Supercuspidal representations in non-defining characteristics

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Abstract

We show that a mod-ℓ-representation of a p-adic group arising from the analogue of Yu's construction is supercuspidal if and only if it arises from a supercuspidal representation of a finite reductive group. This has been previously shown by Henniart and Vigneras under the assumption that the second adjointness holds.

1 Introduction

The exhaustive explicit construction and parametrization of supercuspidal irreducible representations of p-adic groups with complex coefficients plays a key role in the complex representation theory of p-adic groups and beyond. For number theoretic applications it is often desirable to obtain analogous results for representations whose coefficients are valued in an algebraically closed field R of characteristic ℓ different from p. In that setting one needs to distinguish between cuspidal and supercuspidal representations. An exhaustive construction of the former, more general notion, is known if the p-adic group is tame and p does not divide the order of the Weyl group ([Fin22]). This paper concludes the exhaustive construction of supercuspidal irreducible representations for p-adic groups in the same setting by determining which of the cuspidal representations are supercuspidal.

More precisely, we show that if an R-representation arising from the analogue of Yu's construction is supercuspidal, then the representation of a finite reductive group that forms part of the input for the construction has to be supercuspidal as well (Theorem 1). Combined with the reverse implication proved by Henniart and Vigneras [HV, Theorem 6.10 and §6.4.2]

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and the result that all cuspidal R-representations arise from the analogue of Yu's construction under the above assumptions ([Fin22, Theorem 4.1]), we obtain an exhaustive explicit construction of all supercuspidal irreducible R-representations.

Theorem 1 has previously been proven by Henniart and Vigneras ([HV, Theorem 6.10 and §6.4.2]) using different techniques, but only under the assumption that the second adjointness holds in this setting. When the first version of this paper was posted on arxiv.org, the second adjointness was only proven for depth-zero representations or if the p-adic group is a general linear group, a classical group (with $p \neq 2$) or a group of relative rank 1 ([Dat09]). Recently Dat, Helm, Kurinczuk and Moss ([DHKM]) proved that second adjointness holds for general reductive groups, but their proof relies on the results of Fargues and Scholze ([FS21]). We therefore believe that it is still desirable to have an entirely representation theoretic and drastically shorter proof that does not rely on the second adjointness to hold, which is the case for our approach.

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2 The main theorem and corollaries

Let F be a non-archimedian local field of residual characteristic p with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}_q$. Let G be a (connected) reductive group over F. Let R be an algebraically closed field of characteristic ℓ different from p. All representations in this paper are representations with coefficient field R.

For a point x in the enlarged Bruhat–Tits building $\mathscr{B}(G, F)$ of G over F, we write $[x]$ for the image of the point x in the reduced Bruhat–Tits building, $G_{[x]}$ for the stabilizer of $[x]$, $G_{x,0}$ for the parahoric subgroup, and $G_{x,r}$ for the Moy–Prasad filtration subgroup of depth r for a positive real number r.

Let $((G_i)_{1\leq i\leq n+1}, x \in \mathscr{B}(G_{n+1}, F) \subset \mathscr{B}(G, F), (r_i)_{1\leq i\leq n}, \rho, (\phi_i)_{1\leq i\leq n})$ be an input for the construction of a cuspidal R-representation as in [Fin22, Section 2.2], i.e. following Yu's construction [Yu01] adapted to the mod-ℓ setting. If $n > 0$, then we assume that G_{n+1}, and hence also G, splits over a tamely ramified extension of F and $p \neq 2$, as in Yu's construction. In the case of $n = 0$, the construction recovers the depth-zero representations and we allow G to be wildly ramified and/or $p = 2$. The irreducible R-representation ρ of $(G_{n+1})_{[x]}$ is trivial on $(G_{n+1})_{x,0+}$ and its restriction to $(G_{n+1})_{x,0}$, a maximal parahoric subgroup of $G_{n+1}(F)$, is a cuspidal representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$. Note that the restriction of ρ to $(G_{n+1})_{x,0}$ is semisimple because $(G_{n+1})_{[x]}$ normalizes $(G_{n+1})_{x,0}$. Hence it makes sense to talk about the restriction being supercuspidal or not.

