

CUBIC FOURFOLDS WITH AN INVOLUTION

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ABSTRACT. There are three types of involutions on a cubic fourfold; two of anti-symplectic type, and one symplectic. Here we show that cubics with involutions exhibit the full range of behaviour in relation to rationality conjectures. Namely, we show a general cubic fourfold with a symplectic involution has no associated $K3$ surface and is conjecturally irrational. In contrast, a cubic fourfold with a particular anti-symplectic involution has an associated $K3$, and is in fact rational. We show such a cubic is contained in the intersection of all non-empty Hassett divisors; we call such a cubic Hassett maximal. We study the algebraic and transcendental lattices for cubics with an involution both lattice theoretically and geometrically.

1. INTRODUCTION

Cubic fourfolds are one of the central objects of interest in algebraic geometry. In particular, they are intensely studied with respect to rationality. It is expected that the general cubic fourfold is irrational; in contrast, many examples of cubic fourfolds are known to be rational [Has00], [Kuz16], [BRS19], [RS19]. The moduli and geometry of a cubic fourfold are controlled by a period map, similar to the theory of $K3$ surfaces. A standard approach to rationality questions is Hodge theory [Zar90], [Iza99], [Has00]. It is conjectured that a cubic fourfold is not rational if it does not have an associated $K3$ surface (i.e. the transcendental cohomology is not induced from a $K3$ surface) [Has00], [Kuz16]. In a different direction, cubic fourfolds are closely related to the study and construction of hyperkähler manifolds (e.g. see [BD85], [LSV17], [LLSvS17]).

More recently, there has been renewed interest in the classification of automorphisms of a cubic fourfold. The model for this study is the case of $K3$ surfaces; involutions of $K3$ surfaces have been completely classified by Nikulin [Nik79a]. This was extended to higher order symplectic automorphisms of $K3$ surfaces by Mukai and Kondō [Muk88], [Kon98]. We say an automorphism is symplectic if the automorphism preserves the symplectic form. Similar techniques were adapted to the case of hyperkähler manifolds of $K3^{[n]}$ type, and a classification of symplectic prime order automorphisms was achieved [Mon16]. Fixed loci of these involutions have also been identified [KMO18]. One can further adapt these techniques to the case of automorphisms of a cubic fourfold X . The Strong Torelli theorem [Voi86], [Zhe19] asserts that automorphisms of X are equivalent to (polarized) Hodge isometries of the middle cohomology $H^4(X, \mathbb{Z})$. Such an isometry in turn determines the lattice of algebraic primitive cycles $A(X)_{\text{prim}} := H^{2,2}(X, \mathbb{C}) \cap H^4(X, \mathbb{Z})_{\text{prim}}$ contained in the cubic X . Similarly to the hyperkähler situation, an automorphism of a cubic

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fourfold is symplectic if the induced automorphism on $H^4(X, \mathbb{Z})$ acts trivially on $H^{3,1}(X)$. Possible cyclic groups of symplectic automorphisms were classified [Fu16], followed by a complete classification of possible symplectic automorphism groups [LZ22]. Further the lattice $A(X)_{\text{prim}}$ was identified in each symplectic case.

The purpose of this paper is to study the case of involutions of a cubic fourfold X . Any automorphism of X can be lifted to one of the ambient projective spaces: there are three possible involutions of X , denoted ϕ_1, ϕ_2, ϕ_3 . Here ϕ_i is uniquely characterised by the dimension of the fixed linear subspaces of $\mathbb{P}^5$; ϕ_i fixes complementary linear spaces of codimension $i, 6-i$ respectively. The involutions ϕ_1, ϕ_3 are easily seen to be anti-symplectic, whereas ϕ_2 is symplectic. It is easy to write down the equation for a cubic fourfold with specified involution ϕ_i . Identifying the lattice $A(X)_{\text{prim}}$ and the transcendental lattice $T(X) := (A(X)_{\text{prim}})^\perp$ is more subtle; it roughly corresponds to identifying a basis of algebraic cycles. The main result is the classification of these lattices.

Theorem 1.1. *Let X be a general cubic fourfold with ϕ_i an involution of X fixing a linear subspace of $\mathbb{P}^5$ of codimension i . Then either:*

- (1) $i = 1$, ϕ_1 is anti-symplectic and $A(X)_{\text{prim}} \cong E_6(2)$, $T(X) \cong U^2 \oplus D_4^3$. The algebraic lattice is spanned by classes of planes contained in X ;
- (2) $i = 2$, ϕ_2 is symplectic and $A(X)_{\text{prim}} \cong E_8(2)$, $T(X) \cong A_2 \oplus U^2 \oplus E_8(2)$. The algebraic lattice is spanned by classes of cubic scrolls contained in X ;
- (3) $i = 3$, ϕ_3 is anti-symplectic and

$$A(X)_{\text{prim}} \cong M, T(X) \cong U \oplus A_1 \oplus A_1(-1) \oplus E_8(2).$$

The algebraic lattice contains an index 2 sublattice spanned by classes of planes contained in X .

Here M is the unique rank 10 even lattice obtained as an index 2 overlattice of $D_9(2) \oplus \langle 24 \rangle$, as described in Theorem 5.8.

Case (1) is studied in detail in [LPZ18]; the existence of the involution ϕ_1 is equivalent to the cubic having an Eckardt point. In case (2), the lattice $A(X)_{\text{prim}} \cong E_8(2)$ was identified as part of [LZ22] by purely lattice theoretic considerations. Here we show that this lattice is generated by classes corresponding to cubic scrolls contained in X ; in particular X contains 120 pairs of families of cubic scrolls. Case (3) was previously unknown, we approach this case using both lattice theoretic and geometric techniques.

Our interest in cubics with an involution is twofold. On the one hand, they are relevant to the study of birational transformations of hyperkähler manifolds, especially of *OG10* type by using the [LSV17] construction; see [Mar22] for a recent discussion in the case of involutions. One could also study induced automorphisms on the Fano variety of lines (see [Cam12]). On the other hand, they are interesting in terms of rationality questions for cubic fourfolds. Hassett [Has00] defined a countably infinite union of irreducible divisors $\mathcal{C}_d$ in the moduli space of cubic fourfolds, parametrising *special* cubic fourfolds (§2.3). For cubics $X \in \mathcal{C}_d$ for some known values of d [Has00], there exists an *associated* K3 surface whose primitive cohomology can be embedded Hodge-isometrically in $H^4(X, \mathbb{Z})_{\text{prim}}$ (up to a sign and a Tate twist) [Has00]. It was conjectured by Harris that if the transcendental cohomology $T(X)$ is not induced from a K3 surface, then X is not rational; evidence was given by Hassett [Has99, Has16]. Indeed, all known rational cubic fourfolds have an associated K3 surface. Kuznetsov [Kuz10a] proposed another criterion for

rationality via the derived category, which was verified for the known examples of rational cubic fourfolds [Kuz16]. This criterion is satisfied exactly when X has an associated $K3$ [AT14], [BLM⁺21]. A more generalised notion is an associated *twisted* $K3$ surface (S, α) , where $\alpha \in Br(S)$ (§6.1) [Huy17], [Bra20]. Our next result identifies which cubic fourfolds with an involution have an associated $K3$ or twisted $K3$ surface.

Theorem 1.2. *Let X be a cubic fourfold with an involution ϕ fixing a linear subspace of $\mathbb{P}^5$ of codimension i .*

- (1) *For the symplectic involution $\phi := \phi_2$, X does not have an associated $K3$ surface, or a visible twisted $K3$ surface.*
- (2) *For the anti-symplectic involution $\phi := \phi_1$, X has an associated twisted $K3$ surface (S, α) for $\alpha \in Br(S)_2$, but does not have an associated $K3$ surface.*
- (3) *For the anti-symplectic involution $\phi := \phi_3$, X has an associated $K3$ surface.*

Remark 1.3. Cubic fourfolds with large algebraic lattice

$$A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z})$$

have rich geometry, and thus are natural candidates for rationality questions. Particularly interesting are those that do not have an associated $K3$, as they are potential counterexamples to the Hassett and Kuznetsov rationality conjectures. Cubic fourfolds with automorphisms naturally have large algebraic lattices, but unfortunately very few symmetric cubics do not have an associated $K3$ surface. This follows from [Laz21, Proposition 2.9]; if $\text{rank}(A(X)) > 10$, the transcendental lattice $T(X)$ embeds into the $K3$ lattice. Further, if the group of symplectic automorphisms of X is not trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$, then there exists an associated $K3$ surface [Ouc21]. Thus the case of the involutions ϕ_1, ϕ_2 is rather special in this sense; conjecturally irrational, but with large algebraic lattices.

Cases (1) and (2) of Theorem 1.1 are conjecturally irrational, whereas a cubic X with involution ϕ_3 is potentially rational. We indeed verify rationality by examining which Hassett divisors such a cubic fourfold belongs to.

Definition 1.4. We say that a cubic fourfold X is *Hassett maximal* if

$$X \in \bigcap_{\substack{d > 6 \\ d \equiv 0, 2 \pmod{6}}} \mathcal{C}_d.$$

We denote the locus of Hassett maximal cubic fourfolds by $\mathcal{Z}$.

Lemma 1.5. *A Hassett maximal cubic fourfold is rational.*

Proof. A Hassett maximal cubic fourfold necessarily belongs to the divisor $\mathcal{C}_{14}$; we can conclude that such a cubic is rational by results of [BD85] and the fact that rationality specialises in families [KT19]. $\square$

For a cubic X with the involution ϕ_3 , we describe the lattice $A(X)_{\text{prim}} \cong M$ explicitly. We can consider the moduli space of cubic fourfolds with the involution ϕ_3 as M -polarized cubic fourfolds [YY21]. Denote this moduli space $\mathcal{M}_{\phi_3}$; it is irreducible and 10-dimensional. It turns out that $\mathcal{M}_{\phi_3}$ is contained in the locus of Hassett maximal cubic fourfolds.

Corollary 1.6. *Let $\mathcal{M}_{\phi_3}$ be the moduli space of cubic fourfolds with the involution of type ϕ_3 . Then $\mathcal{M}_{\phi_3}$ is contained in the locus of Hassett maximal cubic fourfolds $\mathcal{Z}$. In particular, $[X] \in \mathcal{M}_{\phi_3}$ is rational.*

The intersection of all Hassett divisors $\mathcal{C}_d$ first appears in [YY21]. They exhibit a component of dimension at least 13 by lattice theoretic considerations. However, the only geometric example of a Hassett maximal cubic, identified in [YY21], is the Fermat cubic. Our example of cubic fourfolds with anti-symplectic involution of type ϕ_3 gives a 10 dimensional geometrically meaningful locus of Hassett maximal cubics.

Structure of the paper. In §2, we recall the necessary results on automorphisms of cubic fourfolds and set up notation. In §3 we investigate the geometry of a cubic with an involution. We begin the proof of Theorem 1.1 by identifying some algebraic cycles contained in a cubic fourfold X with an involution. Exploiting the geometry of the involutions, we show that the lattice $A(X)$ is generated (over $\mathbb{Q}$) by the classes of planes in the case of an anti-symplectic involution. In §4, we show that in contrast, a general cubic with symplectic involution cannot contain a plane, and instead we describe $A(X)_{\text{prim}}$ in terms of cubic scrolls.

In §5 we restrict our study to the remaining anti-symplectic involution ϕ_3 . We identify $A(X)_{\text{prim}}, T(X)$ by first identifying a primitive sublattice generated by differences of classes of planes inside X . Using Nikulin's theory of overlattices (see Appendix A) we show the existence of a class $x \in A(X)$ whose existence is not geometrically obvious. This allows us to identify both $A(X)$ and $A(X)_{\text{prim}}$. Finally in §6 we discuss the implications of rationality for this family of cubic fourfolds. Some relevant results concerning lattices are reviewed in Appendix A. Appendix B discusses the intersection of two cubic scrolls from a geometric perspective.

2. PRELIMINARIES

In this section we review the relevant results that will allow us to study the involutions of cubic fourfolds. In §2.1 we review the period map for cubic fourfolds, and in §2.2 the Torelli Theorem for cubic fourfolds X , and reduce the classification of involutions of X to the classification of involutions of Hodge structures. This is essentially a lattice theoretic question; we follow the approach as in the case of K3 surfaces (Nikulin, Mukai, Kondō and others). Next, in §2.4 we review lattice theoretic properties of involutions. In §2.5 we recall the geometric classification of involutions.

2.1. The period map for cubic fourfolds. Let X be a smooth cubic fourfold. The middle cohomology $H^4(X, \mathbb{Z})$ with the natural intersection pairing is the unique unimodular odd lattice of signature $(21, 2)$. Recall that we denote by $\eta_X \in H^4(X, \mathbb{Z})$ the square of the hyperplane class of X . The primitive cohomology $H^4(X, \mathbb{Z})_{\text{prim}} := \langle \eta_X \rangle^\perp$ carries a polarized Hodge structure of K3 type (i.e. Hodge numbers $(0, 1, 20, 1, 0)$). As a lattice, $H^4(X, \mathbb{Z})_{\text{prim}} \cong L$, where

$$L := (E_8)^2 \oplus U^2 \oplus A_2.$$

Denote by $\mathcal{M}$ the moduli space of smooth cubic fourfolds, constructed using GIT [Laz09]. Denote by $\mathcal{D}/\Gamma = \{x \in \mathbb{P}(L \otimes \mathbb{C}) \mid x^2 = 0, x \cdot \bar{x} < 0\}^+/\Gamma$ the global period domain for cubic fourfolds. Here the global monodromy group Γ is the subgroup of $O(L)$ that preserves the period domain $\mathcal{D}$ and acts trivially on the discriminant

group $A_L \cong \mathbb{Z}/3\mathbb{Z}$. Starting with a cubic fourfold X , we associate the Hodge structure on its middle cohomology, obtaining a period map:

$$\mathcal{P} : \mathcal{M} \rightarrow \mathcal{D}/\Gamma.$$

In order to discuss the image of this map, we define two divisors in $\mathcal{D}/\Gamma$.

Definition 2.1.

- (1) A norm 2 vector $v \in L$ is called a **short root**. The set of short roots in L determines a Γ -invariant hyperplane arrangement $\mathcal{H}_6$ in $\mathcal{D}$. Let $\mathcal{C}_6 := \mathcal{H}_6/\Gamma \subset \mathcal{D}/\Gamma$ be the associated divisor.
- (2) A norm 6 vector $v \in L$ with divisibility 3 is called a **long root**. The set of long roots in L determines a Γ -invariant hyperplane arrangement $\mathcal{H}_2$ in $\mathcal{D}$. Let $\mathcal{C}_2 := \mathcal{H}_2/\Gamma \subset \mathcal{D}/\Gamma$.

