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The Moduli Space of Cubic Threefolds with a Non-Eckardt Type Involution via Intermediate Jacobians

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There are two types of involutions on a cubic threefold: the Eckardt type (which has been studied by the first named and the third named authors) and the non-Eckardt type. Here we study cubic threefolds with a non-Eckardt type involution, whose fixed locus consists of a line and a cubic curve. Specifically, we consider the period map sending a cubic threefold with a non-Eckardt type involution to the invariant part of the intermediate Jacobian. The main result is that the global Torelli Theorem holds for the period map. To prove the theorem, we project the cubic threefold from the pointwise fixed line and exhibit the invariant part of the intermediate Jacobian as a Prym variety of a (pseudo-)double cover of stable curves. The proof relies on a result of Ikeda and Naranjo–Ortega on the injectivity of the related Prym map. We also describe the invariant part of the intermediate Jacobian via the projection from a general invariant line and show that the two descriptions are related by the bigonal construction.

1 Introduction

Moduli spaces of cubic hypersurfaces are a central object of moduli theory, as they are one of the first examples one can study via a Hodge theoretic period map. Clemens, Griffiths [13], Mumford [34], and Beauville [8] proved the global Torelli Theorem for cubic threefolds—namely, a cubic threefold is determined up to isomorphism by its

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intermediate Jacobian. Based on the work of Voisin, Hassett, Laza, and Looijenga on the period map for cubic fourfolds, Allcock, Carlson, and Toledo [3] and Looijenga and Swierstra [27] have exhibited the moduli space of cubic threefolds as a ball quotient. More specifically, this is achieved via the (eigen)period map for cubic fourfolds admitting an automorphism of order 3, which are obtained as triple covers of $\mathbb{P}^4$ branched along a cubic threefold. Furthermore, Kudla and Rapoport [25] (see also [42]) have interpreted the above construction as a certain map of stacks taking values in a moduli stack of abelian varieties of Picard type; in this way they are also able to describe the field of definition of the period map. It is worth noting that cubic hypersurfaces with additional automorphisms are related to other interesting moduli problems, and have been key ingredients for constructing several new period maps. Besides the moduli of cubic threefolds [3, 27], examples include moduli of cubic surfaces [2] (via cubic threefolds with an order 3 automorphism), moduli of cubic threefold pairs [26], and cubic surface pairs [16] (via cubic fourfolds and cubic threefolds admitting an Eckardt type involution, respectively). In a different direction, cubic threefolds with extra symmetry provide examples of unlikely intersections in the intermediate Jacobian locus [16].

Involutions on cubic fourfolds have been recently studied in [26], [41], and [31] (see also [29]). In this paper, we focus on cubic threefolds admitting a (biregular) involution—these have been classified in for instance [23]. In particular, there are two types of involutions for a cubic threefold; admitting one type is equivalent to having an Eckardt point. The moduli space of cubic threefolds admitting an Eckardt type involution has been studied in [16]; the main result is that the period map sending an Eckardt cubic threefold to the anti-invariant part of the intermediate Jacobian is injective. The purpose of this paper is to study the analogous situation for the remaining involution.

More concretely, we study the moduli space $\mathcal{M}$ of cubic threefolds $X \subset \mathbb{P}^4$ with an involution τ of non-Eckardt type, whose fixed locus in X consists of a line L and a cubic curve C. We define JX^τ to be the invariant part of the intermediate Jacobian JX with respect to the induced involution τ. The abelian subvariety $JX^\tau \subset JX$ is of dimension 3 and inherits a polarization of type $(1,2,2)$, and thus we obtain a period map:

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathcal{A}_3^{(1,2,2)}$$

$$(X, \tau) \mapsto JX^\tau.$$

Our main result is the following global Torelli theorem for $\mathcal{P}$.

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Theorem 0.1 (Global Torelli for cubic three-folds with a non-Eckardt type involution; Theorem 3.1). The period map $\mathcal{P} : \mathcal{M} \to \mathcal{A}_3^{(1,2,2)}$, which sends a cubic threefold X with a non-Eckardt type involution τ to the invariant part $JX^\tau \subset JX$, is injective.

We also prove that the infinitesimal Torelli theorem holds for $\mathcal{P} : \mathcal{M} \to \mathcal{A}_3^{(1,2,2)}$ over an open subset $\mathcal{M}_0 \subset \mathcal{M}$ (this is analogous to the situation for the moduli space of smooth curves of genus greater than 2, where the infinitesimal Torelli theorem holds for the non-hyperelliptic locus; see Remark 9).

Proposition 0.2 (Infinitesimal Torelli for cubic threefolds with a non-Eckardt type involution; Proposition 3.4). Let $\mathcal{M}_0 \subset \mathcal{M}$ be the open subset described in § 3.4. The differential $d\mathcal{P}$ of the period map $\mathcal{P}$ is an isomorphism at every point of $\mathcal{M}_0 \subset \mathcal{M}$. Combining this with Theorem 0.1, $\mathcal{P}|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{A}_3^{(1,2,2)}$ is an open embedding.

The strategy for proving the global Torelli theorem for the period map $\mathcal{P}$ is similar to that in [16]. Specifically, we project (X, τ) from the pointwise fixed line $L \subset X$ to realize the invariant part $JX^\tau \subset JX$ as a Prym variety. In particular, we show that JX^τ is isomorphic to the dual abelian variety of the Prym variety $P(\widetilde{C}, C)$ of a double cover $\pi : \widetilde{C} \to C$ of a genus 1 curve C branched in six points (see Theorem 2.5). The crucial element in the proof is the description of Prym varieties for (pseudo-)double covers of stable curves given in [7]. The associated Prym map (recall that $\mathcal{R}_{g,2n}$ is the moduli space of double covers of smooth genus g curves branched in $2n$ distinct points)

$$\mathcal{P}_{1,6} : \mathcal{R}_{1,6} \to \mathcal{A}_3^{(1,1,2)}$$

is known to be injective (cf. [24] or [36]), allowing us to recover $\pi : \widetilde{C} \to C$ from JX^τ. We then apply the reconstruction result in [9] (see also [14]) to prove Theorem 0.1, noting that the line bundle $\mathcal{L}$ associated with the double cover π allows one to embed C into $\mathbb{P}^2$ as a plane cubic (cf. Proposition 2.2).

We also study the fibration in conics obtained via the projection of a (general) cubic threefold X with a non-Eckardt type involution τ from a (general) invariant line $l \subset X$. Invariant but not pointwise fixed lines in X are parameterized by the curve $\widetilde{C}$. In this direction, we prove that the invariant part JX^τ is isomorphic to the Prym variety $P(D_\tau, \overline{D}_l)$ associated with a double cover $b_\tau : D_\tau \to \overline{D}_l$ of a genus 2 curve $\overline{D}_l$ ramified in four points (c.f. Theorem 4.4). The main techniques used in the proof are those developed in [34], [20], and [38] for studying Galois covers of curves with automorphism group the Klein four group. Letting the invariant line l vary, one would expect the generic injectivity

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of the natural map from $\widetilde{C}$ to the generic fiber of the associated Prym map

$$\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3^{(1,2,2)}.$$

However, this is not the case—if two invariant lines l and l' form a coplanar pair (corresponding to a point $c_{l\cup l'} \in C$) meeting the pointwise invariant line L, then the double covers $b_\tau : D_\tau \to \overline{D}_l$ and $b'_\tau : D'_\tau \to \overline{D}_{l'}$ are isomorphic. Indeed, we prove the following result which relates the double covers $b_\tau : D_\tau \to \overline{D}_l$ (respectively, $b'_\tau : D'_\tau \to \overline{D}_{l'}$) and $\pi : \widetilde{C} \to C$ via the bigonal construction (see for example [20]). This allows us to apply the argument in [36] or [22] to show that the generic fiber of $\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3^{(1,2,2)}$ over a general member $JX^\tau \in \mathcal{A}_3^{(1,2,2)}$ is birational to the elliptic curve C (cf. Proposition 4.6).

Proposition 0.3 (Projection from the pointwise fixed line versus projection from a general invariant line; Proposition 4.5). Notation as above. The towers of double covers

$$D_\tau \xrightarrow{b_\tau} \overline{D}_l \xrightarrow{r} \mathbb{P}^1; \quad D'_\tau \xrightarrow{b'_\tau} \overline{D}_{l'} \xrightarrow{r'} \mathbb{P}^1$$

are both bigonally related to the tower of double covers $\widetilde{C} \xrightarrow{\pi} C \xrightarrow{p} \mathbb{P}^1$, where $r : \overline{D}_l \to \mathbb{P}^1$ (respectively, $r' : \overline{D}_{l'} \to \mathbb{P}^1$) denotes the map determined by the unique g^1_2 of the genus 2 curve $\overline{D}_l$ (respectively, $\overline{D}_{l'}$) and $p : C \to \mathbb{P}^1$ is the projection map from the point $c_{l\cup l'} \in C$. In particular, $b_\tau : D_\tau \to \overline{D}_l$ and $b'_\tau : D'_\tau \to \overline{D}_{l'}$ are isomorphic.

Finally, we note that exhibiting the invariant part JX^τ as a Prym variety turns out to be crucial for the ongoing project of the second named author, in applying the LSV construction (cf. [28] and [39]) to cubic fourfolds with a non-Eckardt type involution. This is important in the geometric study of involutions of hyper-Kähler manifolds of $OG10$ type (see [32]), particularly those involutions induced from a cubic fourfold. Recall that the Prym construction of the intermediate Jacobian of a cubic threefold is central to the work in [28], which associates to a cubic fourfold a hyper-Kähler manifold of $OG10$ type.

We now give an outline of the paper; we work throughout over the complex numbers $\mathbb{C}$. In § 1, we introduce our objects of interest, namely cubic threefolds X with a non-Eckardt type involution τ. We also investigate lines that are invariant under such an involution. In § 2, we exhibit such a cubic threefold X as a conic fibration via projection from the pointwise fixed line $L \subset X$ and describe the invariant part JX^τ as a Prym variety. Using this description, we prove global and infinitesimal Torelli theorems for the period map $\mathcal{P} : \mathcal{M} \to \mathcal{A}_3^{(1,2,2)}$ in § 3. Finally, we discuss an alternative description for $J(X)^\tau$, obtained by projecting X from an invariant line different from L in § 4.

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1 Cubic threefolds with a non-Eckardt type involution

In § 1.1, we introduce cubic threefolds X with a non-Eckardt type involution τ. We then define a period map $\mathcal{P}$ for these cubic threefolds in § 1.2. In order to study the period map $\mathcal{P}$, we will project X from an invariant line to exhibit the intermediate Jacobian JX as a Prym variety; we investigate the τ-invariant lines that are contained in X in § 1.3.

1.1 Involutions of cubic threefolds

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Denote by $\mathrm{Aut}(X)$ the group of automorphisms of X, and by $\mathrm{Lin}(X)$ the subgroup of $\mathrm{Aut}(X)$ consisting of automorphisms induced by projective transformations of the ambient projective space leaving X invariant. By [33, Thm. 1 and 2], assuming $n \geq 2, d \geq 3$ we have that $\mathrm{Aut}(X) = \mathrm{Lin}(X)$, except in the case $n = 2, d = 4$. Moreover, $\mathrm{Aut}(X)$ is finite (again excluding the case $n = 2, d = 4$). As a consequence (and specifying to the case $n = d = 3$), one can obtain a complete classification of prime order automorphisms of smooth cubic threefolds (e.g. [23, Thm. 3.5], see also the references in [29, Rmk. 1.6]). In particular, for involutions we have the following classification.