From the tuple $((G_i)_{1\leq i\leq n+1}, x \in \mathscr{B}(G_{n+1}, F) \subset \mathscr{B}(G, F), (r_i)_{1\leq i\leq n}, \rho, (\phi_i)_{1\leq i\leq n})$ we obtain a representation $(\widetilde{\rho} = \rho \otimes \kappa, V_\rho \otimes V_\kappa)$ of

$$\widetilde{K} = (G_1)_{x,\frac{r_1}{2}}(G_2)_{x,\frac{r_2}{2}} \dots (G_n)_{x,\frac{r_n}{2}}(G_{n+1})_{[x]},$$

where ρ also denotes the extension of the depth-zero representation ρ from $(G_{n+1})_{[x]}$ to $\widetilde{K}$ that is trivial on $(G_1)_{x,\frac{r_1}{2}}(G_2)_{x,\frac{r_2}{2}}\dots(G_n)_{x,\frac{r_n}{2}}$ and κ is constructed from the characters ϕ_i using the theory of mod-ℓ Weil–Heisenberg representations ([Fin22, Section 2.3]), such that the compactly induced representation $\text{c-ind}_{\widetilde{K}}^{G}(\rho\otimes\kappa)$ is irreducible and cuspidal.

Theorem 1. *With the above notation, if* $\text{c-ind}_{\widetilde{K}}^{G(F)}(\rho\otimes\kappa)$ *is supercuspidal, then the restriction of* ρ *to* $(G_{n+1})_{x,0}$ *is supercuspidal as a representation of* $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$.

Combining Theorem 1 with the unconditional result of Henniart and Vigneras that $\text{c-ind}_{\widetilde{K}}^{G}(\rho\otimes\kappa)$ is supercuspidal when the restriction of ρ to $(G_{n+1})_{x,0}$ is supercuspidal as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ ([HV, Theorem 6.10 and §6.4.2]), we obtain the following unconditional Corollary 2.

Corollary 2. *With the above notation,* $\text{c-ind}_{\widetilde{K}}^{G(F)}(\rho\otimes\kappa)$ *is supercuspidal if and only if the restriction of* ρ *to* $(G_{n+1})_{x,0}$ *is supercuspidal as a representation of* $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$.

Combined with [Fin22, Theorem 4.1] we obtain the following result.

Corollary 3. *Suppose that* G *splits over a tamely ramified field extension of* F *and that* p *does not divide the order of the (absolute) Weyl group of* G. *Then all supercuspidal irreducible representations of* $G(F)$ *are of the form* $\text{c-ind}_{\widetilde{K}}^{G(F)}(\rho\otimes\kappa)$ *as above where the restriction of* ρ *to* $(G_{n+1})_{x,0}$ *is supercuspidal as a representation of* $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$.

3 Proof of Theorem 1

Let $\pi := \text{c-ind}_{\widetilde{K}}^{G(F)}(\rho\otimes\kappa)$ be a cuspidal irreducible representation as in the previous section. In the present section, Section 3, we will prove that π being supercuspidal implies that ρ is supercuspidal, i.e. we prove Theorem 1. This statement is trivially true if G_{n+1} is anisotropic as in this case $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ is also anisotropic and hence all its semisimple representations are supercuspidal. Thus we assume throughout this section that G_{n+1} is not anisotropic.

We will eventually prove the desired result by assuming that ρ is not supercuspidal and proving that then π is a subquotient of a parabolically induced representation, i.e. is not supercuspidal either. However, we will not make this assumption until the end of this section to first prove a series of results that hold without this assumption and might be useful on its own for other applications.