The two divisors $\mathcal{C}_2, \mathcal{C}_6$ above are the complement of the image of the period map. Geometrically, $\mathcal{C}_6$ corresponds to singular cubic fourfolds, whereas $\mathcal{C}_2$ corresponds to degenerations of cubics to the secant to Veronese surface in $\mathbb{P}^5$. We recall that cubic fourfolds contained in $\mathcal{C}_2$ are determinantal.

Theorem 2.2 (Voisin, Hassett, Laza, Looijenga). *The period map for cubic fourfolds is an isomorphism*

$$\mathcal{P} : \mathcal{M} \rightarrow \mathcal{D}/\Gamma \setminus (\mathcal{C}_2 \cup \mathcal{C}_6).$$

2.2. Strong Torelli and automorphisms. Let $\phi \in \text{Aut}(X)$ be an automorphism of a cubic fourfold X . By considering the induced action ϕ^* on $H^4(X, \mathbb{Z}) \cong L$, we obtain a map

$$\text{Aut}(X) \rightarrow O(L).$$

We note that this map is injective (combine [JL17, Prop. 2.12], [MM64]). Further, the Strong Global Torelli Theorem [Zhe19] holds for cubic fourfolds. In other words, any isomorphism between polarized Hodge structures of two smooth cubic fourfolds is induced by a unique isomorphism between the cubic fourfolds themselves.

Proposition 2.3 (Strong Global Torelli Theorem). *Let X_1, X_2 be two smooth cubic fourfolds. Assume that there is an isomorphism $\phi : H^4(X_2, \mathbb{Z}) \xrightarrow{\cong} H^4(X_1, \mathbb{Z})$ of polarized Hodge structures. Then there exists a unique isomorphism $f : X_1 \xrightarrow{\cong} X_2$ such that $\phi = f^*$. In particular, for any smooth cubic fourfold X ,*

$$\text{Aut}(X) \cong \text{Aut}_{HS}(X, \eta_X),$$

where $\text{Aut}_{HS}(X, \eta_X)$ denotes the group of Hodge isometries fixing the class η_X .

Thus it is natural to study automorphisms of a cubic fourfold via the induced Hodge isometry of $H^4(X, \mathbb{Z})_{\text{prim}}$.

Definition 2.4. We say an automorphism $\phi \in \text{Aut}(X)$ is **symplectic** if ϕ acts trivially on $H^{3,1}(X)$. Otherwise, ϕ is non-symplectic (or anti-symplectic in the case of involutions).

Let X be a smooth cubic fourfold and consider the full automorphism group $\text{Aut}(X)$. The induced action of $\text{Aut}(X)$ on $H^{3,1}(X)$ gives a character $\chi : \text{Aut}(X) \rightarrow \mathbb{C}^\times$. The kernel is the group of symplectic automorphisms, denoted by G . The image of χ is a cyclic group, denoted by $\bar{G}$. We have the following exact sequence of groups:

$$1 \rightarrow G \rightarrow \text{Aut}(X) \rightarrow \bar{G} \rightarrow 1.$$

Proposition 2.5 ([LZ22]). *A prime factor of the order of the automorphism group of a smooth cubic fourfold can only be 2, 3, 5, 7 or 11. Furthermore, a non-symplectic automorphism of prime order can only have order 2 or 3.*

2.3. Moduli of cubics with automorphisms. Let A be a finite group, and let $\mathcal{M}_A$ be the irreducible subvariety parametrising smooth cubic fourfolds with a specified action of A , constructed via GIT (see [YZ20, §2] for details). One can construct a period map for $\mathcal{M}_A$ in the following way. Let $X \in \mathcal{M}_A$ and consider the induced action of A on $H^4(X, \mathbb{C})$, and let ξ be the character corresponding to $H^{3,1}(X)$. Denote by $H^4(X, \mathbb{C})_\xi$ the ξ -eigenspace; it admits a natural hermitian form h induced by the intersection pairing of signature $(n, 2)$ if $\xi = \bar{\xi}$, $(n, 1)$ otherwise.

Theorem 2.6 ([YZ20]). *The Hodge structure on $H^4(X, \mathbb{C})_\xi$ gives an algebraic isomorphism*

$$\mathcal{P}_A : \mathcal{M}_A \xrightarrow{\cong} (\mathcal{D} \setminus \mathcal{H})/\Gamma.$$

Here $\mathcal{D}$ is a complex hyperbolic ball if the signature is $(n, 1)$; a type IV symmetric domain otherwise. The group Γ is an arithmetic group acting properly discontinuously on $\mathcal{D}$, and $\mathcal{H}$ is a Γ -invariant hyperplane arrangement in $\mathcal{D}$.

By the result above, if $A \subset \text{Aut}(X)$ is a group of symplectic automorphisms of X , then $\mathcal{D}$ is a type IV symmetric domain. This is not the case for non-symplectic automorphisms; $\mathcal{D}$ is a complex hyperbolic ball in general. In the case of anti-symplectic involutions, $\xi = \bar{\xi}$, and so the resulting period domain is still a type IV symmetric domain.

Corollary 2.7. *Let $\mathcal{M}_\phi$ be the moduli space of smooth cubic fourfolds with an anti-symplectic involution ϕ . Then we have an isomorphism*

$$\mathcal{P}_\phi : \mathcal{M}_\phi \xrightarrow{\cong} (\mathcal{D} \setminus \mathcal{H})/\Gamma$$

as above, where $\mathcal{D}$ is a component of $\{x \in \mathbb{P}((L \otimes \mathbb{C})_-) \mid x^2 = 0, x \cdot \bar{x} < 0\}$. Here $(L \otimes \mathbb{C})_-$ denotes the -1 eigenspace of the induced action ϕ on $L \otimes \mathbb{C} \cong H^4(X, \mathbb{Z})_{\text{prim}} \otimes \mathbb{C}$.

Similarly to the theory developed by Dolgachev [Dol96] for K3 surfaces, one can also define $\mathcal{M}_\phi$ as a moduli space of lattice polarized cubic fourfolds. More precisely, let M be a positive definite lattice with a fixed primitive embedding into the primitive lattice L . Assume further that M does not contain any short or long roots. One can define a moduli space $\mathcal{M}_M$ of cubics such that $M \subset H^{2,2}(X) \cap H^4(X, \mathbb{Z})_{\text{prim}} \subset H^4(X, \mathbb{Z})_{\text{prim}} \cong L$ and the composition $M \subset L$ is equivalent to the fixed embeddings. In other words, $\mathcal{M}_M$ is the moduli space of cubics with primitive algebraic lattice equivalent to the prescribed lattice M . We say such a cubic is M -polarized [YY21]. Up to passing to a normalization, $\mathcal{M}_M$ is (possibly the complement of some divisors in) a locally symmetric variety $\mathcal{D}_M/\Gamma_M$, where $\mathcal{D}_M$ is the type IV domain associated with the transcendental lattice $T = M^\perp_L$. Thus $\dim \mathcal{M}_M = 20 - \text{rank}(M)$. Further, if $M \subset M' \subset L$ (all primitive embeddings), then $\mathcal{M}_{M'} \subset \mathcal{M}_M$; the more algebraic cycles contained in X , the smaller the moduli. To construct $\mathcal{M}_\phi$, we apply the above construction for $M = (L \otimes \mathbb{C})^\perp$.

In particular, one can consider the loci of *special* cubic fourfolds [Has00] as lattice polarized cubic fourfolds.

Definition 2.8. A cubic fourfold X is **special** if it admits a surface S not homologous to a complete intersection, i.e. $\mathbb{Z}[\eta_X] \subsetneq A(X)$. A **labeling** of X consists of a rank 2 saturated sublattice $K \subset A(X)$ containing η_X . The discriminant of K is the determinant of the intersection form of K , denoted by d .

Theorem 2.9 ([Has00]). *Let $\mathcal{M}$ denote the moduli of smooth cubic fourfolds. Let $\mathcal{C}_d \subset \mathcal{M}$ denote special cubic fourfolds admitting a labeling of discriminant d . Then $\mathcal{C}_d$ is non-empty if and only if $d > 6, d \equiv 0, 2 \pmod{6}$. Moreover, $\mathcal{C}_d$ is an irreducible divisor.*

2.4. Case of involutions: Arithmetic. Let L be any even lattice. An involution of L determines two eigenspaces

$$L_{\pm} := \{v \in L \mid \phi(v) = \pm v\},$$

where L_- and L_+ are primitive, mutually orthogonal lattices. The group

$$H := L/(L_+ \oplus L_-)$$

is a 2-elementary group, i.e. $H = (\mathbb{Z}/2\mathbb{Z})^a$ for some $a \geq 0$, and admits embeddings into the discriminant groups A_{L_+} and A_{L_-} such that the diagonal embedding $H \subset A_{L_+} \oplus A_{L_-}$ is isotropic with respect to the finite quadratic form $q_{L_+} \oplus q_{L_-}$ (see [Nik79b, Sect. 1.5] for more details).

From now on, we consider $L \cong H^4(X, \mathbb{Z})_{\text{prim}}$; this is an even lattice of signature $(20, 2)$ and discriminant group $A_L \cong \mathbb{Z}/3\mathbb{Z}$. In contrast to the $K3$ case, this lattice is not unimodular - however the strategy of [Nik79b] can be adapted to our situation.

We define the following sublattices of $H^4(X, \mathbb{Z})$:

$$A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z}),$$

$$T(X) := (A(X))^{\perp}_{H^4(X, \mathbb{Z})},$$

the lattice of algebraic cycles, and the transcendental lattice, respectively. Note that the integral Hodge conjecture holds for cubic fourfolds [Voi13, Thm 1.4], so every class $v \in A(X)$ is indeed algebraic. We denote by $A(X)_{\text{prim}}$ the lattice $A(X) \cap H^4(X, \mathbb{Z})_{\text{prim}} \subset H^4(X, \mathbb{Z})_{\text{prim}}$.

Proposition 2.10. *Let X be a cubic fourfold and $\phi \in \text{Aut}(X)$. Let r denote the rank of L_- .*

- (1) *The coinvariant sublattice L_- is 2-elementary, with discriminant group of length $0 \leq a \leq \min\{r, 22 - r\}$. Thus*

$$A_{L_-} \cong (\mathbb{Z}/2\mathbb{Z})^a; A_{L_+} \cong \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^a.$$

- (2) *If ϕ is symplectic, L_- has signature $(r, 0)$.*
- (3) *If ϕ is anti-symplectic, L_- has signature $(r - 2, 2)$.*

Proof. Since $\phi \in \text{Aut}(X) \cong \text{Aut}_{HS}(X, \eta_X)$, ϕ acts as the identity on $\langle \eta_X \rangle$, and thus acts trivially on $A_{\langle \eta_X \rangle} \cong \mathbb{Z}/3\mathbb{Z} \cong A_L$. We see that $H^4(X, \mathbb{Z})_- \cong L_-$, and since $H^4(X, \mathbb{Z})$ is a unimodular lattice, L_- is 2-elementary by Lemma A.4. The description of the discriminant group of L_+ follows from Proposition A.1.

Suppose now that ϕ is symplectic; by definition, ϕ acts trivially on $T(X)$, thus $L_- \subset A(X)$. The class $\eta_X \in A(X)$ is clearly invariant under ϕ , and so $L_- \subset A(X) \cap L$. The claim on the signature follows. Similarly, if ϕ is anti-symplectic, then $T(X) \subset L_-$, but η_X is invariant. $\square$

Remark 2.11. Let X be a cubic fourfold with an anti-symplectic involution. Then $L_+ \subset A(X)_{\text{prim}}$, and we can consider such cubics as latticed polarized cubic fourfolds. Thus the moduli space of cubics with the fixed involution can be constructed as $\mathcal{M}_{A(X)}$. We say a cubic fourfold with the specified involution is general if $L_+ \cong A(X)_{\text{prim}}$. The case of ϕ_1 was studied in detail in [LPZ18], and in the symplectic case [YZ20].

2.5. Case of involutions: Geometry. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d . Denote by $\text{Lin}(X)$ the subgroup of automorphisms of X consisting of elements induced by projective transformations of the ambient projective space leaving X invariant. By [MM64, Theorem 1 and 2], assuming $n \geq 2, d \geq 3$ we have that $\text{Aut}(X) = \text{Lin}(X)$, except in the case $n = 2, d = 4$. Moreover, this group is finite (again excluding the case $n = 2, d = 4$).

Using the fact that any automorphism can be linearised, one can obtain a complete classification of cyclic automorphisms of cubic fourfolds [GAL11], and identify which are symplectic [Fu16]. In particular, for involutions we have the following classification.

Proposition 2.12 ([GAL11], [Fu16]). *Let $X = V(F)$ be a smooth cubic fourfold in $\mathbb{P}^5$, $\phi \in \text{Aut}(X)$ an involution of X . Applying a linear change of co-ordinates, we can diagonalize ϕ , so that*

$$\phi : \mathbb{P}^5 \rightarrow \mathbb{P}^5, [x_0, \dots, x_5] \mapsto [(-1)^{a_0}x_0, \dots, (-1)^{a_5}x_5],$$

with $a_i \in \{0, 1\}$. Let $a := (a_0, \dots, a_5)$, and $d = \dim \mathcal{M}_\phi$. Then either:

- (1) $a = (0, 0, 0, 0, 0, 1)$ and $\phi := \phi_1$ fixes linear subspaces of $\mathbb{P}^5$ of codimension 1 and 5 respectively. We have that ϕ_1 is anti-symplectic, $d = 14$, and

$$F = g(x_0, x_1, x_2, x_3, x_4) + x_5^2 l_1(x_0, x_1, x_2, x_3, x_4).$$

- (2) $a = (0, 0, 0, 0, 1, 1)$ and $\phi := \phi_2$ fixes linear subspaces of $\mathbb{P}^5$ of codimension 2 and 4 respectively. We have that ϕ_2 is symplectic, $d = 12$ and

$$F = g(x_0, x_1, x_2, x_3) + x_4^2 l_1(x_0, x_1, x_2, x_3) \\ + x_4 x_5 l_2(x_0, x_1, x_2, x_3) + x_5^2 l_3(x_0, x_1, x_2, x_3).$$

- (3) $a = (0, 0, 0, 1, 1, 1)$ and $\phi := \phi_3$ fixes two linear subspaces of $\mathbb{P}^5$ of codimension 3. We have that ϕ_3 is anti-symplectic, $d = 10$ and

$$F = g(x_0, x_1, x_2) + x_0 q_0(x_3, x_4, x_5) + x_1 q_1(x_3, x_4, x_5) + x_2 q_2(x_3, x_4, x_5).$$

Here l_i, q_i, g denote homogeneous polynomials of degrees 1, 2 and 3, respectively.

