proposition 4.1] again, we see that the restriction of the line bundle $\mathcal{P}\epsilon_-$ to $X^{\wedge y}$ is $\mathcal{O}(1)$ on that scheme. Since the line bundle $\mathcal{O}(\mathbb{P}^1)$ on $T^*\mathbb{P}^1$ is $\mathcal{O}(2)$, the claim of the lemma follows. □

Now we explain how to relate the rational Cherednik algebras to quantizations of $\tilde{\mathcal{P}}$ (instead of $\tilde{\mathcal{P}}^{\mathrm{reg}}$). Let ι denote the embedding $\tilde{X}^{\mathrm{reg}} \hookrightarrow \tilde{X}$. Set $\tilde{\mathcal{P}}_{\hbar,\lambda} := \iota_*(\tilde{\mathcal{P}}^{\mathrm{reg}}_{\hbar,\lambda})$. Since $H^1(\tilde{X}^{\mathrm{reg}}, \tilde{\mathcal{P}}^{\mathrm{reg}}) = 0$, we see that $\tilde{\mathcal{P}}_{\hbar,\lambda}$ is a quantization of $\tilde{\mathcal{P}}$. Similarly, $\tilde{\mathcal{E}}_{\hbar,\lambda} := \iota_*\tilde{\mathcal{E}}^{\mathrm{reg}}_{\hbar,\lambda}$ is the endomorphism sheaf of $\tilde{\mathcal{P}}_{\hbar,\lambda}$ (with opposite multiplication).

Let us proceed to quantizations of $\mathcal{P}$, the Procesi sheaf on X, and its endomorphism sheaf. Similarly to the proof of Lemma 2.3, we see that $R^i\overline{\rho}_*\tilde{\mathcal{E}} = 0$ for $i > 0$. So we get that $\mathcal{E}_{\hbar,\lambda} := \overline{\rho}_*\tilde{\mathcal{E}}_{\hbar,\lambda}$ is a quantization of $\mathcal{E}nd(\mathcal{P})$. Further, we set $\mathcal{P}_{\hbar,\lambda} := \epsilon\mathcal{E}_{\hbar,\lambda}$. This is a quantization of $\mathcal{P}$. Also

$$\mathcal{E}_{\hbar,\lambda} = \mathcal{E}nd_{\mathcal{D}_{\hbar,\lambda}}(\mathcal{P}_{\hbar,\lambda})^{\mathrm{opp}},$$

where $\mathcal{D}_{\hbar,\lambda}$ is the pushforward of $\tilde{\mathcal{D}}_{\hbar,\lambda}$ to X.

In what follows we will write $\mathcal{O}^{\mathrm{reg}}(1) := \mathcal{P}^{\mathrm{reg}}\epsilon_-$. This is a line bundle on X^{reg}.

Note that $\bar{\rho}^*$ induces an isomorphism $\mathrm{Pic}(X^{\mathrm{reg}}) \xrightarrow{\sim} \mathrm{Pic}(\tilde{X}^{\mathrm{reg}})$. This allows us to view $c_1(\mathcal{O}^{\mathrm{reg}}(1))$ as an element of $H^2(\tilde{X}^{\mathrm{reg}}, \mathbb{C})$. If $\lambda \in H^2(\tilde{X}^{\mathrm{reg}}, \mathbb{C})$ corresponds to a Cherednik parameter $c = c_\lambda$, then the Cherednik parameter, say c', corresponding to $\lambda + c_1(\mathcal{O}^{\mathrm{reg}}(1))$ satisfies $c'(s) = c(s) + 1$ for all $s \in S$.

2.4. *Trigonometric Cherednik algebras*

In this section we will discuss the trigonometric Cherednik algebras and their connection to rational Cherednik algebras. Assume that G is a connected reductive group. Recall that T denotes a maximal torus in G.

Let Λ denote the cocharacter lattice of T, and let Λ_0 be the coroot lattice of $\mathfrak{g}$, a sublattice of Λ. Consider the extended affine Weyl group $\widetilde{W} := W \ltimes \Lambda$ that contains the affine Weyl group $W^a := W \ltimes \Lambda_0$ as a normal subgroup. We have the length function $\ell : \widetilde{W} \to \mathbb{Z}_{\geq 0}$. The subgroup of length 0 elements is identified with Λ/Λ_0 under the projection $\widetilde{W} \twoheadrightarrow \widetilde{W}/W^a \cong \Lambda/\Lambda_0$. We have the decomposition $\widetilde{W} = (\Lambda/\Lambda_0) \ltimes W^a$.

The group $\widetilde{W}$ acts on $\Lambda \times \mathbb{Z}$ by

$$w(\mu,a) := (w\mu,a), \qquad \chi(\mu,a) := (\mu + a\chi, a),$$
$$\chi \in \Lambda \subset \widetilde{W}, w \in W \subset \widetilde{W}, \mu \in \Lambda, a \in \mathbb{Z}. \tag{2.5}$$

We consider the dual action of $\widetilde{W}$ on $\mathfrak{t}^* \oplus \mathbb{C}$. It is given by

$$w(y,z) = (wy,z), \qquad \chi(y,z) = \big(y, z + \langle \chi, y\rangle\big), \quad \chi \in \Lambda, w \in W, y \in \mathfrak{t}^*, z \in \mathbb{C}.$$

Let $s_1,\dots,s_r$ denote the simple reflections in W, and let s_0 denote the simple affine reflection. Let $\alpha_1^\vee,\dots,\alpha_r^\vee$ denote the simple coroots, and let $\alpha_0^\vee$ denote the minimal (negative) coroot. Pick a W-invariant function $c : S \to \mathbb{C}$. Set $c(s_0) := c(s_{\alpha_0})$.

The trigonometric Cherednik algebra $H_{\hbar,c}^\times$ is defined as the algebra generated by two subalgebras $\mathbb{C}\widetilde{W}$ and $\mathbb{C}[\mathfrak{t},\hbar]$, subject to the following cross relations:

$$\begin{aligned} s_i y - (s_i.y)s_i &= \hbar c(s_i)\langle y, \alpha_i^\vee\rangle, \quad i = 0,\dots,r, y \in \mathfrak{t}^*,\\ \pi y &= (\pi.y)\pi, \quad y \in \mathfrak{t}^*, \pi \in \Lambda/\Lambda_0 \subset \widetilde{W},\\ x\hbar &= \hbar x, \quad x \in \widetilde{W}. \end{aligned} \tag{2.6}$$

Here we write $x.y$ for the image of $y \in \mathfrak{t}^*$ under $x \in \widetilde{W}$ for the action of $\widetilde{W}$ on $\mathfrak{t} \oplus \mathbb{C}$ described above (with $\hbar$ corresponding to $1 \in \mathbb{C}$).

The algebra $H_{\hbar,c}^\times$ admits an embedding into the algebra $D_\hbar(T^{\vee,\mathrm{reg}})\#W$, where $T^{\vee,\mathrm{reg}}$ denotes the complement to the union of root codimension 1 subgroups in the Langlands dual torus $T^\vee$; we write $D_\hbar$ for the algebra of homogenized differential operators. Namely, let us write e^λ for the function on $T^\vee$ given by λ. The embedding maps $\lambda \in \Lambda \subset \widetilde{W}$ to e^λ, $w \in W$ to $w \in W$, $\hbar$ to $\hbar$, and $y \in \mathfrak{t}$ to the trigonometric Dunkl operator (see [15, (2.12.16)]) defined as follows:

$$D_y^{\mathrm{trig}} = \partial_y + \sum_{\alpha>0} \hbar c(s_\alpha)\frac{\langle \alpha, y\rangle}{1 - e^{-\alpha^\vee}}(1 - s_\alpha) - \Big\langle \sum_{\alpha>0} \hbar c(s_\alpha)\alpha^\vee, y\Big\rangle.$$

This embedding can be used to establish the following well-known result that plays an important role in our paper.

LEMMA 2.6
We have an algebra isomorphism

$$H_{\hbar,c} \otimes_{\mathbb{C}[\mathfrak{t}^*]} \mathbb{C}[\mathfrak{t}^*]^{\wedge_0} \cong H_{\hbar,c}^\times \otimes_{\mathbb{C}[T^\vee]} \mathbb{C}[T^\vee]^{\wedge_1}.$$

Proof
We can identify $\mathbb{C}[\mathfrak{t}^*]^{\wedge_0} \cong \mathbb{C}[T^\vee]^{\wedge_1}$ by sending $x \in \mathfrak{t}$ to e^x. This identification is W-equivariant. It remains to show that the subalgebra in $D_\hbar(\mathfrak{t}^{*,\wedge_0,\mathrm{reg}})\#W$ generated by $\mathbb{C}[\mathfrak{t}^*]^{\wedge_0}\#W$ and the rational Dunkl operators coincides with the subalgebra generated

by $\mathbb{C}[\mathfrak{t}^*]^{\wedge_0}\#W$ and the trigonometric Dunkl operators. This is because the difference between the trigonometric and rational Dunkl operators associated to $y \in \mathfrak{t}^*$ lies in $\mathbb{C}[\mathfrak{t}^*]^{\wedge_0}\#W$. The latter subalgebra lies in both images. □

2.5. Representation theory of rational Cherednik algebras

In this section we will recall several known constructions and facts related to the representation theory of rational Cherednik algebras. Set $H_c := H_{\hbar,c}/(\hbar-1)$; this is a filtered deformation of $\mathbb{C}[T^*\mathfrak{t}^*]\#W$. Let ϵ, ϵ_- denote the trivial and sign idempotents in $\mathbb{C}W$.

We abuse the notation and write $c+1$ for the map $S \to W$, sending s to $c(s)+1$. We start with the following classical result (see, e.g., [3, Proposition 4.6]) that will also be established below, in Lemma 3.7.

LEMMA 2.7
We have a filtered algebra isomorphism $\epsilon H_c \epsilon \cong \epsilon_- H_{c+1} \epsilon_-$ that is the identity on the associated graded algebras.

We say that a parameter c is *ϵ-spherical* if $H_c = H_c \epsilon H_c$. In this case the categories H_c −mod and $\epsilon H_c \epsilon$ −mod are equivalent via the bimodules $H_c\epsilon$, ϵH_c. The following result, due to Bezrukavnikov, is [16, Theorem 5.5].

PROPOSITION 2.8
The parameter c is ϵ-spherical if and only if the algebra $\epsilon H_c \epsilon$ has finite homological dimension.

Similarly, we can talk about ϵ_--spherical parameters. A complete analog of Proposition 2.8 holds. In particular, we can use Lemma 2.7 to prove the following result.

COROLLARY 2.9
The parameter c is ϵ-spherical if and only if $c+1$ is ϵ_--spherical.

We will be interested in two classes of parameters c. The first class is the parameters c with $c(s) \in \mathbb{Z}$ for all s. The following result was obtained in [2, Theorem 1.4, Proposition 1.7].

LEMMA 2.10
If $c(s) \in \mathbb{Z}$ for all s, then the algebra H_c is simple. In particular, c is both ϵ- and ϵ_--spherical.

The second class of parameters is as follows. Assume that $\mathfrak{g}$ is simple. Let h denote the Coxeter number for W and let $d \in \mathbb{Z}_{\geq 0}$. We consider constant functions $c : S \to \mathbb{C}$ such that $c(s) = d + \frac{1}{h}$ for all $s \in S$.

The following result was obtained in [3, Theorem 1.4, Proposition 1.7].

PROPOSITION 2.11
There is a unique irreducible finite dimensional H_c-module. Its dimension is $(dh + 1)^{\dim \mathfrak{t}}$. Moreover, as a W-representation, it is isomorphic to the permutation module $\mathbb{C}(\Lambda_0/(dh+1)\Lambda_0)$, where, recall, Λ_0 is the coroot lattice.

The following is [20, Lemma 4.5].

PROPOSITION 2.12
The parameter $c = d + \frac{1}{h}$ is ϵ-spherical for $d \geq 0$ and ϵ_--spherical for $d > 0$.

3. Deformation of B_d

Let d be a positive integer. The goal of this section is, for a Cherednik parameter c, to define an $H_{\hbar,c+d}$-$H_{\hbar,c}$-bimodule $B_{\hbar,c+d\leftarrow c}$ that is a $\mathbb{C}[\hbar]$-flat deformation of B_d. This is done in Section 3.3. This construction is based on two algebro-geometric results of independent interest, Propositions 3.1 and 3.2.

3.1. Main geometric results

Consider the vector bundle $\mathcal{P}^{\mathrm{reg}}$ on X^{reg}, the restriction of $\mathcal{P}$ from Section 2.2, and the line $\mathcal{O}^{\mathrm{reg}}(1) := \epsilon_- \mathcal{P}^{\mathrm{reg}}$ on X^{reg}. Let ι denote the inclusion $X^{\mathrm{reg}} \hookrightarrow X$. We write $\mathcal{O}^{\mathrm{reg}}(j)$ for the jth tensor power of $\mathcal{O}^{\mathrm{reg}}(1)$.

Here is the first important result in this section.

PROPOSITION 3.1
The following claims hold:
(1) *For all $j > 0$, the sheaf $\iota_*(\mathcal{P}^{\mathrm{reg},*} \otimes \mathcal{O}^{\mathrm{reg}}(j) \otimes \mathcal{P}^{\mathrm{reg}})$ on X is maximal Cohen–Macaulay and its higher cohomology vanishes.*
(2) *In particular, we have $H^i(X^{\mathrm{reg}}, \mathcal{P}^{\mathrm{reg},*} \otimes \mathcal{O}^{\mathrm{reg}}(j) \otimes \mathcal{P}^{\mathrm{reg}}) = 0$ for all $j > 0$ and $i = 1, 2$.*

We will prove this proposition using another major result of this section:

PROPOSITION 3.2
Let $\mathcal{L}$ be a line bundle on X^{reg} and let ι denote the inclusion $X^{\mathrm{reg}} \hookrightarrow X$. Suppose there

is $\ell > 0$ *such that* $\iota_*(\mathcal{L}^{\otimes \ell})$ *is a line bundle on* X. *Then* $\iota_*\mathcal{L}$ *is a Cohen–Macaulay sheaf.*

Proposition 3.2 will be proved in Section 3.2. Now we prove Proposition 3.1 assuming Proposition 3.2.

Proof of Proposition 3.1

We note that (1) implies (2): if $\mathcal{F}$ is a maximal Cohen–Macaulay sheaf on X, then $R^i\iota_*(\iota^*\mathcal{F}) = 0$ for $i = 1,2$ because $\operatorname{codim}_X X^{\text{sing}} \geq 4$ (by Proposition 2.1). It follows that $H^i(X^{\text{reg}}, \mathcal{P}^{\text{reg},*} \otimes \mathcal{O}^{\text{reg}}(j) \otimes \mathcal{P}^{\text{reg}}) = H^i(X, \iota_*(\mathcal{P}^{\text{reg},*} \otimes \mathcal{O}^{\text{reg}}(j) \otimes \mathcal{P}^{\text{reg}}))$ for $i = 1,2$. The right-hand side vanishes by (1).

The proof of (1) is in several steps.

Step 1. Note that $\mathcal{O}^{\text{reg}}(2k) \cong \mathcal{O}(kD)$ for the divisor $D \subset X^{\text{reg}}$ from Section 2.1 and some $k > 0$. This follows from Lemma 2.5. By (ii) of Proposition 2.1, there is a positive integer ℓ such that $\mathcal{O}^{\text{reg}}(\ell)$ is obtained by restricting an ample line bundle on X that will be denoted by $\mathcal{O}(\ell)$. So, for each coherent sheaf $\mathcal{F}$ on X, there is a positive integer $d(\mathcal{F})$ such that $H^i(X, \mathcal{F} \otimes \mathcal{O}(d\ell)) = 0$ for all $i > 0$ and all $d > d(\mathcal{F})$. Set d_0 to be the maximum of $d(\mathcal{F})$, where $\mathcal{F}$ runs over $\iota_*\mathcal{O}^{\text{reg}}(j)$ for $j = 0, \ldots, \ell - 1$. Now, by Proposition 3.2, each of the sheaves $(\iota_*\mathcal{O}^{\text{reg}}(j)) \otimes \mathcal{O}(d\ell)$ is a maximal Cohen–Macaulay $\mathcal{O}_X$-module. We conclude that $\iota_*(\mathcal{O}^{\text{reg}}(j)) \otimes \mathcal{O}(d\ell)$ is Cohen–Macaulay and has vanishing higher cohomology for all d sufficiently large and all $j = 0, \ldots, \ell - 1$.

Step 2. Note that X and $\mathcal{O}^{\text{reg}}(1)$ are defined over a finite localization of a ring of algebraic integers, say R (cf the discussion after [38, Lemma 2.3]). After a further finite localization of R we can achieve the following:

- $(X^{\text{reg}})_R$ is regular and $\mathcal{O}^{\text{reg}}(1)$ is a base change of a line bundle, $\mathcal{O}_R^{\text{reg}}(1)$, on X_R^{reg},
- $\mathcal{O}_R(\ell) := \iota_*\mathcal{O}_R^{\text{reg}}(\ell)$ is an ample line bundle on X_R,
- $\iota_*\mathcal{O}_R^{\text{reg}}(j)$ is a maximal Cohen–Macaulay sheaf on X_R for all $j = 0, \ldots, \ell - 1$.

Using these properties we see that $\iota_*\mathcal{O}_R^{\text{reg}}(j) \otimes \mathcal{O}_R(d\ell)$ is a maximal Cohen–Macaulay sheaf with vanishing higher cohomology for all d sufficiently large, say $d \geq d_1$, and all $j = 0, \ldots, \ell - 1$. We conclude that for any field $\mathbb{F}$ that is an R-algebra, we have

(♡) $\iota_*\mathcal{O}_{\mathbb{F}}^{\text{reg}}(j) \otimes \mathcal{O}_{\mathbb{F}}(d\ell)$ is a maximal Cohen–Macaulay sheaf on $X_{\mathbb{F}}$ with vanishing higher cohomology for all $d \geq d_1$.

Step 3. We will use property (♡) to establish (1) in this and subsequent step. Recall that we write $\mathcal{E}$ for endomorphism sheaf of $\mathcal{P}$. Consider the scheme $X_{\mathbb{F}}^{\wedge}$ defined analogously to (2.1). It is enough to show the direct analog of (1) over $\mathbb{F}$ for $p := \operatorname{char} \mathbb{F} \gg 0$. Let $\mathcal{E}_{\mathbb{F}}^{\wedge}$ denote restriction of $\mathcal{E}_{\mathbb{F}}$ to $X_{\mathbb{F}}^{\wedge}$ and, similarly, let $\mathcal{O}_{\mathbb{F}}^{\wedge,\text{reg}}$

denote the restriction of $\mathcal{O}_{\mathbb{F}}^{\mathrm{reg}}$ to $X_{\mathbb{F}}^{\wedge,\mathrm{reg}}$. Similarly to Step 1 in the proof of [39, Lemma 3.4], we see that the restriction of $\iota_*(\mathcal{E}_{\mathbb{F}}^{\mathrm{reg}} \otimes \mathcal{O}_{\mathbb{F}}^{\mathrm{reg}}(j))$ to $X_{\mathbb{F}}^{\wedge}$ coincides with

$$\iota_*^{\wedge}\big(\mathcal{E}_{\mathbb{F}}^{\wedge,\mathrm{reg}} \otimes \mathcal{O}_{\mathbb{F}}^{\wedge,\mathrm{reg}}(j)\big), \tag{3.1}$$

where we write $\iota^{\wedge}$ for the inclusion $X_{\mathbb{F}}^{\wedge,\mathrm{reg}} \hookrightarrow X_{\mathbb{F}}^{\wedge}$. So we need to show that (3.1) is maximal Cohen–Macaulay with vanishing higher cohomology for all $j > 0$. It is enough to do this after a Frobenius twist. In the notation of Section 2.2, (3.1) becomes

$$\iota_*^{\wedge}\big(\mathcal{A}_{\mathbb{F}}|_{X_{\mathbb{F}}^{(1),\wedge,\mathrm{reg}}} \otimes \mathcal{O}_{\mathbb{F}}^{(1),\wedge,\mathrm{reg}}(j)\big). \tag{3.2}$$

Recall that $\mathcal{A}_{\mathbb{F}}$ has the same direct summands as a quantization $\mathcal{D}_{\mathbb{F}}^{\wedge}$ of $X_{\mathbb{F}}^{\wedge}$. So we need to show that

$$\iota_*^{\wedge}\big(\mathcal{D}_{\mathbb{F}}^{\wedge}|_{X_{\mathbb{F}}^{(1),\wedge,\mathrm{reg}}} \otimes \mathcal{O}_{\mathbb{F}}^{(1),\wedge,\mathrm{reg}}(j)\big) \tag{3.3}$$

is maximal Cohen–Macaulay with vanishing higher cohomology. Again, arguing as in Step 1 of the proof of [39, Lemma 3.4], we see that this sheaf is the restriction to $X_{\mathbb{F}}^{(1),\wedge}$ of

$$\iota_*\big(\mathcal{D}_{\mathbb{F}}|_{X_{\mathbb{F}}^{(1),\mathrm{reg}}} \otimes \mathcal{O}_{\mathbb{F}}^{(1),\mathrm{reg}}(j)\big). \tag{3.4}$$

So it is enough to show that (3.4) is maximal Cohen–Macaulay with vanishing higher cohomology. We will do this in the next step.