Let T be a maximally split, maximal torus of G_{n+1} such that x is contained in the apartment $\mathscr{A}(T,F)$ of T. Let λ be a cocharacter of T, i.e. an F-homomorphism from $\mathbb{G}_m$ to T. We write $P_{G_i}(\lambda)$ for the parabolic subgroup of G_i $(1\leq i\leq n+1)$ attached to λ as in Section 2.1 and 2.2, in particular Proposition 2.2.9, of [CGP15]. This means $P_{G_i}(\lambda)(F)$ consists of the elements $g\in G_i(F)$ for which the limit of $\lambda(t)g\lambda(t)^{-1}$ as t goes to zero exists (i.e. extends to a map from the affine line to G_i). Then the centralizer $Z_{G_i}(\lambda)$ of λ is a Levi subgroup of $P_{G_i}(\lambda)$, which we also denote by $M_{G_i}(\lambda)$. Let $U_{G_i}(\lambda)$ be the unipotent radical of $P_{G_i}(\lambda)$ and $\bar{U}_{G_i}(\lambda)$ the unipotent radical of the opposite parabolic $\bar{P}_{G_i}(\lambda)$ of $P_{G_i}(\lambda)$ with respect to $M_{G_i}(\lambda)$.

Let $\epsilon > 0$ be sufficiently small so that $G_{x,s+} \subset G_{y,s} \subset G_{x,s}$ for $s \in \{\frac{r_1}{2}, \frac{r_2}{2}, \ldots, \frac{r_n}{2}, 0\}$ and $y = x + \epsilon\lambda \in \mathscr{A}(T, F)$. While there is in general no canonical embedding of the Bruhat–Tits building of $M_{n+1} := M_{G_{n+1}}(\lambda)$ into the Bruhat–Tits building of G_{n+1}, the embedding is unique up to translation by $X_*(Z(M_{n+1})) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X_*(Z(M_{n+1}))$ denotes the (F-)cocharacters of the center $Z(M_{n+1})$ of M_{n+1}. Note that the dimension of the real vector space $X_*(Z(M_{n+1})) \otimes \mathbb{R}$ is equal to the rank of the maximal split torus in the center of M_{n+1}. We fix an embedding of Bruhat–Tits buildings throughout the paper to view $\mathscr{B}(M_{n+1}, F)$ as a subset of $\mathscr{B}(G_{n+1}, F)$. More generally, we will fix a compatible system of embeddings of the Bruhat–Tits buildings of all the below occurring twisted Levi subgroups of G to view all Bruhat–Tits buildings over F as subsets of $\mathscr{B}(G, F)$, i.e.

$$\begin{array}{ccccccc}
\mathscr{B}(M_{G_{n+1}}(\lambda), F) & \hookrightarrow & \mathscr{B}(M_{G_n}(\lambda), F) & \hookrightarrow & \ldots & \hookrightarrow & \mathscr{B}(M_G(\lambda), F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathscr{B}(G_{n+1}, F) & \hookrightarrow & \mathscr{B}(G_n, F) & \hookrightarrow & \ldots & \hookrightarrow & \mathscr{B}(G, F)
\end{array} .$$

Then we have $y = x + \epsilon\lambda \in \mathscr{B}(M_{n+1}, F) \subset \mathscr{B}(G_{n+1}, F) \subset \mathscr{B}(G, F)$, and

$$(G_{n+1})_{y,0}/(G_{n+1})_{y,0+} \simeq (M_{n+1})_{y,0}/(M_{n+1})_{y,0+}. \tag{1}$$

For $z \in \{x, y\}$ and $s \in \{0, 0+\}$, we set

$$K_{z,s} = (G_1)_{z,\frac{r_1}{2}}(G_2)_{z,\frac{r_2}{2}} \ldots (G_n)_{z,\frac{r_n}{2}}(G_{n+1})_{z,s},$$

$$K_+ = (G_1)_{x,\frac{r_1}{2}+}(G_2)_{x,\frac{r_2}{2}+} \ldots (G_n)_{x,\frac{r_n}{2}+}(G_{n+1})_{x,0+}$$

We might abbreviate the groups $K_{x,0}$ and $K_{x,0+}$ by K_0 and K_{0+}, respectively, and write $P = P_G(\lambda)$, $M = M_G(\lambda)$, $U = U_G(\lambda)$, $\bar{P} = \bar{P}_G(\lambda)$ and $\bar{U} = \bar{U}_G(\lambda)$. Using that $y = x + \epsilon\lambda$ with ϵ sufficiently small we obtain that $K_{y,0} \subset K_0$ and

$$\begin{aligned}
K_{y,0} \cap U(F) &= K_{y,0+} \cap U(F) = K_0 \cap U(F), & (2)\\
K_{y,0} \cap \bar{U}(F) &= K_{y,0+} \cap \bar{U}(F) = K_+ \cap \bar{U}(F), & (3)\\
K_{y,0} &= (K_{y,0} \cap \bar{U}(F))(K_{y,0} \cap M(F))(K_{y,0} \cap U(F)). & (4)
\end{aligned}$$

Here Equations (2) and (3) follow from the definitions of $U = U_G(\lambda)$ and $\bar{U} = \bar{U}_G(\lambda)$, and Equation (4) follows from (2) and (3).