Proof. The classification and dimension of the families of such cubics can be found in either [GAL11] or [Fu16]. Let $\Omega = \sum_i (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_5$. A basis for $H^{3,1}(X)$ is given by $\{\text{Res}_{\frac{\Omega}{F^2}}\}$, where F is the defining equation for X . One can check whether this is invariant or not for each involution above. $\square$

Corollary 2.13. *Let X be a general cubic fourfold with an involution ϕ . Then:*

- (1) *If ϕ is anti-symplectic, the sublattice $A(X)_{\text{prim}} \cong L_+ \hookrightarrow L$ is a positive definite lattice of rank*

$$r(L_+) = \begin{cases} 6, & \text{if } \phi = \phi_1, \\ 10, & \text{if } \phi = \phi_3. \end{cases}$$

- (2) If ϕ is symplectic, the sublattice $A(X)_{\text{prim}} \cong L_- \hookrightarrow L$ is a positive definite lattice of rank 8.

Proof. The claims follow immediately from Proposition 2.10 and Proposition 2.12. $\square$

3. GEOMETRY OF ANTI-SYMPLECTIC INVOLUTIONS

The purpose of this paper is to identify both the lattice of algebraic cycles and transcendental lattice for a generic cubic fourfold with an involution. We approach this problem via a mixture of geometric considerations and lattice theory. In this section, we prove that in the anti-symplectic case the algebraic lattice is spanned (over $\mathbb{Q}$) by the classes of planes. It is straightforward to see that a cubic fourfold with an anti-symplectic involution contains many invariant planes.

Lemma 3.1. *Let X be a general cubic fourfold, ϕ an anti-symplectic involution. Then X contains an invariant plane P . Further, (X, P) determines a plane sextic curve C_P and a theta-characteristic on C_P . More precisely:*

- (1) if $\phi = \phi_1$, $C_P = L \cup Q$ where L is a line, Q is a smooth quintic curve, κ a non-trivial odd theta characteristic on Q .
- (2) if $\phi = \phi_3$ and $C_P = C \cup D$ where C, D are smooth cubic curves, κ a non-trivial two torsion line bundle on D .

Conversely, such a triple (L, Q, κ) (respectively (C, D, κ)) determines a cubic fourfold with an anti-symplectic involution of type (X, ϕ_1) (respectively (X, ϕ_3)).

Proof. Keeping the notation of Proposition 2.12, we see that we can choose coordinates for $\mathbb{P}^5$ such that ϕ is either ϕ_1 or ϕ_3 and the equation for X is as in the theorem.

For the involution ϕ_1 , studied in [LPZ18], X contains the cone over a cubic surface S , where S is fixed by the involution. Each of the 27 lines on the cubic surface gives a plane contained in X that is invariant under ϕ_1 .

In the case of ϕ_3 , one can see from the equation of X that the plane $V(x_0, x_1, x_2)$ is contained in X , and is point-wise fixed by ϕ_3 .

Choose this invariant plane P . The linear projection with center P expresses X as a quadric fibration over $\mathbb{P}^2$; indeed, $\mathbb{P}^2$ parametrises the space of $\mathbb{P}^3$ -sections of X containing P (see for example [Voi86]). The blow up $\mathbb{P}_P^5$ of the ambient projective space along P gives a commutative diagram:

$$\begin{array}{ccc} X_P & \longrightarrow & \mathbb{P}_P^5 \\ & \searrow \pi & \downarrow \tau \\ & & \mathbb{P}^2 \end{array}$$

where X_P is the strict transform of X , and τ and π are the linear projections with center P . The generic fiber of π is a smooth quadric surface; the degenerate fibers of π are parametrised by a plane sextic C_P , the discriminant curve, where $C_P = V(\det A_i)$ for a matrix A_i depending on ϕ_i for $i = 1, 3$ respectively. A simple analysis of the cases gives:

$$A_1 = \begin{pmatrix} l_3 & l_{34} & 0 & q_3 \\ l_{34} & l_4 & 0 & q_4 \\ 0 & 0 & l_1 & 0 \\ q_3 & q_4 & 0 & f \end{pmatrix}; A_3 = \begin{pmatrix} l_3 & l_{34} & l_{35} & 0 \\ l_{34} & l_4 & l_{45} & 0 \\ l_{35} & l_{45} & l_5 & 0 \\ 0 & 0 & 0 & g \end{pmatrix},$$

where each l_i, l_{ij} is linear, q_i quadratic and f, g cubic polynomials. We see that C_P is the union of two curves of the correct degree respectively. We discuss the case of ϕ_3 in detail; the other case is similar.

Let X be a cubic fourfold with the involution ϕ_3 - we can rewrite the equation of X as:

$$(3.1) \quad l_3x_3^2 + l_4x_4^2 + l_5x_5^2 + 2l_{34}x_3x_4 + 2l_{35}x_3x_5 + 2l_{45}x_4x_5 + g = 0,$$

where l_i, l_{ij}, g are homogeneous polynomials in x_0, x_1, x_2 , with l_i, l_{ij} linear and g degree 3. The discriminant sextic C_P is given by the determinant of the matrix A_3 , and so C_P is the union of two smooth cubic plane curves C and D , where

$$C = V(g),$$

$$D = V(l_3l_4l_5 - l_{45}^2l_3 + 2l_{34}l_{35}l_{45} - l_{34}^2l_5 - l_{35}^2l_4).$$

Since D is a determinantal curve, by [Bea00, Prop 4.2] this determines an even theta characteristic κ on the corresponding curve. In particular, since D is an elliptic curve $\kappa^{\otimes 2} = \mathcal{O}_D$. Conversely, suppose we have the triple (C, D, κ) . Then by [Bea00, Prop 4.2] we can write D as the determinant of a 3×3 matrix of linear forms in 3 variables. Using this matrix and $C = V(g)$, we can write an equation for a cubic fourfold of the form of equation (3.1). This is clearly invariant under the involution ϕ_3 . $\square$

The general fiber of $\pi : X_P \rightarrow \mathbb{P}^2$ is a smooth quadric surface, and the fiber over a smooth point of the discriminant curve C_P is a quadric cone. We claim that the fiber over a node of C_P is the union of two planes. This follows from the following elementary lemma, we omit the proof.

Lemma 3.2. *Let $P \subset X$ be a cubic fourfold containing a plane, $\pi : X_P \rightarrow \mathbb{P}^2$ be the quadric fibration obtained via projection from P . Suppose that $\pi^{-1}(p)$ was a plane with multiplicity 2 (a double plane). Then X is a singular cubic fourfold.*

3.1. The involution ϕ_1 . One sees that we have 5 pairs of planes corresponding to the singular points of the discriminant curve $C_P = L \cup Q$. In fact, a cubic fourfold X with the involution ϕ_1 contains more planes, and we have the following result:

Theorem 3.3 ([LPZ18]). *For a cubic fourfold with an involution (X, ϕ_1) as in [LPZ18], the following hold:*

- (1) X is geometrically equivalent to a pair (Y, H) where Y is a cubic threefold, H a hyperplane in $\mathbb{P}^4$.
- (2) X contains 27 planes Π_i passing through the Eckardt point p , corresponding to the 27 lines on the cubic surface $Y \cap H$;
- (3) The primitive algebraic cohomology $A(X)_{\text{prim}} \cong E_6(2)$ (spanned by classes $[\Pi_i] - [\Pi_j]$);
- (4) The transcendental cohomology of X is $T \cong (D_4)^3 \oplus U^2$.

Their method uses the description of invariant surface classes as cones over the 27 lines on the cubic surface $S = Y \cap H$ with vertex p . Indeed, the plane P corresponds to a line on the cubic surface, and each pair of planes corresponds to the residual lines of a tritangent plane containing this line.

3.2. The involution ϕ_3 . Denote by $\{F_i, F'_i\}_{i=1}^9$ the 9 pairs of planes occurring as the fibers of $\pi : X_P \rightarrow \mathbb{P}^2$ over the singular points of the discriminant curve $C_P = C \cup D$. Let the corresponding classes in $H^4(X, \mathbb{Z})$ be denoted by $[P], [F_i], [F'_i] \in A(X)$ for $i = 1, \dots, 9$. Notice that

$$\eta_X \sim [P] + [F_i] + [F'_i].$$

Proposition 3.4. *Let X be a cubic fourfold with involution ϕ_3 . Then:*

- (1) *X contains at least 19 invariant distinct planes $P, \{F_i, F'_i\}_{i=1}^9$, where additionally P is point-wise fixed by ϕ_3 .*
- (2) *The classes $\{\eta_X, [P], [F_1], \dots, [F_9]\} \subset H^{2,2}(X)$ span $A(X) \otimes \mathbb{Q}$.*

Proof. One can check that ϕ_3 leaves each plane F_i, F'_i invariant by considering equations for these planes.

We have identified 11 linearly independent algebraic classes, thus $\text{rank} A(X) \geq 11$. By Corollary 2.13, we see this must be equality. Thus $A(X)$ is determined up to finite index by these classes. $\square$

The identification of the lattices $A(X), A(X)_{\text{prim}}$ and $T(X)$ is done in §5. In contrast to the previous case, the lattice spanned by the differences of two planes forms an index 2 sublattice of $A(X)_{\text{prim}}$.

4. GEOMETRY OF SYMPLECTIC INVOLUTIONS

A cubic fourfold X with the symplectic involution ϕ_2 was studied briefly as part of [LZ22], where they identified $A(X)_{\text{prim}} \cong E_8(2)$ via lattice theory. The geometry was not explored - here we make a couple of complementary remarks, distinguishing this case from the anti-symplectic situation. In particular, we prove that such a cubic cannot contain a plane, and further the lattice $A(X)_{\text{prim}}$ is generated by classes that correspond to cubic scrolls contained in X .

4.1. Non-existence of planes. Let X be a general cubic fourfold with a symplectic involution $\phi := \phi_2$; we first show that X contains no planes. We recall a Hodge theoretic characterisation of the condition to contain a plane.

Lemma 4.1 ([Voi86]). *Let P_1, P_2 be planes contained in X , and denote by $p_i = [P_i] \in H^4(X, \mathbb{Z})$. Then:*

- (1) $\eta_X \cdot p = 1$,
- (2) $p^2 = 3$,
- (3) $p_1 \cdot p_2 = \begin{cases} 0 & \text{if } P_1 \cap P_2 = \emptyset, \\ 1 & \text{if } P_1 \cap P_2 = \text{a point}, \\ -1 & \text{if } P_1 \cap P_2 = \text{a line}. \end{cases}$

This has the following important consequence:

Corollary 4.2. *Let $p \in H^4(X, \mathbb{Z})$ such that $p^2 = 3$, and $p \cdot \eta_X = 1$. Then p is represented by a unique plane.*

One can also detect the existence of a plane by studying the discriminant group of $A(X)_{\text{prim}}$.

Lemma 4.3. *Let X be a cubic fourfold containing a plane P . Then P determines a non-trivial class $\bar{\delta} \in A_{A(X)_{\text{prim}}}$ of order 3. Moreover, any two planes P_1, P_2 determine the same class, i.e. $\bar{\delta}_1 = \bar{\delta}_2 \in A_{A(X)_{\text{prim}}}$.*

Proof. Let p denote the class of the plane P in $H^4(X, \mathbb{Z})$. Consider the class $\delta := 3p - \eta_X$; then $\delta \in A(X)_{\text{prim}}$ with $\delta^2 = 24$. Let $\alpha \in A(X)_{\text{prim}}$; we have that $\alpha \cdot (3p - \eta) = 3\alpha \cdot p$. Thus $\text{div}_{A(X)_{\text{prim}}}(\delta) = 3$.

Note that δ is primitive: if $\delta = kw$ for some $w \in A(X)_{\text{prim}}, k \in \mathbb{Z}$, then k must divide $\text{div}_{A(X)_{\text{prim}}}(\delta) = 3$. On the other hand, $k^2 = 9$ does not divide 24; thus $k = 1$.

Thus δ is primitive of divisibility 3, and so $\delta^* = \frac{\delta}{3} \in A(X)_{\text{prim}}^*$, where

$$A(X)_{\text{prim}}^* = \{y \in A(X)_{\text{prim}} \otimes \mathbb{Q} \mid y \cdot x \in \mathbb{Z} \text{ for all } x \in A(X)_{\text{prim}}\}.$$

Let $\bar{\delta}$ be the image of δ^* in $A_{A(X)_{\text{prim}}}$; then $\bar{\delta}$ is a non-trivial element of order three.

Now suppose X contains two planes; denote their cohomology classes by p_1, p_2 . Then in turn this determines δ_1^*, δ_2^* as above. Note that $\delta_1^* - \delta_2^* = p_1 - p_2 \in A(X)_{\text{prim}}$; it follows that the images of δ_1^* and δ_2^* in $A_{A(X)_{\text{prim}}}$ coincide. $\square$

Corollary 4.4. *Let X be a general smooth cubic fourfold with the symplectic involution ϕ . Then X does not contain a plane.*

Proof. From [LZ22], the general such cubic has $A(X)_{\text{prim}} \cong E_8(2)$; the determinant is 2^8 . In particular, there are no non-trivial elements of $A_{A(X)_{\text{prim}}}$ of order 3. $\square$

4.2. Existence of cubic scrolls. Let $v \in A(X)_{\text{prim}} \cong E_8(2)$ be an element with $v^2 = 4$. Then $K = \langle \eta_X, v \rangle$ gives a labeling of X with determinant 12; the lattice $E_8(2)$ has 240 such elements. Indeed, the elements v correspond exactly to the roots of the unscaled lattice E_8 . Thus a general cubic fourfold with a symplectic involution belongs to the Hassett divisor $\mathcal{C}_{12}$; this is the closure of the locus of cubic fourfolds containing a cubic scroll. Equivalently, a general $[X] \in \mathcal{C}_{12}$ has a hyperplane section with at least 6 double points in linear position [HT10, Proposition 23].

Theorem 4.5. *Let X be a general cubic fourfold with the symplectic involution ϕ .*

- (1) *X contains 120 pairs of families of cubic scrolls $\{T_i, T'_i\}_{i=1}^{120}$ whose cohomology classes satisfy $[T_i] + [T'_i] = 2\eta_X$.*
- (2) *The lattice $A(X)_{\text{prim}} \cong E_8(2)$ is generated by classes $\alpha_i := [T_i] - \eta_X$.*

In order to prove Theorem 4.5, we will need to investigate the geometry of such a cubic fourfold X more closely. The involution ϕ fixes a line $l \subset X$, and a cubic surface $S := \Pi \cap X$, where $\Pi \cong \mathbb{P}^3$ is the complementary subspace of $\mathbb{P}^5$ also pointwise fixed by the involution (Proposition 2.12). Let $\pi : Bl_l X \rightarrow \Pi$ be the linear projection from l to the disjoint linear subspace Π ; this is a conic fibration. The discriminant locus parametrising singular fibers is given by:

$$\det \begin{pmatrix} l_1(x_0, \dots, x_3) & l_2(x_0, \dots, x_3) & 0 \\ l_2(x_0, \dots, x_3) & l_3(x_0, \dots, x_3) & 0 \\ 0 & 0 & g(x_0, \dots, x_3) \end{pmatrix} = 0,$$

and is thus the union of the fixed cubic surface $S = V(g) \subset \Pi$ and a quadric surface $Q = V(l_1 l_3 - l_2^2)$. Denote the intersection $S \cap Q = C$; this is then a genus 4 space curve, parametrising the fibers that are double lines.