Step 4. We note that $\mathcal{D}_{\mathbb{F}}$ is a filtered deformation of $\mathrm{Fr}_* \mathcal{O}_{X_{\mathbb{F}}}$. It follows that

$$\mathcal{D}_{\mathbb{F}}|_{X_{\mathbb{F}}^{(1),\mathrm{reg}}} \otimes \mathcal{O}_{\mathbb{F}}^{(1),\mathrm{reg}}(j)$$

is a filtered deformation of

$$(\mathrm{Fr}_* \mathcal{O}_{X_{\mathbb{F}}^{\mathrm{reg}}}) \otimes \mathcal{O}_{\mathbb{F}}^{(1),\mathrm{reg}}(j) \cong \mathrm{Fr}_* \mathcal{O}_{\mathbb{F}}^{\mathrm{reg}}(pj).$$

Since p is sufficiently large, by (♡), $\iota_* \mathrm{Fr}_* \mathcal{O}_{\mathbb{F}}^{\mathrm{reg}}(pj)$ is maximal Cohen–Macaulay with vanishing higher cohomology. Similarly to the derivation of (1)⇒(2) in the beginning of the proof, we see that $R^1\iota_* \mathrm{Fr}_* \mathcal{O}_{\mathbb{F}}^{\mathrm{reg}}(pj) = 0$. It follows that (3.4) is a filtered deformation of $\iota_* \mathrm{Fr}_* \mathcal{O}_{\mathbb{F}}^{\mathrm{reg}}(pj)$. Since the latter is maximal Cohen–Macaulay with vanishing higher cohomology, so is (3.4). This finishes the proof. □

3.2. Cohen–Macaulay property

In this section we prove Proposition 3.2. Let $\mathcal{L}$ be as in Proposition 3.2. We need to prove that $\iota_*\mathcal{L}$ is Cohen–Macaulay. We start with the following lemma.

LEMMA 3.3
Every point $x \in X$ has a Zariski open neighborhood, say U, such that $\mathcal{L}|_{U^{\mathrm{reg}}}$ has a D-module structure.

Proof
Since $H^1(X^{\mathrm{reg}}, \mathcal{O}) = 0$, the line bundle $\mathcal{L}$ quantizes to a $\mathcal{D}_{\hbar,\lambda+c_1(\mathcal{L})}$-$\mathcal{D}_{\hbar,\lambda}$-bimodule for any $\lambda \in H^2(X^{\mathrm{reg}}, \mathbb{C})$ (see, e.g., [10, Proposition 5.2]). Take a Zariski open neighborhood U of x in X such that the line bundle $\iota_*(\mathcal{L}^{\otimes \ell})$ trivializes on U. We can assume that U is affine. Then $H^i(U^{\mathrm{reg}}, \mathcal{O}) = 0$ for $i = 1, 2$ (because U is Cohen–Macaulay and $\operatorname{codim}_U U^{\mathrm{sing}} \geq 4$); hence, the formal quantizations of U^{reg} are classified by their period (see [7, Theorem 1.8]).

Since $\iota_*(\mathcal{L}^{\otimes \ell})$ trivializes on U, it follows that $\mathcal{L}^{\otimes \ell}$ is trivial on U^{reg}; hence, $c_1(\mathcal{L}|_{U^{\mathrm{reg}}}) = 0$. Therefore, we have

$$\mathcal{D}_{\hbar,\lambda}|_{U^{\mathrm{reg}}} \cong \mathcal{D}_{\hbar,\lambda+c_1(\mathcal{L})}|_{U^{\mathrm{reg}}}.$$

A vector bundle that quantizes to a bimodule over the same formal quantization on the left and on the right gets a Poisson structure (see, e.g., [39, Section 2.4]). But over a smooth symplectic variety, a coherent Poisson module is the same thing as a D-module (see, e.g., Step 3 of the proof of [37, Lemma 3.9]). □

Recall the morphism $\underline{\rho} : X \to Y$. Set $y := \underline{\rho}(x)$. Let $v \in \mathfrak{t} \oplus \mathfrak{t}^*$ be a point in the preimage of y. Choose a W_v-stable small disc Z around v. Then Z/W_v is a neighborhood of y in the complex analytic topology. Set $\tilde{Z} := \underline{\rho}^{-1}(Z/W_v)$.

LEMMA 3.4
The group $\pi_1(\tilde{Z}^{\mathrm{reg}})$ is a quotient of W_v.

Proof
Indeed, $(Z/W_v)^{\mathrm{reg}}$ embeds into $\tilde{Z}^{\mathrm{reg}}$ as the complement to a closed complex analytic subspace; hence, $\pi_1((Z/W_v)^{\mathrm{reg}}) \twoheadrightarrow \pi_1(\tilde{Z}^{\mathrm{reg}})$. But $\pi_1((Z/W_v)^{\mathrm{reg}})$ is easily seen to coincide with W_v. □

Proof of Proposition 3.2
What we need to show is that the completion $(\iota_*\mathcal{L})^{\wedge x}$ is Cohen–Macaulay; this implies that the stalk of $\iota_*\mathcal{L}$ is Cohen–Macaulay. The proof is in several steps.

Step 1. Let W^0 denote the kernel of $\pi_1((Z/W_v)^{\mathrm{reg}}) \twoheadrightarrow \pi_1(\tilde{Z}^{\mathrm{reg}})$. Set $Y^0 := (\mathfrak{t} \oplus \mathfrak{t}^*)/W^0$ and $X^0 := Y^0 \times_Y X$. Let η denote the projection $X^0 \twoheadrightarrow X$. The preimage Z^0 of $\tilde{Z}^{\mathrm{reg}}$ in X^0 is smooth and is a simply connected cover of $\tilde{Z}^{\mathrm{reg}}$, by the choice of W^0. The morphism $\eta : Z^0 \to \tilde{Z}^{\mathrm{reg}}$ is étale. It follows that there is a Zariski open neighborhood U_1 of x in U such that η is étale over U_1^{reg}. Define U_1^0 from the Stein decomposition for $\pi^{-1}(U_1^{\mathrm{reg}}) \to U_1$ so that $\eta^{-1}(U_1^{\mathrm{reg}})$ embeds into U_1^0 as an open subset and $U_1^0 \to U_1$ is the quotient morphism for the group $\pi_1(\tilde{Z}^{\mathrm{reg}})$. As in the proof of [39, Lemma 2.5], it follows that U_1^0 has symplectic singularties and hence Cohen–Macaulay.

Step 2. Let $\mathcal{O}_X^{an}$ and $\mathcal{O}_{X^{\mathrm{reg}}}^{an}$ denote the sheaves of analytic functions on X and X^{reg}. Then we have the analytification functor $\bullet^{an} := \mathcal{O}_X^{an} \otimes_{\mathcal{O}_X} \bullet$ from the category of coherent $\mathcal{O}_X$-modules to the category of coherent $\mathcal{O}_X^{an}$-modules, and similarly for X^{reg}. We claim that $(\iota_* \mathcal{L})^{an}$ coincides with the analytic pushforward of $\mathcal{L}^{an}$, to be denoted by $\iota_*^{an} \mathcal{L}^{an}$.

Note that $\mathcal{O}_X^{an}$ is flat over $\mathcal{O}_X$. So we have an isomorphism of functors

$$\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{O}_X)^{an} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X^{an}}(\bullet^{an}, \mathcal{O}_X^{an}) \tag{3.5}$$

(from $\mathrm{Coh}(X)$ to $\mathrm{Coh}(X^{an})$). Since $\operatorname{codim}_X X^{\mathrm{sing}} \geq 2$, the pushforward $\iota_* \mathcal{L}$ is a reflexive $\mathcal{O}_X$-module, that is, it coincides with its double dual. It follows from (3.5) that $(\iota_* \mathcal{L})^{an}$ is a reflexive $\mathcal{O}_X^{an}$-module. Note that $\mathcal{L}^{an}$ coincides with the pullback of $(\iota_* \mathcal{L})^{an}$. Since $(\iota_* \mathcal{L})^{an}$ is reflexive and $\operatorname{codim}_X X^{\mathrm{sing}} \geq 2$, we see that $(\iota_* \mathcal{L})^{an}$ coincides with $\iota_*^{an} \mathcal{L}^{an}$.

Step 3. Recall from Lemma 3.3 that $\mathcal{L}|_{U^{\mathrm{reg}}}$ has a D-module structure. In particular, $\mathcal{L}^{an}|_{\tilde{Z}^{\mathrm{reg}}}$ is a D-module, that is, a vector bundle with a flat connection. It follows that it is the direct sum of $\pi_1(\tilde{Z}^{\mathrm{reg}})$-isotypic component in $\eta_* \mathcal{O}^{an}_{\eta^{-1}(\tilde{Z}^{\mathrm{reg}})}$. Therefore, $(\iota_* \mathcal{L})^{\wedge_x}$ is also the direct sum of isotypic components in the complete ring $\mathbb{C}[\eta^{-1}(X^{\wedge_x})]$. The latter is ring is Cohen–Macaulay because U_1^0 has symplectic singularities. Hence, $(\iota_* \mathcal{L})^{\wedge_x}$ is a Cohen–Macaulay $\mathbb{C}[X]^{\wedge_x}$-module. □

Remark 3.5
We expect that a direct analog of Proposition 3.2 holds for the partial resolutions of general conical symplectic singularities. The proof should be similar to the one we gave above, modulo some technical issues.

3.3. *Construction and properties of* $B_{\hbar, c+d \leftarrow c}$

Let $d > 0$. Recall the space

$$B_d := \Gamma\big(X^{\mathrm{reg}}, \mathcal{P}^{\mathrm{reg},*} \otimes \mathcal{O}^{\mathrm{reg}}(d) \otimes \mathcal{P}^{\mathrm{reg}}\big).$$

We view $\mathcal{P}$ as a right H-module, so B_d becomes an H-bimodule. Moreover, recall that $\mathcal{P}$ has a $(\mathbb{C}^\times)^2$-equivariant structure. The line bundle $\mathcal{O}(1) = \mathcal{P}\epsilon_-$ inherits the $(\mathbb{C}^\times)^2$-equivariant structure. This equips B_d with a $(\mathbb{C}^\times)^2$-equivariant structure. We will mostly consider a part of the action, an action of $\mathbb{C}^\times$ given by $t.(x, y) = (x, t^{-2} y)$ for $x \in \mathfrak{t}$, $y \in \mathfrak{t}^*$.

In this section we produce a deformation of B_d to an $H_{\hbar,c+d}$-$H_{\hbar,c}$-bimodule and study its properties. The deformation of B_d is constructed as follows. Recall from Section 2.3 that $\mathcal{P}^{\mathrm{reg}}$ quantizes to a $\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda} - \mathcal{E}^{\mathrm{reg}}_{\hbar,\lambda}$-bimodule $\mathcal{P}^{\mathrm{reg}}_{\hbar,\lambda}$ for any $\lambda \in H^2(X^{\mathrm{reg}}, \mathbb{C})$. Set $\nu := c_1(\mathcal{O}(1))$. Also $\mathcal{O}^{\mathrm{reg}}(d)$ quantizes to a $\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu}$-$\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda}$-bimodule to be denoted by $\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu\leftarrow\lambda}$ (see [10, Proposition 5.2]). So

$$\Gamma(X^{\mathrm{reg}}, \mathcal{P}^{\mathrm{reg},*}_{\hbar,\lambda+d\nu} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu}} \mathcal{D}_{\hbar,\lambda+d\nu\leftarrow\lambda} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda}} \mathcal{P}^{\mathrm{reg}}_{\hbar,\lambda}) \tag{3.6}$$

becomes a $(\mathbb{C}^\times)^2$-equivariant $H^{\wedge\hbar}_{\hbar,\lambda+d\nu}$-$H^{\wedge\hbar}_{\hbar,\lambda}$-bimodule, where the superscript $\bullet^{\wedge\hbar}$ indicates an $\hbar$-adic completion. We set

$$B_{\hbar,\lambda+d\nu\leftarrow\lambda} := \Gamma(X^{\mathrm{reg}}, \mathcal{P}^{\mathrm{reg},*}_{\hbar,\lambda+d\nu} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu}} \mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu\leftarrow\lambda} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda}} \mathcal{P}^{\mathrm{reg}}_{\hbar,\lambda})^{\mathrm{fin}},$$

where the superscript "fin" means that we take the finite part for our $\mathbb{C}^\times$-action. This is a bigraded $H_{\hbar,\lambda+d\nu}$-$H_{\hbar,\lambda}$-bimodule. Note that for $d = 0$, we recover the regular $H_{\hbar,\lambda}$-bimodule.

Remark 3.6
By the construction, $B_{\hbar,\lambda+d\nu\leftarrow\lambda}$ carries an action of $(\mathbb{C}^\times)^2$. So it is bigraded.

We claim that $B_{\hbar,\lambda+d\nu\leftarrow\lambda}$ is a free-graded $\mathbb{C}[\hbar]$-module with

$$B_{\hbar,\lambda+d\nu\leftarrow\lambda}/(\hbar) = B_d. \tag{3.7}$$

We only need to prove (3.7). By (2) of of Proposition 3.1,

$$H^1\big(\mathcal{P}^{\mathrm{reg},*} \otimes \mathcal{O}^{\mathrm{reg}}(d) \otimes \mathcal{P}^{\mathrm{reg}}\big) = 0.$$

We now use the long exact sequence in cohomology for

$$\begin{aligned} 0 \to\ & \mathcal{P}^{\mathrm{reg},*}_{\hbar,\lambda+d\nu} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu}} \mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu\leftarrow\lambda} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda}} \mathcal{P}^{\mathrm{reg}}_{\hbar,\lambda} \\ \xrightarrow{\hbar}\ & \mathcal{P}^{\mathrm{reg},*}_{\hbar,\lambda+d\nu} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu}} \mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu\leftarrow\lambda} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda}} \mathcal{P}^{\mathrm{reg}}_{\hbar,\lambda} \to \mathcal{P}^{\mathrm{reg},*} \otimes \mathcal{O}^{\mathrm{reg}}(d) \otimes \mathcal{P}^{\mathrm{reg}} \to 0 \end{aligned}$$

and the argument in the last paragraph of the proof of [21, Lemma 5.6.3] to show that

$$H^1(\mathcal{P}^{\mathrm{reg},*}_{\hbar,\lambda+d\nu} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu}} \mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda+d\nu\leftarrow\lambda} \otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,\lambda}} \mathcal{P}^{\mathrm{reg}}_{\hbar,\lambda}) = 0.$$

From the same long exact sequence we deduce that (3.6) is a $\mathbb{C}[[\hbar]]$-flat deformation of B_d. (3.7) easily follows.

We now explain the choice of λ we mostly need: we want c_λ to take the same value (to be denoted by c) on all simple reflections. Then $c_{\lambda+d\nu}$ takes value $c + d$ on every reflection. So we will write $H_{\hbar,c+d}$, $H_{\hbar,c}$ for the algebras and $B_{\hbar,c+d\leftarrow c}$ for the bimodule.

Now we explain an important property of the bimodules $B_{\hbar,c+?\leftarrow c}$. We note that (1) below also follows from Lemma 2.7 but we are not going to check that the isomorphism below coincides with the isomorphism from that lemma (although it does).

LEMMA 3.7

The following claims are true:

(1) *For all $c\in\mathbb{C}$, we have a bigraded algebra isomorphism $\epsilon_- H_{\hbar,c+1}\epsilon_-\cong \epsilon H_{\hbar,c}\epsilon$.*

(2) *Thanks to (1) we can view the $\epsilon_- B_{\hbar,c+d+1\leftarrow c}$ as $\epsilon H_{\hbar,c+d}\epsilon$-$H_{\hbar,c}$-bimodule. For all $c\in\mathbb{C}$ and all $d>0$, we have an $\epsilon H_{\hbar,c+d}\epsilon$-$H_{\hbar,c}$-bilinear bigraded isomorphism*

$$\epsilon_- B_{\hbar,c+d+1\leftarrow c}\cong \epsilon B_{\hbar,c+d\leftarrow c}.$$

Proof

Consider the $\mathcal{E}^{\mathrm{reg}}_{\hbar,c+d+1}$-$\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d}$-bimodule $(\mathcal{P}^{\mathrm{reg}}_{\hbar,c+d+1})^*\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1}}\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1\leftarrow c+d}$. We claim that

$$\epsilon_-\big((\mathcal{P}^{\mathrm{reg}}_{\hbar,c+d+1})^*\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1}}\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1\leftarrow c+d}\big)$$

is the regular $\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d}$-bimodule. Indeed, $\mathcal{P}^{\mathrm{reg}}_{\hbar,c+d+1}\epsilon_-$ is the unique quantization of $\mathcal{O}^{\mathrm{reg}}(1)$ to a left $\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1}$-module; the uniqueness follows from $H^1(X^{\mathrm{reg}},\mathcal{O}_{X^{\mathrm{reg}}})=0$ (cf the proof of [10, Proposition 5.2]). The opposite endomorphism sheaf is $\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d}$. Hence, $\mathcal{P}^{\mathrm{reg}}_{\hbar,c+d+1}\epsilon_-\cong\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1\leftarrow c+d}$ as a bimodule. Our claim follows.

To prove (1) we use the previous paragraph to see that

$$\epsilon_-\mathcal{E}^{\mathrm{reg}}_{\hbar,c+1}\epsilon_-=\mathcal{E}nd_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c+1}}(\mathcal{P}^{\mathrm{reg}}_{\hbar,c+1}\epsilon_-)^{\mathrm{opp}}=\mathcal{D}_{\hbar,c}=\epsilon\mathcal{E}^{\mathrm{reg}}_{\hbar,c}\epsilon. \tag{3.8}$$

Since $\Gamma(X,\mathcal{E}^{\mathrm{reg}}_{\hbar,c})^{\mathrm{fin}}=H_{\hbar,c}$, (3.8) implies (1).

We proceed to (2). We note that

$$\begin{aligned}
&\epsilon_-\mathcal{P}^{\mathrm{reg}*}_{\hbar,c+d+1}\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1}}\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1\leftarrow c}\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c}}\mathcal{P}^{\mathrm{reg}}_{\hbar,c}\\
&\quad=(\epsilon_-\mathcal{P}^{\mathrm{reg}*}_{\hbar,c+d+1}\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1}}\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d+1\leftarrow c+d})\\
&\qquad\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d}}(\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d\leftarrow c}\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d}}\mathcal{P}^{\mathrm{reg}}_{\hbar,c})\\
&\quad=\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d\leftarrow c}\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c}}\mathcal{P}^{\mathrm{reg}}_{\hbar,c}=\epsilon\mathcal{P}^{\mathrm{reg}*}_{\hbar,c+d}\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d}}\mathcal{D}^{\mathrm{reg}}_{\hbar,c+d\leftarrow c}\otimes_{\mathcal{D}^{\mathrm{reg}}_{\hbar,c}}\mathcal{P}^{\mathrm{reg}}_{\hbar,c}.
\end{aligned}$$

Passing to the global sections of the initial and final expressions and taking the $\mathbb{C}^\times$-finite part, we arrive at the statement of (2). □

We will also need a description of $\operatorname{End}_{H_{\hbar,c}}(B_{\hbar,c+d\leftarrow c})$. Note that we have a graded $\mathbb{C}[\hbar]$-algebra homomorphism

$$H_{\hbar,c+d} \to \operatorname{End}_{H_{\hbar,c}}(B_{\hbar,c+d\leftarrow c}). \tag{3.9}$$

LEMMA 3.8
This homomorphism is an isomorphism.

Proof
It is enough to prove that the homomorphism

$$H \to \operatorname{End}_H(B_d) \tag{3.10}$$

is an isomorphism. Indeed, the injectivity of (3.9) will follows because the source is flat over $\mathbb{C}[\hbar]$, while the surjectivity follows from the graded Nakayama lemma (note that $\operatorname{End}_{H_{\hbar,c}}(B_{\hbar,c+d\leftarrow c})/(\hbar) \hookrightarrow \operatorname{End}_H(B_d)$).

Since B_d is the global section of a vector bundle on X^{reg}, and $\mathbb{C}[Y] = \mathbb{C}[X^{\mathrm{reg}}]$, we see that B_d is a torsion-free $\mathbb{C}[Y]$-module. It follows that we have an algebra embedding

$$\operatorname{End}_H(B_d) \hookrightarrow \operatorname{End}_{H^{\mathrm{reg}}}(B_d^{\mathrm{reg}}). \tag{3.11}$$

Here we write H^{reg}, B_d^{reg} for the restrictions of H and B_d to Y^{reg}. So it is enough to show that the composition of (3.11) and (3.10) is an isomorphism. On the other hand, from the definition of the Procesi bundle $\mathcal{P}^{\mathrm{reg}}$ and the construction of B_d, it is easy to see that

$$H^{\mathrm{reg}} \xrightarrow{\sim} \mathcal{E}nd_{H^{\mathrm{reg}}}(B_d^{\mathrm{reg}}). \tag{3.12}$$

The composition $H \to \operatorname{End}_{H^{\mathrm{reg}}}(B_d^{\mathrm{reg}})$ of (3.11) and (3.10) is obtained from (3.12) by passing to global sections. This finishes the proof. □

4. Borel–Moore homology

4.1. General properties of Borel–Moore homology

In this section we recall the notion of equivariant Borel–Moore homology and the necessary properties needed to prove the isomorphism in Theorem 1.1. The main references we use are [32], [11], and [23].