Proposition 1.1. Let $X = V(F)$ be a smooth cubic threefold in $\mathbb{P}^4$ that admits an involution τ. Applying a linear change of coordinates, we can diagonalize τ, so that

$$\tau : \mathbb{P}^4 \to \mathbb{P}^4, \ \ [x_0, \dots, x_4] \mapsto [(-1)^{a_0} x_0, \dots, (-1)^{a_4} x_4],$$

with $a_i \in \{0, 1\}$. Let $a := (a_0, \dots a_4)$, and let D be the dimension of the family of smooth cubic threefolds that admit the involution τ. Then

1. either $a = (0, 0, 0, 0, 1)$ and $\tau = \tau_1$ fixes pointwise a hyperplane section $S \subset X$ and a point $p \in X \setminus S$. We have that $D = 7$ and
$$F = f(x_0, x_1, x_2, x_3) + \ell(x_0, x_1, x_2, x_3) x_4^2,$$
where ℓ is a homogeneous linear polynomial, and f is homogeneous of degree 3.
2. or $a = (0, 0, 0, 1, 1)$ and $\tau = \tau_2$ fixes pointwise a line $l \subset X$ and a plane cubic $C \subset X$. We have that $D = 6$ and
$$F = x_0 q_0(x_3, x_4) + x_1 q_1(x_3, x_4) + x_2 q_2(x_3, x_4) + g(x_0, x_1, x_2),$$
where each q_i is a homogeneous quadratic polynomial and g is a homogeneous cubic polynomial.

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Proof. See [23, Thm. 3.5]. ■

Admitting an involution of type τ_1 is equivalent to the existence of an Eckardt point. Such a cubic is called an *Eckardt* cubic and has been well studied (see for example [26] and [16]). In this paper, we will focus on studying involutions of the type τ_2; we make the following definition.

Definition 1.2. We call an involution on a smooth cubic threefold of type τ_2 (as in Proposition 1.1) an involution of *non-Eckardt type*.

Throughout, $X \subset \mathbb{P}^4$ is a smooth cubic threefold with an involution τ of non-Eckardt type with equation

$$F = x_0 q_0(x_3, x_4) + x_1 q_1(x_3, x_4) + x_2 q_2(x_3, x_4) + g(x_0, x_1, x_2) = 0, \tag{1}$$

where each $q_i(x_3, x_4)$ is homogeneous of degree 2 and $g(x_0, x_1, x_2)$ is homogeneous of degree 3. The involution

$$\tau : [x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, x_1, x_2, -x_3, -x_4] \tag{2}$$

fixes two complementary linear subspaces of $\mathbb{P}^4$ pointwise; the line $L := V(x_0, x_1, x_2)$ and the plane $\Pi := V(x_3, x_4)$. Notice that the line $L \subset X$, and the fixed curve C is given by the intersection $X \cap \Pi$; i.e. $C = V(g(x_0, x_1, x_2), x_3, x_4)$.

Lemma 1.3. Let (X, τ) be a smooth cubic threefold with a non-Eckardt type involution. Then the fixed curve $C \subset X$ as above is smooth.

Proof. Suppose that C is not smooth. Then there exists $a = [a_0, a_1, a_2] \in C \subset \Pi \cong \mathbb{P}^2_{x_0, x_1, x_2}$ such that $\frac{\partial g}{\partial x_i}(a) = 0$ for $i = 0, 1, 2$. Taking partial derivatives of the Equation (1) shows that X is singular at the point $[a_0, a_1, a_2, 0, 0]$. ■

1.2 The period map for cubic threefolds with a non-Eckardt type involution

Let X be a cubic threefold with the involution τ of non-Eckardt type as discussed in the previous subsection. By abuse of notation, we use τ to denote the involution on the principally polarized intermediate Jacobian JX induced by the involution τ of X. Define the invariant part JX^τ and the anti-invariant part $JX^{-\tau}$, respectively, by

$$JX^\tau := \mathrm{Im}(1 + \tau); \;\; JX^{-\tau} := \mathrm{Im}(1 - \tau). \tag{3}$$

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Note that JX^{τ} and $JX^{-\tau}$ are τ-stable complementary abelian subvarieties of JX (cf. [11, Prop. 13.6.1]).

Lemma 1.4. The abelian subvarieties JX^{τ} and $JX^{-\tau}$ have dimensions 3 and 2, respectively. The principal polarization of JX induces polarizations of type $(1,2,2)$ and $(2,2)$ on JX^{τ} and $JX^{-\tau}$, respectively.

Proof. The abelian subvarieties JX^{τ} and $JX^{-\tau}$ correspond to the symmetric idempotents $\frac{1+\tau}{2}$ and $\frac{1-\tau}{2}$ in $\mathrm{End}_{\mathbb{Q}}(JX)$, respectively. Using [11, Prop. 5.3.10], we compute their dimensions by studying the eigenspace decomposition of τ on $H^{1,2}(X)$ or $H^{2,1}(X)$ for a particular smooth cubic threefold with an involution of non-Eckardt type (e.g. $V(x_0x_3^2 + x_0x_4^2 + x_1x_3^2 + x_2x_4^2 - x_0^3 + x_1^3 + x_2^3)$). Identifying the eigenspaces is a standard computation using Griffiths residues (see for instance [15, Thm. 3.2.10]). The claim on the polarization types will be proved later in Theorem 2.5 (see also Theorem 4.4). Note that here the number of 2's in the polarization types for JX^{τ} and $JX^{-\tau}$ are the same, which for instance follows from [16, Lem. 1.13]. ■

Let $\mathcal{M}$ be the moduli space of cubic threefolds X with an involution τ of non-Eckardt type constructed using GIT (see for example [41, §2.2]). Let $\mathcal{A}_3^{(1,2,2)}$ be the moduli space of abelian threefolds with a polarization of type $(1,2,2)$. Note that $\dim \mathcal{M} = 6$ and $\dim \mathcal{A}_3^{(1,2,2)} = 6$. Define a period map (via Lemma 1.4):

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathcal{A}_3^{(1,2,2)};$$

$$(X,\tau) \mapsto JX^{\tau},$$

which sends a smooth cubic threefold X with an involution τ of non-Eckardt type to the invariant part JX^{τ} of the intermediate Jacobian JX.

1.3 Invariant lines

In order to study the period map $\mathcal{P}$ in § 1.2, we will need to understand how the intermediate Jacobian JX of such a cubic threefold (X,τ) decomposes with respect to the involution τ. As in [16], our strategy will be to project X from a τ-invariant line to exhibit JX as the Prym variety of the associated discriminant double cover.

Lemma 1.5. Let X be a smooth cubic threefold with an involution τ of non-Eckardt type, cut out by Equation (1). Let $l \subset X$ be a τ-invariant line. Then either l is pointwise fixed

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by τ (i.e. $l = L = V(x_0, x_1, x_2)$), or l intersects both the fixed line L and the fixed curve $C = V(g(x_0, x_1, x_2), x_3, x_4) \subset X$.

Proof. We use the notation in § 1.1. The fixed locus of the involution τ acting on $\mathbb{P}^4$ consists of the line L and the plane $\Pi = V(x_3, x_4)$. If τ fixes every point of $l \subset C$, then either $l = L$, or l is a component of C. By Lemma 1.3, the curve C is smooth, and so $l = L$. Otherwise, τ fixes two points of $l \subset X$. One of the points needs to be off of the fixed line L, and hence must be a point of C. Thus l intersects both L and C. ■

Observe that the plane $\langle L, l\rangle$ spanned by the pointwise fixed line L and an invariant line $l \neq L$ is itself τ-invariant and therefore must intersect X along a third invariant line l'. Through projecting X from the pointwise fixed line $L \subset X$, we will see in § 2.2 (and also the proof of Proposition 2.4) that the τ-invariant lines $l \subset X$ which are not pointwise fixed are parameterized by a smooth genus 4 curve $\widetilde{C}$ which is a double cover of C. In other words, the fixed locus of τ on the Fano surface $F(X)$ of lines consists of a point corresponding to the pointwise fixed line L and the other curve $\widetilde{C}$ parameterizing other τ-invariant lines l: $F(X)^\tau = \{L\} \cup \widetilde{C}$.

2 Cubic threefolds with an involution of non-Eckardt type as fibrations in conics I: pointwise fixed line

In this section, we study the intermediate Jacobians JX of cubic threefolds (X, τ) with an involution of non-Eckardt type via projections from the pointwise fixed lines $L \subset X$. Some basic facts about cubic threefolds as fibrations in conics are first recalled in § 2.1. We then focus on cubic threefolds with a non-Eckardt type involution and study the fibrations in conics obtained by projecting these cubic threefolds from pointwise fixed lines in § 2.2. An important observation is that the discriminant quintic curves split as the union of smooth cubic curves and transverse quadratic curves. Based on the observation and the results in [7, §0.3], we give a characterization of the invariant and anti-invariant parts of the intermediate Jacobians in § 2.3.

2.1 Cubic threefolds as fibrations in conics

Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold with a line $l \subset X$. The linear projection with center l expresses X as a conic fibration over a complementary plane $\mathbb{P}^2$; indeed, $\mathbb{P}^2$ also parametrizes the space of $\mathbb{P}^2$-sections of X containing l. The blow up $\mathrm{Bl}_l\mathbb{P}^4$ of the ambient

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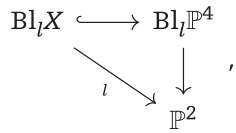
projective space along l gives a commutative diagram



where Bl$_l X$ is the strict transform of X in Bl$_l \mathbb{P}^4$, yielding a fibration in conics $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2$. The discriminant curve is a plane quintic $D \subset \mathbb{P}^2$ which by [7, Prop. 1.2] is stable, and there is an associated pseudo-double cover (Let $\widetilde{D} \to D$ be a double cover of stable curves with the associated covering involution ι. We say that $\widetilde{D} \to D$ is *admissible* if the fixed points of ι are nodes, and the local branches are not interchanged by ι at each fixed node of $\widetilde{D}$. An admissible double cover $\widetilde{D} \to D$ is called *allowable* (see [6, (**), p. 173] and [21, §I.1.3]) if the associated Prym is compact. An allowable double cover $\widetilde{D} \to D$ is said to be a *pseudo-double cover* (cf. [6, (*), p. 157] and [7, Def. 0.3.1]) if the fixed points of ι are exactly the nodes of $\widetilde{D}$.) $\pi : \widetilde{D} \to D$ determined by interchanging the lines in the fiber of π_l over the points of D (cf. [7, Prop. 1.5]). For a projection from a general line $l \subset X$, D is smooth and $\widetilde{D} \to D$ is connected and étale (see for example [13, Appendix C]).

Associated with the discriminant double cover $\pi : \widetilde{D} \to D$ is a rank-1 torsion-free sheaf η_D which is reflexive, i.e. $\mathcal{H}om(\eta_D, \mathcal{O}_D) \cong \eta_D$ (more precisely, $\widetilde{D} \to D$ is constructed as $\underline{\mathrm{Spec}}_D(\mathcal{O}_D \oplus \eta_D) \to D$, where the $\mathcal{O}_D$-algebra structure on $\mathcal{O}_D \oplus \eta_D$ is induced by $\mathcal{H}om(\eta_D, \mathcal{O}_D) \cong \eta_D$). Let $\kappa_D := \eta_D \otimes \mathcal{O}_D(1)$. Then κ_D is an odd theta characteristic satisfying $\mathcal{H}om(\kappa_D, \omega_D) \cong \kappa_D$ and $h^0(D, \kappa_D) = 1$. Note also that κ_D (and therefore η_D) is locally free at a point $d \in D$ if and only if $\widetilde{D} \to D$ is étale over d. By [7, §1.6] and [14, Prop. 4.2], the conic fibration construction gives a one-to-one correspondence between pairs (X, l) consisting of a smooth cubic threefold X and a line $l \subset X$ and pairs (D, κ_D), where D is a stable plane quintic curve and κ_D is a theta characteristic with $h^0(D, \kappa_D) = 1$, both up to projective linear transformations.