Lemma 4.6. *Let X be a cubic fourfold with symplectic involution ϕ , l the pointwise fixed line contained in X . Let $H \subset \mathbb{P}^5$ be a general hyperplane containing l , and denote $Y := H \cap X$ the hyperplane section. Then Y is smooth, and the discriminant*

locus of the linear projection of Y from $l \subset Y$ is the union of a smooth conic Z and a smooth cubic plane curve E .

Proof. Notations as above. Since Π, l are complementary linear subspaces, $\Gamma := H \cap \Pi \cong \mathbb{P}^2$. Note that ϕ induces an involution $\phi|_H : H \rightarrow H$ whose fixed locus is $l \cup \Gamma$. The cubic threefold Y is invariant under the involution, and from the equation of Y one can check that Y is smooth. In particular, Y has equation

$$x_4^2 L_1(x_0, x_1, x_2) + 2x_4 x_5 L_2(x_0, x_1, x_2) + x_5^2 L_3(x_0, x_1, x_2) + G(x_0, x_1, x_2) = 0,$$

where L_i are linear and G has degree 3. Consider the restriction of $\pi : Bl_l X \rightarrow \Pi$ to the proper transform of Y ; we have a conic bundle $\pi_Y : Bl_l Y \rightarrow \Gamma$, with discriminant curve

$$\det \begin{pmatrix} L_1 & L_2 & 0 \\ L_2 & L_3 & 0 \\ 0 & 0 & G \end{pmatrix} = 0.$$

This is the union of the plane cubic $E := V(G(x_0, x_1, x_2)) = S \cap \Gamma$ and $Z := V(L_1 L_3 - L_2^2) := Q \cap \Gamma$; both are smooth for a general hyperplane H . $\square$

We claim that if we choose a hyperplane containing $l \subset H \subset \mathbb{P}^5$ such that the discriminant locus of π_Y consists of curves E, Z that are tangent at a point p , then $Y := X \cap H$ is singular.

Lemma 4.7. *Let Y be a cubic threefold with an involution fixing a line l and consider the discriminant locus $E \cup Z$ of $\pi_Y : Bl_l Y \rightarrow \mathbb{P}^2$ as above. Suppose that the conic Z is tangent to the cubic curve E at a point $p \in \mathbb{P}^2$. Then Y has (at least) two nodes interchanged by the involution.*

Proof. Without loss of generality, assume that Z is given by $x_0 x_1 - x_2^2 = 0$, and suppose that Z and E are tangent at the point $p = [p_0, p_1, p_2]$. Note that this implies that:

$$(4.1) \quad \frac{\partial G}{\partial x_0}(p) = p_1, \frac{\partial G}{\partial x_1}(p) = p_0, \frac{\partial G}{\partial x_2}(p) = -2p_2.$$

The equation for Y is given as

$$x_4^2 x_0 + 2x_4 x_5 x_2 + x_5^2 x_1 + G(x_0, x_1, x_2) = 0.$$

Taking partial derivatives, we see that Y has two nodes, interchanged by the involution, at the points $[p_0, p_1, p_2, \pm\sqrt{-p_1}, \pm\sqrt{-p_0}]$. $\square$

Proposition 4.8. *Let X be a general smooth cubic fourfold with a symplectic involution. Then X contains 120 pairs of families of cubic scrolls T_i, T'_i whose classes satisfy $[T_i] + [T'_i] = 2\eta_X$.*

Proof. Notations as above. Let $C \subset \Pi$ be the intersection of $Q \cap S \subset \Pi \subset \mathbb{P}^5$, the genus 4 curve as above. Then there are 120 tritangent planes to C , denoted by $\Gamma_i \subset \Pi \cong \mathbb{P}^3$ with $\Gamma_i \cong \mathbb{P}^2$. Since the intersection points of $\Gamma_i \cap C$ are the intersection points $\Gamma \cap Q \cap S$, we must have that the conic $Z := \Gamma \cap Q$ is tangent to the cubic curve $E := \Gamma \cap S$ in three points. Let $H_i = \text{span}\{l, \Gamma_i\} \subset \mathbb{P}^5$; by Lemma 4.7, $Y_i := H_i \cap X$ has three pairs of nodes $\{p_i, q_i\}$ with $\iota(p_i) = q_i$. Using the involution, one can check that these nodes are in general position, and so the existence of the cubic scrolls follows by [HT10, Proposition 23]. $\square$

Remark 4.9. The existence of a cubic scroll implies the existence of a rational curve on the Fano variety of lines $F(X)$ parametrising the lines in the ruling of T_i . The induced symplectic involution on $F(X)$ was studied in [Cam12].

Proposition 4.10. *Let M be the lattice spanned by $\{\alpha_i\}_{i=1}^{120}$, where $\alpha_i := [T_i] - \eta_X$. Then the lattice M is isomorphic to $A(X)_{\text{prim}} \cong E_8(2)$.*

In order to prove Proposition 4.10, we must first look at the possible intersection numbers of two cubic scrolls contained in a cubic fourfold. Note two homologous cubic scrolls are necessarily contained in the same hyperplane section (see [Has96, Lemma 2.11]).

Lemma 4.11. *Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold containing two non-homologous cubic scrolls T_1, T_2 . Then $[T_1] \cdot [T_2] = \tau$ for $\tau \in \{1, 3, 5\}$.*

Proof. The cubic fourfold X has a sublattice $K_\tau := \langle \eta_X, T_1, T_2 \rangle \subset A(X)$, with Gram matrix:

$$\begin{array}{ccccc} & \eta_X & T_1 & T_2 & \\ \eta_X & 3 & 3 & 3 & \\ T_1 & 3 & 7 & \tau & \\ T_2 & 3 & \tau & 7 & \end{array}$$

for some $\tau \in \mathbb{Z}$ depending on X . The lattice $A(X)$ is positive definite; it follows that the discriminant of K_τ should be positive. We see that $d(K_\tau) = 3(7 - \tau^2 + 6\tau)$, the only values ensuring this is positive are $\tau \in \{0, 1, 2, 3, 4, 5, 6\}$.

Let $\alpha_1 := \eta_X - T_1, \alpha_2 := \eta_X - T_2$; this is a basis for $\langle \eta_X \rangle_{M_\tau}^\perp$. Note that $\alpha_i^2 = 4$, and $\alpha_1 \cdot \alpha_2 = \tau - 3$. Let $v = x\alpha_1 + y\alpha_2$ with $x, y \in \mathbb{Z}$. We will exclude $\tau = 0, 2, 4, 6$ by exhibiting either a short or long root in $\langle \eta_X \rangle^\perp \subset M_\tau$. We have that

$$(4.2) \quad v^2 = 2(2x^2 + 2y^2 + xy(\tau - 3)).$$

Let $\tau = 0$, then $v = \alpha_1 + \alpha_2$ has $v^2 = 2$. Similarly for $\tau = 6$, $v = \alpha_1 - \alpha_2$ is a short root. Now let $\tau = 2$; we see that $v = \alpha_1 + \alpha_2$ satisfies $v^2 = 6$. Note that $v = 2\eta_X - T_1 - T_2$; it is easy to check that v has divisibility 3, and is thus a long root. Similarly, for $\tau = 4$ we see that $v = \alpha_1 - \alpha_2$ is also a long root.

For the remaining values of τ , one can check that $\langle \eta_X \rangle^\perp \subset M_\tau$ contains no long or short roots by using standard Diophantine equation techniques. $\square$

Remark 4.12. Up to orientation, the possible intersection numbers τ correspond to the intersection numbers of two twisted cubics lying on a cubic surface (see Appendix B). Notice that the cubic scrolls were obtained by considering odd theta-characteristics of a genus 4 canonical curve. There is a notion of syzygetic/azygyetic theta-characteristics; $\tau = 2$ corresponds to when T_1, T_2 are obtained from syzygetic odd theta-characteristics (or syzygetic twisted cubics).

Proof of Lemma 4.10. Recall that M is the lattice spanned by the classes $\alpha_i := [T_i] - \eta_X, \alpha'_i = [T'_i] - \eta_X$. Since the classes $[T_i], [T'_i]$ satisfy $[T_i] + [T'_i] = 2\eta_X$, we see that $\alpha'_i = -\alpha_i$. Thus M is generated by the classes α_i . Each class is contained in $A(X)_{\text{prim}}$, and M is an even lattice. For each i we have $\alpha_i^2 = 4$, and so each class $\alpha_i \neq 0$. Suppose that $\alpha_i = \alpha_j$. This implies that $[T_i] = [T_j]$, the two cubic scrolls are homologous. By [Has96, Lemma 2.11] this implies that both T_i, T_j are contained in the same hyperplane section of X . By the construction in Proposition 4.8, T_i and T_j are contained in different hyperplane sections $H_i \neq H_j$, a contradiction. Thus we have 120 distinct classes $\{\alpha_i\}_{i=1}^{120}$ generating M .

We next show that the lattice $R := M(\frac{1}{2})$ is a well-defined integral lattice. For $i \neq j$ we see that

$$(4.3) \quad \alpha_i \cdot \alpha_j = \begin{cases} -2, \\ 0, \\ 2; \end{cases}$$

this follows from Lemma 4.11. Recall that $\alpha_i^2 = 4$; thus the integral lattice $R := M(\frac{1}{2})$ is well-defined.

The lattice R generated by the 120 classes α_i , where $(\alpha_i \cdot \alpha_j)_R := \frac{1}{2}(\alpha_i \cdot \alpha_j)_M$, and is positive definite. Since $(\alpha_i)_R^2 = 2$, we can conclude that R is a root lattice with 240 roots, corresponding to $\pm\alpha_i$. Note that the rank of M is at most 8. Since $M = R(2)$, it remains to show that $R \cong E_8$.

The lattice E_8 has 240 roots; suppose for contradiction that $R \not\cong E_8$. If R is an irreducible root system, then since $\text{rank}(R) \leq 8$, the number of roots of R is at most the number of roots of D_8 , which is 112. Thus R must be a reducible root system. Let $n(L)$ denote the number of roots of a lattice L . Suppose that $R = R_1 \oplus R_2$; we have that $n(R) = n(R_1) + n(R_2)$. One can check by direct verification that $n(R) < 240$; for example, the number of roots of $D_7 \oplus A_1$ is 85. Again we have a contradiction, the only lattice of rank at most 8 with 240 roots is $R := E_8$. We conclude that $M = E_8(2) \cong A(X)_{\text{prim}}$. □

5. THE LATTICES $A(X)_{\text{prim}}$, $A(X)$, AND $T(X)$ FOR ϕ_3

In this section we restrict our attention to the study of the smooth cubic fourfolds X with anti-symplectic involution $\phi := \phi_3$ as in Theorem 2.12. We will prove Theorem 1.1 by identifying the lattices $A(X)$, $L_+ \cong A(X)_{\text{prim}}$, and $T(X)$ for a general such cubic. We briefly outline the strategy. In §5.1 we consider the lattice spanned by planes contained in X . We show that the primitive lattice spanned by differences of planes $\tilde{K}$ is an index 2 sublattice of $A(X)_{\text{prim}}$. In particular, we will show that a class $y = \frac{[P] + \sum_{i=1}^9 [F_i]}{2}$ belongs to $A(X)$. In §5.2 we show that the lattice spanned by $\langle \eta_X, [F_1], \dots, [F_9], y \rangle$ is in fact isomorphic to $A(X)$. This allows us to identify the lattice $A(X)_{\text{prim}}$ and $T(X)$ in §5.3.

For convenience, we collect the lattice invariants for $A(X)$, $A(X)_{\text{prim}}$ and $T(X)$ below - these follow from §2.4, §2.5.

Lemma 5.1. *Let X be a cubic fourfold with the involution $\phi := \phi_3$. Then:*

- (1) $A(X)$ is an odd, positive definite, 2-elementary lattice of rank 11 and discriminant group

$$A_{A(X)} \cong (\mathbb{Z}/2\mathbb{Z})^a,$$

where $1 \leq a \leq 10$.

- (2) $A(X)_{\text{prim}}$ is a positive definite even lattice of rank 10, with discriminant group

$$A_{A(X)_{\text{prim}}} \cong \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^a.$$

- (3) $T(X)$ is a 2-elementary lattice of signature $(10, 2)$, with discriminant group

$$A_{T(X)} \cong (\mathbb{Z}/2\mathbb{Z})^a.$$

5.1. The lattice spanned by the planes. By Lemma 3.1 and Proposition 3.4, X contains the fixed plane P , and the invariant planes $F_1, \dots, F_9$.

Lemma 5.2. *The intersection products of the classes above are given as follows:*

- (1) $[P] \cdot [P] = [F_i] \cdot [F_i] = \eta_X \cdot \eta_X = 3$ for $1 \leq i \leq 9$,
- (2) $\eta_X \cdot [P] = \eta_X \cdot [F_i] = 1$,
- (3) $[P] \cdot [F_i] = -1$,
- (4) $[F_i] \cdot [F_j] = 1$ for $1 \leq i \neq j \leq 9$.

Proof. One can compute the intersections using Lemma 4.1. $\square$

Remark 5.3. Consider the lattice $N = \langle \eta_X, [P], [F_1], \dots, [F_9] \rangle \subset A(X)$. This has determinant 2^{12} , and rank 11. By Lemma 5.1, N cannot be isomorphic to $A(X)$. Indeed, if $N \cong A(X)$, then the number of generators $l(T(X)) = a$ of the discriminant group of $T(X)$ satisfies $l(T(X)) > 10$; this contradicts Corollary 2.13.