Let X be a projective variety. Then we can consider the dualizing sheaf $\omega_X \in D_c^b(X)$, the bounded derived category of constructible sheaves on X. Then we can define:

$$H_*^{\mathrm{BM}}(X) = H^{-*}(\omega_X).$$

Now assume we have an algebraic action of a torus T on X. To consider the equivariant Borel–Moore homology we need to define the Borel–Moore homology of the Borel construction $X \times^T ET$, where $ET \to BT$ is the universal T bundle. Since this is not a finite type variety we need to do this by approximating ET using finite-type varieties, which can be done along the lines of [4].

Note that from the map $X \times^T ET \to BT$, we get a map $\zeta : H_T^{\mathrm{BM}}(X) \to H_T^{\mathrm{BM}}(pt) = \mathbb{C}[\mathfrak{t}]$. Also, there is an action of the constant sheaf $\mathbb{C} \in D_c^b(X)$ on ω_X, which equips $H_T^{\mathrm{BM}}(X)$ with an $H_T^*(X)$-module structure. In particular, $H_T^{\mathrm{BM}}(X)$ becomes a module over $H_T^*(pt) = \mathbb{C}[\mathfrak{t}]$. The map $\zeta : H_T^{\mathrm{BM}}(X) \to H_T^{\mathrm{BM}}(pt)$ is $\mathbb{C}[\mathfrak{t}]$-linear. We get a map

$$H_T^*(X) \to \mathrm{Hom}_{H_T^*(pt)}\big(H_T^{\mathrm{BM}}(X), H_T^*(pt)\big) \tag{4.1}$$

by $\alpha \mapsto [\beta \to \zeta(\alpha\beta)]$. This map is an isomorphism when X is equivariantly formal, which follows from [11, Proposition 1]. Also, when X is equivariantly formal, the dual map

$$H_T^{\mathrm{BM}}(X) \to \mathrm{Hom}_{H_T^*(pt)}\big(H_T^*(X), H_T^*(pt)\big) \tag{4.2}$$

is also an isomorphism.

We further have the following two localization lemmas which follow from [11, Lemma 1].

LEMMA 4.1
Suppose that X has isolated T-fixed points. Consider the inclusion of the fixed points $X^T \hookrightarrow X$. This induces a map

$$H_T^{\mathrm{BM}}(X^T) \to H_T^{\mathrm{BM}}(X).$$

This map is an isomorphism after inverting finitely many characters of T.

A dual result also holds for the cohomology $H_T^*(X)$; that is, we have a natural map

$$H_T^*(X) \to H_T^*(X^T)$$

that is an isomorphism after inverting the same characters as in the above lemma.

LEMMA 4.2
Let $T' \subset T$ and X be a variety with T-action; then we have the localization map

$$H_T^{\mathrm{BM}}(X^{T'}) \to H_T^{\mathrm{BM}}(X)$$

which becomes an isomorphism after inverting those characters of Lemma 4.1 that do not vanish on T'.

Lemma 4.2 follows from Lemma 4.1 applied to the action of T'.

We also have that these two localization maps are compatible with the action of $H_T^*(X)$ on $H_T^{\mathrm{BM}}(X)$ in the sense that we have a commuting diagram

$$\begin{array}{ccccc} H_T^*(X) \otimes H_T^{\mathrm{BM}}(X^T) & & \longrightarrow & & H_T^*(X) \otimes H_T^{\mathrm{BM}}(X) \\ \downarrow & & & & \downarrow \\ H_T^*(X^T) \otimes H_T^{\mathrm{BM}}(X^T) & \longrightarrow & H_T^{\mathrm{BM}}(X^T) & \longrightarrow & H_T^{\mathrm{BM}}(X) \end{array}$$

Further, we can explicitly understand the equivariant Borel–Moore homology under certain conditions of the T-action on the space X, using the map in Lemma 4.1.

We first introduce some notation that we will need to state the result. Consider a 1-dimensional orbit E of T in X. Then consider the action of T on E factors through some character $\chi : T \to \mathbb{G}_m$, such that the kernel of χ is precisely the stabilizer of a point in E. Note that there are two choices here by changing the sign, but this does not make a difference to the conditions in the following proposition. Taking the closure of E, we get two fixed points in the boundary, which we denote by x_0 and x_∞. With this notation we get the following result [11, Corollary 1].

PROPOSITION 4.3

Let X be a proper equivariantly formal variety with a T-action. Assume further that it only has finitely many 1-dimensional orbits. Let $E_i, i = 1, \dots, k$ be these orbits and let $\chi_i, i = 1, \dots, k$ denote the corresponding characters. Then $H_T^{\mathrm{BM}}(X) \subset H_T^{\mathrm{BM}}(X^T) \otimes_{H_T^*(pt)} \mathrm{Frac}(H_T^*(pt))$ coincides with the subset of all tuples $(f_x)_{x \in X^T}$ (with $f_x \in \mathrm{Frac}(H_T^*(pt))$ note that only finitely many f_x are nonzero because we consider BM homology) satisfying the following conditions:

- *Let $x \in X^T$. Let $E_1, \dots, E_k$ be all 1-dimensional orbits whose closure contains x, and let $\chi_1, \dots, \chi_k$ be the corresponding characters. Then $f_x \prod_{i=1}^k \chi_i \in H_T^*(pt)$ for any $x \in X^T$.*
- *Let E be a 1-dimensional T-orbit and let x_0 and x_∞ be the two points in the boundary of E. Let χ be the character corresponding to E. Then*

$$\mathrm{Res}_{\chi=0}(f_{x_0} + f_{x_\infty}) = 0$$

for all 1-dimensional orbits E.

The above results are stated for varieties, but we will need them for ind-schemes. In this setting, the corresponding functors H_T^* and H_T^{BM} can be defined respectively as the limit and colimit over the finite dimensional T-stable subvarieties, and so we can use the above results for varieties to get similar results for ind-schemes.

Remark 4.4
Under Hom, colimits are sent to limits. So we still have an isomorphism

$$H_T^*(X) \xrightarrow{\sim} \mathrm{Hom}_{H_T^*(pt)}\big(H_T^{\mathrm{BM}}(X), H_T^*(pt)\big).$$

Note that in the finite-type scheme case we also have the dual map

$$H_T^{\mathrm{BM}}(X) \to \mathrm{Hom}_{H_T^*(pt)}\big(H_T^*(X), H_T^*(pt)\big)$$

being an isomorphim, but in the ind-scheme case this is only true when we consider continuous Hom with respect to the limit topology.

Remark 4.5
In the case of ind-schemes, we have a direct analog of Proposition 4.3 under the following conditions:

- X is an ind-proper equivariantly formal ind-scheme with a T-action.
- X has isolated fixed points.
- For any two fixed points x, x', there are finitely many 1-dimensional orbits E whose boundary is $\{x, x'\}$.

4.2. Borel–Moore homology of equivalued unramified affine Springer fibers
In this section we will describe some properties of the Borel–Moore homology of our affine Springer fibers. We use the above results, and the main references for this section are [25] and [24].

We use the notation $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$.

We start by recalling the definition of the affine flag variety. For a reductive algebraic group G with root data $(R, \mathbb{X}^* = \Lambda^*, R^\vee, \mathbb{X}_* = \Lambda)$, consider the Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We also consider the arc and loop groups $G(\mathcal{O}) \subset G(\mathcal{K})$ and the Iwahori subgroup $\mathfrak{B} \subset G(\mathcal{O})$. Recall that the latter is defined as the preimage of B under the projection $G(\mathcal{O}) \twoheadrightarrow G$.

Using these we can define the affine flag variety $\mathcal{F}l = G(\mathcal{K})/\mathfrak{B}$, which is an ind-projective variety. This space has actions by T and $T(\mathcal{K})$ given by left multiplication. Further, $\mathbb{C}^\times$ acts by field automorphisms on $\mathcal{K}$ scaling t and so we get an induced action on $\mathcal{F}l$, which is referred to as the loop rotation action.

We write Λ for the cocharacter lattice of T. The fixed points of the action of both T and $T \times \mathbb{C}^\times$ are in bijection with the affine Weyl group $\widetilde{W} = W \ltimes \Lambda$ under the natural embedding $\widetilde{W} \hookrightarrow \mathcal{F}l$. To get this embedding, note that $W \hookrightarrow G/B \hookrightarrow \mathcal{F}l$ and that $T(\mathcal{K})/T(\mathcal{O}) \cong \Lambda$ and $T(\mathcal{O})$ acts trivially on the image of W in $\mathcal{F}l$.

Further, we have an action of the affine Weyl group $\widetilde{W}$ on the extended torus $T \times \mathbb{C}^\times$. The finite Weyl group W acts only on the T factor with the usual action coming from $W = N(T)/T$. The cocharacter lattice Λ acts via

$$t^\lambda : T \times \mathbb{C}^\times \to T \times \mathbb{C}^\times$$
$$(t,h) \mapsto \big(t\lambda(h), h\big).$$

Note that the cocharacter lattice of $T \times \mathbb{C}^\times$ is naturally identified with $\Lambda \times \mathbb{Z}$ and the induced action of $\widetilde{W}$ on $\Lambda \times \mathbb{Z}$ is given by (2.5).

Now we can introduce the affine Springer fibers we will look at. Fix a nonnegative integer d. Consider a regular semisimple element $s \in \mathfrak{t} \hookrightarrow \mathfrak{g}$. Then we can consider $e_d = t^d s \in \mathfrak{g}(\mathcal{O})$ and its associated affine Springer fiber, known as the *equivalued unramified affine Springer fiber*

$$\mathcal{F}l_{e_d} := \big\{ g\mathfrak{B} \in \mathcal{F}l \,\big|\, \mathrm{Ad}(g)^{-1} e_d \in \mathrm{Lie}(\mathfrak{B}) \big\}. \tag{4.3}$$

Note that e_d is fixed by T and thus $\mathcal{F}l_{e_d} \subset \mathcal{F}l$ is T-stable and the loop rotation scales e_d; hence, these Springer fibers $\mathcal{F}l_{e_d}$ are also stable under the loop rotation action. The image of $\widetilde{W}$ is contained in all these affine Springer fibers; thus, these give the T-fixed and $T \times \mathbb{C}^\times$-fixed points for all $\mathcal{F}l_{e_d}$.

We can further consider the 1-dimensional orbits of $T \times \mathbb{C}^\times$. In order to do this, we need some notation. For a root α of $\mathfrak{g}$, we write s_α for the corresponding reflection in W. For an integer k, we write $s_{\alpha,k}$ for $t^{k\alpha} s_\alpha$; this is a reflection in $\widetilde{W}$. A root α gives a character $\alpha : T \to \mathbb{C}^\times$ and so also gives a character of $T \times \mathbb{C}^\times$ by acting trivially on the loop rotation factor. Further, define $\hbar : T \times \mathbb{C}^\times \to \mathbb{C}^\times$ as the projection to the loop rotation factor. We can also act on the characters of $T \times \mathbb{C}^\times$ by $\widetilde{W}$, the action induced from that on $T \times \mathbb{C}^\times$. So we get the character $\alpha + k\hbar$ of $T \times \mathbb{C}^\times$. Let ${}^x(\alpha + k\hbar)$ denote the image of $\alpha + k\hbar$ under the action of $x \in \widetilde{W}$.

The 1-dimensional orbits in $\mathcal{F}l$ can be seen to be given by $\mathbb{P}^1$s connecting the fixed points x and $xs_{\alpha,k}$ for all $x \in \widetilde{W}$, roots α, and integers k. The associated character is given by ${}^x(\alpha + k\hbar)$.

Below we will use the following notation

$$\mathsf{R} := H^*_{T\times\mathbb{C}^\times}(pt), \mathsf{F} := \mathrm{Frac}(\mathsf{R}). \tag{4.4}$$

PROPOSITION 4.6

(1) *For the affine Springer fibers $\mathcal{F}l_{e_d}$, the 1-dimensional orbits are exactly the 1-dimensional orbits in $\mathcal{F}l$ connecting x and $xs_{\alpha,k}$ if $-d \leq k \leq d-1$.*

(2) *The affine Springer fibers* $\mathcal{F}l_{e_d}$ *and the affine flag variety* $\mathcal{F}l$ *with the* $T\times\mathbb{C}^\times$*-action are equivariantly formal.*

(3) $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ *is flat as an* R*-module and we have*

$$H^{\mathrm{BM}}_{T}(\mathcal{F}l_{e_d})\cong H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})\otimes_{H^*_{\mathbb{C}^\times}(pt)}\mathbb{C},$$

$$H^{\mathrm{BM}}(\mathcal{F}l_{e_d})\cong H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})\otimes_{\mathsf{R}}\mathbb{C}.$$

The similar claim holds for $\mathcal{F}l$.

Proof
The first result is worked out in [24, Section 5.11]. The second result follows from the existence of an affine space paving as constructed in [25, Theorem 0.2] for the affine Springer fibers, while for the affine flag variety it follows from the Bruhat decomposition. The last result follows immediately from the second. □

Example 4.7
Let $d=0$. Then $e_0=s$, a regular semisimple element. The Springer fiber $\mathcal{F}l_{e_0}$ is discrete and is identified with the T-fixed point locus, $\widetilde{W}$. Claim (1) of the proposition is manifestly true.

The following claim follows from combining Proposition 4.3 (or, more precisely, its ind-scheme generalization, see Remark 4.5) and Proposition 4.6.

COROLLARY 4.8
The localization homomorphism identifies $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ *with the subset of all elements* $(f_x)_{x\in\widetilde{W}}\in\bigoplus_{\widetilde{W}}\mathsf{F}$ *satisfying the following two conditions:*

(i) *For all* x*, the product*

$$f_x\prod_{\alpha\in R^+}\prod_{k=-d}^{d-1}({}^x\alpha+k\hbar)$$

is an element of R*. Here* R^+ *stands for the system of positive Dynkin roots.*

(ii) *For all* $x\in\widetilde{W}$, $\alpha\in R^+$ *and* k *with* $-d\le k\le d-1$*, we have*

$$\mathrm{Res}_{\,{}^x(\alpha+k\hbar)}(f_x+f_{xs_{\alpha,k}})=0.$$

We will also need the following corollary of (1) of Proposition 4.6. Recall that $e_d=t^ds$, where $s\in\mathfrak{t}^{\mathrm{reg}}$.

COROLLARY 4.9
The image of $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ *in* $\bigoplus_{\widetilde{W}}\mathsf{F}$ *is independent of the choice of a regular semisimple element* $s\in\mathfrak{t}^{\mathrm{reg}}$.

Using this corollary we identify the spaces $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ for different choices of s.

Remark 4.10
We now discuss line bundles on $\mathcal{F}l$. For a weight $\lambda\in\Lambda^*\times\mathbb{Z}$ of $T\times\mathbb{C}^\times$ we can construct a 1-dimensional $T(\mathcal{O})\times\mathbb{C}^\times$-representation $\mathbb{C}_\lambda$, which extends to a $\mathfrak{B}$-representation. The latter gives rise to a $G(\mathcal{K})\rtimes\mathbb{C}^\times$-equivariant line bundle on $\mathcal{F}l$ to be denoted by $\mathcal{L}_\lambda$.

The proof of Proposition 4.6 also implies that the conditions for Proposition 4.3 are satisfied for $\mathcal{F}l_{e_d}$ and $\mathcal{F}l$. We can thus consider the localization homomorphism

$$H^*_{T\times\mathbb{C}^\times}(\mathcal{F}l)\hookrightarrow\prod_{\widetilde{W}}\mathsf{F}.$$

Now we want to compute the images of the Chern classes of the line bundles $\mathcal{L}_\lambda$ under this localization map. To compute the localization to the fixed points of $c_1(\mathcal{L}_\lambda)$, we need to consider the $T\times\mathbb{C}^\times$-representations given by $\mathcal{L}_\lambda$ restricted to a fixed point, $x\in\widetilde{W}$. Note that this gives the 1-dimensional representation $\mathbb{C}_{{}^x\lambda}$ and thus under the map

$$H^*_{T\times\mathbb{C}^\times}(\mathcal{F}l)\to\prod_{x\in\widetilde{W}}\mathsf{F}$$

the Chern class $c_1(\mathcal{L}_\lambda)$ is sent to $({}^x\lambda)_{x\in\widetilde{W}}$.

5. The actions on the Borel–Moore homology

In Section 2.4 we have recalled the trigonometric Cherednik algebras $H^\times_{\hbar,c}$. The goal of this section is to equip $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ with a structure of an $H^\times_{\hbar,d}$-$H^\times_{\hbar,0}$-bimodule and establish some properties of this bimodule. Recall that we write R for $H^*_{T\times\mathbb{C}^\times}(pt)$ and F for $\mathrm{Frac}(\mathsf{R})$.

5.1. Chern–Springer action

In this section we will establish a left action of $H^\times_{\hbar,d}$ on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$. Let ι denote the localization embedding

$$H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})\hookrightarrow\bigoplus_{\widetilde{W}}\mathsf{F}. \tag{5.1}$$

For $x \in \widetilde{W}$, let $\iota(?)_x$ denote the x-component of $\iota(?)$; this is an element of F. We note that the target of (5.1) can be viewed as the space of functions $\widetilde{W} \to \mathsf{F}$ that are zero outside of a finite set.

We start by describing the action of the Chern classes. Note that $\prod_{\widetilde{W}} \mathsf{R}$ naturally acts on $\bigoplus_{\widetilde{W}} \mathsf{R}$. So, for a character λ of $T \times \mathbb{C}^\times$, the element $c_1(\mathcal{L}_\lambda)$ acts on $\operatorname{im}\iota$ as the multiplication with $({}^x\lambda)_{x\in\widetilde{W}}$. This is a consequence of Remark 4.10. So we get an action of $\mathfrak{t}^* \oplus \mathbb{C}\hbar$ on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$. Note that the operators of this action pairwise commute.

The group $\widetilde{W}$ acts on $H^*_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ via the Springer action (see [41], [50], [47]). We will recall the construction in Section A.1.

So we get two actions on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$: the action of $\mathfrak{t}^* \oplus \mathbb{C}\hbar$ by the multplication with Chern classes and the Springer action of $\widetilde{W}$. The former gives rise to an action of the algebra $\mathbb{C}[\mathfrak{t}, \hbar]$, while the latter gives an action of the algebra $\mathbb{C}\widetilde{W}$. Both actions are R-linear and so extend to the localization $\bigoplus_{\widetilde{W}} \mathsf{F}$.

PROPOSITION 5.1

These two actions equip $\bigoplus_{\widetilde{W}} \mathsf{F}$ *with an* $H^\times_{\hbar,d}$*-module structure. The subspace* $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ *embedded via* ι *is a submodule.*

The key tool in the proof is as follows: we write formulas for the actions of simple affine reflections, the elements of $\Lambda/\Lambda_0 \subset \widetilde{W}$, and also the elements of $\mathbb{C}[\mathfrak{t}, \hbar]$ on the image of the embedding ι. Let us state the corresponding result.

LEMMA 5.2

For all $\beta \in H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$, $x \in \widetilde{W}$, *simple affine reflections* $s = s_\alpha$, $\lambda \in \mathfrak{t}^* \oplus \mathbb{C}\hbar$ *and* $\pi \in \Lambda/\Lambda_0 \subset \widetilde{W}$ *we have the following formulas:*

$$\begin{aligned}
\iota(s\beta)_x &= \frac{d\hbar}{{}^x\alpha}\iota(\beta)_x + \frac{{}^{xs}\alpha - d\hbar}{{}^{xs}\alpha}\iota(\beta)_{xs}, \\
\iota(\lambda\beta)_x &= ({}^x\lambda)\iota(\beta)_x, \\
\iota(\pi\beta)_x &= \iota(\beta)_{x\pi}.
\end{aligned} \tag{5.2}$$

Note that the formulas make sense for an arbitrary element of $\bigoplus_{\widetilde{W}} \mathsf{F}$ not just for $\iota(\beta)$. They define an action of $H^\times_{\hbar,d}$ on $\bigoplus_{\widetilde{W}} \mathsf{F}$.

The second equality in (5.2) has already been discussed in the beginning of the section. The last equality easily follows from the construction of the Λ/Λ_0-action to be discussed in Section A.1. The first equality requires more work; it will be established in the appendix, Section A.2.

Proof of Proposition 5.1
It is enough to check the commutation relations of (2.6).