The above construction can also be described in coordinates. We may assume that $l \subset \mathbb{P}^4$ is cut out by $x_0 = x_1 = x_2 = 0$. Since $l \subset X$, the equation of X is of the form

$$\ell_1(x_0, x_1, x_2)x_3^2 + \ell_2(x_0, x_1, x_2)x_4^2 + 2\ell_3(x_0, x_1, x_2)x_3x_4$$
$$+ 2q_1(x_0, x_1, x_2)x_3 + 2q_2(x_0, x_1, x_2)x_4 + c(x_0, x_1, x_2) = 0,$$

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where ℓ_i, q_j and c are homogeneous polynomials of degree 1, 2, and 3, respectively. Let M be the matrix

$$M = \begin{pmatrix} \ell_1(x_0,x_1,x_2) & \ell_3(x_0,x_1,x_2) & q_1(x_0,x_1,x_2) \\ \ell_3(x_0,x_1,x_2) & \ell_2(x_0,x_1,x_2) & q_2(x_0,x_1,x_2) \\ q_1(x_0,x_1,x_2) & q_2(x_0,x_1,x_2) & c(x_0,x_1,x_2) \end{pmatrix}. \tag{4}$$

Then the discriminant quintic curve $D \subset V(x_3,x_4) \cong \mathbb{P}^2$ for the conic fibration $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2$ is cut out by the discriminant of M: $D = V(\det(M))$. In particular, a point $d \in D$ is a smooth point if and only if the corank of M at d is 1 (note that because X is smooth the corank of M is at most 2). Moreover, following [9, Prop. 4.2] and [14, Thm. 4.1] the theta characteristic κ_D admits a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2} \to \kappa_D \to 0.$$

(Indeed, the smooth cubic threefold X and the line $l \subset X$ can be recovered from the above minimal resolution of κ_D up to projective linear transformations, cf. [14, Prop. 4.2].) When the plane discriminant quintic D is smooth and the discriminant double cover $\pi : \widetilde{D} \to D$ is connected and étale, the theta characteristic κ_D corresponds to the divisor $\sqrt{(\ell_1\ell_2 - \ell_3^2 = 0)}$ (which is the unique effective divisor such that twice of the divisor is the divisor $(\ell_1\ell_2 - \ell_3^2 = 0)$ on D), and the étale double cover $\widetilde{D} \to D$ is associated with the 2-torsion line bundle $\eta_D = \kappa_D(-1)$.

Denote the Prym variety of the discriminant pseudo-double cover $\pi : \widetilde{D} \to D$ by $P(\widetilde{D}, D)$, which is defined as

$$P(\widetilde{D}, D) := (\ker(\mathrm{Nm}_\pi : J(\widetilde{D}) \to J(D)))^0$$

(cf. [34, §3] and [6, §3]). For later use, let us give an explicit description of $P(\widetilde{D}, D)$ following [7, §0.3]. Set $\tilde{\nu} : N(\widetilde{D}) \to \widetilde{D}$ (respectively, $\nu : N(D) \to D$) to be the normalization of $\widetilde{D}$ (respectively, D). Denote by $\pi' : N(\widetilde{D}) \to N(D)$ the induced double cover. By [6, Prop. 3.5], there exists an isogeny $\tilde{\nu}^* : P(\widetilde{D}, D) \to P(N(\widetilde{D}), N(D))$. More precisely, denote by $\Theta_{J(N(\widetilde{D}))}$ the principal polarization on $J(N(\widetilde{D}))$ and consider the restriction of $\Theta_{J(N(\widetilde{D}))}$ to $P(N(\widetilde{D}), N(D))$:

$$\Theta_{P(N(\widetilde{D}),N(D))} := \Theta_{J(N(\widetilde{D}))}|_{P(N(\widetilde{D}),N(D))}.$$

By [6, Thm. 3.7], $\Theta_{P(N(\widetilde{D}),N(D))}$ induces twice of a principal polarization Ξ on $P(\widetilde{D}, D)$:

$$(\tilde{\nu}^*)^{-1}\Theta_{P(N(\widetilde{D}),N(D))} \equiv_{\mathrm{alg}} 2\Xi.$$

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In other words, the isogeny above is an isogeny of polarized abelian varieties:

$$\tilde{\nu}^* : (P(\widetilde{D}, D), 2\Xi) \to (P(N(\widetilde{D}), N(D)), \Theta_{P(N(\widetilde{D}),N(D))}).$$

From [11, Prop. 12.1.3], we deduce that

$$(P(N(\widetilde{D}), N(D))^\vee, (\Theta_{P(N(\widetilde{D}),N(D))})^\vee) \cong (J(N(\widetilde{D}))/\pi'^* J(N(D)), \Theta')$$

where Θ' denotes the dual polarization (Here we are using the dual polarization defined in [10, Thm. 2.1] (see also [16, Rmk. 1.14]) which is slightly different from the one used in [11, §14.4]. In particular, for a polarization of type $(d_1, d_2, \ldots, d_g)$ the dual polarization has type $(\frac{d_g}{d_g}, \frac{d_g}{d_{g-1}}, \ldots, \frac{d_g}{d_1})$.). As a result, we get the dual isogeny of polarized abelian varieties

$$(\tilde{\nu}^*)^\vee : (J(N(\widetilde{D}))/\pi'^* J(D), \Theta') \to (P(\widetilde{D}, D)^\vee, (2\Xi)^\vee) \cong (P(\widetilde{D}, D), \Xi).$$

For a pseudo-double cover, the kernel of $(\tilde{\nu}^*)^\vee$ has been described in [7, §0.3] (see also [1, p. 76]). Specifically, let $H' \subset \mathrm{Pic}(N(\widetilde{D}))$ be the subgroup generated by $\mathcal{O}_{N(\widetilde{D})}(s - s')$, where $s, s' \in N(\widetilde{D})$ with $\tilde{\nu}(s) = \tilde{\nu}(s')$. Set H to be the image of $H_0 := H' \cap J(N(\widetilde{D}))$ in the quotient $J(N(\widetilde{D}))/\pi'^* J(N(D))$. Then H is the kernel of the isogeny of polarized abelian varieties

$$0 \to H \to J(N(\widetilde{D}))/\pi'^* J(D) \xrightarrow{(\tilde{\nu}^*)^\vee} P(\widetilde{D}, D) \to 0.$$

By [13, Appendix C] and [7, Thm. 2.1], the conic fibration construction $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2$ induces a canonical isomorphism of principally polarized abelian varieties

$$JX \cong P(\widetilde{D}, D)$$

between the intermediate Jacobian JX of the smooth cubic threefold X and the Prym variety $P(\widetilde{D}, D)$ of the discriminant double cover $\widetilde{D} \to D$.

2.2 Projecting cubic threefolds with a non-Eckardt type involution from the pointwise fixed lines

Let (X, τ) be a smooth cubic threefold with a non-Eckardt type involution as in § 1.1. Let $L \subset X$ be the pointwise fixed line under τ (see Lemma 1.5). We can rewrite the equation of X in Equation (1) as

$$\ell_1(x_0, x_1, x_2)x_3^2 + \ell_2(x_0, x_1, x_2)x_4^2 + 2\ell_3(x_0, x_1, x_2)x_3x_4 + g(x_0, x_1, x_2) = 0, \tag{5}$$

where $\ell_i(x_0, x_1, x_2)$ are linear forms, and $g(x_0, x_1, x_2)$ is homogeneous of degree 3. As in § 2.1 we project X from the fixed line $L = V(x_0, x_1, x_2)$ to the complementary plane

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$\Pi := V(x_3, x_4) \cong \mathbb{P}^2_{x_0,x_1,x_2}$, and obtain a fibration in conics $\pi_L : \mathrm{Bl}_L X \to \Pi$. The plane discriminant quintic $D_L \subset \Pi$ has equation $\det M = 0$ where the matrix M is

$$M = \begin{pmatrix} \ell_1(x_0,x_1,x_2) & \ell_3(x_0,x_1,x_2) & 0 \\ \ell_3(x_0,x_1,x_2) & \ell_2(x_0,x_1,x_2) & 0 \\ 0 & 0 & g(x_0,x_1,x_2) \end{pmatrix}. \tag{6}$$

We are ready to see that D_L is nodal and consists of the following components: a smooth plane cubic

$$C := V(g(x_0,x_1,x_2), x_3, x_4) \subset \Pi,$$

and a (possibly degenerate) plane conic

$$Q := V(\ell_1(x_0,x_1,x_2)\ell_2(x_0,x_1,x_2) - \ell_3^2(x_0,x_1,x_2), x_3, x_4) \subset \Pi.$$

Note that $C = X \cap \Pi$ is the pointwise fixed curve by the involution τ in § 1.1.

Lemma 2.1. Let (X, τ) be a smooth cubic threefold with a non-Eckardt type involution as above, and let $\pi_L : \mathrm{Bl}_L X \to \Pi$ be the projection from the fixed line $L \subset X$. Then the discriminant curve D_L is a union of a cubic curve C and a conic curve Q. Moreover, the cubic component C is smooth, and C meets the conic component Q transversely.

Proof. The claim that C is smooth has been verified in Lemma 1.3. It is shown in [7, Prop. 1.2] that D_L is at worst nodal, and therefore C and Q meet transversally. ■

Denote by $\pi_L : \widetilde{D}_L \to D_L = C \cup Q$ the discriminant double cover of the fibration in conics $\pi_L : \mathrm{Bl}_L X \to \Pi$. Note that $\pi_L : \widetilde{D}_L \to D_L$ is branched at the intersection points $C \cap Q$. Let us also observe that $\widetilde{D}_L = \widetilde{C} \cup \widetilde{Q}$, where $\widetilde{C}$ (respectively, $\widetilde{Q}$) is a double cover of the smooth cubic C (respectively, the conic Q) ramified in the intersection points $C \cap Q$. We now restrict the discriminant double cover $\pi_L : \widetilde{D}_L = \widetilde{C} \cup \widetilde{Q} \to D_L = C \cup Q$ to C and focus on the obtained double cover $\pi_L|_{\widetilde{C}} : \widetilde{C} \to C$ (if no confusion is likely to be caused we will simply write π instead of $\pi_L|_{\widetilde{C}}$). Specifically, we describe the quadruple $(C, \beta, \mathcal{L}, s)$ corresponding to $\pi : \widetilde{C} \to C$, where β is the branch divisor, $\mathcal{L}$ is the associated line bundle on C satisfying $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(\beta)$, and s is a section of $\mathcal{O}_C(\beta)$ vanishing on β. For $\pi : \widetilde{C} \to C$, clearly one has $\beta = C \cap Q$ and $s = \ell_1(x_0,x_1,x_2)\ell_2(x_0,x_1,x_2) - \ell_3^2(x_0,x_1,x_2)$. It remains to determine the associated line bundle $\mathcal{L}$.

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Proposition 2.2. The double cover $\pi : \widetilde{C} \to C$ obtained by restricting the discriminant double cover $\pi_L : \widetilde{D}_L \to D_L$ to the smooth cubic component C is associated with the line bundle $\mathcal{L} = \mathcal{O}_C(1)$.