Proposition 5.4. *Consider the lattice K spanned by the classes $\alpha_i = [F_i] - [F_{i+1}]$, $1 \leq i \leq 8$ and $\alpha_9 = [P] + [F_8] + [F_9] - \eta_X$. Then K is a sublattice of $A(X)_{\text{prim}}$ and is isomorphic to $D_9(2)$.*

Proof. Clearly $\alpha_i \in A(X)_{\text{prim}}$ for $1 \leq i \leq 9$; we compute the intersection matrix for the lattice spanned by the $\{\alpha_i\}$ using Lemma 5.2. In particular, we see that for $1 \leq i \leq j \leq 9$:

$$\alpha_i \cdot \alpha_j = \begin{cases} 4 & \text{if } i = j, \\ -2 & \text{if } j = i + 1, i \neq 8, \\ -2 & \text{if } i = 7, j = 9, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we see the lattice spanned by $\{\alpha_i\}_{i=1}^9$ is isomorphic to $D_9(2)$. $\square$

Consider the class $\delta := \eta_X - 3[P]$; we see that $\delta \in A(X)_{\text{prim}}$, and the lattices $\langle \delta \rangle$ and $K \cong D_9(2)$ are mutually orthogonal in $A(X)_{\text{prim}}$. Thus $A(X)_{\text{prim}}$ is an overlattice of $\langle \delta \rangle \oplus D_9(2)$:

$$\langle \delta \rangle \oplus D_9(2) \subset A(X)_{\text{prim}}.$$

We can calculate that $\delta^2 = 24$; it follows that $\tilde{K} := \langle \delta \rangle \oplus D_9(2) \cong \langle 24 \rangle \oplus D_9(2)$. The discriminant group of $\tilde{K}$ is

$$A_{\tilde{K}} = A_\delta \oplus A_{D_9(2)} \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^8,$$

whereas $A_{A(X)_{\text{prim}}} = \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^a$. By Proposition A.1, $A(X)_{\text{prim}}$ is a non-trivial overlattice of $\tilde{K}$, corresponding to a non-trivial isotropic subgroup $H \subset A_{\tilde{K}}$. Since $D_9(2) \hookrightarrow A(X)_{\text{prim}}$ is a primitive embedding with $D_9(2)^\perp \cong \langle \delta \rangle$, we see that the projections

$$H \rightarrow A_\delta \cong \mathbb{Z}/24\mathbb{Z},$$

$$H \rightarrow A_{D_9(2)} \cong \mathbb{Z}/8\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^8$$

are also embeddings (see §A.3 for a discussion). It follows that $H \cong \mathbb{Z}/4\mathbb{Z}$, or $\mathbb{Z}/8\mathbb{Z}$.

Lemma 5.5. *There exists a non-trivial isotropic subgroup $H \cong \mathbb{Z}/4\mathbb{Z}$ of $A_{\tilde{K}}$ corresponding to an overlattice $\tilde{K} \subset M \subset A(X)_{\text{prim}}$.*

Proof. The discriminant group of $\langle \delta \rangle$ is generated by $\xi = [\frac{2x-3P}{24}]$, and $q_\delta(\xi) = \frac{1}{24}$. Let G_D be the Gram matrix for $D_9(2)$ with respect to the basis $\{\alpha_i\}_{i=1}^9$. To find explicit generators of the discriminant group $A_{D_9(2)}$, we proceed as follows: first consider the inverse matrix G_D^{-1} , given below. We consider the linear combinations of $\alpha_1, \dots, \alpha_9$ with coefficients given by the columns of G_D^{-1} . Denote them by $\alpha_1^*, \dots, \alpha_9^*$, and their image in $A_{D_9(2)}$ by $[\alpha_i^*]$. We see that $\beta := [\alpha_9^*]$ has order 8, thus we can consider β as a generator of $\mathbb{Z}/8\mathbb{Z}$. Note that $q_{D_9(2)}(\beta) = \frac{9}{8}$.

$$G_D^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{4} & \frac{3}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & 2 & 2 & 2 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{4} & \frac{5}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & 3 & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & \frac{7}{2} & \frac{7}{4} & \frac{7}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \frac{5}{4} & \frac{3}{2} & \frac{7}{4} & \frac{9}{8} & \frac{7}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \frac{5}{4} & \frac{3}{2} & \frac{7}{4} & \frac{7}{8} & \frac{9}{8} \end{pmatrix}.$$

Let $H = \mathbb{Z}/4\mathbb{Z} = \langle 6\xi + 2\beta \rangle$; indeed one can see that $q(6\xi + 2\beta) = 0 \pmod{2\mathbb{Z}}$, hence H is an isotropic subgroup, and thus corresponds to some overlattice $\tilde{K} \subset M \subset A(X)_{\text{prim}}$. $\square$

Proposition 5.6. *Notations as above.*

(1) *The class*

$$x = \frac{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7 + [F_9] - [P]}{2}$$

belongs to $M \subset A(X)_{\text{prim}}$; in particular x is an integral algebraic class in $A(X)$.

(2) *The Gram matrix of M (denoted G_M) with respect to the basis $\{x, \alpha_i\}_{i=1}^9$ is given below. In particular, the discriminant group of M is $\mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$.*

$$G_M = \begin{pmatrix} 6 & 2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & 0 \\ 2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & -2 & -2 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 4 \end{pmatrix}.$$

Proof. For the first claim, we have that M is the overlattice of $\tilde{K}$ corresponding to the isotrivial subgroup given by $H = \langle 6\xi + 2\beta \rangle \subset A_{\tilde{K}}$. We see that

$$\begin{aligned} 6\xi + 2\beta &= \frac{\eta_X - 3[P] + 2(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7) - \alpha_8 + \alpha_9}{4} \\ &= \frac{2(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7) + 2[F_9] - 2[P]}{4} \\ &= \frac{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7 + [F_9] - [P]}{2} \pmod{\tilde{K}}. \end{aligned}$$

It follows that class x belongs to M ; in particular, $x \in A(X)_{\text{prim}}$.

For the second claim, we can calculate the intersection matrix with respect to the basis $\{x, \alpha_i\}_{i=1}^9$ using Lemma 5.2. Let G_M be the matrix with respect to this basis. We see that $\det G_M = 3 \times 2^{10}$; in order to compute the discriminant group, we compute the inverse matrix G_M^{-1} , and consider the column vectors x^*, α_i^* . It is clear to see that there are no elements of order 4; thus the discriminant group of $A_M \cong \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$. $\square$

Remark 5.7. Consider the linear combination $y := x + [F_2] + [F_4] + [F_6] + [F_8] + [P] \in A(X)$. We see that

$$y := \frac{[P] + [F_1] + [F_2] + [F_3] + [F_4] + [F_5] + [F_6] + [F_7] + [F_8] + [F_9]}{2}$$

is thus an element of the integral lattice $A(X)$.

We are now ready to identify the lattices $A(X), A(X)_{\text{prim}}, T(X)$; more specifically we prove the following:

Theorem 5.8. *Let N be the lattice generated by $\{\eta_X, y, [F_1], \dots, [F_9]\}$; and M be the lattice generated by $\{x, \alpha_i\}$ for $1 \leq i \leq 9$. Then:*

- (1) *The lattice of algebraic cycles $A(X)$ is isomorphic to N .*
- (2) *The lattice $A(X)_{\text{prim}} \cong L_+$ is isomorphic to the lattice M .*
- (3) *The lattice $T(X)$ is isomorphic to the lattice $E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U$.*

This will conclude the proof of Theorem 1.1.

5.2. The lattice $A(X)$. Consider the lattice $A(X) \subset H^4(X, \mathbb{Z})$; this is an odd, 2-elementary lattice of rank 11. Let N be the lattice spanned by $\eta_X, y, [F_1], \dots, [F_9]$, where y is the class as in Remark 5.7; we claim that $N = A(X)$. Let G_N be the Gram matrix of N (with respect to $\eta_X, y, [F_1], \dots, [F_9]$), with entries calculated according to Lemma 5.2. The inverse matrix is given below. Denote by $\eta^*, y^*, [F_1]^*, \dots, [F_9]^*$ the dual basis of N^* , given as linear combinations of the elements $\{\eta_X, y, [F_1], \dots, [F_9]\}$ with coefficients given by the column vectors of G_N^{-1} . By abuse of notation, $\eta^*, y^*, [F_1]^*, \dots, [F_9]^*$ also denote the corresponding elements in $A_N = N^*/N$. It is straightforward to check that A_N is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{10}$

and $\{\eta^*, [F_1]^*, \dots, [F_9]^*\}$ is a basis.

$$G_N^{-1} := \begin{pmatrix} \frac{3}{2} & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -\frac{5}{2} & 6 & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} \\ 1 & -\frac{5}{2} & \frac{3}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & \frac{3}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} \end{pmatrix}.$$

Lemma 5.9. *The discriminant group A_N of N is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{10}$. The discriminant bilinear form*

$$b_N : A_N \times A_N \rightarrow \mathbb{Q}/\mathbb{Z}$$

is given by the matrix with every entry equal to $1/2$ (with respect to the basis $\{\eta^, [F_1]^*, \dots, [F_9]^*\}$ of A_N).*

Proposition 5.10. *The lattice N generated by $\eta_X, y, [F_1], \dots, [F_9]$ is saturated in $A(X) = H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$.*

Proof. Assume the natural embedding $N \subset A(X)$ is not saturated. Then it factors as $N \subsetneq \text{Sat}(N) = A(X) \subset H^4(X, \mathbb{Z})$, where $\text{Sat}(N)$ denotes the saturation of N in $H^4(X, \mathbb{Z})$. Thus, $\text{Sat}(N)$ is a non-trivial overlattice of N and corresponds to a non-trivial isotropic subgroup of $(A_N, b_N : A_N \times A_N \rightarrow \mathbb{Q}/\mathbb{Z})$. Elements in A_N are given by linear combinations of $\eta^*, [F_1]^*, \dots, [F_9]^*$ with coefficients either 0 or 1; isotropic elements must have an even number of non-zero coefficients. There are 9 cases to consider. For example, let us suppose that $[F_i]^* + [F_j]^*$ is contained in the isotropic subgroup. From the expression of G_N^{-1} it is easy to see that $[F_i]^* + [F_j]^* \equiv \frac{1}{2}([F_i] + [F_j]) \pmod{N}$. It follows that the element $\frac{1}{2}([F_i] + [F_j])$ belongs to $A(X)$ (i.e. $[F_i] + [F_j]$ is divisible by 2). The other cases are similar. In conclusion, $N \neq \text{Sat}(N)$ if and only if at least one of the following elements is 2-divisible in $A(X)$:

- (1) $[F_i] + [F_j]$
- (2) $[F_i] + [F_j] + [F_k] + [F_l]$
- (3) $[F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n]$
- (4) $[F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n] + [F_p] + [F_q]$
- (5) $\eta + [F_i]$
- (6) $\eta + [F_i] + [F_j] + [F_k]$
- (7) $\eta + [F_i] + [F_j] + [F_k] + [F_l] + [F_m]$
- (8) $\eta + [F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n] + [F_p]$
- (9) $\eta + [F_1] + [F_2] + [F_3] + [F_4] + [F_5] + [F_6] + [F_7] + [F_8] + [F_9]$.

for $1 \leq i, j, k, l, m, n, p, q \leq 9$ distinct. Let us do a case by case analysis.

- (1) Write $[F_i] + [F_j] = 2\sigma$ for some $\sigma \in A(X)$. It is easy to see that $\sigma^2 = 2$; by [Voi86, Sect. 4 Prop. 1], this implies that X is a singular cubic fourfold, clearly a contradiction.

- (2) Write $[F_i] + [F_j] + [F_k] + [F_l] = 2\sigma$, and consider the element $2\sigma - 2[F_j] - 2[F_l] \in A(X)$. This is clearly divisible by 2; write $2\tilde{\sigma} = 2\sigma - 2[F_j] - 2[F_l]$. Then $\tilde{\sigma}^2 = 2$, again a contradiction.
- (3) Write $[F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n] = 2\sigma$, and consider the element $2\sigma - 2[F_j] - 2[F_l] - 2[F_n] \in A(X)$. This is clearly divisible by 2; write $2\tilde{\sigma} = 2\sigma - 2[F_j] - 2[F_l] - 2[F_n]$. It is easy to see that $\eta_X \cdot \tilde{\sigma} = 0$, implying that $\tilde{\sigma}$ is even. On the other hand, $\tilde{\sigma}^2 = 3$, a contradiction.
- (4) Let r be the index such that $\{i, j, k, l, m, n, p, q, r\} = \{1, \dots, 9\}$. Write $[F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n] + [F_p] + [F_q] = 2\sigma$; note that $2y - 2\sigma = ([P] + [F_r])$. Thus $[P] + [F_r]$ is divisible by 2; write $[P] + [F_r] = 2\tilde{\sigma}$. It is easy to see that $\tilde{\sigma} \cdot \eta_X = 1 = \tilde{\sigma}^2$. Then X is a special cubic fourfold labeled by the rank 2 lattice generated by η_X and $\tilde{\sigma}$. The corresponding discriminant is $d = 2$. By [Has00, Sect 4.4] this cannot happen for the smooth cubic X .
- (5) Write $\eta + [F_i] = 2\sigma$ for some $\sigma \in A(X)$. Again, $\sigma^2 = 2$, a contradiction.
- (6) Write $\eta + [F_i] + [F_j] + [F_k] = 2\sigma$. It is easy to see that $\eta_X \cdot \sigma = 3, \sigma^2 = 6$. Then X is a special cubic fourfold labeled by the rank 2 lattice generated by η_X, σ , with determinant $d = 9$. Since $d \not\equiv 0, 2 \pmod{6}$, no such X exists, by [Has00].
- (7) Write $\eta + [F_i] + [F_j] + [F_k] + [F_l] + [F_m] = 2\sigma$. Let n, p, r, s be the indices such that $\{i, j, k, l, m, n, p, r, s\} = \{1, \dots, 9\}$. Consider the element

$$2\sigma - 2y + 2[F_s] = \eta - [P] - [F_n] - [F_p] - [F_r] + [F_s];$$
 we see that $\eta - [P] - [F_n] - [F_p] - [F_r] + [F_s]$ must be divisible by 2. Write $\eta - [P] - [F_n] - [F_p] - [F_r] + [F_s] = 2\tilde{\sigma}$. We see that $\eta_X \cdot \tilde{\sigma} = 0$, and $\tilde{\sigma}^2 = 2$, again a contradiction.
- (8) Let r, s be the indices such that $\{i, j, k, l, m, n, p, r, s\} = \{1, \dots, 9\}$, and write $\eta + [F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n] + [F_p] = 2\sigma$. Consider the element $2\sigma - 2y = \eta - [F_r] - [F_s] - [P] \in A(X)$. Thus $\eta - [F_r] - [F_s] - [P]$ is divisible by 2; write $\eta - [F_r] - [F_s] - [P] = 2\tilde{\sigma}$. One can check that $\eta_X \cdot \tilde{\sigma} = 0$, implying that $\tilde{\sigma}$ is even. On the other hand, $\tilde{\sigma}^2 = 1$, a contradiction.
- (9) Write $\eta + [F_1] + [F_2] + [F_3] + [F_4] + [F_5] + [F_6] + [F_7] + [F_8] + [F_9] = 2\sigma$. Then $2\sigma - 2y = \eta - [P]$; thus $\eta - [P]$ is divisible by 2. Write $2\tilde{\sigma} = \eta - [P]$. It is clear that $\eta \cdot \tilde{\sigma} = 1 = \tilde{\sigma}^2$, again a contradiction as in (4).