Lemma 4. *(a) The space of $(K_0 \cap U(F))$-fixed vectors $V_\kappa^{K_0 \cap U(F)}$ of the representation (κ, V_κ) is non-trivial.*

(b) The representation (κ, V_κ) is trivial when restricted to the subgroup $K_{y,0} \cap \bar{U}(F)$.

(c) The subspace $V_\kappa^{K_0 \cap U(F)}$ is preserved under the action of $K_{y,0}$ via κ.

Proof.
If π has depth-zero, κ is the trivial one-dimensional representation, and hence all statements are trivially true. Thus we may assume $n > 0$ and hence that we are in the setting where G_{n+1} splits over a tamely ramified field extension. Moreover, if λ is trivial, the statement is trivial, so we assume that λ is nontrivial. Let E be the splitting field of T. Note that λ viewed as a map from $\mathbb{G}_m$ to T factors through the maximally split subtorus T_{split} of T. Since T is maximally split, T_{split} is a maximal split torus of G_{n+1} and is therefore contained in a maximal torus T' that splits over a tamely ramified extension. By replacing T by T' above, we may assume without loss of generality that T is tamely ramified, i.e. that its splitting field E is a tamely ramified field extension of F. For $1 \leq i \leq n$, we define

$$\begin{aligned} U_i &= G(F) \cap \left\langle U_\alpha(E)_{x,\frac{r_i}{2}} \,|\, \alpha \in \Phi(G_i, T) \setminus \Phi(G_{i+1}, T), \lambda(\alpha) > 0 \right\rangle, \\ U_{n+1} &= G(F) \cap \langle U_\alpha(E)_{x,0} \,|\, \alpha \in \Phi(G_{n+1}, T), \lambda(\alpha) > 0 \rangle, \end{aligned}$$

where $\Phi(G_i, T)$ denotes the root system of G_i with respect to T over the field E (for $1 \leq i \leq n+1$) and $U_\alpha(E)_{x,\frac{r_i}{2}}$ denotes the depth-$\frac{r_i}{2}$ filtration subgroup of the root group $U_\alpha(E)$ of $G(E)$ corresponding to α and normalized with respect to the valuation on E that extends the valuation on F used to define the Moy–Prasad filtration. Then

$$K_0 \cap U(F) = U_1 U_2 \dots U_{n+1}.$$

Following [Fin21a, Section 2.5] we write $V_\kappa = \otimes_{i=1}^n V_{\omega_i}$ so that the action of κ restricted to U_j $(1 \leq j \leq n)$ is given by U_j acting on V_{ω_k} for $k \neq j$ via the character $\hat{\phi}_k$ defined in *loc. cit.* and on V_{ω_j} via a Heisenberg representation. The action of κ restricted to U_{n+1} arises from U_{n+1} acting on V_{ω_k} via ϕ_k tensored with a composition with a Weil representation, see *loc. cit.* for a precise definition. For $1 \leq j < k \leq n$, the restriction of $\hat{\phi}_k$ to U_j is trivial by the construction of $\hat{\phi}_k$. For $1 \leq k < j \leq n+1$, the restriction of $\hat{\phi}_k$ to U_j equals the restriction of the character ϕ_k from $G_{k+1}(F)$ to U_j. Since U_j is contained in the unipotent radical of a parabolic subgroup of $G_{k+1}(F)$, we conclude that the restriction of ϕ_k to U_j is trivial ([Tit64, Tit78]). Thus

$$V_\kappa^{U_1 U_2 \dots U_n} = \bigotimes_{i=1}^n (V_{\omega_i})^{U_i}.$$

Using the same arguments as in the proof of [Fin21a, Theorem 3.1], we obtain that the space $(V_{\omega_i})^{U_i}$ is nontrivial and that U_{n+1} acts on $(V_{\omega_i})^{U_i}$ via the restriction of the character ϕ_i to U_{n+1} for $1 \leq i \leq n$, which we observed above is trivial. Hence

$$V_\kappa^{K_0 \cap U(F)} = V_\kappa^{U_1 U_2 \dots U_n U_{n+1}} = \bigotimes_{i=1}^n (V_{\omega_i})^{U_i} \not\simeq \{0\}.$$