Proof. We may assume that the conic Q is smooth (note that this is the case for a general cubic threefold with a non-Eckardt type involution; a similar argument applies to the case when Q has rank 2). As in § 2.1, set η_{D_L} to be the rank-1 torsion-free sheaf which is associated with the discriminant double cover $\widetilde{D}_L \to D_L$ and satisfies $\mathcal{H}om(\eta_{D_L}, \mathcal{O}_D) \cong \eta_{D_L}$. Let $\kappa_{D_L} = \eta_{D_L} \otimes \mathcal{O}_{D_L}(1)$ be the theta characteristic on D_L. By a result of Beauville (see [9, Prop. 4.2] and [14, Thm. 4.1]), there exists a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2} \to \kappa_{D_L} \to 0,$$

where $\mathbb{P}^2$ denotes the plane $\Pi = V(x_3, x_4) \cong \mathbb{P}^2_{x_0,x_1,x_2}$ and M is the matrix in Equation (6). Restricting the above exact sequence to the smooth cubic component C, one gets the following sequence which coincides with the closed subscheme sequence for $C = V(g) \subset \mathbb{P}^2$:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\cdot g} \mathcal{O}_{\mathbb{P}^2} \to (\kappa_{D_L}|_C)/\text{torsion} \to 0.$$

As a consequence, we get $(\kappa_{D_L}|_C)/\text{torsion} \cong \mathcal{O}_C$. Similarly, from [9, Prop. 4.2] one deduces that $(\kappa_{D_L}|_Q)/\text{torsion}$ is $\theta_Q := \mathcal{O}_Q(-pt)$. Since $\eta_{D_L} = \kappa_{D_L} \otimes \mathcal{O}_{D_L}(-1)$, we have that $(\eta_{D_L}|_C)/\text{torsion} \cong \mathcal{O}_C(-1)$ and that $(\eta_{D_L}|_Q)/\text{torsion} \cong \theta_Q(-1)$. Since $\pi_L : \widetilde{D}_L \to D_L$ is associated with the rank-1 reflexive sheaf η_{D_L}, it holds that $\pi_{L*}\mathcal{O}_{\widetilde{D}_L} \cong \mathcal{O}_{D_L} \oplus \eta_{D_L}$ (also compare [18, Prop. 2.5]). Pulling back to C, one gets

$$\pi_*\mathcal{O}_{\widetilde{C}} \cong (\pi_{L*}\mathcal{O}_{\widetilde{D}_L}|_C)/\text{torsion} \cong \mathcal{O}_C \oplus (\eta_{D_L}|_C)/\text{torsion}.$$

The isomorphism on the left arises as follows. The pull-back and push-forward functors induce a natural morphism $\pi_{L*}\mathcal{O}_{\widetilde{D}_L}|_C \to \pi_*(\mathcal{O}_{\widetilde{D}_L}|_{\widetilde{C}}) = \pi_*\mathcal{O}_{\widetilde{C}}$. Since $\pi_*\mathcal{O}_{\widetilde{C}}$ is a rank-2 vector bundle, the morphism factors through the quotient by the torsion sub-sheaf, and then a local computation at the nodes shows the morphism is an isomorphism. Now, because $(\eta_{D_L}|_C)/\text{torsion} \cong \mathcal{O}_C(-1)$, the line bundle $\mathcal{L}$ determining the double cover $\pi : \widetilde{C} \to C$ is then $((\eta_{D_L}|_C)/\text{torsion})^\vee \cong \mathcal{O}_C(1)$. ∎

Remark 7. We give the following characterizations of the rank-1 torsion-free reflexive sheaf η_{D_L}. Without loss of generality, we assume the conic component Q of the discriminant quintic D_L is smooth.

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1. Let $\nu : C \coprod Q \to D_L = C \cup Q$ be the normalization map. Denote the preimages of the six intersection points $d_1, \ldots, d_6 \in C \cap Q$ on C (respectively, Q) by $c_1, \ldots, c_6$ (respectively, $q_1, \ldots, q_6$). Note that η_{D_L} is not locally free at the intersection points $C \cap Q$. Pulling back η_{D_L} via the normalization map ν gives a line bundle $\mathcal{L}'$ (denote the corresponding geometric line bundle by $\mathbb{L}'$) on $C \coprod Q$ together with gluing maps along fibers $\alpha_{CQ,i} = 0 : \mathbb{L}'_{c_i} \to \mathbb{L}'_{q_i}$ which are all zero for $1 \leq i \leq 6$. Equivalently, we could also describe η_{D_L} using the data $(\mathcal{L}'' := \mathcal{L}'(\Sigma_i c_i - \Sigma_i q_i), \alpha_{QC,i} = 0 : \mathbb{L}''_{q_i} \to \mathbb{L}''_{c_i})$ (again $\mathbb{L}''$ denotes the geometric line bundle corresponding to $\mathcal{L}''$). From the proof of Proposition 2.2, we deduce that $\mathcal{L}'|_C = (\eta_{D_L}|_C)/\text{torsion} \cong \mathcal{O}_C(-1)$ and $\mathcal{L}''|_Q = (\eta_{D_L}|_Q)/\text{torsion} \cong \theta_Q(-1)$. In other words, η_{D_L} corresponds to the data $(\mathcal{L}_C \cong \mathcal{O}_C(-1), \mathcal{L}_Q(\Sigma_i q_i) \cong \theta_Q(2), \alpha_{CQ,i} = 0)$, or equivalently, to the data $(\mathcal{L}_C(\Sigma_i c_i) \cong \mathcal{O}_C(1), \mathcal{L}_Q \cong \theta_Q(-1), \alpha_{QC,i} = 0)$. In particular, it holds that $\eta_{D_L} \cong \mathcal{H}om(\eta_{D_L}, \mathcal{O}_D)$.
2. Alternatively, we describe the rank-1 torsion-free sheaf η_{D_L} using line bundles over a semistable model of $D_L = C \cup Q$. Specifically, set $T := \underline{\text{Proj}}_{D_L} \text{Sym}^\bullet \eta_{D_L}$. Note that η_{D_L} is not locally free at the nodes $C \cap Q$. Then T is a semistable curve obtained by replacing every intersection point $d_i \in C \cap Q$ with $1 \leq i \leq 6$ by a smooth rational component E_i which meets the component C at c_i and the component Q at q_i. Moreover, T admits a natural map $\text{st} : T \to D_L$ contracting the exceptional components $E_1, \ldots, E_6$. Letting ξ be the tautological invertible sheaf on T which has total degree 0, we have that $\text{st}_* \xi = \eta_{D_L}$ and $\xi|_{E_i} \cong \mathcal{O}_{E_i}(1)$. From $(\eta_{D_L}|_C)/\text{torsion} \cong \mathcal{O}_C(-1)$ and $(\eta_{D_L}|_Q)/\text{torsion} \cong \theta_Q(-1)$, one gets that $\xi|_C \cong \mathcal{O}_C(-1)$ and $\xi|_Q \cong \theta_Q(-1)$. Now set $\gamma : \xi^{\otimes 2} \to \mathcal{O}_T$ to be the homomorphism which vanishes on the exceptional components $E_1, \ldots, E_6$ and coincides with $(\xi|_C)^{\otimes 2} \cong \mathcal{O}_C(-\Sigma_i c_i) \hookrightarrow \mathcal{O}_C$ on C and $(\xi|_Q)^{\otimes 2} \cong \mathcal{O}_Q(-\Sigma_i q_i) \hookrightarrow \mathcal{O}_Q$ on Q. Then (T, ξ, γ) is a Prym curve in the sense of [5, Def. 1] (see also [17]). Moreover, under the isomorphism between the moduli space $\overline{Pr}_g^+$ of non-trivial genus g Prym curves and the moduli space $\overline{\mathcal{R}}_g$ of admissible double covers of stable curves of genus g described in [5, Prop. 5], the Prym curve (T, ξ, γ) corresponds to the admissible discriminant double cover $\widetilde{D}_L \to D_L$.

We summarize the discussion in the below proposition.

Proposition 2.3. Let (X, τ) be a smooth cubic threefold with an involution τ of non-Eckardt type, and set $L \subset X$ to be the pointwise fixed line. Then the discriminant plane

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quintic D_L for the projection $\pi_L : \mathrm{Bl}_L \to \Pi \cong \mathbb{P}^2$ is the union $D_L = C \cup Q$ of a smooth plane cubic C and a transverse conic Q. Moreover, the restriction of the discriminant double cover $\pi_L : \widetilde{D}_L \to D_L$ to the cubic component C is a double cover $\pi : \widetilde{C} \to C$ branched in the six intersection points $C \cap Q$ and associated with the line bundle $\mathcal{O}_C(1)$.

2.3 The intermediate Jacobians of cubic threefolds with a non-Eckardt type involution via the projections from the pointwise fixed line

Let (X, τ) be a smooth cubic threefold with an involution of non-Eckardt type as in Equations (1) and (2). Consider as in the previous subsection the conic fibration $\pi_L : \mathrm{Bl}_L X \to \Pi = V(x_3, x_4) \cong \mathbb{P}^2_{x_0,x_1,x_2}$ obtained by projecting X from the unique pointwise fixed line $L \subset X$. Denote the discriminant double cover by $\pi_L : \widetilde{D}_L \to D_L$, where $\widetilde{D}_L = \widetilde{C} \cup \widetilde{Q}$ and $D_L = C \cup Q$. Also let $\pi : \widetilde{C} \to C$ be the restriction of π_L to C. Since L is fixed, there exists an involution on $\mathrm{Bl}_L X$ induced by τ, which further induces an involution on $\widetilde{D}_L$ commuting with $\pi_L : \widetilde{D}_L \to D_L$. By abuse of notation, we still denote by τ the involution on $\widetilde{D}_L$ (and also the involution on the Prym variety $P(\widetilde{D}_L, D_L)$) induced by the non-Eckardt type involution on X. Define the invariant part $P(\widetilde{D}_L, D_L)^\tau := \mathrm{Im}(1 + \tau)$ and the anti-invariant part $P(\widetilde{D}_L, D_L)^{-\tau} := \mathrm{Im}(1 - \tau)$. As recalled in § 2.1, the intermediate Jacobian JX is canonically isomorphic to the Prym variety $P(\widetilde{D}_L, D_L)$. We now give an explicit description of $P(\widetilde{D}_L, D_L)$ following [7, §0.3] which allows us to study the induced involution τ on $JX \cong P(\widetilde{D}_L, D_L)$.