□

Corollary 5.11. *We have that $A(X) \cong N$ where N is the lattice given above. In particular,*

$$A_{A(X)} \cong (\mathbb{Z}/2\mathbb{Z})^{10}.$$

5.3. Proof of Theorem 5.8. The above description of $A(X)$ allows us to identify the discriminant group of $A(X)_{\text{prim}}$, and in turn the lattice almost immediately.

Proposition 5.12. *The discriminant group of $A(X)_{\text{prim}}$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$. Further, the lattice $A(X)_{\text{prim}}$ is isomorphic to the lattice M .*

Proof. By definition, $A(X)_{\text{prim}} = \langle \eta_X \rangle^\perp \subset A(X) \cong N$. Since $A_N \cong (\mathbb{Z}/2\mathbb{Z})^{10}$ and $A_{(\eta)} \cong \mathbb{Z}/3\mathbb{Z}$, the first claim follows by Proposition A.1. For (2); if $M \neq A(X)_{\text{prim}}$, then $A(X)_{\text{prim}}$ is a non-trivial overlattice of M , corresponding to a non-trivial isotropic subgroup $H \subset A_M \cong \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$ by Proposition A.1. This would imply that $A_{A(X)_{\text{prim}}} \cong H_A^\perp/H$, which is impossible. □

Next we study the transcendental lattice $T(X) \cong L_-$. Recall that for a 2-elementary lattice S , we define the invariant $\delta(S) \in \{0, 1\}$ to be 0 if $q_S : A_S \rightarrow \mathbb{Q}/2\mathbb{Z}$ takes values in $\mathbb{Z}$, and 1 otherwise.

Lemma 5.13. *The invariants of the transcendental lattice $T(X)$ are computed as follows:*

- (1) $T(X)$ is an even lattice of rank 12. The signature is $(10, 2)$.
- (2) $A_{T(X)} \cong (\mathbb{Z}/2\mathbb{Z})^{10}$. In particular, $T(X)$ is 2-elementary and $\text{discr}(T(X)) = 1024$.
- (3) $T(X)$ has $\delta(T(X)) = 1$.

The lattice $T(X)$ is isomorphic to the orthogonal direct sum $E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U$.

Proof. The first claim follows from Lemma 5.1. Recall that $A_{A(X)_{\text{prim}}} = \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$, and that $A(X)_{\text{prim}} \hookrightarrow A_2 \oplus U^3 \oplus E_8^2$ primitively. By Lemma A.2, this is equivalent to

$$\begin{aligned} q_{A(X)_{\text{prim}}} \big|_{\mathbb{Z}/3\mathbb{Z}} &\cong q_{A_2}, \\ q_{A(X)_{\text{prim}}} \big|_{(\mathbb{Z}/2\mathbb{Z})^{10}} &\cong -q_{T(X)}; \end{aligned}$$

in particular $A_{T(X)} \cong (\mathbb{Z}/2\mathbb{Z})^{10}$. We see that $\delta(T(X)) = 1$ by computing the values of $q_T = -q_{A(X)_{\text{prim}}} \big|_{(\mathbb{Z}/2\mathbb{Z})^{10}}$.

The indefinite, 2-elementary lattice $T(X)$ is classified uniquely up to isomorphism by the signature, $\delta(T(X))$, and $l(T(X))$, where $l(T(X)) = 10$ is the minimum number of generators of A_T (Theorem A.6). The lattice $E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U$ has the same invariants, and thus they are isomorphic. $\square$

6. ASSOCIATED $K3$ SURFACES, HASSETT DIVISORS, AND RATIONALITY

In this section we investigate the consequences of Theorem 1.1 in terms of rationality. In §6.1 we investigate the existence of associated $K3$ surfaces; a cubic fourfold is conjectured to be rational if and only if such an associated $K3$ surface exists. Next in §6.2 we investigate whether a cubic with the anti-symplectic involution ϕ_3 is trivially rational, i.e. contains two disjoint planes. Finally in §6.3 we show that such a cubic fourfold is Hassett maximal; in particular such a cubic is rational.

6.1. Associated and twisted $K3$ surfaces. Let X be a smooth cubic surface with a labeling $K_d \subset A(X)$, as in Definition 2.8.

Definition 6.1. A polarized $K3$ surface (S, L) of degree d is **associated** to X if there exists an isomorphism of Hodge structures

$$K_d^\perp \cong H^2(S, \mathbb{Z})_{\text{prim}},$$

where $H^2(S, \mathbb{Z})_{\text{prim}}$ is orthogonal to L in $H^2(S, \mathbb{Z})$.

A cubic fourfold $X \in \mathcal{C}_d$ has such an associated $K3$ surface if and only if d satisfies the following condition [Has00]:

$$(6.1) \quad d \text{ even and not divisible by } 4, 9 \text{ or any odd prime } p \equiv 2 \pmod{3}.$$

Notice that this implies that as lattices $T(S) \cong T(X)(-1)$, and so a necessary condition for the existence of an associated $K3$ surface is that $T(X)$ embeds primitively into the $K3$ lattice $\Lambda_{K3} \cong U^3 \oplus E_8^2$. It is conjectured that a cubic fourfold

is rational if and only if there exists an associated $K3$ surface [Has00, Kuz10a], we investigate the existence of such $K3$'s for cubics with involutions.

Lemma 6.2. *Let X be a general cubic fourfold with involution either ϕ_1 or ϕ_2 . Then there does not exist an associated $K3$ surface.*

Proof. Consider the involution ϕ_1 ; the statement is proved in [Laz21, Theorem 1.8]. In fact, the lattice $A(X)_{\text{prim}}$ for such a cubic X is maximal in a certain sense.

Consider next the involution ϕ_2 ; we show there does not exist a primitive embedding $T(X) \hookrightarrow \Lambda_{K3}$. Recall that in this case $A(X)_{\text{prim}} \cong E_8(2)$, and so $T(X)$ has signature $(2, 12)$ with discriminant group $A_{T(X)} \cong (\mathbb{Z}/2\mathbb{Z})^8 \oplus \mathbb{Z}/3\mathbb{Z}$. We also have that $q_{T(X)}|_{\mathbb{Z}/3\mathbb{Z}} = q|_{A_2}$, and $q_{T(X)}|_{(\mathbb{Z}/2\mathbb{Z})^8} = -q_{E_8(2)}$.

Suppose that there exists such a primitive embedding. Since Λ_{K3} is the unique even unimodular lattice with signature $(3, 19)$, by [Nik79b, Prop 1.15.1] the existence of this embedding is equivalent to the existence of an even lattice K of signature $(1, 7)$, discriminant group $A_K \cong A_{T(X)}$ such that $q_K = -q_{T(X)}$. Since $(\mathbb{Z}/2\mathbb{Z})^8 \subset A_K$, by Lemma A.7 the lattice $M := K(1/2)$ is a well-defined integral lattice. Now M is a rank $(1, 7)$ lattice with $A_M \cong \mathbb{Z}/3\mathbb{Z}$.

First, suppose that M is an odd lattice. Then by Lemma A.9, there exists a generator $\xi \in \mathbb{Z}/2\mathbb{Z} \subset A_K$ such that $q_K(\xi) \notin \mathbb{Z}/2\mathbb{Z}$. Since $K^\perp = E_8(2)$ is a 2-elementary lattice such that $q_{E_8(2)}(v) \in \mathbb{Z}/2\mathbb{Z}$ for all $[v] \in A_{E_8(2)}$, this is a contradiction - thus M must be an even lattice. By Theorem A.10, M is an even 3-elementary lattice and is uniquely determined by the rank and $l(A_M) = 1$. Consider the lattice $U \oplus E_6$. This is a 3-elementary lattice with the same invariants as M ; thus $M \cong U \oplus E_6$. Hence $K \cong U(2) \oplus E_6(2)$; however $q_M|_{(\mathbb{Z}/3\mathbb{Z})} \neq -q_{A_2}$; it follows that no such lattice K exists. $\square$

The Kuznetsov conjecture would then imply that a cubic fourfold X with involution ϕ_1 or ϕ_2 is irrational. On the other hand, for X with involution ϕ_3 , we can construct a $K3$ surface with transcendental lattice $T(S) \cong T(X)(-1)$ geometrically.

Lemma 6.3. *Let X be a general cubic fourfold with involution ϕ_3 . Then there exists a primitive embedding $T(X) \hookrightarrow \Lambda_{K3}$.*

Proof. First, assume that $\phi = \phi_3$. We show there exists such a primitive embedding by exhibiting a $K3$ surface with transcendental lattice $T(S) \cong T(X)(-1)$. Let $C \subset \mathbb{P}^2$ be an irreducible general sextic curve with 9 nodes. Let $\pi : \tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at the 9 nodes, and let $r : S \rightarrow \tilde{\mathbb{P}}^2$ be the double cover ramified along the strict transform of C . Then S is a $K3$ surface; we claim $T(S) \cong E_8(-2) \oplus U \oplus A_1 \oplus A_1(-1)$. Let $h = r^*\pi^*(l)$, where l is the class of a line in $\mathbb{P}^2$, and e_i for $i = 1, \dots, 9$ the pullback of the exceptional curves of π . Let $D \in \tilde{\mathbb{P}}^2$ be the strict transform of C , and denote also by D the ramified curve on S . In particular, $2D \sim 6h - \sum_{i=1}^9 2e_i$, and the $NS(S)$ is spanned by the classes $h, e_1, \dots, e_9$. Notice that $e_i^2 = -2, h^2 = 2$ and $h \cdot e_i = 0$. Thus the lattice $NS(S)$ is a 2-elementary lattice, with signature $(1, 9)$, $l(A_{NS(S)}) = 10$, and $\delta = 1$ (see Theorem A.6). This determines $NS(S)$ uniquely, and $NS(S) \cong E_8(2) \oplus A_1 \oplus A_1(-1)$. Now by definition $T(S) = (NS(S))^\perp$, and by Lemma A.3 has signature $(2, 10)$, $l(A_{T(S)}) = 10, \delta = 1$. This uniquely determines $T(S)$; in particular, $T(S) \cong E_8(-2) \oplus U \oplus A_1 \oplus A_1(-1)$. $\square$

Recently, Brakkee considered instead associated twisted $K3$ surfaces [Bra20]. Recall that the Brauer group of a scheme S is the group of sheaves of Azumaya

algebras modulo Morita equivalence, with multiplication given by tensor product. For references, see [Huy05, Huy09].

For S a complex $K3$ surface, we have that

$$\mathrm{Br}(S) \cong H^2(S, \mathcal{O}_S^*)_{tors} \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho(X)}.$$

Definition 6.4. A **twisted $K3$ surface** is a pair (S, α) where S is a $K3$ surface, $\alpha \in \mathrm{Br}(S)$.

More precisely, consider the following condition:

$$(6.2) \quad d' = dr^2 \text{ for some } d, r \text{ satisfying condition (6.1).}$$

Theorem 6.5 ([Bra20, Theorem 2]). *A cubic fourfold X belongs to the divisor $\mathcal{C}_{d'}$ for d' satisfying (6.2) if and only if for every decomposition $d' = dr^2$, X has an associated polarized twisted $K3$ surface (S, L, α) of degree d and order r .*

Here $r := \mathrm{order}(\alpha) \in \mathrm{Br}(S)$ and the triple (S, L, α) is associated to X if there is a Hodge isometry $K_d^\perp \cong \mathrm{Ker} f_\alpha$.

Consider the case $d = r = 2$, so $d' = 8$. The cubic fourfolds contained in $\mathcal{C}_8$ contain a plane, and there is an associated twisted $K3$ surface (S, α) . As discussed in [Kuz10a], there is a geometric construction for (S, α) obtained by projecting the cubic fourfold from the plane, and letting S be the double cover of $\mathbb{P}^2$ branched in the discriminant sextic. We call this twisted $K3$ the **visible** twisted $K3$ surface associated to $P \subset X$. By Lemma 3.1, a cubic fourfold with anti-symplectic involution contains a plane - we immediately see the existence of such a $K3$ surface.

Corollary 6.6. *Let X be a cubic fourfold with anti-symplectic involution ϕ . Then there exists an associated visible twisted $K3$ surface (S, α) with $\mathrm{order}(\alpha) = 2$.*

On the other hand, let X be a general cubic fourfold with the symplectic involution ϕ_2 . By Corollary 4.4, we cannot associate to X a visible twisted $K3$ surface.

6.2. The divisor $\mathcal{C}_8$. The cubic fourfolds that contain a plane have been well studied and are central to the original proof of the Torelli theorem [Voi86]. They have a quadric bundle structure, and rationality would follow provided the bundle has a rational section. Here, we note that this is not the case for cubics X with the involution ϕ_3 .

Definition 6.7. A cubic fourfold X containing a plane P is called **trivially rational** (see [Gal17]) if the associated quadric bundle $\pi : X_P := Bl_P X \rightarrow \mathbb{P}^2$ has a rational section.

In particular, one sees immediately that if a cubic fourfold X contains two disjoint planes, then we have such a section and X is trivially rational. Theorem 6.8 gives a criterion for the existence of such a section.

Theorem 6.8 ([Has99, Theorem 3.1]). *A cubic fourfold X containing a plane is trivially rational if and only if there exists a class $T \in A(X)$ with $T \cdot Q$ odd for a smooth fiber Q of π .*

We can use this criterion in our situation; our complete description of the lattice $A(X)$ allows us to conclude no such class exists.

Corollary 6.9. *Let X be a general cubic fourfold with the involution $\phi = \phi_3$, and let $[P]$ be the class of the plane as in §5. Then X is not trivially rational with respect to the quadric fibration $\pi : X_P \rightarrow \mathbb{P}^2$.*

Proof. Let Q denote a general fiber of π , i.e. a smooth quadric surface. The class $[Q]$ in $H^4(X, \mathbb{Z})$ satisfies $[P] + [Q] = \eta_X$. One can easily see that a $\mathbb{Z}$ -linear combination of the basis of $A(X)$ intersects the quadric $[Q]$ evenly, using the intersection matrix in Proposition 5.10. $\square$

Let $X \in \mathcal{C}_8$, and suppose that the discriminant curve C_P associated to $X_P \rightarrow \mathbb{P}^2$ is smooth. Following Voisin [Voi86], we let (S, α) be associated visible $K3$ surface, where $\alpha \in \text{Br}(S)[2]$. This in turn defines a 2-torsion Brauer class $\alpha_X \in \text{Br}(X)$; we have the following result of Kuznetsov [Kuz16, Sect. 4.3].