For (b) recall that κ restricted to K_+ acts via the character $\prod_{1 \leq i \leq n} \hat{\phi}_i$ (times identity). For $1 \leq i \leq n$, we define

$$\begin{aligned} \bar{U}_i^+ &= G(F) \cap \left\langle U_\alpha(E)_{x, \frac{r_i}{2}+} \,|\, \alpha \in \Phi(G_i, T) \setminus \Phi(G_{i+1}, T), \lambda(\alpha) < 0 \right\rangle, \\ \bar{U}_{n+1}^+ &= G(F) \cap \langle U_\alpha(E)_{x,0+} \,|\, \alpha \in \Phi(G_{n+1}, T), \lambda(\alpha) < 0 \rangle . \end{aligned}$$

Then

$$K_{y,0} \cap \bar{U}(F) = K_+ \cap \bar{U}(F) = \bar{U}_1^+ \bar{U}_2^+ \ldots \bar{U}_{n+1}^+.$$

For $1 \leq j \leq i \leq n$, the restriction of $\hat{\phi}_i$ to $\bar{U}_j^+$ is trivial by the construction of $\hat{\phi}_i$ and the definition of $\bar{U}_j^+$. For $1 \leq i < j \leq n+1$, the restriction of $\hat{\phi}_i$ to $\bar{U}_j^+$ equals the restriction of the character ϕ_i from $G_{i+1}(F)$ to $\bar{U}_j^+$. Since $\bar{U}_j^+$ is contained in the unipotent radical of a parabolic subgroup of $G_{k+1}(F)$, the restriction of ϕ_i to $\bar{U}_j^+$ is trivial ([Tit64, Tit78]). Hence the restriction of (κ, V_κ) to $K_{y,0} \cap \bar{U}(F) = \bar{U}_1^+ \bar{U}_2^+ \ldots \bar{U}_{n+1}^+$ is trivial.

Claim (c) follows now from Equation (4) and the observation that $K_{y,0} \cap M(F)$ normalizes $K_{y,0} \cap U(F) = K_0 \cap U(F)$.

□

Lemma 5. *Let $(\rho', V_{\rho'})$ be a representation of $K_{y,0}K_{0+}$ that is trivial on K_{0+}. Then there exists a surjection of $\widetilde{K}$-representations*

$$\mathrm{pr} : \text{c-ind}_{K_{y,0}}^{\widetilde{K}} \left(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)} \right) \twoheadrightarrow \left(\text{c-ind}_{K_{y,0}K_{0+}}^{\widetilde{K}} V_{\rho'} \right) \otimes V_\kappa.$$

We thank Guy Henniart for suggesting to replace our previous explicit proof by the following shorter one.

Proof of Lemma 5.
Since $\left(\text{c-ind}_{K_{y,0}K_{0+}}^{\widetilde{K}} V_{\rho'} \right) \otimes V_\kappa \simeq \left(\text{c-ind}_{K_{y,0}K_{0+}}^{\widetilde{K}} V_{\rho'} \otimes V_\kappa \right)$, it suffices to prove that there exists a surjection

$$\text{c-ind}_{K_{y,0}}^{K_{y,0}K_{0+}} \left(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)} \right) \twoheadrightarrow V_{\rho'} \otimes V_\kappa$$

of $K_{y,0}K_{0+}$-representations. The inclusion $V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)} \hookrightarrow V_{\rho'} \otimes V_\kappa$ yields via Frobenius reciprocity a morphism $\text{c-ind}_{K_{y,0}}^{K_{y,0}K_{0+}} \left(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)} \right) \to V_{\rho'} \otimes V_\kappa$ of $K_{y,0}K_{0+}$-representations whose image contains $V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}$. Since the restriction of (κ, V_κ) to K_{0+} is irreducible and K_{0+} acts trivially on $V_{\rho'}$, the image also contains $V_{\rho'} \otimes V_\kappa$, i.e. the above map is surjective. □

This allows us to prove the following key lemma for the proof of Theorem 1.