The second and third equalities in (2.6) are immediate from Lemma 5.2. In the remainder of the proof we will check the first equality. That is, for a simple reflection $s := s_\alpha$ and $\lambda \in \mathfrak{t}^*$, we should check the following relation:

$$s\lambda - {}^s\lambda s = d\langle \lambda, \alpha^\vee\rangle \hbar. \tag{5.3}$$

To check this, we apply the summands of the left-hand side to an element $\xi \in \bigoplus_{\widetilde{W}} \mathsf{F}$:

$$(s\lambda\xi)_x = \frac{d\hbar}{{}^x\alpha}(\lambda\xi)_x + \frac{{}^{xs}\alpha - d\hbar}{{}^{xs}\alpha}(\lambda\xi)_{xs} = \frac{d\hbar}{{}^x\alpha}\,{}^x\lambda\xi_x + \frac{{}^{xs}\alpha - d\hbar}{{}^{xs}\alpha}\,{}^{xs}\lambda\xi_{xs},$$
$$({}^s\lambda s\xi)_x = {}^{xs}\lambda\Big(\frac{d\hbar}{{}^x\alpha}\xi_x + \frac{{}^{xs}\alpha - d\hbar}{{}^{xs}\alpha}\xi_{xs}\Big).$$

So

$$(s\lambda\xi - {}^s\lambda s\xi)_x = ({}^x\lambda - {}^{xs}\lambda)\frac{d\hbar}{{}^x\alpha}\xi_x = \langle\lambda,\alpha^\vee\rangle\,{}^x\alpha\frac{d\hbar}{{}^x\alpha}\xi_x = d\langle\lambda,\alpha^\vee\rangle\hbar\xi_x.$$

This proves the first equality in (2.6) and finishes the proof. □

5.2. Equivariant-centralizer-monodromy action
The goal of this section is to define an action of $H^\times_{\hbar,0}$ on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$. We will view $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}) \hookrightarrow \bigoplus_{\widetilde{W}} \mathsf{F}$ as right R-modules; this structure on the former space was discussed in the general situation in Section 4.1.

Define a right action of $\widetilde{W}$ on $\bigoplus_{\widetilde{W}} \mathsf{F}$ by

$$(fy)_x = {}^{y^{-1}}f_{yx}, \quad x, y \in \widetilde{W}, (f_x) \in \bigoplus_{\widetilde{W}} \mathsf{F}. \tag{5.4}$$

LEMMA 5.3
The right actions of $\mathsf{R} = \mathbb{C}[\mathfrak{t}][\hbar]$ *and* $\widetilde{W}$ *on* $\bigoplus_{\widetilde{W}} \mathsf{F}$ *constitute a right action of* $H^\times_{\hbar,0}$. *Moreover,* $\operatorname{im}\iota$ *is a submodule.*

Proof
We start by proving that we indeed get an action of $H^\times_{\hbar,0}$. The only missing relation is the commutation relations of the $\widetilde{W}$ action and the R action, that is,

$$y\lambda - {}^y\lambda y = 0, \quad y \in \widetilde{W}, \lambda \in \mathfrak{t}^*.$$

For $(f_x) \in \bigoplus_{\widetilde{W}} \mathsf{F}$, we get

$$(fy\lambda)_x = \lambda(fy)_x = \lambda\,{}^{y^{-1}}f_{yx} = {}^{y^{-1}}({}^y\lambda f_{yx}) = (f\,{}^y\lambda y)_x.$$

This completes the proof of the claim that the actions of R and $\widetilde{W}$ constitute an action of $H^{\times}_{\hbar,0}$.

The claim that the image of ι is $H^{\times}_{\hbar,0}$-stable is immediate from the formulas defining the action and the description of the image in Corollary 4.8. □

Remark 5.4

The action of $\Lambda \subset \widetilde{W}$ on $H^{\mathrm{BM}}_{T\times\mathbb{C}^{\times}}(\mathcal{F}l_{e_d})$ comes from the action of $T(\mathcal{K})$ on $\mathcal{F}l_{e_d}$. The action of $W \subset \widetilde{W}$ is more tricky. Recall from Corollary 4.9 that the spaces $H^{\mathrm{BM}}_{T\times\mathbb{C}^{\times}}(\mathcal{F}l_{e_d})$ are identified for all choices of s via ι. So the action of W can be interpreted as the monodromy action. However, we do not know a way to identify the BM homology space for various s without the GKM description. So it is easier just to define the action on the localized BM homology spaces.

The resulting action of $H^{\times}_{\hbar,0}$ will be called the ECM (equivariant-centralizer-monodromy) action.

COROLLARY 5.5

The CS action of $H^{\times}_{\hbar,d}$ on $\bigoplus_{\widetilde{W}} \mathsf{F}$ commutes with the ECM action of $H^{\times}_{\hbar,0}$. Hence, these actions also commute on $H^{\mathrm{BM}}_{T\times\mathbb{C}^{\times}}(\mathcal{F}l_{e_d})$.

Proof

The actions of generators are specified in Lemma 5.2 for the CS action and in Lemma 5.3 for the ECM action. One directly checks that the generators of $H^{\times}_{\hbar,d}$ commute with the generators of $H^{\times}_{\hbar,0}$. □

So $H^{\mathrm{BM}}_{T\times\mathbb{C}^{\times}}(\mathcal{F}l_{e_d})$ becomes an $H^{\times}_{\hbar,d}$-$H^{\times}_{\hbar,0}$-bimodule.

Example 5.6

Consider the example of $d = 0$, where $\mathcal{F}l_{e_0} \xrightarrow{\sim} \widetilde{W}$ by Example 4.7. The image of ι is just $\bigoplus_{\widetilde{W}} \mathsf{R}$ that naturally identifies with $H^{\times}_{\hbar,0}$. The bimodule structure on $H^{\mathrm{BM}}_{T\times\mathbb{C}^{\times}}(pt)$ is that of the regular bimodule, as seen directly from the formulas in Lemmas 5.2 and 5.3.

5.3. *Properties of the bimodule*

The goal of this section is to prove some properties of the $H^{\times}_{\hbar,d}$-$H^{\times}_{\hbar,0}$-bimodule $H^{\mathrm{BM}}_{T\times\mathbb{C}^{\times}}(\mathcal{F}l_{e_d})$ that are analogous to those of the $H_{\hbar,d}$-$H_{\hbar,0}$-bimodule $B_{\hbar,d\leftarrow 0}$ in Lemma 3.7.

LEMMA 5.7
For $d \geq 0$, *we have a graded* $H^{\times}_{\hbar,0}$*-linear isomorphism* $\epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}) \cong \epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}})$ *(where we shift the grading on one of the sides).*

Proof
The proof is in several steps.

Step 1. Let $\beta \in H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}})$. Set $(f_x) := \iota(\beta)$. The condition that $\beta \in \epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}})$ is equivalent to $f_x = -f_{xs}$ for all simple Dynkin reflections s. This follows from Lemma 5.2.

Now let $\beta' \in H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$. Set $(f'_x) = \iota(\beta')$. Thanks to Lemma 5.2, we have $\beta' \in \epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ if and only if $({}^x\alpha + d\hbar) f_{xs} = ({}^x\alpha - d\hbar) f_x$ for all simple Dynkin reflections $s = s_\alpha$.

Step 2. We want to define mutually inverse maps between $\iota(\epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}}))$ and $\iota(\epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}))$. Define the element $\upsilon \in \mathbb{C}[\mathfrak{t}][\hbar] = \mathsf{R}$ by

$$\upsilon := \prod_{\alpha\in R^+} (\alpha + d\hbar),$$

where we write R^+ for the system of positive Dynkin roots. Define an endomorphism of $\bigoplus_{\widetilde{W}} \mathsf{F}$ by

$$\Upsilon : (f_x) \mapsto (g_x) := ({}^x\upsilon f_x). \tag{5.5}$$

Note that Υ is invertible. Also note that υ can be viewed as an element of $H^{\times}_{\hbar,d}$ (see Proposition 5.1 and Lemma 5.2). From Corollary 5.5 we deduce that Υ is $H^{\times}_{\hbar,0}$-linear. The element υ has degree $|R^+|$ so we can shift the grading and assume Υ is graded. It remains to show that

$$\Upsilon\big(\iota\big(\epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}})\big)\big) \subset \iota\big(\epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})\big), \tag{5.6}$$

$$\Upsilon^{-1}\big(\iota\big(\epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})\big)\big) \subset \iota\big(\epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}})\big). \tag{5.7}$$

Step 3. We start by proving (5.6) in this step and the next two. Assume $(f_x) \in \iota(\epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}}))$. We need to check that $(g_x) \in \iota(\epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}))$. We begin by checking $(g_x) \in \iota(H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}))$. This will be done using Corollary 4.8 (for both d and $d+1$).

We first check (i) for d, that is, that

$$g_x \prod_{\alpha\in R^+} \prod_{k=-d}^{d-1} ({}^x\alpha + k\hbar) = \Big[\Big(f_x \prod_{\alpha\in R^+} ({}^x\alpha + d\hbar)\Big) \prod_{\alpha\in R^+} \prod_{k=-d}^{d-1} ({}^x\alpha + k\hbar)\Big] \in \mathsf{R}.$$

By (i) applied to $d+1$ and the point x in Corollary 4.8, we have

$$f_x \prod_{\alpha\in R^+}\prod_{k=-d-1}^{d} ({}^x\alpha + k\hbar) \in \mathsf{R}.$$

It remains to show that f_x (hence g_x) cannot have poles along ${}^x\alpha - (d+1)\hbar$ for any positive roots α. Note $f_x = -f_{xs}$ so it can only have poles along $({}^{xs}\alpha + k\hbar)$ for $k = -d-1, \dots, d$. But, for $s = s_\alpha$, $({}^{xs}\alpha + k\hbar) = -({}^x\alpha - k\hbar)$. So f_x indeed has no pole along $({}^x\alpha - (d+1)\hbar)$. This establishes (i) of Corollary 4.8 for d.

Step 4. Now we need to check that (ii) of Corollary 4.8 holds for (g_x):

$$\mathrm{Res}_{{}^x\beta+k\hbar}(g_x + g_{xs_{\beta,k}}) = 0$$

for all $x \in \widetilde{W}$, $\beta \in R^+$, and $k = -d, \dots d-1$. Note that

$${}^xF \equiv {}^{xs_{\beta,k}}F \mod {}^x\beta + k\hbar, \quad \forall F \in \mathsf{R}.$$

In particular,

$$\prod_{\alpha\in R^+} ({}^x\alpha + d\hbar) \equiv \prod_{\alpha\in R^+} ({}^{xs_{\beta,k}}\alpha + d\hbar) \mod {}^x\beta + k\hbar. \tag{5.8}$$

Recall that f_x has at most simple pole at ${}^x\beta + k\hbar$. It follows that

$$\mathrm{Res}_{{}^x\beta+k\hbar}\Big(f_x \prod_{\alpha\in R^+} ({}^x\alpha + d\hbar) - f_x \prod_{\alpha\in R^+} ({}^{xs_{\beta,k}}\alpha + d\hbar)\Big) = 0. \tag{5.9}$$

Since

$$\mathrm{Res}_{{}^x\beta+k\hbar}(f_x + f_{xs_{\beta,k}}) = 0,$$

for all $\beta \in R^+$ and all $k = -d, \dots, d-1$ (this is a part of (ii) of Corollary 4.8) for $d+1$, we deduce from (5.9) that

$$\mathrm{Res}_{{}^x\beta+k\hbar}(g_x + g_{xs_{\beta,k}}) = 0$$

for β and k in the same range. This is exactly (ii) of Corollary 4.8. This finishes the proof of $\Upsilon(\iota(\epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}}))) \subset \iota(H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}))$.

Step 5. We finally check that $\epsilon(g_x) = (g_x)$, equivalently $s_\beta(g_x) = (g_x)$ for each Dynkin simple root β. This will finish the proof of (5.6).

Using the formula for the Springer action of s_β, Lemma 5.2, and the construction of (g_x), we see that the equality $s_\beta(g_x) = (g_x)$ is equivalent to

$$({}^x\beta - d\hbar)\Big(\prod_{\alpha\in R^+} ({}^x\alpha + d\hbar)\Big) f_x = ({}^x\beta + d\hbar) \prod_{\alpha\in R^+} ({}^{xs}\alpha + d\hbar) f_{xs} \tag{5.10}$$

for all $x \in \widetilde{W}$.

Rearranging the factors, we get

$$({}^x\beta - d\hbar)\prod_{\alpha\in R^+}({}^x\alpha + d\hbar) = -({}^x\beta + d\hbar)\prod_{\alpha\in R^+}({}^{xs}\alpha + d\hbar). \tag{5.11}$$

Since $(f_x) \in \iota(\epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}}))$, we have $f_x = -f_{xs}$. Combining this with (5.11), we get (5.10). This finishes the proof of (5.6).

Step 6. Now we check (5.7). Let $(g_x) \in \iota(\epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}))$. Set $f_x := (g_x \prod_{\alpha\in R^+}({}^x\alpha + d\hbar)^{-1})$. We need to show that

- $f_x = -f_{xs}$ for all $x \in \widetilde{W}$ and simple Dynkin reflection s;
- and the collection (f_x) satisfies (i) and (ii) of Corollary 4.8 for $d+1$.

The first bullet is checked by reversing the argument of Step 5. In the remainder of the proof we will check the second bullet.

Step 7. We start by checking (i). Note that, by condition (i) for d, g_x has at most simple poles along $({}^x\alpha + k\hbar)$ for $\alpha \in R^+$, $k = -d, \dots d-1$. Hence, f_x has at most simple poles along $({}^x\alpha + k\hbar)$ for $\alpha \in R^+$, $k = -d, \dots d$. This verifies condition (i) for $d+1$.

Step 8. Now we just need to check condition (ii):

$$\mathrm{Res}_{{}^x\beta + k\hbar}(f_x + f_{xs_{\beta,k}}) = 0 \tag{5.12}$$

for $\beta \in R^+$ and $k = -(d+1), \dots, d$. Step 7 implies that f_x has no pole along the roots $({}^x\alpha - (d+1)\hbar)$. (5.12) for $k = -(d+1)$ and all β follows.

Now we establish (5.12) for $k = -d, \dots, d-1$. The function $(\prod_{\alpha\in R^+}({}^x\alpha + d\hbar))^{-1}$ has no poles along $({}^x\beta + k\hbar)$ for $k \neq d$. Using this and (5.8), we easily deduce (5.12) from condition (ii) of Corollary 4.8 for the collection (g_x).

It remains to establish (5.12) for $k = d$. Note that, by Step 6, $f_x + f_{xs_{\beta,d}} = -f_{xs_\beta} - f_{xs_{\beta,-d}}$. So (5.12) for $k = d$ follows from the equation for $k = -d$ (with x replaced with xs_β). The latter has been established in the previous paragraph. □

6. Proofs of the main theorems

In this section we will prove Theorems 1.1 and 1.2.

6.1. Isomorphism of deformations

Now we state the main result of this section that implies Theorem 1.1. We write $H^\wedge_{\hbar,c}$ for the isomorphic algebras in Lemma 2.6. Set

$$B^\wedge_{\hbar,d\leftarrow 0} := B_{\hbar,d\leftarrow 0} \otimes_{\mathbb{C}[\mathfrak{t}^*]} \mathbb{C}[\mathfrak{t}^*]^{\wedge 0},$$

$$H^{T\times\mathbb{C}^\times}_{\mathrm{BM}}(\mathcal{F}l_{e_d})^\wedge := H^{T\times\mathbb{C}^\times}_{\mathrm{BM}}(\mathcal{F}l_{e_d}) \otimes_{\mathbb{C}[T^\vee]} \mathbb{C}[T^\vee]^{\wedge 1}.$$

Both $B^{\wedge}_{\hbar,d\leftarrow 0}$ and $H^{T\times\mathbb{C}^\times}_{\mathrm{BM}}(\mathcal{F}l_{e_d})^{\wedge}$ are graded $H^{\wedge}_{\hbar,d}$-$H^{\wedge}_{\hbar,0}$-bimodules that are flat over $\mathbb{C}[\hbar]$. This follows from Section 3.3 for the former bimodule, and from Corollary 5.5 and (2) of Proposition 4.6 for the latter bimodule.

THEOREM 6.1
We have a graded $H^{\wedge}_{\hbar,d}$-$H^{\wedge}_{\hbar,0}$-bimodule isomorphism $B^{\wedge}_{\hbar,d\leftarrow 0} \xrightarrow{\sim} H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})^{\wedge}$.

Let us explain key ideas of the proof. We use induction on d. Note that for $d=0$ both sides are isomorphic to the regular $H^{\wedge}_{\hbar,0}$-bimodule: for the left-hand side this follows from the construction in Section 3.3. For the right-hand side the claim follows from Example 5.6. The case $d=0$ is our induction base.

Note that $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})^{\wedge}$ is flat over $\mathbb{C}[\mathfrak{t},\hbar]$ for the right action; this follows from (3) of Proposition 4.6. With this in mind, the induction step is based on Lemmas 3.7, 5.7, and the next proposition.

PROPOSITION 6.2
Let $d>0$. Let $M_\hbar$ be a graded $H^{\wedge}_{\hbar,d}$-$H^{\wedge}_{\hbar,0}$-bimodule that is flat over $\mathbb{C}[\hbar]$. Moreover, assume $M_\hbar/(\hbar)$ is torsion-free over $\mathbb{C}[\mathfrak{t}]$. Any graded $\epsilon_- H^{\wedge}_{\hbar,d}\epsilon_-$-$H^{\wedge}_{\hbar,0}$-linear isomorphism

$$\epsilon_- M_\hbar \xrightarrow{\sim} \epsilon_- B^{\wedge}_{\hbar,d\leftarrow 0} \tag{6.1}$$

uniquely extends to a graded $H^{\wedge}_{\hbar,d}$-$H^{\wedge}_{\hbar,0}$-linear isomorphism

$$M_\hbar \xrightarrow{\sim} B^{\wedge}_{\hbar,d\leftarrow 0}.$$

The proof will be given after a construction and a lemma.

We can view $H^{\wedge}_{\hbar,c}$ as a filtered algebra (with $\deg\hbar = \deg\mathbb{C}[\mathfrak{t}]^{\wedge_0} = \deg W = 0$, $\deg\mathfrak{t}=1$). Formally, the filtered algebra $H^{\wedge}_{\hbar,c}$ is obtained as $\mathbb{C}[\hbar']\otimes_{\mathbb{C}[\hbar]} H^{\wedge}_{\hbar,c}$, where the homomorphism $\mathbb{C}[\hbar]\to\mathbb{C}[\hbar']$ sends $\hbar$ to $\hbar'$, but $\hbar'$ is treated as a degree 0 element. In what follows we write $\hbar$ instead of $\hbar'$. Note that the resulting filtration on $H^{\wedge}_{\hbar,c}$ is $\mathbb{C}^\times$-stable (for the $\mathbb{C}^\times$-action induced by the grading). We have $\operatorname{gr} H^{\wedge}_{\hbar,c} = H^{\wedge}\otimes\mathbb{C}[\hbar]$. This follows from the triangular decomposition, $\mathbb{C}[\mathfrak{t}^*]^{\wedge_0}[\hbar]\otimes\mathbb{C}W\otimes\mathbb{C}[\mathfrak{t}] \xrightarrow{\sim} H^{\wedge}_{\hbar,c}$, which is an easy consequence of the analogous decomposition for $H_{\hbar,c}$.

Set

$$\mathfrak{t}^{*\wedge} := \operatorname{Spec}\bigl(\mathbb{C}[\mathfrak{t}^*]^{\wedge_0}\bigr), \qquad Y^{\wedge} := \mathfrak{t}^{*\wedge}/W \times_{\mathfrak{t}^*/W} Y, \qquad Y^{\wedge}_{\hbar} := Y^{\wedge}\times\operatorname{Spec}\bigl(\mathbb{C}[\hbar]\bigr),$$

where we recall that $Y=(\mathfrak{t}\oplus\mathfrak{t}^*)/W$. The scheme $Y^{\wedge}_{\hbar}$ is the spectrum of the center of $\operatorname{gr} H^{\wedge}_{\hbar,c}$ because Y is the spectrum of the center of H. So the algebra $H^{\wedge}_{\hbar,c}$ can be *microlocalized* to $Y^{\wedge}_{\hbar}$.