Let $\widetilde{\nu} : \widetilde{C} \coprod \widetilde{Q} \to \widetilde{C} \cup \widetilde{Q}$ and $\nu : C \coprod Q \to C \cup Q$ be the normalizations of $\widetilde{D}_L = \widetilde{C} \cup \widetilde{Q}$ and $D_L = C \cup Q$, respectively. Let

$$\pi'_L : \widetilde{C} \coprod \widetilde{Q} \to C \coprod Q$$

be the double cover induced by the discriminant double cover $\pi_L : \widetilde{C} \cup \widetilde{Q} \to C \cup Q$. Denote the ramification points of $\widetilde{C} \to C$ (respectively, $\widetilde{Q} \to Q$) by $\tilde{c}_1, \ldots \tilde{c}_6 \in \widetilde{C}$ (respectively, $\tilde{q}_1, \ldots \tilde{q}_6 \in \widetilde{Q}$). Note that $\widetilde{\nu}(\tilde{c}_i) = \widetilde{\nu}(\tilde{q}_i)$ for $i = 1, \ldots 6$. Following [7, §0.3] (see also § 2.1), we set H' to be the subgroup of $\mathrm{Pic}(\widetilde{C} \coprod \widetilde{Q})$ generated by $\mathcal{O}(\tilde{c}_i - \tilde{q}_i)$ for $1 \leq i \leq 6$. Let H denote the image of $H_0 := H' \cap J(\widetilde{C} \coprod \widetilde{Q})$ in the quotient abelian variety

$$J(\widetilde{C} \coprod \widetilde{Q})/\pi'^*_L J(C \coprod Q) \cong \left(J(\widetilde{C})/\pi^* J(C)\right) \times J(\widetilde{Q}).$$

By [7, Exer. 0.3.5], H consists of 2-torsion elements and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. Furthermore, H is the kernel of the isogeny of polarized abelian varieties

$$\phi := (\tilde{\nu}^*)^\vee : J(\widetilde{C} \coprod \widetilde{Q})/\pi'^*_L J(C \coprod Q) \cong (J(\widetilde{C})/\pi^* J(C)) \times J(\widetilde{Q}) \to P(\widetilde{D}_L, D_L).$$

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More precisely, $P(\widetilde{D}_L, D_L)$ admits the principal polarization defined in [6, Thm. 3.7]. Following [11, Prop. 12.1.3], $J(\widetilde{C} \coprod \widetilde{Q})/\pi_L'^* J(C \coprod Q)$ is dual to $P(\widetilde{C} \coprod \widetilde{Q}, C \coprod Q) \cong P(\widetilde{C}, C) \times P(\widetilde{Q}, Q)$, and the dual polarization on it corresponds to the product of the following polarizations on $J(\widetilde{C})/\pi^* J(C)$ and on $J(\widetilde{Q})$: $J(\widetilde{C})/\pi^* J(C)$ is the dual abelian variety (See for instance [11, Prop. 12.1.3]. Let us also recall the following. Let $\pi : \widetilde{C} \to C$ be a connected double cover of smooth curves branched in $2r$ points. Then the principal polarization on $J(\widetilde{C})$ induces a polarization on the Prym variety $P(\widetilde{C}, C)$ which is of type $(1, \ldots, 1, 2, \ldots, 2)$ with 1's repeated $\max\{0, r-1\}$ times. As the dual abelian variety, $J(\widetilde{C})/\pi^* J(C)$ is equipped with a dual polarization of type $(1, \ldots, 1, 2, \ldots, 2)$ with 2's repeated $\max\{0, r-1\}$ times (compare [10, §2]).) to $P(\widetilde{C}, C)$ and therefore comes with a dual polarization, and $J(\widetilde{Q})$ is equipped with twice of the canonical principal polarization.

Proposition 2.4. Notation as above. There exists an isogeny of polarized abelian varieties

$$\phi : \left(J(\widetilde{C})/\pi^* J(C)\right) \times J(\widetilde{Q}) \to P(\widetilde{D}_L, D_L)$$

with kernel $H \cong (\mathbb{Z}/2\mathbb{Z})^4$. Moreover, with respect to the action τ on $P(\widetilde{D}_L, D_L)$ induced by the non-Eckardt type involution on X, the isogeny induces isomorphisms of polarized abelian varieties

$$P(\widetilde{D}_L, D_L)^{\tau} \cong J(\widetilde{C})/\pi^* J(C); \;\; P(\widetilde{D}_L, D_L)^{-\tau} \cong J(\widetilde{Q}).$$

Proof. The proof is quite similar to that of [16, Prop. 3.10]. The existence of the isogeny ϕ and the description of the kernel H is the content of [7, Prop. 0.3.3] (see also § 2.1). It suffices to prove the assertion regarding the invariant and anti-invariant abelian subvarieties $P(\widetilde{D}_L, D_L)^{\tau}$ and $P(\widetilde{D}_L, D_L)^{-\tau}$.

Let $\iota = \iota_L$ be the covering involution associated with the double cover $\pi_L : \widetilde{D}_L = \widetilde{C} \cup \widetilde{Q} \to D_L = C \cup Q$. Consider the involution $\tau : \widetilde{C} \cup \widetilde{Q} \to \widetilde{C} \cup \widetilde{Q}$ induced by the non-Eckardt type involution on X. We claim that the action of τ on $\widetilde{C}$ is trivial, while the action on $\widetilde{Q}$ coincides with ι. Recall that the curve $\widetilde{D}_L = \widetilde{C} \cup \widetilde{Q}$ parametrizes the residual lines to L in a degenerate fiber of the conic fibration $\pi_L : \mathrm{Bl}_L X \to \Pi = V(x_3, x_4) \cong \mathbb{P}^2_{x_0, x_1, x_2}$. Take a point $x \in \Pi$, and consider the plane it corresponds to, namely, the span $\langle L, x \rangle \subset \mathbb{P}^4$. Since both L and $x \in \Pi$ are fixed by $\tau : [x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, x_1, x_2, -x_3, -x_4]$, the plane $\langle L, x \rangle$ is invariant under τ. Assume now that x is a point on C or Q but not both (otherwise, the claim clearly holds). Then the $\mathbb{P}^2$-section $\langle L, x \rangle \cap X$ is three distinct lines, $L \cup M \cup M'$. The lines M, M' correspond to distinct points on $\widetilde{C} \cup \widetilde{Q}$. The involution τ either interchanges

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M and M', or it leaves both fixed. In other words, τ either acts as the identity on a point of $\widetilde{C} \cup \widetilde{Q}$, or by the covering involution ι. The curve $C \subset \mathbb{P}^2$ parametrizes pairs of lines such that each line is invariant under the involution of non-Eckardt type on X (Lemma 1.5), and so the points of $\widetilde{C}$ are fixed by the action of τ. The lines M, M' parametrized by $x \in Q \subset \mathbb{P}^2$ are not preserved and hence must be interchanged by the involution of non-Eckardt type on X; thus τ acts by the covering involution ι on $\widetilde{Q}$ as claimed.

It follows that the isogeny ϕ is equivariant with respect to the involution $\sigma := (1, \iota)$ on the product $(J(\widetilde{C})/\pi^* J(C)) \times J(\widetilde{Q})$. Since $\mathrm{Im}(1+\iota) = (\pi_L|_{\widetilde{Q}})^* J(Q) = 0$ on $J(\widetilde{Q})$, the invariant part $\mathrm{Im}(1+\sigma) = (J(\widetilde{C})/\pi^* J(C)) \times \{0\}$. Similarly, the anti-invariant part $\mathrm{Im}(1-\sigma) = \{0\} \times J(\widetilde{Q})$. We then deduce that ϕ induces isogenies

$$\big(J(\widetilde{C})/\pi^* J(C)\big) \times \{0\} \to P(\widetilde{D}_L, D_L)^{\tau}; \;\; \{0\} \times J(\widetilde{Q}) \to P(\widetilde{D}_L, D_L)^{-\tau},$$

which are isogenies of polarized abelian varieties since ϕ preserves the polarizations. It is not difficult to show that $H \cap ((J(\widetilde{C})/\pi^* J(C)) \times \{0\}) = H \cap (\{0\} \times J(\widetilde{Q})) = \{(0,0)\}$ using the description of the kernel H given earlier in this subsection. As a result, the above isogenies are isomorphisms which completes the proof of the proposition. ■

Putting the above discussion together, we obtain the following theorem.

Theorem 2.5 (Intermediate Jacobian via projecting from the pointwise fixed line). Let (X, τ) be a smooth cubic threefold with an involution τ of non-Eckardt type fixing pointwise a line $L \subset X$. Let JX be the intermediate Jacobian which is principally polarized. Projecting X from L, we obtain a fibration in conics $\pi_L : \mathrm{Bl}_L X \to \Pi \cong \mathbb{P}^2$. Let $\pi_L : \widetilde{D}_L = \widetilde{C} \cup \widetilde{Q} \to D_L = C \cup Q$ be the discriminant double cover, where C is a smooth cubic and Q is a conic intersecting C transversely. We identify $J(\widetilde{C})/\pi^* J(C)$ as the dual abelian variety to $P(\widetilde{C}, C)$ with the dual polarization. We also equip the Jacobian $J(\widetilde{Q})$ with twice of the canonical principal polarization. Then there exists an isogeny of polarized abelian varieties

$$\phi : \big(J(\widetilde{C})/\pi^* J(C)\big) \times J(\widetilde{Q}) \to JX$$

with $\ker \phi \cong (\mathbb{Z}/2\mathbb{Z})^4$. Moreover, with respect to the action of τ on JX, the isogeny ϕ induces isomorphisms of polarized abelian varieties

$$JX^{\tau} \cong J(\widetilde{C})/\pi^* J(C); \;\; JX^{-\tau} \cong J(\widetilde{Q}).$$

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Proof. Since the intermediate Jacobian JX is canonically isomorphic to the Prym variety $P(\widetilde{D}_L, D_L)$, and the involutions are both induced from the non-Eckardt type involution on X, the theorem follows from Proposition 2.4. ■

3 Global Torelli for cubic threefolds with an involution of non-Eckardt type

Let $\mathcal{M}$ be the moduli space of cubic threefolds with a non-Eckardt type involution, and set $\mathcal{A}_3^{(1,2,2)}$ to be the moduli space of abelian threefolds with a polarization of type $(1,2,2)$. Recall that we have defined in § 1.2 the following period map:

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathcal{A}_3^{(1,2,2)};$$

$$(X, \tau) \mapsto JX^{\tau},$$

which sends a smooth cubic threefold X with a non-Eckardt type involution τ to the invariant part JX^{τ} of the intermediate Jacobian. The goal of this section is to prove the following global Torelli theorem for $\mathcal{P}$.

Theorem 3.1 (Global Torelli for cubic threefolds with a non-Eckardt type involution). The period map $\mathcal{P} : \mathcal{M} \to \mathcal{A}_3^{(1,2,2)}$ for smooth cubic threefolds with a non-Eckardt type involution is injective.

Let us briefly outline the strategy. Let (X, τ) be a smooth cubic threefold with an involution of non-Eckardt type. We have shown in § 2.2 that projecting X from the unique pointwise fixed line $L \subset X$ (as in Lemma 1.5) determines a double cover $\pi : \widetilde{C} \to C$ which is the restriction of the discriminant double cover $\pi_L : \widetilde{D}_L = \widetilde{C} \cup \widetilde{Q} \to D_L = C \cup Q$ to the smooth cubic component C. Equivalently, the fibration in conics $\pi_L : \mathrm{Bl}_L X \to \mathbb{P}^2$ gives a quadruple $(C, \beta, \mathcal{L}, s)$ consisting of the branch divisor $\beta = C \cap Q$, the line bundle $\mathcal{L} = \mathcal{O}_C(1)$ (see Proposition 2.2) associated with the double cover and a section $s = \ell_1(x_0, x_1, x_2)\ell_2(x_0, x_1, x_2) - \ell_3^2(x_0, x_1, x_2)$ (see Equation (5)) of $\mathcal{O}_C(\beta)$ vanishing on $\beta = C \cap Q$. Conversely, we will prove in § 3.1 that (X, τ) can be reconstructed (up to projective equivalence) from the double cover $\pi : \widetilde{C} \to C$, or equivalently, from the quadruple $(C, \beta, \mathcal{L}, s)$. Note that here we can also view C as a smooth genus 1 curve whose embedding into $\mathbb{P}^2$ as a smooth cubic curve is given by the linear system $|\mathcal{L}| = |\mathcal{O}_C(1)|$ (such that the branch divisor β lies on a quadratic curve Q). In particular, our reconstruction of the cubic threefold (X, τ) with a non-Eckardt type involution can be thought of as a degenerate case of [7, §1.6] and [14, Prop. 4.2]. The other key ingredient, which we will

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recall in § 3.2, is the injectivity of the Prym map $\mathcal{P}_{1,6} : \mathcal{R}_{1,6} \to \mathcal{A}_3^{(1,1,2)}$ for double covers of smooth genus 1 curves ramified in six distinct points due to Ikeda [24] and Naranjo and Ortega [36]. The proof of Theorem 3.1 will be completed in § 3.3 using Theorem 2.5 and the above results. The infinitesimal Torelli theorem will be discussed in § 3.4.