Lemma 6. *If ρ is not supercuspidal, then there exists a maximally split, maximal torus T of G_{n+1} whose apartment contains x, a cocharacter λ of T and a representation $(\rho', V_{\rho'})$ of $K_{y,0}$ (with $y = x + \epsilon\lambda$ as above) that is trivial on $K_{y,0+}$ such that the representation $(\rho \otimes \kappa, V_\rho \otimes V_\kappa)$ is a subquotient of* $\text{c-ind}_{K_{y,0}}^{\widetilde{K}} (V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)})$.

The cocharacter λ can be chosen so that $M := M_G(\lambda)$ is a proper subgroup of G and the centralizer of the maximal split torus in the center of $M_{n+1} := M_{G_{n+1}}(\lambda)$. The number $\epsilon > 0$ can be chosen so that the point $y = x + \epsilon\lambda \in \mathscr{B}(M_{n+1}, F) \subset \mathscr{B}(G, F)$ is contained in a facet of minimal dimension of $\mathscr{B}(M_{n+1}, F)$ and

$$\sum_{i=1}^{n} \left(\dim\left((G_i)_{y,\frac{r_i}{2}}/(G_i)_{y,r\frac{r_i}{2}+}\right) - \dim\left((M_{G_i}(\lambda))_{y,\frac{r_i}{2}}/(M_{G_i}(\lambda))_{y,\frac{r_i}{2}+}\right)\right) = 0. \tag{5}$$

Proof.
Suppose ρ is not supercuspidal. Recall that $\rho|_{(G_{n+1})_{x,0}}$ is semisimple as $(G_{n+1})_{x,0}$ is normal inside $(G_{n+1})_{[x]}$. Moreover, since at least one of the irreducible quotients of $\rho|_{(G_{n+1})_{x,0}}$ is not supercuspidal, we obtain that none of the irreducible quotients is supercuspidal. Let ρ_1 be an irreducible quotient of $\rho|_{(G_{n+1})_{x,0}}$, viewed as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$. We denote by $\mathbf{G}$ the connected reductive group over $\mathbb{F}_q$ that satisfies for any unramified field extension E of F with residue field $\mathfrak{f}_E$ that $\mathbf{G}(\mathfrak{f}_E) = G_{n+1}(E)_{x,0}/G_{n+1}(E)_{x,0+}$. Let $\mathbf{P}$ be a proper parabolic subgroup of $\mathbf{G}$ with Levi subgroup $\mathbf{M}$, and ρ' a representation of $\mathbf{M}(\mathbb{F}_q)$ such that ρ_1 is a subquotient of the parabolic induction $\mathrm{Ind}_{\mathbf{P}(\mathbb{F}_q)}^{(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}} \rho'$. Let $\mathbf{S}$ be a maximal split torus of $\mathbf{M}$ and $\mathscr{S}$ the split torus defined over $\mathcal{O}$ contained in the parahoric group scheme attached to G_{n+1} and x such that the special fiber of $\mathscr{S}$ is $\mathbf{S}$. We denote the generic fiber $\mathscr{S}_F$ of $\mathscr{S}$ by S. Note that S is a maximal split torus of G_{n+1}. Let $\mathscr{C}$ be the split subtorus of $\mathscr{S}$ whose special fiber $\mathbf{C} := \mathscr{C}_{\mathbb{F}_q}$ is the maximal split torus in the center of $\mathbf{M}$. Let M be the centralizer of $C := \mathscr{C}_F$ in G. Then M is a Levi subgroup of a proper parabolic subgroup of G and there exists a cocharachter $\lambda \in X_*(S)$ such that $M = M_G(\lambda)$ (e.g. by [CGP15, Proposition 2.2.9] combined with the fact that Levi subgroups of a fixed parabolic are rationally conjugate). Choosing a maximally split, maximal torus T of G_{n+1} containing S, we can perform the above constructions to obtain a parabolic subgroup $P_{G_i}(\lambda)$ of G_i $(1 \leq i \leq n+1)$ with Levi subgroup $M_i := M_{G_i}(\lambda) = Z_{G_i}(\lambda)$ and a point $y = x + \epsilon\lambda$ in the apartment $\mathscr{A}(T, F)$. Note that M_{n+1} is the centralizer of C in G_{n+1}, because M is the centralizer of C in G. Hence by Equation (1) and [MP96, Proposition 6.4(1)], the point y is a minimal facet of the building $\mathscr{B}(M_{n+1}, F)$. Moreover, since $\mathbf{C}$ is the maximal split torus in the center of $\mathbf{M}$ and $\mathbf{M}(\mathbb{F}_q) = (M_{n+1})_{y,0}/(M_{n+1})_{y,0+}$, the torus C is the maximal split torus in the center of M_{n+1}. Hence M is the centralizer of the maximal split torus in the center of M_{n+1}, as desired. Moreover, by the definition of $M_{G_i}(\lambda)$, Equation (5) is satisfied by all but finitely many ϵ in the open interval $(0, 1)$. Hence we may choose $\epsilon > 0$ such that Equation (5) holds true.