Lemma 6.10. *Let X be a cubic fourfold containing a plane P as above. The following are equivalent:*

- (1) *there exists a rational section of the quadric fibration $\pi : X_P \rightarrow \mathbb{P}^2$;*
- (2) *the associated Brauer class is trivial, i.e. $\alpha_X = 1 \in \text{Br}(X)$.*

Moreover, the conditions above imply that X is rational.

Corollary 6.11. *Let X be a cubic fourfold with anti-symplectic involution ϕ_3 . Then the associated Brauer class is non-trivial.*

6.3. The intersection of the Hassett divisors. Cubic fourfolds with involutions ϕ_1 or ϕ_2 have no associated $K3$ surfaces, and are conjecturally irrational. On the other hand, a cubic fourfold X with the involution ϕ_3 has transcendental lattice coming from a $K3$ surface, and is potentially rational. Despite the rationality not following from the obvious quadric bundle structure, we will establish that X is indeed rational by investigating which Hassett divisors such an X belongs to. Recall Definition 6.12:

Definition 6.12. We say that a cubic fourfold X is **Hassett maximal** if

$$X \in \bigcap_{\substack{d > 6 \\ d \equiv 0, 2 \pmod{6}}} \mathcal{C}_d.$$

We denote the locus of Hassett maximal cubic fourfolds by $\mathcal{Z}$.

Cubics contained in the Hassett maximal locus $\mathcal{Z}$ are rational (see Lemma 1.5). In this section we prove the following result:

Theorem 6.13. *Let $\mathcal{M}_{\phi_3}$ be the moduli space of cubic fourfolds with the involution of type ϕ_3 . Then $\mathcal{M}_{\phi_3}$ is contained in the Hassett maximal locus $\mathcal{M}_{\phi_3} \subset \mathcal{Z}$. In particular, $X \in \mathcal{M}_{\phi_3}$ is rational.*

It is known that $\mathcal{Z}$ is non-empty; it contains the Fermat cubic fourfold [YY21, Theorem 1.2]. Further, the authors show that $\dim \mathcal{Z} \geq 13$, by illustrating a lattice M of rank 7 such that the moduli of M -polarized cubic fourfolds $\mathcal{M}_M$ is non-empty, and M contains a labeling of determinant d for every $d > 6, d \equiv 0, 2 \pmod{6}$. We adapt the method in [YY21] to our situation.

In order to prove Theorem 6.13, we will need the following classical results of number theory.

Lemma 6.14 (Lagrange's 4-square theorem). *Any positive integer can be expressed in the form $x^2 + y^2 + z^2 + u^2$ for some integers x, y, z, u .*

Lemma 6.15 (Ramanujan). *Any positive integer except for 1 and 17 can be expressed in the form $2x^2 + 2y^2 + 2z^2 + 3u^2$ for some integers x, y, z, u .*

Proof of Theorem 6.13. We will exhibit a primitive sublattice $\eta_X \in K_d \hookrightarrow A(X)$ with determinant d for every $d > 6, d \equiv 0, 2 \pmod{6}$. Recall that a basis for $A(X)$ is given by

$$\{\eta_X, y, [F_1], [F_2], \dots, [F_9]\}$$

keeping notations of §5. Denote by $\alpha_1 = [F_1] - [F_2], \alpha_3 = [F_3] - [F_4], \alpha_5 = [F_5] - [F_6], \alpha_7 = [F_7] - [F_8], \beta = [\eta_X] - [P] - [F_9]$, and $\gamma = y - [F_5] - [F_6] - [F_7] - [F_8] - [F_9]$. It is easy to see that the sublattice $\langle \eta_X, \alpha_1, \alpha_3, \alpha_5, \alpha_7, \beta, \gamma \rangle \subset A(X)$ is primitive; indeed, writing each class in the basis of $A(X)$ we see they are linearly independent.

Suppose that $v = x_1\alpha_1 + x_3\alpha_3 + x_5\alpha_5 + x_7\alpha_7 + s\beta + t\gamma$ for integers $x_1, \dots, x_7, s, t$. One can see that $\eta_X \cdot v = s$, and

$$v^2 = 4x_1^2 + 4x_3^2 + 4x_5^2 + 4x_7^2 + 3s^2 + 6t^2.$$

Consider the rank two sublattice $\langle \eta_X, v \rangle \subset A(X)$: its discriminant is given by:

$$d = 3(4x_1^2 + 4x_3^2 + 4x_5^2 + 4x_7^2 + 6t^2) + 8s^2.$$

We will show we can obtain every $d \equiv 0, 2 \pmod{6}$.

Case 1. $d = 6k$, for $k \geq 2$. Let $s = 0$. We need to find suitable integers such that

$$k = 2x_1^2 + 2x_3^2 + 2x_5^2 + 2x_7^2 + 3t^2.$$

- If $k = 2m$, let $x_1 = 1$. Then the lattice $\langle \eta_X, v \rangle$ is primitive, and by Ramanujan's theorem we can find suitable integers such that

$$2(m - 1) = 2x_3^2 + 2x_5^2 + 2x_7^2 + 3t^2.$$

- If $k = 2m + 1$, let $t = 1$. Then the lattice $\langle \eta_X, v \rangle$ is primitive, and by Lagrange's 4-square theorem we can find suitable integers such that

$$m - 1 = x_1^2 + x_3^2 + x_5^2 + x_7^2.$$

Case 2. $d = 6k + 2$. Since we know $[X] \in \mathcal{C}_8$ (X contains a plane), we can assume that $k \geq 2$. Let $s = 1$. The lattice $\langle \eta_X, v \rangle$ is primitive, and we need to find suitable integers such that

$$k - 1 = 2x_1^2 + 2x_3^2 + 2x_5^2 + 2x_7^2 + 3t^2.$$

For $k \geq 3$ this reduces to Case 1 - we deal with $k = 2$ below.

Case 3. $d = 14$. Consider the class $v = y - [F_2] - [F_4] - [F_6] - [F_8]$; then $v^2 = 5$ and $\eta_X \cdot v = 1$. Thus the lattice $\langle \eta_X, v \rangle$ is primitive and of determinant $d = 14$. It is well known that any $[X] \in \mathcal{C}_{14}$ is rational [BD85].

□

6.4. Low degree classes. We have seen that a cubic X with the involution ϕ_3 is rational by showing it belongs to the Hassett maximal locus. In particular, such a cubic belongs to $\mathcal{C}_{14}$, the closure of the locus of Pfaffian cubic fourfolds. The Pfaffian locus has been well studied; Beauville and Donagi [BD85] showed that Pfaffian cubic fourfolds are rational. Further, the Pfaffian condition is equivalent to X containing a smooth degree 5 del Pezzo surface [Bea00]. In this section, we show that such a cubic does indeed belong to the Pfaffian locus inside of $\mathcal{C}_{14}$. Our argument is lattice theoretic; it would be interesting to realise the rationality of a cubic fourfold X with the involution ϕ_3 geometrically.

The complement of the Pfaffian locus has been studied by Auel in [Aue22]. More precisely, the complement of the Pfaffian locus inside $\mathcal{C}_{14}$ is contained in

the irreducible locus of cubic fourfolds containing two disjoint planes. Using this, description we can prove our next result:

Proposition 6.16. *Let X be a general cubic fourfold with the involution $\phi = \phi_3$. Then X is Pfaffian.*

Proof. Suppose that such a cubic fourfold X is not Pfaffian. Then by [Aue22, Theorem 1], X contains two disjoint planes P, P' . Consider the class $v = [P] - [P']$; clearly $v \in A(X)_{\text{prim}}$. Thus we can write $v = a_0x + \sum_{i=1}^9 a_i\alpha_i$, where $\{x, \alpha_1, \dots, \alpha_9\}$ is a basis for $A(X)_{\text{prim}} \cong M$ as in Theorem 5.8. Let $w \in A(X)_{\text{prim}}$; we can also write $w = b_0x + \sum_{i=1}^9 b_i\alpha_i$. Notice that $x \cdot w$ and $\alpha_i \cdot w$ are even, by using the intersection matrix in Proposition 5.6. Thus $v \cdot \alpha \in 2\mathbb{Z}$ for all $\alpha \in A(X)_{\text{prim}}$.

Consider the element $\delta := 3[P] - \eta_X$, we see $\delta \in A(X)_{\text{prim}}$. Since P and P' are disjoint, $[P] \cdot [P'] = 0$ and so $v \cdot \delta = 3[P]^2 = 9$, a contradiction. Thus no two planes are disjoint. $\square$

The cubic fourfolds X admitting the involution ϕ_3 thus belong to the intersection of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, first studied in [ABBVA14] and later in [BRS19]. In particular, $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has 5 irreducible components, indexed by the value $[P] \cdot [T] \in \{-1, 0, 1, 2, 3\}$, where $P \subset X$ is a plane and $[T]$ is the class of a small OADP surface (for a general point in $\mathbb{P}^5$ there exists a unique secant line to T ; see [BRS19, Def. 1.5]) such that $[T]^2 = 10$ and $[T] \cdot \eta_X = 4$ [BRS19, Theorem 3.4].

Corollary 6.17 ([BRS19, Corollary 3.5]). *Let $X \in \mathcal{C}_{14}$, and $[T] \in H^{2,2}(X, \mathbb{Z})$ such that $[T] \cdot \eta_X = 4$ and $[T]^2 = 10$. Then $[T]$ is represented by a small OADP surface T contained in X .*

Theorem 6.18. *Let X be a cubic fourfold with an involution $\phi := \phi_3$. Then X contains a smooth quartic rational normal scroll.*

Proof. Let $[T] = 2\eta_X - y + [F_7] + [F_8] + [F_9]$. One can easily compute that $[T] \cdot \eta_X = 4$ and $[T]^2 = 10$; thus $[T]$ is represented by a small OADP surface. By the proof of [BRS19, Theorem 3.4], there are three possibilities for T :

- (1) $T = S \cup P'$ where S is a cubic rational normal scroll and P' is a plane;
- (2) T contains only irreducible components of degree less than or equal to 2;
- (3) T is an irreducible smooth quartic rational normal scroll.

Let $[P]$ be the class of any plane in X ; thus $[P]$ is equivalent to a $\mathbb{Z}$ -linear combination of the basis of $A(X)$ given in Proposition 5.10. One can compute that $[P] \cdot T = 2k$ for some integer k . We will use this to rule out (1) and (2) above.

- (1) Suppose that $T = S \cup P'$. Since T is a small OADP, the surfaces S and P' intersect along a line; thus $[S] \cdot [P'] = 0$. Hence we see that $[P'] \cdot [T] = [P']^2 = 3$, a contradiction.
- (2) Suppose that T contains only irreducible components of degree less than or equal to 2. By the proof of [BRS19, Theorem 3.4], this implies that X contains a pair of skew planes. One can compute that for one such plane $[P] \cdot [T] = -1$, again a contradiction.

Thus $[T]$ is represented by a smooth quartic rational normal scroll. $\square$

Remark 6.19. In fact, it follows that a cubic fourfold X with an involution ϕ_3 is contained in the intersection of two of these components. Indeed, for the fixed plane $P \subset X$, we see that $[P] \cdot [T] = 2$, whereas $[F_1] \cdot [T] = 0$.

APPENDIX A. LATTICE THEORY

A.1. Notation. A lattice of rank r is a free finitely generated $\mathbb{Z}$ -module $N \cong \mathbb{Z}^r$ equipped with a non-degenerate symmetric bilinear pairing $N \otimes N \rightarrow \mathbb{Z}$, denoted $e \otimes f \mapsto e \cdot f$. We write $e^2 = e \cdot e$. We assume that N is even ($e^2 \in 2\mathbb{Z}$ for each $e \in N$) unless otherwise stated.

The lattice $N(n)$ is obtained from N by multiplying the pairing by n . The lattice $\langle n \rangle$ denotes the lattice $\mathbb{Z}$ with $e^2 = n$ for a generator e . The lattices A_k, D_k, E_k denote the standard positive definite root lattices of rank k , and U denotes the hyperbolic plane.

The finite abelian group $A_N := N^*/N$ is called the discriminant group of N , with induced quadratic form $q_N : A_N \rightarrow \mathbb{Q}/2\mathbb{Z}$ (or symmetric bilinear form $b_N : A_N \times A_N \rightarrow \mathbb{Q}/\mathbb{Z}$ if N is odd).

A.2. Overlattices. Let L be an even lattice. An **overlattice** of L is an even lattice Γ where $L \hookrightarrow \Gamma$ is an embedding for which Γ/L is a finite abelian group.

Let $H_\Gamma = \Gamma/L$. We have a chain of embeddings:

$$L \hookrightarrow \Gamma \hookrightarrow \Gamma^* \hookrightarrow L^*,$$

and so $H_\Gamma \subset L^*/L = A_L$, and $(\Gamma^*/L)/H_\Gamma = A_\Gamma$.

We have the following result due to Nikulin [Nik79b, Props. 1.4.1, 1.4.2].

Proposition A.1.

- (1) The correspondence $\Gamma \mapsto H_\Gamma$ determines a bijection between even overlattices of L and isotropic subgroups of A_L (a subgroup $H \subset A_L$ is isotropic if $q_L|_H = 0$).
- (2) We have $(H_\Gamma)^\perp = \Gamma^*/L \subset A_L$ and $(q_L|_{(H_\Gamma)^\perp})/H_\Gamma = q_\Gamma$.
- (3) Two even overlattices $L \hookrightarrow \Gamma_1$ and $L \hookrightarrow \Gamma_2$ are isomorphic if and only if the isotropic subgroups $H_{\Gamma_1}, H_{\Gamma_2} \subset A_L$ are conjugate under some automorphism of L .

A.3. Primitive embeddings. An embedding of lattices $M \subset S$ is called *primitive* if L/M is a free group; we denote this by $M \hookrightarrow L$, and denote $N = M^\perp$ the orthogonal complement of M in L . Then L is an overlattice of $M \oplus N$, and $H_L = L/(M \oplus N) \subset A_M \oplus A_N$ is isotropic subgroup corresponding to $q_M \oplus q_N$. The primitive embeddings $M \hookrightarrow L$ and $N \hookrightarrow L$ are equivalent to the condition that the projections $H_L \rightarrow A_M$ and $H_L \rightarrow A_N$ are embeddings.

Lemma A.2 ([Nik79b, Prop 1.15.1]). *Primitive embeddings of M with signature (m_+, m_-) into an even lattice S with signature (s_+, s_-) are determined by the sets $(H_M, H_S, \gamma, K, \gamma_K)$ where*

- H_M, H_S are subgroups of A_M, A_S respectively and $\gamma : q|_{A_M|H_M} \rightarrow -q|_{A_S|H_S}$ is an isometry;
- K is an even lattice with signature $(s_+ - m_+, s_- - m_-)$ and discriminant form $-\delta$ where $\delta \equiv (q_{A_M} \oplus -q_{A_S})|_{\Gamma_\gamma^\perp/\Gamma_\gamma}$ and $\gamma_K : q_K \rightarrow (-\delta)$ is an isometry.