3.1 Reconstructing cubic threefolds with a non-Eckardt type involution

We keep notation as in § 2.2. From [7, §1.6] and [14, Prop. 4.2] (see also § 2.1), we know that the data of a smooth cubic threefold together with a line determine the data of a stable plane quintic curve and an odd theta characteristic (up to projective equivalence), and vice versa. We have a similar result for cubic threefolds with a non-Eckardt type involution.

Theorem 3.2 (Reconstructing a cubic threefold with a non-Eckardt type involution from a ramified double cover of a genus 1 curve). Given a double cover $\widetilde{C} \to C$ of a smooth genus 1 curve C branched in six distinct points, one can associate a smooth cubic threefold (X, τ) with an involution of non-Eckardt type, such that $\widetilde{C} \to C$ is obtained by projecting X from the unique pointwise fixed line $L \subset X$ (i.e. we restrict the discriminant double cover of the conic fibration $\pi_L : \mathrm{Bl}_L X \to \Pi \cong \mathbb{P}^2$ to the smooth cubic component).

Proof. The first observation is that a double cover $\widetilde{C} \to C$ of a smooth genus 1 curve C branched in six distinct points determines an admissible double cover $\widetilde{D} \to D$ of a nodal plane quintic D containing C as a component. More precisely, the data of a double cover $\widetilde{C} \to C$ of a smooth genus 1 curve C ramified in six distinct points can be described equivalently as a quadruple $(C, \beta, \mathcal{L}, s)$, where β is the branch divisor on C consisting of six distinct points, $\mathcal{L}$ is a line bundle on C with $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(\beta)$, and s is a section of $\mathcal{O}_C(\beta)$ vanishing on β. We claim that the data of such a quadruple $(C, \beta, \mathcal{L}, s)$ are equivalent to the data of a pair $(C \subset \mathbb{P}^2, Q \subset \mathbb{P}^2)$ recording the embedding of a smooth cubic C and a transverse conic Q in $\mathbb{P}^2$. Indeed, the line bundle $\mathcal{L}$ is very ample (since $\deg \mathcal{L} \geq 2g(C) + 1$), and hence defines an embedding $C \hookrightarrow \mathbb{P}^2$ of C as a plane cubic, such that $\mathcal{O}_{\mathbb{P}^2}(1)|_C \cong \mathcal{L}$. From the identification $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \xrightarrow{\sim} H^0(C, \mathcal{O}_{\mathbb{P}^2}(2)|_C) = H^0(C, \mathcal{O}_C(\beta))$, we see that there is a unique conic Q in the plane so that $Q \cap C = \beta$. Since β consists of six distinct points, Q is reduced. In summary, we let $D := C \cup Q$ which is a nodal plane quintic, and set $\widetilde{D} \to D$ to be the unique double cover branched at the nodes of D, such that the restriction to C is the cover $\widetilde{C} \to C$ (note that, since Q is rational, only one irreducible double cover of it exists, branched at $C \cap Q$).

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If we knew that the cover $\widetilde{D} \to D$ were odd, then [7, §1.6] (see also [14, Prop. 4.2]) would provide a smooth cubic threefold X and a line $L \subset X$ so that projection from L gave the discriminant cover $\widetilde{D} \to D$. However, we would still need to show that X was a cubic threefold with non-Eckardt type involution, and that L was the unique pointwise fixed line.

For this reason, we instead construct directly the smooth cubic threefold X with a non-Eckardt type involution τ so that projection from the pointwise fixed line gives the desired cover. To begin, since the rank of Q is 2 or 3, it is given by

$$\det\begin{pmatrix} \ell_1(x_0,x_1,x_2) & \ell_3(x_0,x_1,x_2) \\ \ell_3(x_0,x_1,x_2) & \ell_2(x_0,x_1,x_2) \end{pmatrix} = 0$$

for some choice of coordinates on $\mathbb{P}^2$ and some linear forms $\ell_i(x_0,x_1,x_2)$. Let $g(x_0,x_1,x_2) = 0$ be the equation for C. Take X to be the cubic threefold in $\mathbb{P}^4$ given by

$$\ell_1(x_0,x_1,x_2)x_3^2 + \ell_2(x_0,x_1,x_2)x_4^2 + 2\ell_3(x_0,x_1,x_2)x_3x_4 + g(x_0,x_1,x_2) = 0, \tag{8}$$

which admits an involution $\tau : [x_0,x_1,x_2,x_3,x_4] \mapsto [x_0,x_1,x_2,-x_3,-x_4]$ of non-Eckardt type. Let $L := V(x_0,x_1,x_2)$ and $\Pi := V(x_3,x_4) \cong \mathbb{P}^2$. Taking the determinant of the matrix in Equation (4), one has that $D = C \cup Q$ is the discriminant curve of the conic fibration obtained by projecting X from L (compare § 2.2).

It remains to show that the cubic threefold X is smooth, and that the associated discriminant double cover obtained by projection from L is the cover $\widetilde{D} \to D$ constructed above. Assuming X is smooth, the latter assertion is Proposition 2.2. The smoothness of X will follow from the fact that the cubic $C \subset \mathbb{P}^2$ is smooth and the conic Q intersects C transversely. Similar arguments are made in the proof of [14, Prop. 4.2]. Specifically, suppose that $a = [a_0,a_1,a_2,a_3,a_4]$ is a singular point of X, and consider first the case $a \notin L = V(x_0,x_1,x_2)$. After a change of coordinates fixing x_0,x_1,x_2, we may assume that $a_3 = a_4 = 0$. Analyzing the partial derivatives of Equation (8), we see that the cubic C is singular at the point $[a_0,a_1,a_2]$, which is a contradiction. Next, let us consider the case when the point a lies in the fixed line L. We may assume that $a = [0,0,0,1,0]$. A direct calculation using Equation (8) shows that the conic Q is non-reduced, which is absurd. ■

3.2 Injectivity of the Prym map for double covers of genus 1 curves branched in six points

Consider the Prym map $\mathcal{P}_{g,2n} : \mathcal{R}_{g,2n} \to \mathcal{A}_{g-1+n}^{(1,\dots,1,2,\dots,2)}$ (with 2's repeated g times); recall that $\mathcal{R}_{g,2n}$ is the moduli space of double covers of smooth genus g curves branched in

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$2n$ distinct points, and $\mathcal{A}_{g-1+n}^{(1,\dots,1,2,\dots,2)}$ is the moduli space of abelian varieties of dimension $g-1+n$ with a polarization of type $(1,\dots,1,2,\dots,2)$. In [24], Ikeda studies double covers of elliptic curves and proves the injectivity of the Prym map $\mathcal{P}_{1,2n}$ with $n \geq 3$. More generally, Naranjo and Ortega prove the Prym–Torelli theorem for $\mathcal{P}_{g,2n}$ with $g > 0$ and $n \geq 3$ in [36].

Theorem 3.3 ([24] Theorem 1.2; [36] Theorem 1.1). If $n \geq 3$, then the Prym map $\mathcal{P}_{1,2n} : \mathcal{R}_{1,2n} \to \mathcal{A}_n^{(1,\dots,1,2)}$ is injective. In particular, $\mathcal{P}_{1,6} : \mathcal{R}_{1,6} \to \mathcal{A}_3^{(1,1,2)}$ is injective.

Let us very briefly review the proof of the above theorem following [24] and [36]. Let $(P(\widetilde{C}, C), \Xi)$ denote the Prym variety for a double cover $\widetilde{C} \to C$ of a smooth genus 1 curve C branched in $2n$ distinct points. For any member $\Sigma \in |\Xi|$, Ikeda studies the Gauss map Ψ_Σ (more precisely, the branch locus of the restriction $\Psi_\Sigma|_{\mathrm{BS}|\Xi|}$ of Ψ_Σ to the base locus $\mathrm{BS}|\Xi|$ of $|\Xi|$), and uses the information to specify a divisor $\Sigma_0 \in |\Xi|$ which allows him to reconstruct the double cover $\widetilde{C} \to C$. Ikeda's proof is a generalization of the proof of the Torelli theorem for hyperelliptic curves due to Andreotti (cf. [4]). The approach of Naranjo and Ortega is different and relies on the description of the base locus of $|\Xi|$ given in [35] and on a generalized Torelli theorem proved by Martens in [30]. To be specific, Naranjo and Ortega recover a certain Brill–Noether locus on $\mathrm{Pic}^0(\widetilde{C})$ together with an involution through a birational model of the base locus $\mathrm{BS}|\Xi|$; by Martens' generalized Torelli, the Brill–Noether locus determines the double cover $\widetilde{C} \to C$. More generally, Naranjo and Ortega's argument can be used to prove the injectivity of the Prym map $\mathcal{P}_{g,2n}$ with $g > 0$ and $n \geq 3$.

3.3 Proving global Torelli for cubic threefolds with a non-Eckardt type involution

Proof of Theorem 3.1. Let (X_1, τ_1) and (X_2, τ_2) be two cubic threefolds with involutions of non-Eckardt type. We will prove that if the invariant parts of the intermediate Jacobian are isomorphic to each other $JX_1^{\tau_1} \cong JX_2^{\tau_2}$, then (X_1, τ_1) is projectively equivalent to (X_2, τ_2). Recall from Proposition 2.3 that each involution τ_i fixes pointwise a line $L_i \subset C_i$, such that when projecting X_i from L_i we get a double cover $\pi_i : \widetilde{C}_i \to C_i$ in $\mathcal{R}_{1,6}$. To be more precise, the double cover $\widetilde{C}_i \to C_i$ is obtained by projecting X_i from L_i and restricting the discriminant pseudo-double cover $\pi_i : \widetilde{C}_i \cup \widetilde{Q}_i \to C_i \cup Q_i$ over the smooth cubic component C_i. By Theorem 2.5, we have $JX_i^{\tau_i} \cong (P(\widetilde{C}_i, C_i))^\vee$ as polarized abelian varieties which implies that $P(\widetilde{C}_1, C_1) \cong P(\widetilde{C}_2, C_2)$ (see also [10, Thm. 3.1]). From Theorem 3.3, we know that the two double covers $\pi_i : \widetilde{C}_i \to C_i$ are equivalent. By Theorem 3.2,

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we can reconstruct (X_i, τ_i) from the data of the double cover $\widetilde{C}_i \to C_i$ branched in six points. It follows that the cubic threefolds (X_1, τ_1) and (X_2, τ_2) are projectively equivalent. ■

3.4 Infinitesimal Torelli for cubic threefolds with an involution of non-Eckardt type

Recall that $\mathcal{M}$ is the moduli space of cubic threefolds (X, τ) with an involution of non-Eckardt type. Let us consider the open subset $\mathcal{M}_0 \subset \mathcal{M}$ parametrizing those (X, τ) satisfying that when projecting X from the unique fixed line $L \subset X$ (as in § 2.2), the conic component Q of the discriminant curve D_L is smooth. More concretely, the equation of a member in the complement $\mathcal{M}\backslash\mathcal{M}_0$ can be written as Equation (5) with $\ell_3(x_0, x_1, x_2) \equiv 0$ (i.e. no terms containing x_3x_4).