Since we have

$$\mathbf{M}(\mathbb{F}_q) = K_{y,0}/K_{y,0+} = K_{y,0}K_{0+}/K_{y,0+}K_{0+},$$

we may view ρ' as a representation of $K_{y,0}K_{0+}$ via inflation. Note that the image of $K_{y,0}K_{0+}$ in $K_{x,0}/K_{0+} \simeq (G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ is $\mathbf{P}(\mathbb{F}_q)$. Viewing ρ and ρ_1 as representations of $\widetilde{K}$ and $K_{x,0}$, respectively, by asking them to be trivial on $(G_1)_{x,\frac{r_1}{2}}(G_2)_{x,\frac{r_2}{2}} \ldots (G_n)_{x,\frac{r_n}{2}}$, we have by Frobenius reciprocity that the irreducible representation ρ is a quotient of $\text{c-ind}_{K_{x,0}}^{\widetilde{K}} \rho_1$ and therefore a subquotient of

$$\text{c-ind}_{K_{x,0}}^{\widetilde{K}} \text{c-ind}_{K_{y,0}K_{0+}}^{K_{x,0}} \rho' = \text{c-ind}_{K_{y,0}K_{0+}}^{\widetilde{K}} \rho'.$$

Therefore $(\rho \otimes \kappa, V_\rho \otimes V_\kappa)$ is a subquotient of $((\text{c-ind}_{K_{y,0}K_{0+}}^{\widetilde{K}} \rho') \otimes \kappa, (\text{c-ind}_{K_{y,0}K_{0+}}^{\widetilde{K}} V'_\rho) \otimes V_\kappa)$. From Lemma 5, we deduce that $(\rho \otimes \kappa, V_\rho \otimes V_\kappa)$ is a subquotient of $\text{c-ind}_{K_{y,0}}^{\widetilde{K}}(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)})$. □

Proof of Theorem 1.
Suppose ρ is not supercuspidal. We need to prove that $\pi := \text{c-ind}_{\widetilde{K}}^{G(F)}(\rho \otimes \kappa)$ is not supercuspidal. We let λ be as given by Lemma 6, which provides us with a point $y = x + \epsilon\lambda$ and a proper parabolic subgroup $P = P_G(\lambda)$ of G with Levi $M = M_G(\lambda)$ and unipotent radical U as above. Then the representation $(\rho \otimes \kappa, V_\rho \otimes V_\kappa)$ is a subquotient of $\text{c-ind}_{K_{y,0}}^{\widetilde{K}}(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)})$. Hence π is a subquotient of $\text{c-ind}_{K_{y,0}}^{G(F)}(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)})$. We will show that the latter is isomorphic to a parabolic induction of a smooth representation from $P(F)$, which will imply that π is not supercuspidal and hence finish the proof.