We will also use the following simplified version of Lemma A.2:

Lemma A.3. *Let M be an even lattice of signature (m_+, m_-) . The existence of a primitive embedding of M into some unimodular lattice L of signature (l_+, l_-) is equivalent to the existence of a lattice N of signature (n_+, n_-) and discriminant form q_{A_N} such that the following are satisfied:*

- $m_+ + n_+ = l_+$ and $m_- + n_- = l_-$.
- $A_N \equiv A_M$ and $q_{A_N} = -q_{A_M}$.

A.4. 2-Elementary lattices. Let L be any even lattice. An involution $\iota \in O(L)$ determines two eigenspaces

$$L_{\pm} := \{v \in L : \iota(v) = \pm v\}.$$

The following hold:

- L_- and L_+ are primitive, mutually orthogonal lattices
- $H := L/(L_+ \oplus L_-)$ is a 2-elementary group (i.e. $H = (\mathbb{Z}/2)^a$);
- H admits embeddings into the discriminant groups A_{L_+} and A_{L_-} such that the diagonal embedding $H \subset A_{L_+} \oplus A_{L_-}$ is isotropic with respect to the finite quadratic form $q_{L_+} \oplus q_{L_-}$ [Nik79b, Sect. 1.5]

Lemma A.4. *If Λ is a unimodular lattice and $\iota \in O(\Lambda)$ is an involution, then Λ_- and Λ_+ are 2-elementary lattices.*

Lemma A.5. *Let L be a lattice, and $\iota \in O(L)$ an involution acting as the identity on A_L . Then L_- is a 2-elementary lattice.*

Proof. Let Λ be a unimodular lattice such that there exists a primitive embedding $L \hookrightarrow \Lambda$, and let $R = L^{\perp}$. Since ι acts as the identity on A_L , we can extend ι by the identity on R . Since $R_- = 0$, it holds that $L_- \cong \Lambda_-$. We deduce by Lemma A.4 that L_- is 2-elementary. $\square$

Let M be a 2-elementary lattice. Nikulin proved that M is classified by three natural invariants: the signature (t_+, t_-) , the number of generators a of A_M and $\delta \in \{0, 1\}$ with $\delta = 0$ if and only if the finite quadratic form $q_M : A_M \rightarrow \mathbb{Q}/2\mathbb{Z}$ takes values in $\mathbb{Z}/2\mathbb{Z}$.

Theorem A.6 (Classification of 2-elementary lattices). *The genus of an even 2-elementary lattice is determined by the invariants $\delta_M, l(M)$ and $\text{sign}(M)$. If M is indefinite, then the genus consists of one isomorphism class.*

An even 2-elementary lattice M with $\delta_M = \delta$, $l(M) = a$ and $\text{sign}(M) = (t_+, t_-)$ exists if and only if the following conditions are satisfied:

- (1) $t_+ + t_- \geq a$;
- (2) $t_+ + t_- + a \equiv 0 \pmod{2}$;
- (3) $t_+ - t_- \equiv 0 \pmod{4}$ if $\delta = 0$;
- (4) $\delta = 0$, $t_+ - t_- \equiv 0 \pmod{8}$ if $a = 0$;
- (5) $t_+ - t_- \equiv 1 \pmod{8}$ if $a = 1$;
- (6) $\delta = 0$ if $a=2$, $t_+ - t_- \equiv 4 \pmod{8}$;
- (7) $t_+ - t_- \equiv 0 \pmod{8}$ if $\delta = 0$ and $a = t_+ + t_-$.

Lemma A.7. *Let M be an even lattice of rank r such that $(\mathbb{Z}/2\mathbb{Z})^r \subset A_M$. Then $M(1/2)$ is a well-defined integral lattice.*

Proof. For $T := M(1/2)$ to be well-defined, we must check that the induced symmetric bilinear form takes integral values. Recall that $(x, y)_T = \frac{1}{2}(x, y)_M$, and thus this is equivalent to showing that $\text{div}_M(x)$ is divisible by 2 for all $x \in M$. Let G be the Gram matrix of M ; it has integral entries, and has Smith Normal form

$$\text{diag}(d_1, \dots, d_r),$$

with $d_1 d_2 \dots d_r \neq 0$, and $d_i | d_{i+1}$ for $i = 1, \dots, r-1$. Further, $d_1 d_2 \dots d_i$ is the greatest common divisor of all the $i \times i$ minors of G . In particular, d_1 is the gcd of all entries of G . Thus in order to prove that $\text{div}_M(x)$ is divisible by 2 for all $x \in M$ it suffices to show that $2 | d_1$.

Since M is a free module over a PID, there exists a basis $u_1, \dots, u_r$ of M^* such that $v_1 = d_1 u_1, \dots, v_r = d_r u_r$ is a basis of M . Hence

$$M^*/M = \frac{\mathbb{Z}u_1 \oplus \dots \mathbb{Z}u_r}{\mathbb{Z}v_1 \oplus \dots \mathbb{Z}v_r} \cong (\mathbb{Z}/d_1\mathbb{Z}) \oplus \dots \oplus (\mathbb{Z}/d_r\mathbb{Z}).$$

Since $(\mathbb{Z}/2\mathbb{Z})^r$ is a subgroup of A_M , we must have that d_1 is divisible by 2; if not, A_M can only contain $(\mathbb{Z}/2\mathbb{Z})^{r-1}$. $\square$

Lemma A.8. *Let M be an even, 2-elementary lattice with $l(A_M) = r$ where r is the rank and $\delta = 1$. Consider the lattice $T := M(\frac{1}{2})$. Then T is an odd unimodular lattice.*

Proof. Assume that $T := M(\frac{1}{2})$ is even. Then for all $x \in T$, we have that $(x \cdot x)_T = 2k$ for some integer k . Thus $(x \cdot x)_M = 4k$.

Since $\delta = 1$, there exists $\gamma \in A_M$ of order 2 with $q(\gamma)_M \in (\mathbb{Q} \setminus \mathbb{Z})/2\mathbb{Z}$. We can write $\gamma = \alpha + M$ for $\alpha \in M^*$, and so $(\alpha \cdot \alpha)_M \in \mathbb{Q} \setminus \mathbb{Z}$. Since γ has order 2, we have that $2\gamma \in M$, i.e. $2\alpha \in M$ and so $\alpha = \frac{x}{2}$ for some $x \in M$. Finally we see that

$$(\alpha \cdot \alpha)_M = \frac{(x \cdot x)_M}{4} = k$$

with k an integer, providing a contradiction. $\square$

Lemma A.9. *Let T be an odd lattice. Then $M := T(2)$ has $\mathbb{Z}/2\mathbb{Z} \subset A_M$ and $q_M(\xi) \notin \mathbb{Z}/2\mathbb{Z}$, where ξ is a generator of $\mathbb{Z}/2\mathbb{Z} \subset A_M$.*

Proof. Since T is odd, there exists an element $v \in T$ with $(v, v)_T = 2k+1$ for some integer k . Thus $(v, v)_M = 4k+2 \equiv 2 \pmod{4}$.

Write $\text{div}_N(v) = d$; then $\text{div}_M(v) = 2d$. Let $v^* = \frac{v}{2d}$, and consider the class $[v^*] \in A_M$. Then $d[v^*] = [\frac{v}{2}]$ is an element of order 2 in A_M , and thus generates a subgroup

$$\langle d[v^*] \rangle \cong \mathbb{Z}/2\mathbb{Z} \subset A_M.$$

Now

$$q(d[v^*]) = d^2 q(v^*) = d^2 \frac{(v, v)_M}{4d^2} = \frac{4k+2}{4} \notin \mathbb{Z}/2\mathbb{Z}.$$

$\square$

Theorem A.10 ([RS81, Section 1]).

- (1) *An even, hyperbolic, p -elementary lattice M of rank r with $p \neq 2$, $r \geq 3$ is uniquely determined by the number $l(A_M)$.*

- (2) For $p \neq 2$, a hyperbolic p -elementary lattice with invariants $r, a := l(A_M)$ exists if and only if the following conditions are satisfied: $a \leq r$, $r \equiv 0 \pmod{2}$, and

$$\begin{cases} \text{if } a \equiv 0 \pmod{2}, \text{ then } r \equiv 2 \pmod{4}, \\ \text{if } a \equiv 1 \pmod{2}, \text{ then } p \equiv (-1)^{r/2-1} \pmod{4}. \end{cases}$$

Moreover, if $r \not\equiv 2 \pmod{8}$, then $r > a > 0$.

APPENDIX B. CUBIC SCROLLS AND THEIR INTERSECTION

The relative position of two planes contained in a cubic fourfold has been well studied [Voi86]. The case of a cubic fourfold containing two cubic scrolls does not seem to appear in the literature. Here we make some comments on the intersection of two such cubic scrolls. This is relevant in the context of §4.2, especially Theorem 4.5. Arithmetically we can compute the intersection of two cubic scrolls $T_1, T_2 \subset X$ as $[T_1] \cdot T_2 = \tau$ for $\tau \in \{1, 3, 5\}$ (Lemma 4.11); here we note that geometrically this corresponds to a syzygetic/azygetic arrangement of twisted cubic curves.

Let $T_1, T_2 \subset X$ be two scrolls belonging to two different families in X . Then $\text{span}(T_1) = H_1$, $\text{span}(T_2) = H_2$ with $H_i \cong \mathbb{P}^4$ and $H_1 \neq H_2$. We are interested in the intersection $T_1 \cap T_2$; this is thus contained in $H_1 \cap H_2 \cap X = S_{12}$, a cubic surface.

Note that $H_1 \cap H_2 \cap T_i$ has degree 3, since $\eta_X \cdot [T_i] = 3$. Since $H_1 \cap H_2 \cong \mathbb{P}^3$, it is clear that $H_1 \cap H_2 \cap T_i = C_i$ is a twisted cubic curve, necessarily belonging to S_{12} . We can thus reduce the problem to studying the intersection of two twisted cubic curves on a cubic surface.

B.1. Twisted cubics on a cubic surface. There are 72 linear systems of twisted cubic curves on a cubic surface S ; these are in 1-1 correspondence with sets of mutually disjoint six lines i.e. a *sixer* (see for example [BK07]).

Consider a twisted cubic $C \subset S$; this corresponds to a sixer denoted $E_1, \dots, E_6 \in S$ such that $E_i^2 = -1$, and $E_i \cdot E_j = 0$. Blowing down these lines E_i determines $\pi : S \rightarrow \mathbb{P}^2$, and C is the strict transform of a line $l \subset \mathbb{P}^2$, i.e. $C \in |\pi^*(l)|$.

By [Dol12, Lemma 9.1.2], to each sixer there exists a unique root $\alpha \in E_6$ such that $E_i \cdot \alpha = 1$ for all $i = 1, \dots, 6$. Further, in this basis

$$\alpha = 2C - (E_1 + \dots + E_6).$$

Note that if $\{F_1, \dots, F_6\}$ is another sixer such that $\{E_1, \dots, E_6\}, \{F_1, \dots, F_6\}$ form a *double six* then the associated root for $\{F_1, \dots, F_6\}$ is $-\alpha$.

Definition B.1 ([Dol12, Lemma 9.1.5]). Let $C, C' \in S$ be twisted cubic curves such that $|C| \neq |C'|$, α, β their associated roots. We say that C and C' are a syzygetic duad if and only if $(\alpha, \beta) \in 2\mathbb{Z}$.

Lemma B.2. Let $C \in S$ be a twisted cubic, $E_1, \dots, E_6$ the associated sixer corresponding to the blow up $\pi : S \rightarrow \mathbb{P}^2$. Let $C' \in S$ be another twisted cubic such that $|C'| \neq |C|$, with associated sixer $\{F_1, \dots, F_6\}$. Then:

$$C \cdot C' = \begin{cases} 2 & \text{if } C, C' \text{ are azygetic and } (\alpha, \beta) = -1, \\ 3 & \text{if } C, C' \text{ form a syzygetic duad,} \\ 4 & \text{if } C, C' \text{ are azygetic and } (\alpha, \beta) = 1, \\ 5 & \text{if } \{E_1, \dots, E_6\}, \{F_1, \dots, F_6\} \text{ form a double sixer.} \end{cases}$$

Proof. By [Dol12, Theorem 9.1.2], one can write down the list of 36 double sixers with corresponding associated roots. If C' corresponds to the sixer $\{F_1, \dots, F_6\}$, then

$$C' \sim \frac{-K_S + \sum F_1 + \dots + F_6}{3},$$

where $K_S \sim 3C - (E_1 + \dots + E_6)$. One checks the intersection numbers for the corresponding sixers in Dolgachev's list. $\square$

B.2. Theta-characteristics of $g = 4$ curve. The 120 pairs of cubic scrolls as in Theorem 4.5 are in 1-1 correspondence with the 120 tritangent planes to a genus 4 degree six canonical curve $C \subset \mathbb{P}^3$. We can also compute the intersection numbers by investigating how these planes intersect.

Let $J(C)_2$ denote the points of order 2 of the Jacobian of C . Then $J(C)_2 \cong (\mathbb{F}_2)^8$ equipped with the Weyl pairing. Then the space of theta-characteristics of C is an affine space over $J(C)_2$.

Each odd theta-characteristic η defines a tritangent hyperplane H_η .

Definition B.3. We say that four theta-characteristics $\theta_1, \theta_2, \theta_3, \theta_4 := \theta_1 + \theta_2 + \theta_3$ are syzygetic if $\sum_{i=1}^4 D_{\theta_i}$ is cut out by a quadric.

Lemma B.4 ([BS20, Lemma 1.8]). *Let $\epsilon \in J(C)_2$ be a two torsion point. Let H_1, H_2 be two tritangent planes with contact divisors D_1, D_2 such that $H_i \cdot C = 2D_i$ and $D_1 + D_2 \in |\omega_C + \epsilon|$. Then there exists a twisted cubic T passing through the six points in the support of $D_1 + D_2$.*

Fixing an $\epsilon \in J(C)_2$, the automorphism group of the 120 tritangent planes can be identified with $O(8, \mathbb{F}_2)^+$ preserving the quadratic form. In particular, if the unique quadric containing C is singular, then C has a **vanishing theta null** - a distinguished even theta characteristic vanishing at the vertex. Note that these curves correspond to del Pezzo surfaces of degree 1; E_8 acts on the 240 lines in pairs. Each pair corresponds to a tritangent plane of C ; thus the automorphism group of the tritangent planes is isomorphic to $W(E_8)/\pm 1$.

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