Now we recall some basics on the microlocalization. The result of microlocalization of $H^{\wedge}_{\hbar,c}$ is a sheaf of algebras on $Y^{\wedge}_{\hbar}$ whose sections are defined on $\mathbb{C}^{\times}$-stable open subsets of $Y^{\wedge}_{\hbar}$ (for the $\mathbb{C}^{\times}$-action, that is the original action on $Y^{\wedge}$ and is trivial on $\operatorname{Spec}(\mathbb{C}[\hbar])$; we call this the *modified* $\mathbb{C}^{\times}$-action—note that we also have the initial $\mathbb{C}^{\times}$-action for which $\hbar$ has degree 1). It is enough to define sections over principal open subsets of $Y^{\wedge}_{\hbar}$. Pick a homogeneous element $f \in \mathbb{C}[Y^{\wedge}_{\hbar}]$ for the modified $\mathbb{C}^{\times}$-action. Consider the Rees algebra $R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})$, where h is a variable of degree 1 for the modified $\mathbb{C}^{\times}$-action (and degree 0 for the initial $\mathbb{C}^{\times}$-action). Lift f to a homogeneous element $\tilde{f} \in R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})$. Then $\{\tilde{f}^k \mid k \geq 0\}$ is an Ore subset in each quotient $R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})/(\mathsf{h}^n)$. The localization is easily seen to be independent of the lift $\tilde{f}$; denote it by $R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})/(\mathsf{h}^n)[f^{-1}]$. This is completely analogous to [18, Construction of A_S]. The localizations inherit gradings from the grading on $R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})/(\mathsf{h}^n)$ that comes from the Rees construction. The graded algebras $R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})/(\mathsf{h}^n)[f^{-1}]$ form a projective system with respect to n. So we can consider the inverse limit in the category of graded algebras. Denote this inverse limit by $R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})[f^{-1}]$. Set

$$H^{\wedge}_{\hbar,c}[f^{-1}] := R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})[f^{-1}]/(\mathsf{h}-1).$$

By the construction, the algebra $H^{\wedge}_{\hbar,c}[f^{-1}]$ is filtered, and the filtration is complete and separated. Also by the construction, the algebra $R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})[f^{-1}]$ is flat over $\mathbb{C}[\mathsf{h}]$. It follows that

$$\operatorname{gr} H^{\wedge}_{\hbar,c}[f^{-1}] \cong R_{\mathsf{h}}(H^{\wedge}_{\hbar,c})[f^{-1}]/(\mathsf{h}) = H^{\wedge}[\hbar][f^{-1}].$$

The algebras $H^{\wedge}_{\hbar,c}[f^{-1}]$ form a presheaf of filtered algebras. It is a sheaf because the filtration is complete and separated and the associated graded presheaf (that of algebras $H^{\wedge}[\hbar][f^{-1}]$) is a sheaf. We denote the resulting sheaf of filtered algebras (with sections on $\mathbb{C}^{\times}$-stable open subsets for the modified $\mathbb{C}^{\times}$-actions) by $H^{\wedge,\mathrm{loc}}_{\hbar,c}$. This sheaf is complete and separated with respect to the filtration. Its algebra of global sections coincides with $H^{\wedge}_{\hbar,c}$: we have a filtered algebra homomorphism $H^{\wedge}_{\hbar,c} \to \Gamma(H^{\wedge,\mathrm{loc}}_{\hbar,c})$ that is the identity on the associated graded algebras. We note that if f is homogeneous with respect to the initial $\mathbb{C}^{\times}$-action on $H^{\wedge}_{\hbar,c}$, then $H^{\wedge}_{\hbar,c}[f]^{-1}$ inherits this action. So $H^{\wedge,\mathrm{loc}}_{\hbar,c}$ is a $\mathbb{C}^{\times}$-equivariant sheaf of filtered algebras for the initial action.

Now consider a graded $H^{\wedge}_{\hbar,d}$-$H^{\wedge}_{\hbar,0}$-bimodule $\mathcal{B}$. We can view it as a filtered $H^{\wedge}_{\hbar,d}$-$H^{\wedge}_{\hbar,0}$-bimodule by doing the same base change as with the algebra. Consider the microlocalization $\mathcal{B}^{\mathrm{loc}}$ of $\mathcal{B}$, a microlocal filtered sheaf on $Y^{\wedge}_{\hbar}$, defined similarly to the $H^{\wedge,\mathrm{loc}}_{\hbar,c}$. The sections are defined on $\mathbb{C}^{\times}$-stable Zariski open subsets, while the filtration is complete and separated. In particular, the space of sections on any open $\mathbb{C}^{\times}$-stable subset inherits the filtration, and this filtration is complete and separated.

Note that $\mathcal{B}^{\mathrm{loc}}$ is a sheaf of $H_{\hbar,d}^{\wedge,\mathrm{loc}}$-$H_{\hbar,0}^{\wedge,\mathrm{loc}}$-bimodules. We have an isomorphism $\mathcal{B} \xrightarrow{\sim} \Gamma(\mathcal{B}^{\mathrm{loc}})$ because $Y_\hbar^\wedge$ is an affine scheme (cf [8, Lemma 2.10]).

We note that, similarly to $H_{\hbar,c}^{\wedge,\mathrm{loc}}$, the sheaf $\mathcal{B}^{\mathrm{loc}}$ still carries a natural $\mathbb{C}^\times$-action that turns it into a $\mathbb{C}^\times$-equivariant $H_{\hbar,d}^{\wedge,\mathrm{loc}}$-$H_{\hbar,0}^{\wedge,\mathrm{loc}}$-bimodule (for the initial $\mathbb{C}^\times$-action).

Set

$$Y_\hbar^{\wedge,0} := Y_\hbar^\wedge \setminus \big[Y^{\wedge,\mathrm{sing}} \times \{0\}\big]. \tag{6.2}$$

Let $\mathcal{B}^0$ denote the restriction of $\mathcal{B}^{\mathrm{loc}}$ to (6.2). We get a natural homomorphism $\mathcal{B} \to \Gamma(\mathcal{B}^0)$.

LEMMA 6.3

We have the following properties:

(1) *For any ϵ_--spherical parameter c, the microlocal sheaves of algebras $H_{\hbar,c}^{\wedge,0}$ and $\epsilon_- H_{\hbar,c}^{\wedge,0}\epsilon_-$ are Morita equivalent via the bimodule $H_{\hbar,c}^{\wedge,0}\epsilon_-$.*

(2) *For $\mathcal{B} = B_{\hbar,d\leftarrow 0}^\wedge$, the homomorphism $\mathcal{B} \to \Gamma(\mathcal{B}^0)$ is an isomorphism.*

Before we get to the proof, we comment on (2). It is a well-known property that for a vector bundle on a regular scheme its global sections coincide with the sections over any open subset whose complement has codimension at least 2. By the construction, $B_{\hbar,d\leftarrow 0}^\wedge$ is the global section of a *quantization* of a vector bundle, and (2) is a quantum version of the property explained in the previous sentence. And we need the microlocalization procedure explained before the lemma to make this work in the quantum setting.

Proof

Let us prove (1). The claim is equivalent to $H_{\hbar,c}^{\wedge,0}\epsilon_- H_{\hbar,c}^0 = H_{\hbar,c}^{\wedge,0}$, which, in its turn, is equivalent to the claim that $H_{\hbar,c}^\wedge / H_{\hbar,c}^\wedge \epsilon_- H_{\hbar,c}^\wedge$ is supported on $Y^{\wedge,\mathrm{sing}} \times \{0\}$. First, the condition that c is ϵ_--spherical is equivalent to the claim that $H_{\hbar,c}/H_{\hbar,c}\epsilon_- H_{\hbar,c}$ is $\hbar$-torsion. So the support of $H_{\hbar,c}^\wedge / H_{\hbar,c}^\wedge \epsilon_- H_{\hbar,c}^\wedge$ is contained in $Y^\wedge \times \{0\}$. (1) follows because $H/H\epsilon_- H$ is supported on Y^{sing}—this is because H is Azumaya over Y^{reg} (e.g., this is an easy special case of [17, Theorem 1.7]).

Let us prove (2). Both $\mathcal{B}$ and $\Gamma(\mathcal{B}^0)$ come with complete and separated filtrations: the filtration on $\mathcal{B}$ was specified in the discussion preceding the lemma, and it induces a filtration on $\Gamma(\mathcal{B}^0)$. The homomorphism $\mathcal{B} \to \Gamma(\mathcal{B}^0)$ is that of filtered bimodules. To show that it is an isomorphism, it is enough to check that the associated graded homomorphism

$$\operatorname{gr} \mathcal{B} \to \operatorname{gr} \Gamma(\mathcal{B}^0) \tag{6.3}$$

is an isomorphism. We have $\operatorname{gr}\mathcal{B} = B^{\wedge}_{d\leftarrow 0} \otimes \mathbb{C}[\hbar]$ by the construction of the filtration on $\mathcal{B}$. Also we have a natural inclusion

$$\operatorname{gr}\Gamma(\mathcal{B}^0) \hookrightarrow \Gamma(\operatorname{gr}\mathcal{B}^0), \tag{6.4}$$

and the composition of (6.3) and (6.4) is the natural homomorphism

$$\operatorname{gr}\mathcal{B} \to \Gamma(\operatorname{gr}\mathcal{B}^0). \tag{6.5}$$

So, (2) will follow if we show that (6.5) is an isomorphism.

Set

$$X^{\wedge} := Y^{\wedge} \times_Y X, \qquad X^{\wedge}_{\hbar} := X^{\wedge} \times \operatorname{Spec}\bigl(\mathbb{C}[\hbar]\bigr), \qquad X^{\wedge,0}_{\hbar} := Y^{\wedge,0}_{\hbar} \times_{Y^{\wedge}_{\hbar}} X^{\wedge}_{\hbar}.$$

We have

($\diamondsuit$) the complement to $X^{\wedge,0}_{\hbar}$ in $X^{\wedge}_{\hbar}$ has codimension 2.

Note that $\operatorname{gr}\mathcal{B}$ is the global section of the vector bundle

$$\bigl(\mathcal{P}^{\mathrm{reg},*} \otimes \mathcal{O}^{\mathrm{reg}}(d) \otimes \mathcal{P}^{\mathrm{reg}}\bigr) \boxtimes \mathcal{O}_{\operatorname{Spec}(\mathbb{C}[\hbar])}$$

on $X^{\wedge,\mathrm{reg}}_{\hbar}$, while $\Gamma(\operatorname{gr}\mathcal{B}^0)$ is the global section of the same vector bundle restricted to $X^{\wedge,\mathrm{reg}}_{\hbar} \cap X^{\wedge,0}_{\hbar}$. Because of the codimension condition ($\diamondsuit$), (6.5) is indeed an isomorphism. $\square$

Proof of Proposition 6.2

The proof is in several steps.

Step 1. We are going to produce a homomorphism $M_{\hbar} \to B^{\wedge}_{\hbar,d\leftarrow 0}$. Consider the isomorphism

$$\epsilon_- M^0_{\hbar} \xrightarrow{\sim} \epsilon_- B^{\wedge,0}_{\hbar,d\leftarrow 0}$$

induced by (6.1). Thanks to (1) of Lemma 6.3, this isomorphism gives rise to

$$M^0_{\hbar} \xrightarrow{\sim} B^{\wedge,0}_{\hbar,d\leftarrow 0}. \tag{6.6}$$

Note that this isomorphism is $\mathbb{C}^{\times}$-equivariant, by the construction. So we have homomorphisms

$$M^0_{\hbar} \to \Gamma(M^0_{\hbar}) \xrightarrow{\sim} \Gamma(B^{\wedge,0}_{\hbar,d\leftarrow 0}) \xrightarrow{\sim} B^{\wedge}_{\hbar,d\leftarrow 0}.$$

The first homomorphism is the natural one (see the discussion before Lemma 6.3), and the second is obtained from (6.6) by passing to the global sections, while the third is the inverse of the isomorphism in (2) of Lemma 6.3. The composed homomorphism is graded and $H^{\wedge}_{\hbar,d}$-$H^{\wedge}_0$-bilinear by the construction. We need to show that it is an isomorphism.

Step 2. Our proof of this is based on the following easy general fact: let M, N by two $\mathbb{Z}_{\geq 0}$-filtered vector spaces. Let $\varphi : M \to N$ be an isomorphism mapping $M^{\leq i} \to N^{\leq i}$ for all i. If $\operatorname{gr}\varphi : \operatorname{gr} M \to \operatorname{gr} N$ is injective, then it is an isomorphism (and hence φ intertwines the filtrations).

Step 3. We apply the observation of Step 2 to the homomorphism

$$M_\hbar \to B^{\wedge}_{\hbar,d\leftarrow 0}$$

specialized at $\hbar = 1$. Denote this specialization by φ. It is an isomorphism by Lemma 2.10 applied to $c = d$ and is filtered by the construction. To show that $\operatorname{gr}\varphi$ is injective, we recall that $M_\hbar/(\hbar) = \operatorname{gr}(M_\hbar/(\hbar-1))$ is torsion-free over $\mathbb{C}[\mathfrak{t}]$. By the assumption of the proposition, $\operatorname{gr}\varphi$ gives an isomorphism between the sign-invariant parts. It follows from (1) of Lemma 6.3 that $\operatorname{gr}\varphi$ is an isomorphism over the localizations of the full bimodules to $(Y^{\wedge})^{\mathrm{reg}}$, and, in particular, to its open subscheme $(\mathfrak{t}/W)^{\mathrm{reg}} \times_{\mathfrak{t}/W} Y^{\wedge}$. Since $M_\hbar/\hbar M_\hbar$ is torsion-free over $\mathbb{C}[\mathfrak{t}]$, we see that $\operatorname{gr}\varphi$ is injective.

Thanks to Step 2, this completes the proof. □

Proof of Theorem 6.1
We prove the theorem by induction on d. We have an isomorphism

$$B^{\wedge}_{\hbar,0\leftarrow 0} \xrightarrow{\sim} H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_0})^{\wedge}$$

by the remark after the theorem. The proof of the theorem is now in several steps.

Step 1. Suppose we already have a graded bimodule isomorphism

$$B^{\wedge}_{\hbar,d\leftarrow 0} \xrightarrow{\sim} H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})^{\wedge}$$

for some $d \geq 0$. Multiply by ϵ on the left. Thanks to Lemma 3.7, we have a graded algebra isomorphism $\epsilon_- H^{\wedge}_{\hbar,d+1}\epsilon_- \cong \epsilon H^{\wedge}_{\hbar,d}\epsilon$ and a graded $\epsilon H^{\wedge}_{\hbar,d}\epsilon$-$H^{\wedge}_{0,\hbar}$-bimodule isomorphism $\epsilon_- B^{\wedge}_{\hbar,d+1\leftarrow 0} \xrightarrow{\sim} \epsilon B^{\wedge}_{\hbar,d\leftarrow 0}$. On the other hand, by Lemma 5.7, we have a graded $H^{\wedge}_{\hbar,0}$-linear isomorphism

$$\epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}})^{\wedge} \xrightarrow{\sim} \epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})^{\wedge}. \tag{6.7}$$

We are not going to check that this isomorphism is also $\epsilon H^{\wedge}_{\hbar,d}\epsilon$-linear. Instead, we will see that it is semilinear with respect to an automorphism of $\epsilon H^{\wedge}_{\hbar,d}\epsilon$ given by conjugation with an invertible element of $\mathbb{C}[\mathfrak{t}^*]^{\wedge_0}$.

Step 2. We claim that the homomorphism

$$\epsilon H^{\wedge}_{\hbar,d}\epsilon \to \operatorname{End}_{H^{\wedge}_{\hbar,0}}(\epsilon B^{\wedge}_{\hbar,d\leftarrow 0}) \tag{6.8}$$

is an isomorphism. From Lemma 3.8 we deduce that

$$\epsilon H_{\hbar,d}\epsilon \xrightarrow{\sim} \operatorname{End}_{H_{\hbar,0}}(\epsilon B_{\hbar,d\leftarrow 0}). \tag{6.9}$$

Note that $B_{\hbar,d\leftarrow 0}$ is a finitely generated right $H_{\hbar,0}$-module. Using this and the fact that $\mathbb{C}[\mathfrak{t}^*]^{\wedge 0}$ is a flat $\mathbb{C}[\mathfrak{t}^*]$-module, we see that (6.9) implies (6.8).

Recall that

$$\epsilon B^{\wedge}_{\hbar,d\leftarrow 0} \xrightarrow{\sim} \epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})^{\wedge}.$$

It follows from (6.8) that isomorphism (6.7) becomes $\epsilon H^{\wedge}_{\hbar,d}\epsilon$-linear after we twist one of the actions by a uniquely determined graded $\mathbb{C}[\hbar]$-linear automorphism of the algebra $\epsilon H^{\wedge}_{\hbar,d}\epsilon$. We denote this automorphism by ζ. We claim that there is an invertible element $F \in \mathbb{C}[\mathfrak{t}^{*\wedge}]^W$ such that ζ is the conjugation with F.

Step 3. The formula for Υ in the proof of Lemma 5.7 implies that Υ modulo $\hbar$ is $\mathbb{C}[T^*T^\vee]^W$-linear. It follows that ζ is the identity modulo $\hbar$. So $\zeta = \exp(\hbar\partial)$, where ∂ is a derivation of $\epsilon H^{\wedge}_{\hbar,d}\epsilon$ that has degree -1 with respect to the grading. We have $\partial = \frac{1}{\hbar}[f,\cdot]$ for some $f \in \epsilon H^{\wedge}_{\hbar,d}\epsilon$. This follows because every Poisson derivation of $\mathbb{C}[Y^\wedge]$ is restricted from a W-equivariant Poisson derivation of $\mathbb{C}[T^*\mathfrak{t}^{*,\wedge}]$ (cf the proof of [17, Lemma 2.23]) and hence is inner. Then $f \in \mathbb{C}[\mathfrak{t}^{*\wedge}]^W$ because f has degree 0. Subtracting a scalar from f we can assume that it lies in the maximal ideal of $\mathbb{C}[\mathfrak{t}^{*\wedge}]^W$. Set $F := \exp(f)$. Then we can compose (6.7) with the multiplication by F and achieve that (6.7) is a graded bimodule isomorphism.

Step 4. We now have graded $\epsilon_- H^{\wedge}_{\hbar,d+1}\epsilon_-$-$H^{\wedge}_{\hbar,0}$-bimodule isomorphisms

$$\epsilon_- B^{\wedge}_{\hbar,d+1\leftarrow 0} \xrightarrow{\sim} \epsilon B^{\wedge}_{\hbar,d\leftarrow 0} \xrightarrow{\sim} \epsilon H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})^{\wedge} \xrightarrow{\sim} \epsilon_- H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}})^{\wedge}.$$

Applying Proposition 6.2 to $M_\hbar = H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})^{\wedge}$, which satisfies the assumptions of that proposition thanks to (3) of Proposition 4.6, we extend the composed isomorphism to a graded $H^{\wedge}_{\hbar,d+1}$-$H^{\wedge}_{\hbar,0}$-bimodule isomorphism

$$B^{\wedge}_{\hbar,d+1\leftarrow 0} \xrightarrow{\sim} H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_{d+1}})^{\wedge}.$$

This finishes the proof of the induction step and hence of the theorem. □

COROLLARY 6.4
$B_{\hbar,d}$ is flat over $\mathbb{C}[\mathfrak{t}][\hbar]$.

Proof
The bimodule $B_{\hbar,d}$ is bigraded (Remark 3.6) and, thanks to Theorem 6.1 combined with Proposition 4.6, $B^{\wedge 0}_{\hbar,d}$ is flat over $\mathbb{C}[\mathfrak{t},\hbar]$. The claim of the corollary follows. □

Remark 6.5
In fact, the proof of Theorem 6.1 gives us a characterization of the family of bimodules $B_{\hbar,d}$ for $d \geq 0$. Suppose we have another family of finitely generated graded $H_{\hbar,d}$-$H_{\hbar,0}$-bimodules $B'_{\hbar,d}$ satisfying the following conditions:
(i) $B'_{\hbar,d}$ is flat over $\mathbb{C}[\hbar]$ and $B'_{\hbar,d}/(\hbar)$ is torsion-free over $\mathbb{C}[\mathfrak{t}]$ for all d.
(ii) $B'_{\hbar,0}$ is isomorphic to $H_{\hbar,0}$ as a graded $H_{\hbar,0}$-bimodule.
(iii) We have an isomorphism of graded right $H_{\hbar,0}$-modules $\epsilon B'_{\hbar,d} \cong \epsilon_- B'_{\hbar,d+1}$.
Then the argument of the proof of Theorem 6.1 shows that for all d we have a graded $H_{\hbar,d}$-$H_{\hbar,0}$-bimodule isomorphism $B'_{\hbar,d} \xrightarrow{\sim} B_{\hbar,d}$. Moreover, if we require the isomorphisms in (ii) and (iii) to be bigraded, then we get a bigraded isomorphism $B'_{\hbar,d} \xrightarrow{\sim} B_{\hbar,d}$. In fact, the proof simplifies: ζ from Step 2 of the proof of Theorem 6.1 is automatically the identity.

6.2. *Proof of Theorem 1.2*
Recall that we are going to prove that

$$B_d \otimes_H \mathbb{C}_{\mathrm{triv}} \cong H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d}) \otimes_{H^\times} \mathbb{C}_{\mathrm{triv}} = \mathbb{C}\big(\Lambda_0/(dh+1)\Lambda_0\big),$$

where h denotes the Coxeter number of W, and Λ_0 is the root lattice of $\mathfrak{g}$. We write $\mathbb{C}_{\mathrm{triv}}$ for the 1-dimensional trivial W-module, and we assume that $\mathbb{C}[\mathfrak{t}^* \oplus \mathfrak{t}] \subset H$ acts on $\mathbb{C}_{\mathrm{triv}}$ via the specialization to 0, while $\mathbb{C}[T^\vee \times \mathfrak{t}] \subset H^\times$ acts on $\mathbb{C}_{\mathrm{triv}}$ via the specialization to $(1, 0)$.

We already know that the dimensions are the same thanks to Theorem 1.1. We will prove that

$$B_d \otimes_H \mathbb{C}_{\mathrm{triv}} \twoheadrightarrow \mathbb{C}\big(\Lambda_0/(dh+1)\Lambda_0\big), \tag{6.10}$$

$$\dim H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d}) \otimes_{H^\times} \mathbb{C}_{\mathrm{triv}} \leq (dh+1)^{\dim \mathfrak{t}}. \tag{6.11}$$

This will prove Theorem 1.2.

We first establish (6.10).

PROPOSITION 6.6
We have $B_d \otimes_H \mathbb{C}_{\mathrm{triv}} \twoheadrightarrow \mathbb{C}(\Lambda_0/(dh+1)\Lambda_0)$, an epimorphism of W-modules.

This proposition is inspired by [22, Theorem 1.8] and [20, Theorem 1.4].