Proposition 3.4 (Infinitesimal Torelli for cubic threefolds with a non-Eckardt type involution). Let $\mathcal{P} : \mathcal{M} \to \mathcal{A}_3^{(1,2,2)}$ be the period map for cubic threefolds with a non-Eckardt type involution. The differential $d\mathcal{P}$ is an isomorphism at every point of $\mathcal{M}_0 \subset \mathcal{M}$. (As a consequence, $\mathcal{P}|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{A}_3^{(1,2,2)}$ is an open embedding.)

Proof. The infinitesimal computation is quite similar to that of [16, §5.1] (see also [15, §8.1, §8.3]). Specifically, let (X, τ) be a smooth cubic threefold with an involution τ of non-Eckardt type cut out by $F = 0$ as in Equation (1). On one side, we have

$$T_{(X,\tau)}\mathcal{M} = (R_F^3)^\tau,$$

where R_F^i $(i \geq 0)$ denotes degree i part of the Jacobian ring of F and the superscript means taking the τ-invariant subspace. On the other side, we also have

$$T_{JX^\tau}\mathcal{A}_3^{(1,2,2)} = \mathrm{SymHom}(H^{2,1}(X)^\tau, H^{1,2}(X)^\tau) = \mathrm{SymHom}((R_F^1)^\tau, (R_F^4)^\tau).$$

Furthermore, the differential $d\mathcal{P}_{(X,\tau)}$ of the period map $\mathcal{P} : \mathcal{M} \to \mathcal{A}_3^{(1,2,2)}$ can be identified with the following map, which is induced by cup product:

$$(R_F^3)^\tau \to \mathrm{SymHom}((R_F^1)^\tau, (R_F^4)^\tau).$$

Using Macaulay's theorem to identify R_F^4 with $(R_F^1)^\vee$, the differential is

$$d\mathcal{P}_{(X,\tau)} : (R_F^3)^\tau \to \mathrm{Sym}^2((R_F^1)^\tau)^\vee$$

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and the codifferential is

$$d\mathcal{P}^*_{(X,\tau)} : H^0(\mathbb{P}^2_{x_0,x_1,x_2}, \mathcal{O}(2)) \cong \mathrm{Sym}^2((R^1_F)^\tau) \to (R^2_F)^\tau.$$

The kernel of the codifferential $d\mathcal{P}^*_{(X,\tau)}$ is $J^2_F \cap \mathrm{Sym}^2\mathbb{C}[x_0, x_1, x_2]$, where J^2_F denotes degree 2 part of the Jacobian ideal. Now we suppose that $(X, \tau) \in \mathcal{M}_0$. Rewrite the equation F of X in the form of Equation (5)

$$\ell_1(x_0, x_1, x_2)x_3^2 + \ell_2(x_0, x_1, x_2)x_4^2 + 2\ell_3(x_0, x_1, x_2)x_3x_4 + g(x_0, x_1, x_2) = 0.$$

A direct calculation then shows that $J^2_F \cap \mathrm{Sym}^2\mathbb{C}[x_0, x_1, x_2]$ is not trivial if and only if $\ell_1(x_0, x_1, x_2), \ell_2(x_0, x_1, x_2)$, and $\ell_3(x_0, x_1, x_2)$ have a common zero. However, it is impossible since otherwise the conic component Q of the discriminant curve D_L for the fibration in conics $\pi_L : \mathrm{Bl}_L X \to \Pi$ will be singular. Note also that $\dim \mathrm{Sym}^2((R^1_F)^\tau) = \dim(R^2_F)^\tau = 6$. It follows that the (co)differential of the period map $\mathcal{P} : \mathcal{M} \to \mathcal{A}_3^{(1,2,2)}$ at every point of $\mathcal{M}_0 \subset \mathcal{M}$ is an isomorphism. ■

Remark 9. We remark that the (co)differential of the period map $\mathcal{P} : \mathcal{M} \to \mathcal{A}_3^{(1,2,2)}$ has one-dimensional kernel at a point (X, τ) of the divisor $\mathcal{M}\backslash\mathcal{M}_0$ (note that a quadratic curve being degenerate is a codimension 1 condition) and hence the infinitesimal Torelli fails. A similar phenomenon happens for the hyperelliptic locus when considering the period map for smooth curves of genus greater than 2 (see for instance [12, §1]). We give the following explanation using [40, §1]. Observe that a member $(X, \tau) \in \mathcal{M}\backslash\mathcal{M}_0$ is cut out by

$$\ell_1(x_0, x_1, x_2)x_3^2 + \ell_2(x_0, x_1, x_2)x_4^2 + g(x_0, x_1, x_2) = 0$$

and hence admits extra automorphisms $\sigma : x_3 \mapsto -x_3$ and $\sigma' : x_4 \mapsto -x_4$ (note that $\tau = \sigma \circ \sigma' = \sigma' \circ \sigma$). By [40, §1] (see also [12, §1]), the differential $d\mathcal{P}_{(X,\tau)} : T_{(X,\tau)}\mathcal{M} \to T_{JX^\tau}\mathcal{A}_3^{(1,2,2)}$ is equivariant with respect to the action of σ (and also σ'). Using the identifications in the proof of Proposition 3.4, it is not difficult to see that $T_{JX^\tau}\mathcal{A}_3^{(1,2,2)}$ is fixed by σ, while $T_{(X,\tau)}\mathcal{M}$ splits as the direct sum of the $(+1)$ eigenspace $T^+_{(X,\tau)}\mathcal{M}$ (which is five-dimensional and corresponds to infinitesimal deformations inside $\mathcal{M}\backslash\mathcal{M}_0$) and the (-1) eigenspace $T^-_{(X,\tau)}\mathcal{M}$ (which has dimension 1 and represents infinitesimal deformations in the normal directions). As a result, the (-1) eigenspace $T^-_{(X,\tau)}\mathcal{M}$ must be contained in the kernel of $d\mathcal{P}_{(X,\tau)}$ and therefore the infinitesimal Torelli fails.

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4 Cubic threefolds with an involution of non-Eckardt type as fibrations in conics II: invariant lines

In this section, we aim to characterize the intermediate Jacobians JX of general cubic threefolds X with an involution τ of non-Eckardt type via projections from general invariant lines $l \subset X$ that are not pointwise fixed. Specifically, we project a cubic threefold (X, τ) with a non-Eckardt type involution from a τ-invariant line l and study the obtained conic fibration in § 4.1. Among others, we prove that the involution of non-Eckardt type on X induces an involution on the discriminant quintic curve which is generically smooth. A different description of the invariant part JX^{τ} and the anti-invariant part $JX^{-\tau}$ for a general (X, τ, l) is then given in § 4.2 using the techniques developed in [34], [20], and [38]. We conclude by giving an application of our results in the study of the generic fiber of the Prym map $\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3^{(1,2,2)}$ in § 4.3. In particular, the discussion in [36, §4] and [22, §5] will play a crucial role.

4.1 Projecting cubic threefolds with a non-Eckardt type involution from invariant lines

We keep notation as in § 1.1. Let (X, τ) be a smooth cubic threefold with a non-Eckardt type involution as in Equations (1) and (2)

$$x_0 q_0(x_3, x_4) + x_1 q_1(x_3, x_4) + x_2 q_2(x_3, x_4) + g(x_0, x_1, x_2) = 0;$$

$$\tau : [x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, x_1, x_2, -x_3, -x_4].$$

Let $L = V(x_0, x_1, x_2) \subset X$ denote the pointwise fixed line under τ. When studying the projection of X from L, it is more convenient to rewrite the above equation in the form of Equation (5)

$$\ell_1(x_0, x_1, x_2)x_3^2 + \ell_2(x_0, x_1, x_2)x_4^2 + 2\ell_3(x_0, x_1, x_2)x_3x_4 + g(x_0, x_1, x_2) = 0.$$

Let $C = V(g(x_0, x_1, x_2), x_3, x_4) \subset \Pi = V(x_3, x_4) \cong \mathbb{P}^2_{x_0,x_1,x_2}$ be the fixed plane section. By Lemma 1.5, an invariant line $l \subset X$ that is different from L intersects both L and $C \subset \Pi \cong \mathbb{P}^2_{x_0,x_1,x_2}$. From the discussion in § 2.2 (see also the proof of Proposition 2.4), we know that l is contained in a singular fiber of the projection $\pi_L : \mathrm{Bl}_L X \to \Pi$ over a point of the cubic component C of the discriminant curve. In other words, denote the restriction of the discriminant double cover to C by $\pi : \widetilde{C} \to C$; invariant lines in X that are not pointwise fixed are then parametrized by $\widetilde{C}$ (which is a smooth curve of genus 4).

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Let $l \neq L$ be an invariant line in X. Without loss of generality, we assume that l is contained in the fiber of $\pi_L : \mathrm{Bl}_L X \to \Pi \cong \mathbb{P}^2_{x_0,x_1,x_2}$ over $[0,0,1] \in C \subset \Pi$. Note that the equation $g(x_0,x_1,x_2)$ of $C \subset \Pi$ has no term x_2^3. Also, the quadratic polynomial $q_2(x_3,x_4)$ in Equation (1) can be factored as $q_2(x_3,x_4) = (\alpha x_3 + \beta x_4)(\gamma x_3 + \delta x_4)$ with α, β, γ and δ constants. Consider the fiber of $\pi_L : \mathrm{Bl}_L X \to \Pi$ over the point $[0,0,1] \in C \subset \Pi$. The equations for the invariant lines l and l' contained in the fiber are, respectively,

$$l = V(x_0,x_1,\alpha x_3 + \beta x_4); \;\; l' = V(x_0,x_1,\gamma x_3 + \delta x_4).$$

Let us set $z := \alpha x_3 + \beta x_4$ and apply the change of coordinates

$$[x_0,x_1,x_2,x_3,x_4] \mapsto [x_0,x_1,x_2,z,x_4].$$

Then the line $l = V(x_0,x_1,z)$ and the involution τ acts by

$$\tau : [x_0,x_1,x_2,z,x_4] \mapsto [x_0,x_1,x_2,-z,-x_4].$$

Now we project X from the invariant line $l = V(x_0,x_1,z)$ to the complementary plane $\mathbb{P}^2_l := V(x_2,x_4) \cong \mathbb{P}^2_{x_0,x_1,z}$ and study the corresponding fibration in conics $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2_l$. Specifically, let us rewrite the equation of X as

$$\begin{aligned} &L_1(x_0,x_1,z)x_2^2 + L_2(x_0,x_1,z)x_4^2 + 2L_3(x_0,x_1,z)x_2x_4 \\ &+ 2Q_1(x_0,x_1,z)x_2 + 2Q_2(x_0,x_1,z)x_4 + G(x_0,x_1,z) = 0, \end{aligned} \tag{10}$$

where $L_i(x_0,x_1,z)$ are linear polynomials, $Q_j(x_0,x_1,z)$ are quadratic forms and $G(x_0,x_1,z)$ is homogeneous of degree 3. We can simplify Equation (10) in the following way.