Recall from Equations (2), (3) and (4) that

$$K_{y,0} = (K_{y,0} \cap \bar{U}(F))(K_{y,0} \cap M(F))(K_{y,0} \cap U(F))$$

and that $K_{y,0} \cap \bar{U}(F) = K_{y,0+} \cap \bar{U}(F)$ and $K_{y,0} \cap U(F) = K_{y,0+} \cap U(F)$. Moreover, by Lemma 4(b) and since $K_0 \supset K_{y,0}$ and $(\rho', V_{\rho'})$ is trivial on $K_{y,0+}$, the restriction of $V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}$ to $K_{y,0} \cap \bar{U}(F)$ and to $K_{y,0} \cap U(F)$ is trivial. Hence the pair

$$(K_{y,0}, (\rho' \otimes \kappa, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}))$$

is decomposed over the pair

$$(K_{y,0} \cap M(F), ((\rho' \otimes \kappa)|_{K_{y,0} \cap M(F)}, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}))$$

with respect to $\bar{P}$ as in the notation of [Blo05, p. 245].

We write

$$K_{y,+} = (G_1)_{y,\frac{r_1}{2}+}(G_2)_{y,\frac{r_2}{2}+} \dots (G_n)_{y,\frac{r_n}{2}+}(G_{n+1})_{y,0+}$$

and note that the action of $K_{y,+}$ on $V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}$ via $\rho' \otimes \kappa$ is given by $\prod_{1 \le i \le n} \hat{\phi}_i$ (times identity). Let (π', V') be an irreducible smooth representation of $G(F)$. Then we write $V'^{(K_{y,+}, \prod \hat{\phi}_i)}$ for the subspace of V' on which $K_{y,+}$ acts via $\prod_{1 \le i \le n} \hat{\phi}_i$. Since y is contained in a facet of minimal dimension of $\mathscr{B}(M_{n+1}, F)$ and Equation (5) holds by Lemma 6 (which ensures that the embedding of the Bruhat–Tits buildings is $(0, \frac{r_n}{2}, \dots, \frac{r_1}{2})$-generic relative to y as defined by [KY17, 3.5 Definition], see also [Fin21b, p. 341]) we can apply the proof of [KY17, 6.3 Theorem] to obtain that the restriction of the Jacquet functor with respect to $\bar{U}$ to the subspace $V'^{(K_{y,+}, \prod \hat{\phi}_i)}$ is injective. Note that while Kim and Yu work with complex coefficients in [KY17], their proof and [MP96, Proposition 6.7], on which the proof relies, also work with coefficients in the field R. Hence the Jacquet functor with respect to $\bar{U}$ is also injective when restricted to the maximal subspace of V' that is isomorphic to a direct sum of copies of $(\rho' \otimes \kappa, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)})$ as a $K_{y,0}$-representation. Therefore, the pair

$$(K_{y,0}, (\rho' \otimes \kappa, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}))$$

is a cover of

$$(K_{y,0} \cap M(F), ((\rho' \otimes \kappa)|_{K_{y,0} \cap M(F)}, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}))$$

with respect to $\bar{P}$ as in [Blo05, p. 246 and Corollaire de Proposition 2]. Thus by [Blo05, Théorème 2] we have an isomorphism of $G(F)$-representations

$$\text{c-ind}_{K_{y,0}}^{G(F)}(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}) \simeq \text{c-ind}_{(K_{y,0} \cap M(F))U(F)}^{G(F)}(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}),$$

where $U(F)$ acts trivially on $V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}$. Therefore π is a subquotient of

$$\begin{aligned} \text{c-ind}_{(K_{y,0} \cap M(F))U(F)}^{G(F)}(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}) &\simeq \text{c-ind}_{P(F)}^{G(F)} \text{c-ind}_{(K_{y,0} \cap M(F))U(F)}^{M(F)U(F)}(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}) \\ &\simeq \text{Ind}_{P(F)}^{G(F)} \left(\text{c-ind}_{K_{y,0} \cap M(F)}^{M(F)}(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}) \right), \end{aligned}$$

where $\text{Ind}_{P(F)}^{G(F)}$ denotes the (unnormalized) parabolic induction. This is a contradiction to π being supercuspidal. □

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