Proof
Consider the $H_{d+1/h}$-$H_{1/h}$-bimodule $B_{\hbar,d+1/h\leftarrow 1/h}$. By the construction in Section 3.3, this is a $\mathbb{C}[\hbar]$-flat bimodule with

$$B_{\hbar,d+1/h\leftarrow 1/h}/\hbar B_{\hbar,d+1/h\leftarrow 1/h} \xrightarrow{\sim} B_d, \tag{6.12}$$

which is a special case of (3.7). Set

$$B_{d+1/h\leftarrow 1/h} := B_{\hbar,d+1/h\leftarrow 1/h}/(\hbar-1)B_{\hbar,d+1/h\leftarrow 1/h}. \tag{6.13}$$

Since $B_{\hbar,d+1/h\leftarrow 1/h}$ is flat over $\mathbb{C}[\hbar]$, (6.12) is equivalent to $\operatorname{gr} B_{d+1/h\leftarrow 1/h} = B_d$. Recall from Proposition 2.11 that $H_{d+1/h}$ has a unique finite dimensional representation to be denoted by $L_{d+1/h}$. By that proposition, this representation is isomorphic to $\mathbb{C}(\Lambda_0/(dh+1)\Lambda_0)$ as a W-representation. In particular, $L_{1/h}$ is the trivial 1-dimensional representation of W. The subspaces $\mathfrak{t},\mathfrak{t}^* \subset H_{1/h}$ act by 0 on $L_{1/h}$, for example, thanks to the presence of the grading element in $H_{1/h}$ (see, e.g., (4) in [19, Section 3.1]). Equip $B_{d+1/h\leftarrow 1/h}\otimes_{H_{1/h}} L_{1/h}$ with the tensor product filtration. Then we have

$$B_d \otimes_H \mathbb{C}_{\mathrm{triv}} \twoheadrightarrow \operatorname{gr}(B_{d+1/h\leftarrow 1/h}\otimes_{H_{1/h}} L_{1/h}).$$

To show that $\dim B_d \otimes_H \mathbb{C}_{\mathrm{triv}} \twoheadrightarrow \mathbb{C}(\Lambda_0/(dh+1)\Lambda_0)$, it is therefore sufficient to show that

$$B_{d+1/h\leftarrow 1/h}\otimes_{H_{1/h}} L_{1/h} \cong L_{d+1/h}. \tag{6.14}$$

Thanks to Proposition 2.11, (6.14) will follow once we show that $B_{d+1/h\leftarrow 1/h}$ is a Morita equivalence bimodule. We will prove this by induction on d starting with $d=0$, where $B_{1/h\leftarrow 1/h} = H_{1/h}$ and the claim is vacuous.

Suppose we already know that $B_{d+1/h\leftarrow 1/h}$ is a Morita equivalence bimodule. Since $d+1/h$ is ϵ-spherical (see Proposition 2.12), we see that $\epsilon B_{d+1/h\leftarrow 1/h}$ is a Morita equivalence bimodule between $H_{1/h}$ and $\epsilon H_{d+1/h\leftarrow 1/h}\epsilon$. It follows from (2) of Lemma 3.7 that we have a bimodule isomorphism

$$\epsilon_- B_{d+1+1/h\leftarrow 1/h} \cong \epsilon B_{d+1/h\leftarrow 1/h}.$$

So $\epsilon_- B_{d+1+1/h\leftarrow 1/h}$ is a Morita equivalence bimodule between $\epsilon_- H_{d+1+1/h}\epsilon_-$ and $H_{1/h}$. But, according to Proposition 2.12, $d+1+1/h$ is ϵ_--spherical, so $B_{d+1+1/h\leftarrow 1/h}$ is also a Morita equivalence bimodule between $H_{d+1+1/h}$ and $H_{1/h}$. This finishes the proof. □

Now we proceed to the upper bound. The proof here is an easy generalization of a proof due to the first named author joint with Bezrukavnikov, Shan, and Vasserot in [5, Proposition 2.9], but we include it here for completeness.

PROPOSITION 6.7
We have $\dim H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d}) \otimes_{H^\times} \mathbb{C}_{\mathrm{triv}} \le (dh+1)^{\dim \mathfrak{t}}$.

Proof

The proof is in several steps. Note that it is enough to assume that G is simply connected and hence $\widetilde{W} = W^a$. For example, this follows from the isomorphism $H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d}) \otimes_{H^\times} \mathbb{C}_{\mathrm{triv}} \cong B_d \otimes_H \mathbb{C}_{\mathrm{triv}}$ as the right-hand side manifestly depends only on W.

Step 1. We note that $H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d}) \otimes_{H^\times} \mathbb{C}_{\mathrm{triv}}$ is nothing else as the space of coinvariants $H^{\mathrm{BM}}(\mathcal{F}l_{e_d})_{\widetilde{W}}$ for the action of the affine Weyl group $\widetilde{W}$ on $H^{\mathrm{BM}}(\mathcal{F}l_{e_d})$. Recall from Proposition 4.6 that the affine Springer fiber $\mathcal{F}l_{e_d}$ has a paving by affine cells. Each cell is the intersection of $\mathcal{F}l_{e_d}$ with a Schubert cell by [25, Theorem 0.2]. This gives a basis in $H^{\mathrm{BM}}(\mathcal{F}l_{e_d})$ consisting of the fundamental classes of cells.

We will study the action of $\widetilde{W}$ on this basis to get a spanning set of $H^{\mathrm{BM}}(\mathcal{F}l_{e_d})_{\widetilde{W}}$ with $(dh+1)^{\dim \mathfrak{t}}$ elements.

Step 2. Let us introduce some notation. In this proof $\mathfrak{b}$ will denote the Lie algebra of the Iwahori subgroup $\mathfrak{B} \subset G(\mathcal{K})$. For the Schubert cell $\mathfrak{B}x\mathfrak{B}/\mathfrak{B}$, we denote the corresponding basis element in $H^{\mathrm{BM}}(\mathcal{F}l_{e_d})$ (or $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$) by φ_x. For $x \in \widetilde{W}$ we will write ${}^x\mathfrak{b}$ for $\mathrm{Ad}(\dot{x})\mathfrak{b}$ for a lift $\dot{x}$ of x to the normalizer of $T(\mathcal{K})$. Also for a $T(\mathcal{O})$-stable subset $Z \subset \mathcal{F}l$ we use the notation ${}^xZ$ for $\dot{x}Z$; this is well- defined. Finally, we set $e_d^x := \mathrm{Ad}(\dot{x})^{-1}(e_d)$. We note that ${}^x\mathcal{F}l_{e_d} = \mathcal{F}l_{e_d^x}$ for all $x \in \widetilde{W}$. Finally, for $x \in \widetilde{W}$ we will write A_x for the corresponding (closed) alcove in $\mathfrak{t}_{\mathbb{R}}$.

Step 3. For $w \in \widetilde{W}$, consider the subvariety $\mathcal{F}l_{e_d}^{\le w} = \mathcal{F}l_{e_d} \cap \sqcup_{x\le w}\mathfrak{B}x\mathfrak{B}/\mathfrak{B}$ of $\mathcal{F}l_{e_d}$. It is $T\times\mathbb{C}^\times$-stable. The Borel–Moore homology $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{\le w}) \subset H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ is spanned by the classes φ_x for $x \le w$ as a $H^*_{T\times\mathbb{C}^\times}(pt)$-module. The image of $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{\le w})$ under ι from (5.1) is given by

$$\iota\big(H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{\le w})\big) = \big\{(g_y)_{y\in\widetilde{W}} \in \iota\big(H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})\big) \;\big|\; g_y \neq 0 \Rightarrow y \le w\big\}. \tag{6.15}$$

This follows by applying Proposition 4.3 to the space $\mathcal{F}l_{e_d}^{\le w}$. Further, note that we have the long exact sequence

$$\cdots \to H^{\mathrm{BM},i}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{<w}) \to H^{\mathrm{BM},i}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{\le w}) \to H^{\mathrm{BM},i}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d} \cap \mathfrak{B}w\mathfrak{B}) \to \cdots,$$

where the superscript i indicates the cohomological grading.

Note that odd homology vanishes as all spaces involved have affine pavings and so the long exact sequence breaks up into short exact sequences. Assembling these exact sequences for all degrees we get

$$0 \to H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{<w}) \to H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{\le w}) \to H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d} \cap \mathfrak{B}w\mathfrak{B}) \to 0.$$

Further, by construction φ_w is mapped to the basis element spanning $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d} \cap \mathfrak{B}w\mathfrak{B})$. Using the compatibility with the localization map from (6.15) and the description of the image ι in Corollary 4.8, we see that that $\iota(\varphi_w)_w = \frac{1}{\prod \chi}$, where the product is over all characters χ appearing in the $T\times\mathbb{C}^\times$-representation $\mathcal{F}l_{e_d} \cap \mathfrak{B}w\mathfrak{B}$.

Step 4. Pick a simple affine reflection $s := s_\alpha$ at a root α. We want to get a necessary and sufficient condition on x for $\varphi_{sx} = s\varphi_x + l.o.t.$ when $sx > x$ in the Bruhat order. Here "$l.o.t.$" indicates an $H^*_{T\times\mathbb{C}^\times}(pt)$-linear combination of the elements φ_y with $y < sx$ in the Bruhat order. We claim that this equality holds if the cells ${}^s(\mathcal{F}l_{e_d} \cap \mathfrak{B}x\mathfrak{B}/\mathfrak{B})$ and $\mathcal{F}l_{e_d^s} \cap \mathfrak{B}sx\mathfrak{B}/\mathfrak{B}$ are equal. Indeed, if ${}^s(\mathcal{F}l_{e_d} \cap \mathfrak{B}x\mathfrak{B}/\mathfrak{B}) = \mathcal{F}l_{e_d^s} \cap \mathfrak{B}sx\mathfrak{B}/\mathfrak{B}$, then $\iota(s\varphi_x)_{sx} = \iota(\varphi_{sx})_{sx}$ and so $\varphi_{sx} - s\varphi_x$ is a class in $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l^{<sx}_{e_d})$ and thus, by Step 3, a combination of φ_y with $y < sx$. We conclude that the equality $\varphi_{sx} = s\varphi_x + l.o.t.$ also holds in $H^{\mathrm{BM}}(\mathcal{F}l_{e_d})$.

Note that, for all x, one of ${}^s(\mathfrak{B}x\mathfrak{B}/\mathfrak{B})$ and $\mathfrak{B}sx\mathfrak{B}/\mathfrak{B}$ contains the other. Therefore one of the two cells ${}^s(\mathcal{F}l_{e_d} \cap \mathfrak{B}x\mathfrak{B}/\mathfrak{B})$ and $\mathcal{F}l_{e_d^s} \cap \mathfrak{B}sx\mathfrak{B}/\mathfrak{B}$ contains the other. Note that both cells are contracting loci for suitable tori actions. So they coincide if and only if their tangent spaces at their common $T\times\mathbb{C}^\times$-fixed point sx are the same, equivalently, and have the same dimension.

Note that the tangent space of $\mathcal{F}l_{e_d} \cap \mathfrak{B}x\mathfrak{B}/\mathfrak{B}$ at the fixed point x is $T\times\mathbb{C}^\times$—equivariantly isomorphic to

$$\frac{\mathfrak{b} \cap t^{-d}({}^x\mathfrak{b})}{\mathfrak{b} \cap {}^x\mathfrak{b}}. \tag{6.16}$$

So the tangent spaces of interest are

$$\frac{\mathfrak{b} \cap t^{-d}({}^{sx}\mathfrak{b})}{\mathfrak{b} \cap {}^{sx}\mathfrak{b}}, \qquad {}^s\Big(\frac{\mathfrak{b} \cap t^{-d}({}^x\mathfrak{b})}{\mathfrak{b} \cap {}^x\mathfrak{b}}\Big). \tag{6.17}$$

The roots that appear as weights of (6.16) are exactly from

$$R^+_{\mathrm{aff}} \cap x\Big(\bigsqcup_{1\le r\le d} (R^+ - r\delta) \sqcup \bigsqcup_{0\le r\le d-1} (R^- - r\delta)\Big), \tag{6.18}$$

where we write R^+_{aff} for the set of positive affine roots, R^+, R^- for the sets of positive and negative Dynkin roots, and δ for the indecomposable imaginary root. Note that every element in $R^+_{\mathrm{aff}} \setminus \{\alpha\}$ appears as a weight in one of the spaces in (6.17) if and only if it appears in the other. On the other hand, $-\alpha$ does not appear as a weight in the first space and α does not appear as a weight of the second space. It thus follows that

$${}^s(\mathcal{F}l_{e_d} \cap \mathfrak{B}x\mathfrak{B}/\mathfrak{B}) = \mathcal{F}l_{e_d^s} \cap \mathfrak{B}sx\mathfrak{B}/\mathfrak{B}$$

if and only if

$$\alpha \notin x\Big(\bigsqcup_{1\le r\le d} (R^+ - r\delta) \sqcup \bigsqcup_{0\le r\le d-1} (R^- - r\delta)\Big). \tag{6.19}$$

Step 5. In particular, if (6.19) holds, the projection of φ_{sx} (for $s = s_\alpha$) to $H^{\mathrm{BM}}(\mathcal{F}l_{e_d})_{\widetilde{W}}$ coincides with a linear combination of projections of φ_y with $y < sx$.

Consider the equivalence relation on $\widetilde{W}$ generated by the relation $x \to s_\alpha x$ for α satisfying (6.19).

In the next step we will prove that

(*) each equivalence class has a representative x satisfying $\langle \alpha_i, A_x \rangle \geq -d$ and $\langle \alpha_0, A_w \rangle \leq d+1$,

where we write α_i for the simple Dynkin roots and α_0 for the longest root.

Showing (*) will finish the proof of the proposition because the set of alcoves A satisfying $\langle \alpha_i, A \rangle \geq -d$ and $\langle \alpha_0, A \rangle \leq d+1$ forms a poset ideal in the Bruhat order and has exactly $(dh+1)^{\dim \mathfrak{t}}$ elements. To see the latter we argue as follows. Shifting by $d\rho^\vee$ we can instead consider the set of alcoves A' satisfying $\langle \alpha_i, A' \rangle \geq 0$ and $\langle \alpha_0, A \rangle \leq d+1+d(h-1) = dh+1$. There are exactly $(dh+1)^{\dim \mathfrak{t}}$ such alcoves.

Step 6. Fix an equivalence class for the equivalence relation specified in Step 5 and pick a representative x that is minimal with respect to the Bruhat order. To show (*), it is enough to check that if $\langle \alpha_i, A_x \rangle \leq -d$, then (6.19) holds for x and α_i (and that the similar claim holds for the affine simple reflection). Indeed, since $\langle \alpha_i, A_x \rangle \leq -d \leq 0$, we see that $s_i x$ is less than x in the Bruhat order, while $s_i x$ is equivalent to x. This will give a contradiction with the choice of x.

We will only consider the case of simple Dynkin roots; the remaining case is similar.

Assume $x = wt^\beta$ for $w \in W$ and $\beta \in \Lambda$. Then $\langle \alpha_i, A_x \rangle = \langle w^{-1}(\alpha_i), A_1 + \beta \rangle$; thus, $\langle \alpha_i, A_x \rangle \leq -d$ holds if and only if one of the following conditions hold:

- $w^{-1}(\alpha_i) \in R^+$ and $\langle w^{-1}(\alpha_i), \beta \rangle \leq -d-1$
- $w^{-1}(\alpha_i) \in R^-$ and $\langle w^{-1}(\alpha_i), \beta \rangle \leq -d$.

(6.19) follows from $x^{-1}(\alpha_i) = w^{-1}(\alpha_i) + \langle x^{-1}(\alpha_i), \beta \rangle \delta$. □

Remark 6.8

In our proof of Proposition 6.7 we study the action of simple reflections $s (= s_\alpha)$ on the basis of Schubert cells $\mathcal{F}l_{e_d} \cap \mathfrak{B}x\mathfrak{B}/\mathfrak{B}$ under the assumption that (6.19) holds. This is enough to give a spanning set of $H_T^{\mathrm{BM}}(\mathcal{F}l_{e_d}) \otimes_{H^\times} \mathbb{C}_{\mathrm{triv}}$ with $(dh+1)^{\dim \mathfrak{t}}$ elements.

However, showing directly that this spanning set is a basis would involve studying the action of simple reflections s where (6.19) fails. This is more subtle, as $\varphi_{sx} = s\varphi_x + l.o.t.$ may fail as well.

So the proof of the dimension formula in Theorem 1.2 requires both sides of the isomorphism of Theorem 1.1.

7. Applications

The goal of this section is to obtain some corollaries of Theorems 1.1 and 1.2 for $d = 1$, mostly in type A. We will write e for e_1.

7.1. Statements of the results

Until the further notice, $\mathfrak{g} = \mathfrak{sl}_n$. Then $W = S_n$, X is the normalized version of the Hilbert scheme (of dimension $2n-2$), and $\mathcal{P}$ is the restriction of Haiman's Procesi bundle to $X \subset \mathrm{Hilb}_n(\mathbb{C}^2) = X \times \mathbb{C}^2$. Let $\tilde{X}$ denote the preimage of X in the isospectral Hilbert scheme; in other words, $\tilde{X}$ is $(\mathfrak{t} \oplus \mathfrak{t}^*) \times_Y X$ with its reduced scheme structure. Let ζ denote the natural finite morphism $\widetilde{X} \to X$. Haiman's $n!$ theorem says that $\tilde{X}$ is a Cohen–Macaulay scheme, equivalently, ζ is flat (of degree $n!$). The bundle $\mathcal{P}$ can be obtained as $\zeta_* \mathcal{O}_{\widetilde{X}}$. That the bundle we consider coincide with Haiman's follows, for example, from the main result of [36].

Set $B^{\mathrm{sgn}} := \Gamma(\mathcal{P} \otimes \mathcal{P})$, this is an $H^{\otimes 2}$-module (equivalently, an H-bimodule). It follows from Haiman's construction—or the main result of [36]—that $\mathcal{P} \cong \mathcal{P}^* \otimes \mathcal{O}(1)$, a $(\mathbb{C}^\times)^2 \times S_n$-equivariant isomorphism, where the action of $(\mathbb{C}^\times)^2 \times S_n$ on $\mathcal{O}(1)$ comes from the isomorphism $\mathcal{O}(1) \cong \mathcal{P}\epsilon_-$. It follows that B^{sgn} is obtained from $B(:= B_1)$ by twisting the left S_n-action with the sign.

In particular, B^{sgn} has an algebra structure; in fact, this is the algebra $\mathbb{C}[\tilde{X} \times_X \tilde{X}]$. Our first goal is to describe this algebra structure.

Consider the algebra

$$\tilde{B} := \mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*] \otimes_{\mathbb{C}[Y]} \mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*].$$

Note that both B^{sgn} and $\tilde{B}$ are graded $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]^{\otimes 2}$-algebras.

THEOREM 7.1

Let $\mathfrak{g} = \mathfrak{sl}_n$. We have a graded $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]^{\otimes 2}$-algebra isomorphism $B^{\mathrm{sgn}} \cong \tilde{B} / \operatorname{rad} \tilde{B}$.

We can also describe the $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]$-bimodule structure on B^{sgn}.

THEOREM 7.2

Let $\mathfrak{g} = \mathfrak{sl}_n$. We have a graded $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]$-bimodule isomorphism $B^{\mathrm{sgn}} \cong H\epsilon H$, where the latter is viewed as a subbimodule in H.

Remark 7.3

We note that Theorems 7.2, 1.1, and Proposition 3.1 imply [12, Conjecture 3.7]. Namely, their M is $H\epsilon H$, the higher cohomology of $\mathcal{P} \otimes \mathcal{P}$ vanish thanks to Proposition 3.1, and the claim that B is flat over $\mathbb{C}[\mathfrak{t}]$ follows from Corollary 6.4. To our knowledge the claim that $H\epsilon H$ is free over $\mathbb{C}[\mathfrak{t}]$ is new.

Now we proceed to prospective applications to the center of the principal block of the small quantum group. For now we assume that $\mathfrak{g}$ is an arbitrary simple Lie algebra—but we still get more complete results in type A.

Recall the notations Z, $G^\vee$, $T^\vee$ from the introduction. Also recall that $H^*_T(\mathcal{F}l_e) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H^{\mathrm{BM}}_T(\mathcal{F}l_e), \mathbb{C}[\mathfrak{t}])$ (see Remark 4.4). This gives a $\widetilde{W}$-action on $H^*_T(\mathcal{F}l_e)$ corresponding to the centralizer-monodromy action on $H^{\mathrm{BM}}_T(\mathcal{F}l_e)$. The $\widetilde{W}$-action on $H^*_T(\mathcal{F}l_e)$ gives rise to a W-action on $H^*(\mathcal{F}l_e)^\Lambda$.

The following conjecture is due to the first named author joint with Bezrukavnikov, Shan, and Vasserot in [5, Conjecture A].

CONJECTURE 7.4
For any semisimple Lie algebra $\mathfrak{g}$, there is an algebra isomorphism $H^*(\mathcal{F}l_e)^\Lambda \xrightarrow{\sim} Z^{T^\vee}$. This isomorphism is W-equivariant, where on the left-hand side we have the action described above and on the right-hand side the action comes from the identification $W = N_{G^\vee}(T^\vee)/T^\vee$.

In fact, [5, Theorem 4.12] establishes the existence of a W-equivariant algebra monomorphism $H^*(\mathcal{F}l_e)^\Lambda \hookrightarrow Z^{T^\vee}$. The conjectural part is that this monomorphism is surjective.

Here is our result on the structure of Z.

THEOREM 7.5
Assume Conjecture 7.4 holds. Then the following claims are true:
(1) *For any semisimple Lie algebra $\mathfrak{g}$, the dimension of the subalgebra of $N_{G^\vee}(T^\vee)$-invariants in Z is $(h+1)^{\dim \mathfrak{t}}$.*
(2) *If $\mathfrak{g} = \mathfrak{sl}_n$, then the $G^\vee$-action on Z is trivial. In particular,* $\dim Z = (n+1)^{n-1}$.

Note that (2) confirms a conjecture from [34].

The following result is used to prove Theorems 7.1 and 7.2 as well as (2) of Theorem 7.5. Consider the 1-dimensional representation $\mathbb{C}_0$ of $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]$ corresponding to the point $0 \in \mathfrak{t} \oplus \mathfrak{t}^*$.

PROPOSITION 7.6
For $\mathfrak{g} = \mathfrak{sl}_n$, we have $B \otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]} \mathbb{C}_0 = (B \otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]} \mathbb{C}_0)\epsilon$.

7.2. *Proposition 7.6 and $n!$ theorem*

In this section we prove Proposition 7.6. In fact, we will show that Proposition 7.6 is equivalent to the $n!$ theorem of Haiman [28]. We need some preparation for the proof.

For a partition μ on n, let x_μ denote the fixed point in X labeled by μ and let $\mathcal{P}_\mu$ denote the fiber of $\mathcal{P}$ at x_μ. This is a $(\mathbb{C}^\times)^2$-equivariant H-module of dimension $n!$. The following is a consequence of the $n!$ theorem (see, e.g., [29, Corollary 5.2.2]).

(A) For each μ, the head of the H-module $\mathcal{P}_\mu$ is a trivial S_n-module.

In fact, more is true. If we use the Bezrukavnikov–Kaledin construction of $\mathcal{P}$ as a definition, then (A) is equivalent to the $n!$ theorem (that, recall, is the claim that $\tilde{X}$ is Cohen–Macaulay). Indeed, (A) implies the similar claim for all fibers of $\mathcal{P}$. In particular, $\mathcal{P}$ acquires a sheaf of algebras structure. Once we know that the head of each fiber of $\mathcal{P}$ is the trivial 1-dimensional module, we see that the relative spectrum of $\mathcal{P}$ embeds into $\tilde{X}$ as a closed subvariety. The embedding is an isomorphism because it is so over Y^{reg}. Moreover, the relative spectrum of $\mathcal{P}$ is Cohen–Macaulay because it is flat (of degree $n!$) over the smooth variety X.

We will give several equivalent formulations of (A). We will prove that they are equivalent but will not prove any of them unconditionally, hence getting several equivalent statements of the $n!$ theorem but not its new proof.

Consider the adjoint pair

$$\mathsf{Loc} := \mathcal{P} \otimes_H \bullet : H\text{-mod} \rightleftarrows \mathrm{Coh}(X) : \tilde{\Gamma} := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}, \bullet).$$

Note that the derived functors $L\mathsf{Loc}$ and $R\tilde{\Gamma}$ are mutually quasi-inverse equivalences (see, e.g., [7, Proposition 2.2]).

Note also that we can view every irreducible representation τ of S_n as an irreducible H-module by making $\mathfrak{t} \oplus \mathfrak{t}^*$ act by 0.

LEMMA 7.7

(A) *is equivalent to the following claim:*

(B) *For a nontrivial irreducible representation τ of S_n, we have* $\mathsf{Loc}(\tau) = 0$.

Proof

Let us write $\mathbb{C}_\mu$ for the skyscraper sheaf at x_μ. Then $\mathcal{P}_\mu^* = \tilde{\Gamma}(\mathbb{C}_\mu)$. Therefore

$$\mathrm{Hom}_H(\tau, \mathcal{P}_\mu^*) = \mathrm{Hom}_{\mathcal{O}_X}\big(\mathsf{Loc}(\tau), \mathbb{C}_\mu\big).$$

So (A) is equivalent to the claim that $\mathrm{Hom}_{\mathcal{O}_X}(\mathsf{Loc}(\tau), \mathbb{C}_\mu) = 0$ for all μ as long as $\tau \neq \mathrm{triv}$. Hence, (B)$\Rightarrow$(A). To show the implication in the opposite direction, we must show that for a nonzero $(\mathbb{C}^\times)^2$-equivariant coherent sheaf $\mathcal{F}$ on X there is a partition μ such that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathbb{C}_\mu) \neq 0$. The action of $(\mathbb{C}^\times)^2$ contains a contracting 1-dimensional subtorus whose fixed points are precisely the points x_μ for all μ. So if the fiber $\mathcal{F}_{x_\mu}$ is zero for all μ, then every fiber of $\mathcal{F}$ is zero. (A)$\Rightarrow$(B) follows. □

To prove Proposition 7.6 we now need to show that (A)$\Leftrightarrow$(B) is equivalent to the following condition:

(C) For a nontrivial irreducible representation τ of S_n, we have $B^{\mathrm{sgn}} \otimes_{H_n} \tau = 0$.

Indeed, (C) is equivalent to the claim of Proposition 7.6.

In the proof we will need to following lemma.

LEMMA 7.8
We have an isomorphism of endofunctors of $D^b(H-\text{mod})$,

$$R\tilde{\Gamma}\big(L\mathsf{Loc}(\bullet)(1)\big) \cong B^{\mathrm{sgn}} \otimes_H^L \bullet.$$

Proof
This is standard: the left-hand side is the derived tensor product with

$$R\tilde{\Gamma}\big(L\mathsf{Loc}(H)(1)\big) = R\,\mathrm{Hom}_X\big(\mathcal{P}, \mathcal{P}(1)\big).$$

The right-hand side in the last equation is B^{sgn}. □

Proof of Proposition 7.6
We will show more: that (B) and (C) are equivalent. This will follow if we show that for an irreducible representation τ of S_n, we have $B^{\mathrm{sgn}} \otimes_H \tau = \{0\}$ if and only if $\mathsf{Loc}(\tau) = \{0\}$.

Assume first that $B^{\mathrm{sgn}} \otimes_H \tau = \{0\}$. Note that since the algebra H has finite homological dimension, only finitely many of homologies of $L\mathsf{Loc}(\tau)$ are nonzero. Pick m large enough so that the homology sheaves $H_i(L\mathsf{Loc}(\tau))(m)$ are generated by their global sections and their higher cohomology groups vanish. By Lemma 7.8,

$$R\tilde{\Gamma}\big(L\mathsf{Loc}(\tau)(m)\big) = (B^{\mathrm{sgn}})^{\otimes_H^L m}\tau.$$

The zeroth homology group of the right-hand side is zero. By our choice of m this implies that $\mathsf{Loc}(\tau) = 0$.

Now assume that $\mathsf{Loc}(\tau) = 0$. By the previous paragraph, for some m we have $(B^{\mathrm{sgn}})^{\otimes_H m}\tau = 0$. Let S denote the set of all irreducible S_n-representations τ such that $B^{\mathrm{sgn}} \otimes_H \tau \neq \{0\}$. Note that $\tau \in S$ if and only if
(*) τ appears in the S_n-module $B^{\mathrm{sgn}}/B^{\mathrm{sgn}}(\mathfrak{t} \oplus \mathfrak{t}^*)$ (where S_n acts from the right).
But the H-actions on $B^{\mathrm{sgn}} = \Gamma(\mathcal{P} \otimes \mathcal{P})$ from the left and from the right are completely symmetric. So (*) is equivalent to the condition that τ appears in $B^{\mathrm{sgn}}/(\mathfrak{t} \oplus \mathfrak{t}^*)B^{\mathrm{sgn}}$ (where S_n acts from the left). The latter condition in its turn is equivalent to $\mathrm{Hom}_H(B^{\mathrm{sgn}}, \tau) \neq 0$. So we see that $\tau \in S$ if and only if τ appears in the head of some H-module of the form $B^{\mathrm{sgn}} \otimes_H \tau'$ (where τ' is automatically in S). This shows that $\tau \in S$ if and only if $(B^{\mathrm{sgn}})^{\otimes_H m}\tau \neq 0$ for all m.

This finishes the proof of (B)⇔(C) and hence shows that the proposition is equivalent to (A), that is, to the $n!$ theorem of Haiman. □

7.3. Proofs of Theorems 7.1 and 7.2

Proof of Theorem 7.1

Step 1. Here we prove that B^{sgn} is a reduced algebra. First of all, note that $B^{\mathrm{sgn}} = \Gamma(\mathcal{P} \otimes \mathcal{P})$ is nothing else but the algebra $\mathbb{C}[\tilde{X} \times_X \tilde{X}]$. The scheme $\tilde{X} \times_X \tilde{X}$ is flat and finite over the Cohen–Macaulay scheme $\tilde{X}$, hence is Cohen–Macaulay. It is generically reduced and therefore reduced. The algebra of regular functions on a reduced scheme is always reduced.

Step 2. Here we produce an algebra homomorphism $\varphi : \tilde{B} \to B^{\mathrm{sgn}}$. This comes as the pullback of the morphism

$$\tilde{X} \times_X \tilde{X} \to (\mathfrak{t} \oplus \mathfrak{t}^*) \times_Y (\mathfrak{t} \oplus \mathfrak{t}^*)$$

induced by the morphisms $\tilde{X} \to \mathfrak{t} \oplus \mathfrak{t}^*$, $X \to Y$. Note that φ is the unique $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]^{\otimes 2}$-algebra homomorphism $\tilde{B} \to B^{\mathrm{sgn}}$.

Step 3. We show that the homomorphism $\varphi : \tilde{B} \to B^{\mathrm{sgn}}$ is surjective. This is a crucial step in the proof that uses Proposition 7.6. Namely, note that both $\tilde{B}$ and B^{sgn} are finitely generated graded $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]^{\otimes 2}$-modules. Let $\tilde{B}_0$ and B_0^{sgn} denote the specializations of $\tilde{B}$ and B^{sgn} to $(0, 0) \in (\mathfrak{t} \oplus \mathfrak{t}^*)^2$. We need to show that the induced algebra homomorphism $\tilde{B}_0 \to B_0^{\mathrm{sgn}}$ is surjective. Clearly, $\tilde{B}_0$ is 1-dimensional. Now consider B_0^{sgn}. This space is acted by S_n on the left and on the right. Proposition 7.6 implies that the action from the right is trivial. By symmetry, the action on the left is trivial as well. By Theorem 1.2, we have $B^{\mathrm{sgn}} \otimes_H \mathbb{C}_{\mathrm{triv}} \cong \mathrm{sgn} \otimes \mathbb{C}(\Lambda_0/(n+1)\Lambda_0)$. The space of S_n-invariants in the latter module is 1-dimensional. So $\dim B_0^{\mathrm{sgn}} = 1$ and our claim follows.

Step 4. It is easy to see that $\varphi : \tilde{B} \to B^{\mathrm{sgn}}$ is an isomorphism over Y^{reg}. Since φ is surjective and B^{sgn} is reduced, we conclude that φ induces an isomorphism $\tilde{B}/\operatorname{rad}\tilde{B} \xrightarrow{\sim} B^{\mathrm{sgn}}$. This completes the proof of Theorem 7.1. □

Proof of Theorem 7.2

We need to prove that $B \cong H\epsilon_- H$.

According to [29, Proposition 6.1.5], the $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]$-module $\Gamma(\mathcal{P} \otimes \mathcal{O}(1))$ is identified with the ideal J in $\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]$ generated by the sgn-invariant polynomials. Therefore we get a graded bimodule homomorphism

$$B \to \mathrm{Hom}_{\mathbb{C}[Y]}\big(\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*], J\big)$$

from the global sections of the sheaf Hom to the Hom between the global sections. Composing this with the inclusion $J \hookrightarrow \mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^*]$, we get a bimodule homomorphism

$$B \to \mathrm{End}_{\mathbb{C}[\underline{x}, \underline{y}]^{S_n}}\big(\mathbb{C}[\underline{x}, \underline{y}]\big) = H. \tag{7.1}$$

For the latter equality, see, for example, [17, Theorem 1.5]. By the construction, B is torsion-free as a module over $\mathbb{C}[Y]$. Also over the localization $\mathbb{C}[\mathfrak{t}^*]^{\mathrm{reg}}$ of $\mathbb{C}[\mathfrak{t}^*]$ at the Vandermond determinant, (7.1) becomes an isomorphism. We conclude that (7.1) is injective. So B is a two-sided ideal in H.

It follows from Theorem 7.1 that the $\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]$-bimodule B is generated by a single element in degree 0 that is sign invariant. The corresponding element in $\Gamma(\mathcal{P}\otimes\mathcal{P})$ is the image of the identity under the inclusion of $\mathbb{C}[Y]$ arising from the direct summand $\mathcal{O}$ of $\mathcal{P}\otimes\mathcal{P}$. So the element in $B=\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P},\mathcal{P}(1))$ we need is described as the composition $\mathcal{P}\twoheadrightarrow\mathcal{O}(1)\hookrightarrow\mathcal{P}(1)$, where the first map is ϵ_- and the second is the inclusion of $\mathcal{O}(1)$ into $\mathcal{P}$. The image of this element in H is ϵ_-. We conclude that $B\cong H\epsilon_-H$. □

Remark 7.9

By [29, Proposition 6.1.5], we have $\Gamma(\mathcal{P}\otimes\mathcal{O}(d))=J^d$. For the same reason as in the proof of the proposition, we get $B_d\hookrightarrow\mathrm{Hom}_{\mathbb{C}[Y]}(\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*],J^d)$.

7.4. *Proof of Theorem 7.5*

In the proof we will need the following three lemmas.

LEMMA 7.10

For any $\mathfrak{g}$, we have a W-equivariant identification (for the right action)

$$\big(H^*(\mathcal{F}l_e)^\Lambda\big)^*\cong B\otimes_{\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]}\mathbb{C}_0.$$

Proof

Recall from Theorem 1.1 that we have an $H^\wedge$-bimodule isomorphism $H_T^{\mathrm{BM}}(\mathcal{F}l_e)^\wedge\cong B^\wedge$. Also $H_T^{\mathrm{BM}}(\mathcal{F}l_e)/H_T^{\mathrm{BM}}(\mathcal{F}l_e)\mathfrak{t}^*\cong H^{\mathrm{BM}}(\mathcal{F}l_e)$. Next, we have an identification $H^*(\mathcal{F}l_e)\cong H^{\mathrm{BM}}(\mathcal{F}l_e)^*$; this was discussed in Section 4.1 (in the equivariant setting). This identification is $\widetilde{W}$-equivariant. It follows that $H^*(\mathcal{F}l_e)^\Lambda\cong(H^{\mathrm{BM}}(\mathcal{F}l_e)_\Lambda)^*$, where the subscript Λ indicates taking the coinvariants. Note that

$$H^{\mathrm{BM}}(\mathcal{F}l_e)_\Lambda\xrightarrow{\sim}H^{\mathrm{BM}}(\mathcal{F}l_e)^\wedge/H^{\mathrm{BM}}(\mathcal{F}l_e)^\wedge\mathfrak{t}.$$

The claim of the lemma follows. □

Part (1) of Theorem 7.5 follows from Lemma 7.10 combined with Theorem 1.2.

In the remainder of this section we will prove (2) of Theorem 7.5. Recall that $G=\mathrm{SL}_n$ and hence $G^\vee=\mathrm{PGL}_n$.

LEMMA 7.11

Let μ be a highest weight of $G^\vee$ in the center of $\mathfrak{u}_\epsilon$. Let ℓ denote the order of ϵ and *recall that it is an odd number. Then we have $\ell\mu \leq 2(\ell-1)\rho$ in the usual order on the dominant weights.*

Proof
We note that $2(\ell-1)\rho = (\ell-1)\sum_{\alpha>0}\alpha$ is the maximal weight in $\mathfrak{u}_\epsilon$. The action of the Lusztig form $\dot{U}_\epsilon$ on Z factors through the quantum Frobenius epimorphism to give an action of $G^\vee$. The pullback inflates the weights ℓ times. This gives the required inequality. □

LEMMA 7.12
Let V be an irreducible PGL_n-module with the following property: the action of S_n on the weight zero subspace, V_0, is trivial. Then $V \cong S^{2kn}(\mathbb{C}^n)$ or $S^{2kn}(\mathbb{C}^n)^*$ for some $k \in \mathbb{Z}_{\geq 0}$.

Proof
In what follows it will be convenient to view V as a representation of GL_n. Our proof of the lemma is by induction on n.

The base is $n=2$, where our claim is easy. Now suppose it is proved for $n-1$, we are going to prove it for n. Let $\mu = (\mu_1, \dots, \mu_n)$ be the highest weight of V. The GL_{n-1}-module with highest weight $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ occurs in the restriction of V if and only if

$$\mu_1 \geq \lambda_1, \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_n. \tag{7.2}$$

And this GL_{n-1}-module intersects the zero weight space for PGL_n if and only if

$$\frac{\lambda_1 + \cdots + \lambda_{n-1}}{n-1} = \frac{\mu_1 + \cdots + \mu_n}{n}. \tag{7.3}$$

Clearly, at least one λ satisfying (7.2) and (7.3) exists.

Let I be the set of indices $i \in \{1, \dots, n-1\}$ such that $\mu_i > \mu_{i+1}$. Assume that $|I| > 1$. The claim that a solution λ to (7.2) and (7.3) satisfies the induction assumption easily implies that one of the following possibilities holds:

(1) $\lambda_i = \mu_i$ for all $i \in I$,

(2) $\lambda_i = \mu_{i+1}$ for all $i \in I$.

Indeed, otherwise we can increase one component of λ by 1 and decrease another by 1 so that (7.2) continues to hold. But if λ is the highest weight of $S^{2k(n-1)}(\mathbb{C}^{n-1})$ or its dual (up to a twist with a power of the determinant), then the modification is not of that form.

Replacing V with V^* if necessary we can assume that (1) holds. Also if $i \notin I$, then $\lambda_i = \mu_i (= \mu_{i+1})$. So we have $\lambda_i = \mu_i$ for all $i = 1, \dots, n-1$. From (7.3) we deduce

$$\mu_n = \frac{\mu_1 + \cdots + \mu_{n-1}}{n-1}.$$

Together with $\mu_1 \geq \cdots \geq \mu_n$, this implies $\mu_1 = \cdots = \mu_n$, a contradiction with $|I| > 1$.

So $|I| = 1$ meaning that μ has two different entries. Since λ is the highest weight of $S^{2k(n-1)}(\mathbb{C}^{n-1})$ or its dual, this implies that $I = \{1\}$ or $I = \{n-1\}$, which, in turn, easily implies the claim of the lemma. □

Proof of (2) of Theorem 7.5

Recall that we have a W-equivariant isomorphism $H^*(\mathcal{F}l_e)^\Lambda \xrightarrow{\sim} Z^{T^\vee}$ by Conjecture 7.4. Using Lemma 7.10 combined with Proposition 7.6, we see that S_n acts trivially on $Z^{T^\vee}$. By Lemma 7.12, all irreducible summands of the PGL_n-module Z are of the form $S^{2kn}(\mathbb{C}^n)$ or $S^{2kn}(\mathbb{C}^n)^*$. But for $k > 0$, the highest weights μ of these modules do not satisfy the inequality of Lemma 7.11. It follows that Z is a trivial PGL_n-module, implying the claim of the theorem. □

Appendix. Springer action on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$

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In this Appendix we include some of constructions and proofs for Section 5.

A.1. Reminder on the affine Springer action

In this section we recall the generalities on the affine Springer action. We use the notation from Section 4.2.

The action of W^a was constructed in [41, Section 5.4]. To construct the operators corresponding to simple affine reflections we introduce certain auxiliary spaces. For a parahoric subgroup $\mathfrak{P}$ of $G(\mathcal{K})$ containing $\mathfrak{B}$, we can consider the partial affine flag variety

$$\mathcal{F}l^{\mathfrak{P}} = G(\mathcal{K})/\mathfrak{P}.$$

Using this space, we can introduce the affine Springer fibers in the partial flag variety $\mathcal{F}l^{\mathfrak{P}}$:

$$\mathcal{F}l^{\mathfrak{P}}_{e_d} := \{g\mathfrak{P} \in \mathcal{F}l \mid \mathrm{Ad}(g)^{-1}e_d \in \mathrm{Lie}(\mathfrak{P})\}.$$

Now we introduce certain stacks. To do this we need some notation. Let L be the standard Levi subgroup of $\mathfrak{P}$. Let B_L denote the image of $\mathfrak{B}$ in L; this is a Borel subgroup of L. We write $\mathfrak{l}$, $\mathfrak{b}_L$ for the Lie algebras of these groups.

With this notation, we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{F}l_{e_d} & \xrightarrow{q_1} & \mathfrak{b}_L/B_L \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ \mathcal{F}l_{e_d}^{\mathfrak{P}} & \xrightarrow{q_2} & \mathfrak{l}/L \end{array} \tag{A.1}$$

The map $\mathcal{F}l_{e_d}^{\mathfrak{P}} \to \mathfrak{l}/L$ sends $g\mathfrak{P}$ to the image of $\mathrm{Ad}(g)^{-1}e_d$ and $\mathrm{Lie}(\mathfrak{P}) \to \mathfrak{l}$. The map $\mathcal{F}l_{e_d} \to \mathfrak{b}_L/B_L$ is defined in a similar way.

Note that we have the following canonical isomorphisms of objects in the $T \times \mathbb{C}^\times$-equivariant derived category:

$$(\pi_2)_*(\omega_{\mathcal{F}l_{e_d}}) \xrightarrow{\sim} (\pi_2)_*\big(q_1^!(\mathbb{C}_{\mathfrak{b}_L/B_L})\big) \xrightarrow{\sim} q_2^!(\pi_1)_*(\mathbb{C}_{\mathfrak{b}_L/B_L}).$$

Using these isomorphisms we can define the action of W^a on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ (see [41] and [47, Construction 7.1.3]). Namely, fix a simple affine reflection $s \in W^a$. If s is a reflection in the Weyl group W_L of L, then we can define an action of s on $(\pi_1)_*(\mathbb{C}_{\mathfrak{b}_L/B_L})$ via the usual finite dimensional Springer correspondence. This gives rise to an action of s on

$$(\pi_2)_*(\omega_{\mathcal{F}l_{e_d}}) = q_2^!(\pi_1)_*(\mathbb{C}_{\mathfrak{b}_L/B_L}). \tag{A.2}$$

Since q_1 and q_2 are $T \times \mathbb{C}^\times$-equivariant, we get an action of s on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ (via pushforward of the left-hand side of (A.2) to the point). This action of s is independent on the choice of L. To check that the actions of the simple affine reflections satisfy the braid relations, it is enough to consider two simple reflections at a time, which reduces to the finite case, because any two simple reflections lie in W_L for some choice of $\mathfrak{P}$.

To extend the W^a-action on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ to an action of $\widetilde{W}$, recall that $\widetilde{W} = (\Lambda/\Lambda_0) \ltimes W^a$. We note that Λ/Λ_0 acts on $\mathcal{F}l$. This action is constructed as follows. Take a lift of $\pi \in \Lambda/\Lambda_0 \subset \widetilde{W}$ to $\dot{\pi}$ in the normalizer of $T(\mathcal{K})$ and define the map $\mathcal{F}l \to \mathcal{F}l$ by $\mathfrak{g}\mathfrak{B} \to g\dot{\pi}\mathfrak{B}$. This is well defined as the lift of any element of $(\Lambda/\Lambda_0) \subset \widetilde{W}$ normalizes $\mathfrak{B}$ and the map is independent of the chosen lift. From the definition of $\mathcal{F}l_{e_d}$, see, for examnple, (4.3), it follows that this action preserves $\mathcal{F}l_{e_d}$. So we get an action of Λ/Λ_0 on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$.

Recall that we write R for $H^*_{T\times\mathbb{C}^\times}(pt)$ and F for $\mathrm{Frac}(\mathsf{R})$.

LEMMA A.1

(1) *The actions of W^a and Λ/Λ_0 give an action of the affine Weyl group $\widetilde{W}$ on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$.*

(2) *This $\widetilde{W}$-action is by R-linear automorphisms.*

(3) *The action of $\widetilde{W}$ preserves the homological grading on* R.

Proof

(1) follows from [50, Theorem 3.3.3] or [47, Theorem 7.1.5].

(2) is a direct consequence of the construction.

(3) follows from the construction of the action in [47, Construction 7.13, 7.14]. □

Remark A.2

In this remark we recall a classical description of the connected components of $\mathcal{F}l$ and $\mathcal{F}l_{e_d}$.

The connected components of the affine flag variety $\mathcal{F}l$ are in a natural bijection with $\pi_1(G)$. Namely, recall the decomposition $\widetilde{W} = (\Lambda/\Lambda_0) \ltimes W^a$. The union of Schubert cells corresponding to the left W^a-orbits in $\widetilde{W}$ give the connected components. The group Λ/Λ_0 acts on $\mathcal{F}l$ as recalled above in this section. This action induces a simply transitive action on the set of components.

Let $\widetilde{G}$ be the simply connected cover of the derived subgroup $(G, G) \subset G$. Its extended affine Weyl group is W^a. In fact, $\mathcal{F}l_{\widetilde{G}}$ is isomorphic to any of the connected components of $\mathcal{F}l_G$. To see this, note that we have a natural map $\mathcal{F}l_{\widetilde{G}} \to \mathcal{F}l_G$. This map is injective because the kernel of $\widetilde{G} \to G$ is contained in the center and thus contained in any Iwahori subgroup. The image contains precisely the T-fixed points given by W^a. The $\mathfrak{B}$-orbits coincide with the orbits of the pro-unipotent radical of $\mathfrak{B}$. Thus we see the image is precisely one connected component of $\mathcal{F}l_G$. Since all connected components are isomorphic via the Λ/Λ_0-action the result follows.

Moreover, the action of Λ/Λ_0 preserves $\mathcal{F}l_{e_d}$. The embedding $\mathcal{F}l_{\widetilde{G}} \hookrightarrow \mathcal{F}l_G$ restricts to an embedding of the Springer fibers associated to e_d. This embedding realizes the Springer fiber for $\widetilde{G}$ as a connected component of the Springer fiber for G. It follows that every connected component of $\mathcal{F}l_{e_d}$ for G is identified with the Springer fiber of e_d for $\widetilde{G}$.

A.2. Springer action vs localization

The goal of this section is to prove Lemma A.5, which is the hard part of Lemma 5.2.

Recall that we write ι for the localization homomorphism

$$H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}) \to \bigoplus_{\widetilde{W}} \mathsf{F}.$$

In the proof we will need an explicit description of $\operatorname{im}\iota$ for SL_2. We identify $\mathbb{Z}$ with $\widetilde{W}_{SL_2}$ via $2m \mapsto t^{m\alpha}$, $2m+1 \mapsto t^{m\alpha}s$, where α is the finite simple root of SL_2 and s is the corresponding simple reflection. Let y be the basis element in $\mathfrak{t}^*$ corresponding to the simple root. So $\mathsf{R} = \mathbb{C}[y, \hbar]$.

Pick an element $r \in \{0, \dots, d\}$. For $k, m \in \mathbb{Z}$ set

$$f_k^{r,(m)} = \prod_{i=1}^{r} \big(y + (k + m + i - 1)\hbar\big). \tag{A.3}$$

We then define elements

$$b_k^r = (b_{k,\ell}^r)_{\ell \in \mathbb{Z}} \in \bigoplus_{\mathbb{Z}} \mathbb{C}(y, \hbar)$$

for r, k as above as follows. For $r = 0$, we set $b_{k,\ell}^0 := \delta_{k,\ell}$, the Kroneker delta. For $r \in \{1, \dots, d\}$, define $m \in \mathbb{Z}$, $\epsilon \in \{0, 1\}$ by $\ell = k + 2m + \epsilon$ and set

$$b_{k,\ell}^r := (-1)^{m+\epsilon} \binom{r}{m} (f_k^{r,(m)})^{-1}. \tag{A.4}$$

LEMMA A.3
Let $G = SL_2$. Then $\iota(H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})) \subset \bigoplus_{\mathbb{Z}} \mathbb{C}(y, \hbar)$ has a basis over $\mathbb{C}[y, \hbar]$ given by b_k^r where
- *either $k = 0, 1$ and $r = 0, \dots d - 1$*
- *or $r = d$ and $k \in \mathbb{Z}$.*

Proof
The elements b_k^r are indeed in $\operatorname{im} \iota$: condition (i) of Corollary 4.8 is immediate, while condition (ii) is straightforward.

Now we check that the elements b_k^r for r, k as in the statement of the lemma span that $\mathbb{C}[y, \hbar]$-module $\operatorname{im} \iota$. Pick $(g_k) \in \operatorname{im} \iota$.

Replacing (g_k) with its sum with a linear combination of the elements b_k^d we can assume that (g_k) is supported between 0 and $2d - 2$. To see this, assume that $g_r \neq 0$ for some $r < 0$ and let k be the minimal such number r. Then the entry g_k can have at most the same singularities as $1/f_k^{d,(0)}$ by Corollary 4.8 and so is a multiple of this. Hence we can subtract a multiple of b_k^d from (g_k), such that the index of the minimal nonzero entry is bigger than k. Thus by induction we can assume that for all negative k we have $g_k = 0$.

A similar argument works for nonzero entries of (g_k) for $k > 2d - 2$. Here the inequality $k > 2d - 2$ comes from the fact that b_k^d has support exactly between k and $k + 2d - 1$. So, subtracting the elements b_k^d for $k \geq 0$ from (g_r) doesn't change the condition that $g_r = 0$ for $r < 0$. So we can assume that $g_k \neq 0 \Rightarrow 0 \leq k \leq 2d - 2$.

Now using b_k^r for $k = 0, 1$ and $r = 0, \dots d - 1$, we can continue reducing the support and using conditions (i) and (ii) of Corollary 4.8 to ensure the maximal entries are indeed multiples of those of the b_k^r we are considering. Indeed if (g_k) is supported between 0 and $2r-\epsilon$, $\epsilon \in \{0,1\}$ and $0 \le r \le d-1$, then $g_{2r-\epsilon}$ has at most the singularities of $1/f_{1-\epsilon}^{r,(r)}$ by the conditions of Corollary 4.8.

It follows that the elements b_k^r for k, r as described in the statement of the lemma span the R-module $\operatorname{im}\iota$.

To check that our elements are linearly independent (hence form a basis) we use a partial order on $\mathbb{Z}$. Consider the partial order given by

- $k \preceq r$ if $0 > k \ge r$,
- $k \preceq r$ if $2d-1 \le k \le r$,
- and $0 \preceq 1 \preceq \cdots \preceq 2d-2 \preceq k$ for all $k \notin \{0,\dots,2d-2\}$.

For each element b_k^r with r, k as in the statement of the lemma, there is a unique maximal $\ell(=\ell(b_k^r))$ in the poset order such that $b_{k,l}^r \neq 0$, namely,

$$\ell(b_k^r) = \begin{cases} k & \text{if } k < 0, \\ k+2r-1 & \text{else.} \end{cases}$$

It is clear that $(k,r) \mapsto \ell(b_k^r)$ identifies the set r, k in the statement of the lemma with $\mathbb{Z}$. Now we use induction on the above partial order to show that the elements b_k^r are linearly independent. □

Remark A.4

For a general semisimple rank 1 group G, we have a similar basis for each connected component of $\mathcal{F}l$ as described in Remark A.2. In that basis we use the polynomials $f_k^{r,(m)}$ replacing y with the unique root $\alpha \in \mathfrak{t}^* \subset \mathsf{R}$ of G. Indeed, the 1-dimensional $T \times \mathbb{C}^\times$-orbit all have characters $\alpha + k\hbar$, where $k \in \mathbb{Z}$ and α is the positive root of G. Each connected component of the affine Springer fiber for G is isomorphic to the one for SL_2 by Remark A.2. Then the same proof as for the SL_2 case gives a similar basis.

LEMMA A.5

We have

$$\iota(s\beta)_x = \frac{d\hbar}{{}^{x}\alpha}\iota(\beta)_x + \frac{{}^{xs}\alpha - d\hbar}{{}^{xs}\alpha}\iota(\beta)_{xs}. \tag{A.5}$$

Proof

Our proof is in several steps.

Step 1. Recall that the Springer action is by R-linear automorphisms and preserves the degrees by Lemma A.1.

Step 2. Let $\beta \in H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ and let s be a simple affine reflection. In this step we will prove that, for every $x \in \widetilde{W}$, the element $\iota(s\beta)_x$ only depends on $\iota(\beta)_x$ and $\iota(\beta)_{xs}$. To do this we describe the localization morphism in terms of maps of sheaves. We use the notation from the construction of the Springer action in Section A.1.

For an arbitrary parahoric $\mathfrak{P}$ (including $\mathfrak{B}$), let $i^{\mathfrak{P}}$ denote the inclusion

$$\pi_2^{-1}\big((\mathcal{F}l_{e_d}^{\mathfrak{P}})^{T\times\mathbb{C}^\times}\big) \to \mathcal{F}l_{e_d}.$$

By adjunction applied to

$$\omega_{\mathcal{F}l_{e_d}^{T\times\mathbb{C}^\times}} \to i^{\mathfrak{B}!}\omega_{\mathcal{F}l_{e_d}},$$

we get a morphism of sheaves

$$i_*^{\mathfrak{B}}(\omega_{\mathcal{F}l_{e_d}^{T\times\mathbb{C}^\times}}) \to \omega_{\mathcal{F}l_{e_d}} \tag{A.6}$$

in the $T\times\mathbb{C}^\times$-equivariant derived category. The localization map

$$H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{T\times\mathbb{C}^\times}) \to H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$$

is obtained from (A.6) by passing to cohomology.

The same construction as in Section A.1 establishes an action of the Weyl group W_L of L on

$$(\pi_2)_* i_*^{\mathfrak{P}}(\omega_{\pi_2^{-1}((\mathcal{F}l_{e_d}^{\mathfrak{P}})^{T\times\mathbb{C}^\times})}). \tag{A.7}$$

Note that the space $\pi_2^{-1}((\mathcal{F}l_{e_d}^{\mathfrak{P}})^{T\times\mathbb{C}^\times})$ decomposes as the disjoint union of subspaces indexed by $\widetilde{W}/W_L$ so that the subspace indexed by xW_L contains exactly the fixed points labeled by the elements from xW_L. This decomposition is compatible with Cartesian diagram (A.1). Hence this decomposition yields the decomposition of (A.7) into the direct sum with summands indexed by $\widetilde{W}/W_L$. Each summand is W_L-stable.

Note that (A.6) factors as

$$i_*^{\mathfrak{B}}(\omega_{\mathcal{F}l_{e_d}^{T\times\mathbb{C}^\times}}) \to i_*^{\mathfrak{P}}(\omega_{\pi_2^{-1}((\mathcal{F}l_{e_d}^{\mathfrak{P}})^{T\times\mathbb{C}^\times})}) \to \omega_{\mathcal{F}l_{e_d}}.$$

The induced maps in cohomology,

$$H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{T\times\mathbb{C}^\times}) \to H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}\big(\pi_2^{-1}\big((\mathcal{F}l_{e_d}^{\mathfrak{P}})^{T\times\mathbb{C}^\times}\big)\big) \to H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}),$$

become an isomorphism after inverting some characters by a direct analog of Lemma 4.1 for ind-varieties.

Apply the last observation to the minimal Levi subgroup corresponding to the reflection s. Consider the classes of the points x and xs in $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{T\times\mathbb{C}^\times})$. The subspace (over F) in the localization of $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\pi_2^{-1}((\mathcal{F}l_{e_d}^{\mathfrak{P}})^{T\times\mathbb{C}^\times}))$ spanned by their images is s-stable. Therefore the same conclusion is true if we consider the images in the localization of $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$. This claim is equivalent to the claim of that $\iota(s\beta)_x$ only depends on $\iota(\beta)_x$ and $\iota(\beta)_{xs}$.

Step 3. For $x \in \widetilde{W}$ consider the element $a^x := (\delta_{x,y})_{y\in\widetilde{W}} \in \bigoplus_{\widetilde{W}} \mathsf{F}$, where we recall that $\delta_{x,y}$ is the Kroneker delta. Note that in $a^x \in \operatorname{im}\iota$, one can see this, for example, from Corollary 4.8. Then the element $s(a^x)$ has the following properties:

(I) $s(a^x) = A_{x,x}a^x + A_{x,xs}a^{xs}$ for some $A_{x,x}, A_{x,xs} \in \mathsf{F}$. This follows from Step 2.

(II) $A_{x,x}$, $A_{x,xs}$ are homogeneous of degree 0. This is because a^y is of degree 0 for all y and the Springer action preserves the grading (Step 1).

(III) ${}^x\alpha A_{x,x}$, ${}^x\alpha A_{x,xs}$ are linear functions. Indeed, from (i) of Corollary 4.8 it follows that ${}^x\alpha A_{x,x}, {}^x\alpha A_{x,xs} \in \mathsf{R}$. Now our claim follows from (II).

(IV) $A_{x,x} + A_{x,xs}$ has no pole so is an element of $\mathbb{C}$. This follows from (III) and condition (ii) of Corollary 4.8.

(V) The elements ${}^x\alpha A_{x,x}$, ${}^x\alpha(A_{x,xs} - 1)$ are divisible by $\hbar$. This follows from $sa^x = a^{xs}$ modulo $\hbar$, which is a consequence of [24, Section 14.4].

Combining (III), (IV) and (V), we see that

$$A_{x,x} = \frac{z\hbar}{{}^x\alpha}, \qquad A_{x,xs} = \frac{{}^x\alpha - z\hbar}{{}^x\alpha} \tag{A.8}$$

for some $z \in \mathbb{C}$. Note that we have $\iota(s\beta)_x = A_{x,x}\iota(\beta)_x + A_{xs,x}\iota(\beta)_{xs}$. So the lemma amounts to showing that $z = d$.

Step 4. We will use the case of SL_2 for the computation of the elements $A_{x,x}$ and $A_{x,xs}$. For an affine root β let $\bar{\beta}$ be the projection of β to $\mathfrak{t}^*$. Equivalently, $\bar{\beta}$ is the unique root such that $\beta = \bar{\beta} + k\delta$ for some integer k, where, recall that δ is the indecomposable imaginary root. Set $\beta := {}^x\alpha$. Let $T_{\bar{\beta}} \subset T$ denote the kernel of $\bar{\beta}$ viewed as a homomorphism $T \to \mathbb{C}^\times$. We write $\widetilde{W}_{\bar{\beta}}$ for the subgroup of $\widetilde{W}$ generated by the reflection s_β and $t^{\bar{\beta}^\vee}$. Note that $\mathcal{F}l^{T_{\bar{\beta}}}$ is the affine flag variety of the semisimple rank 1 subgroup $G_\beta := Z_G(T_{\bar{\beta}})$ given by considering orbits of the loop group of G_β at points $\widetilde{W} \subset \mathcal{F}l$. The connected components of $\mathcal{F}l^{T_{\bar{\beta}}}$ are labeled by the cosets $\widetilde{W}_{\bar{\beta}} \setminus \widetilde{W}$. Each component is isomorphic to the affine flag variety of SL_2 and contains the T-fixed points labeled by points in the corresponding coset. A similar decomposition holds for $\mathcal{F}l_{e_d}^{T_{\bar{\beta}}}$: it is the union of connected components labeled by $\widetilde{W}_{\bar{\beta}} \setminus \widetilde{W}$.

Now we can localize $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ at all characters that do not vanish on $T_{\bar{\beta}}$, which includes $\gamma + k\hbar$ for $\gamma \in R^+ \setminus \{\bar{\beta}\}$ and $k \in \mathbb{Z}$, but not ${}^x\alpha + k\hbar$ for any k. By the ind-variety analog of Lemma 4.2, this localized BM homology is naturally isomorphic to the same localization of

$$H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d}^{T_{\bar{\beta}}}).$$

So the localization $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ breaks up as the direct sum of copies of the localized Borel–Moore homology of $\mathcal{F}l_{e_d}(G_\beta)$, the equivalued unramified affine Springer fiber for the semisimple rank 1 group G_β.

Step 5. Recall that the Springer action of $\widetilde{W}$ on $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ is R-linear (Step 1). So it lifts to the localization considered in Step 4. Consider the connected component in $\mathcal{F}l_{e_d}^{T_{\bar\beta}}$ whose T-fixed points are the coset $\widetilde{W}_{\bar\beta}x$. Note that $\widetilde{W}_{\bar\beta}x = \widetilde{W}_{\bar\beta}xs$ because $\beta = {}^x\alpha$. It follows that the summand in the localization of $H^{\mathrm{BM}}_{T\times\mathbb{C}^\times}(\mathcal{F}l_{e_d})$ corresponding to this component is fixed by the Springer action of s. Lemma A.3, or, more precisely, its generalization discussed in Remark A.4 give a basis in the summand we consider. Note that there is a unique element $k \in \mathbb{Z}$ such that the basis element b^d_k of this summand is 0 at xs and nonzero at x. Then $s(b^d_k)$ will only have singularities along the affine root hyperplanes ${}^x\alpha + p\hbar$ with $p \in \mathbb{Z}$. Further, by the description of the basis in Lemma A.3, we have $(b^d_k)_x = \frac{1}{f}$ for $f := ({}^x\alpha - \hbar)({}^x\alpha - 2\hbar)\cdots({}^x\alpha - d\hbar)$.

Note that $(s(b^d_k))_{xs} = A_{x,xs}\frac{1}{f}$ and $(s(b^d_k))_w = 0$ for all w of the form $xss_0s\ldots$, where $s_0 := t^{-\alpha}s$. It thus follows that $A_{x,xs}/f$ only has singularities along ${}^{xs}\alpha, \ldots, {}^{xs}\alpha + (d-1)\hbar$. Since ${}^x\alpha A_{x,xs}$ is a linear polynomial, we see that $A_{x,xs}$ is proportional to

$$\frac{{}^x\alpha - d\hbar}{{}^x\alpha}.$$

We apply (V) to see that

$$A_{x,xs} = \frac{{}^x\alpha - d\hbar}{{}^x\alpha}. \tag{A.9}$$

This implies the claim of the lemma by the last sentence of Step 3. □

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