- Since the equation of X is preserved under τ, both of the terms $L_1(x_0,x_1,z)$ and $L_2(x_0,x_1,z)$ must have no z term; in other words, $L_1(x_0,x_1,z) = L_1(x_0,x_1)$ and $L_2(x_0,x_1,z) = L_2(x_0,x_1)$. For the same reason, $L_3(x_0,x_1,z)$ must be linear in z, i.e. $L_3(x_0,x_1,z) = Az$ for some constant A. Furthermore, when $l = l'$ (equivalently, $\pi : \widetilde{C} \to C$ is ramified at the point corresponding to the invariant line l), one deduces from the above calculation that $A = 0$.
- The terms $Q_1(x_0,x_1,z)x_2$ and $Q_2(x_0,x_1,z)x_4$ must be invariant under τ, so $Q_1(x_0,x_1,z)$ can only contain terms with even powers of z and $Q_2(x_0,x_1,z) = zN(x_0,x_1)$ for a linear form $N(x_0,x_1)$.

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- Similarly, the cubic polynomial $G(x_0, x_1, z)$ has no monomials with odd powers of z.

Thus Equation (10) becomes

$$\begin{aligned} &L_1(x_0,x_1)x_2^2 + L_2(x_0,x_1)x_4^2 + 2Azx_2x_4 \\ &+ 2Q_1(x_0,x_1,z)x_2 + 2zN(x_0,x_1)x_4 + G(x_0,x_1,z) = 0; \end{aligned} \tag{11}$$

in particular, $Q_1(x_0, x_1, z)$ and $G(x_0, x_1, z)$ only contain monomials with even powers of z. The matrix associated with the fibration in conics $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2_l \cong \mathbb{P}^2_{x_0,x_1,z}$ is

$$M = \begin{pmatrix} L_1(x_0,x_1) & Az & Q_1(x_0,x_1,z) \\ Az & L_2(x_0,x_1) & zN(x_0,x_1) \\ Q_1(x_0,x_1,z) & zN(x_0,x_1) & G(x_0,x_1,z) \end{pmatrix}, \tag{12}$$

and the discriminant quintic curve $D_l \subset \mathbb{P}^2_l \cong \mathbb{P}^2_{x_0,x_1,z}$ is cut out by

$$\begin{aligned} &\det(M) = L_1(x_0,x_1)L_2(x_0,x_1)G(x_0,x_1,z) + 2Az^2N(x_0,x_1)Q_1(x_0,x_1,z) \\ &- L_2(x_0,x_1)Q_1^2(x_0,x_1,z) - z^2L_1(x_0,x_1)N^2(x_0,x_1) - A^2z^2G(x_0,x_1,z) = 0. \end{aligned}$$

The involution τ of non-Eckardt type on X induces an involution on $\mathbb{P}^2_l \cong \mathbb{P}^2_{x_0,x_1,z}$, given by $\tau_{\mathbb{P}^2_l} : [x_0, x_1, z] \mapsto [x_0, x_1, -z]$. Because $l \subset X$ is an invariant line, there is an induced involution on $\mathrm{Bl}_l X$, which we also denote using τ, making $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2_l$ equivariant. Let $\pi_l : \widetilde{D}_l \to D_l$ be the discriminant double cover. Note that the equation of the discriminant curve D_l only contains terms with even powers of z and hence is preserved by $\tau_{\mathbb{P}^2_l}$. As a result, the restrictions of τ and $\tau_{\mathbb{P}^2_l}$ to $\widetilde{D}_l$ and D_l, respectively, induce involutions

$$\tau : \widetilde{D}_l \to \widetilde{D}_l; \quad \tau_{D_l} : D_l \to D_l$$

with respect to which $\pi_l : \widetilde{D}_l \to D_l$ is equivariant.

We summarize the discussion above in the following proposition.

Proposition 4.1. Let (X, τ) be a smooth cubic threefold with an involution τ of non-Eckardt type. Choose an invariant (but not pointwise fixed) line $l \subset X$ and project X from l. Denote the obtained discriminant double cover by $\pi_l : \widetilde{D}_l \to D_l$. Then the non-Eckardt type involution on X induces involutions τ and τ_{D_l} on $\widetilde{D}_l$ and D_l, respectively, making $\pi_l : \widetilde{D}_l \to D_l$ equivariant.

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4.2 The intermediate Jacobians of cubic threefolds with a non-Eckardt type involution via the projections from invariant lines

In what follows, we suppose that (X, τ) is a general cubic threefold admitting a non-Eckardt type involution. It is not difficult to show that there exists an invariant line $l \subset X$ such that the rank of the matrix M in Equation (12) never drops to 1 (e.g. consider $V(\det(M_{1,1}), \det(M_{2,3}), \det(M_{3,3}))$, where $M_{i,j}$ denotes the (i,j)-minor of M), and therefore the discriminant curve D_l is smooth and the discriminant double cover $\pi_l : \widetilde{D}_l \to D_l$ is connected and étale. Recall that $\widetilde{C}$, viewed as a curve on the Fano surface $F(X)$, parametrizes τ-invariant lines l that are not pointwise fixed. It follows that $\widetilde{C}$ is not a component of the curve $R \subset F(X)$ corresponding to lines of second type (in the sense of [13, Def. 6.6]) or a component of the curve $R' \subset F(X)$ parametrizing lines residual to lines of second type. We will call such an invariant line $l \neq L$ a general invariant line (in particular, $\langle L, l\rangle \cap X \neq L + 2l$ and thus the coefficient $A \neq 0$ in Equation (11)).

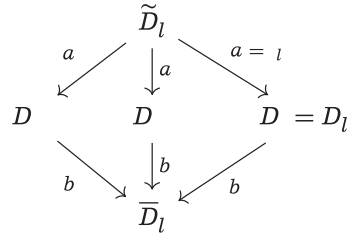
Suppose that (X, τ, l) is general as above. We wish to study the intermediate Jacobian JX via the Prym variety $P(\widetilde{D}_l, D_l)$. The key observation is that the covering curve $\widetilde{D}_l$ admits two commuting involutions: the involution τ in Proposition 4.1 induced from the non-Eckardt type involution on X, and the covering involution $\iota = \iota_l$ associated with $\pi_l : \widetilde{D}_l \to D_l$. In other words, the automorphism group $\mathrm{Aut}(\widetilde{D}_l)$ contains the Klein four group $\langle \tau, \iota\rangle$. This allows us to apply the techniques that go back to [34], and explored in more depth in [20] and [38], to decompose $P(\widetilde{D}_l, D_l)$. In particular, we follow closely the analogous case for Eckardt cubic threefolds as discussed in [16, §7.1].

We will take the quotient of $\widetilde{D}_l$ by an element g of the Klein four group $\langle \tau, \iota\rangle \subset \mathrm{Aut}(\widetilde{D}_l)$; let us denote the quotient curve by

$$D_g := \widetilde{D}_l / \langle g \rangle,$$

noting that $D_\iota = D_l$. Also set $\overline{D}_l := D_l / \tau_{D_l}$.

Proposition 4.2. We have the following commutative diagram.



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Moreover,

1. The map a_τ is a double covering map ramified in four points, the map $a_{\tau\iota}$ is a double cover branched in eight points, whereas the map a_ι is the discriminant double covering map π_l which is étale.
2. The map b_τ is a double covering map ramified in four points, the map $b_{\tau\iota}$ is a double cover branched in two points, and b_ι is a ramified double cover with six branch points.
3. The curves are all smooth and their genera are as follows: $g(\widetilde{D}_l) = 11$, $g(D_\tau) = 5$, $g(D_{\tau\iota}) = 4$, $g(D_l) = 6$ and $g(\overline{D}_l) = 2$.

Proof. We choose coordinates and project a general cubic threefold (X, τ) admitting an involution of non-Eckardt type from a general invariant line $l \subset X$ as in § 4.1. In particular, the equations of X and the discriminant quintic $D_l \subset \mathbb{P}^2_{x_0,x_1,z}$ are given in Equations (11) and (12), respectively. Note that D_l is smooth and the discriminant double cover $\pi_l : \widetilde{D}_l \to D_l$ is connected and étale. Also, the involution $\tau_{D_l} : [x_0, x_1, z] \mapsto [x_0, x_1, -z]$ fixes six points on D_l: the point $[0, 0, 1]$ and the five intersection points of D_l and the line $z = 0$ which satisfy the equation $z = L_2(x_0, x_1)(L_1(x_0, x_1)G(x_0, x_1, z) - Q_1^2(x_0, x_1, z)) = 0$ and are distinct since (X, τ, l) is general. The six pairs of lines in the fibers of $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2_{x_0,x_1,z}$ that lie over the fixed points on D_l correspond to the ramification points of $a_\tau : \widetilde{D}_l \to D_\tau$ and $a_{\tau\iota} : \widetilde{D}_l \to D_{\tau\iota}$, depending on whether they are fixed or switched by τ. By a straightforward computation, one verifies that the four lines corresponding to the preimages of the points $[0, 0, 1]$ and $V(z, L_2(x_0, x_1))$ under $\pi : \widetilde{D}_l \to D_l$ are fixed by τ, while the remaining eight lines are interchanged pairwise by τ. For example, the lines lying over $[0, 0, 1] \in D_l$ are the pointwise fixed line L and another invariant line $l' \neq l$. The remainder of the assertions follow directly from Riemann–Hurwitz and [38, Thm. 3.2]. ∎

Applying [38, Thm. 3.2] (see also [16, Prop. 7.12]), we obtain the following description of the invariant part $P(\widetilde{D}_l, D_l)^\tau := \mathrm{Im}(1 + \tau)$ and anti-invariant part $P(\widetilde{D}_l, D_l)^{-\tau} := \mathrm{Im}(1 - \tau)$ for the induced involution τ on $P(\widetilde{D}_l, D_l)$.

Proposition 4.3 ([38] Theorem 3.2). Notation as in Proposition 4.2. Consider the Prym variety $P(\widetilde{D}_l, D_l)$ which is principally polarized. There is an isogeny of polarized abelian varieties

$$\phi_l : P(D_\tau, \overline{D}_l) \times P(D_{\tau\iota}, \overline{D}_l) \to P(\widetilde{D}_l, D_l), \quad (y_1, y_2) \mapsto a_\tau^*(y_1) + a_{\tau\iota}^*(y_2)$$

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with $\ker(\phi_l) \cong (\mathbb{Z}/2\mathbb{Z})^4$, where a_τ^* and $a_{\tau\iota}^*$ denote the pull-back maps between the appropriate Jacobians. Moreover, with respect to the action of τ on $P(\widetilde{D}_l, D_l)$, the isogeny ϕ_l induces isomorphisms of polarized abelian varieties

$$P(\widetilde{D}_l, D_l)^\tau \cong P(D_\tau, \overline{D}_l); \;\; P(\widetilde{D}_l, D_l)^{-\tau} \cong P(D_{\tau\iota}, \overline{D}_l).$$

Putting it together, we obtain the following theorem.

Theorem 4.4 (Intermediate Jacobian via projecting from a general invariant line). Let (X, τ) be a general cubic threefold with an involution τ of non-Eckardt type, and choose a general invariant line $l \subset X$. Project X from l and denote the discriminant double cover by $\pi_l : \widetilde{D}_l \to D_l$, and keep notation as in Proposition 4.2. There is an isogeny of polarized abelian varieties

$$\phi_l : P(D_\tau, \overline{D}_l) \times P(D_{\tau\iota}, \overline{D}_l) \to JX$$

with $\ker\phi \cong (\mathbb{Z}/2\mathbb{Z})^4$. Moreover, with respect to the action of τ on the principally polarized intermediate Jacobian JX, the isogeny ϕ_l induces isomorphisms of polarized abelian varieties

$$JX^\tau \cong P(D_\tau, \overline{D}_l); \;\; JX^{-\tau} \cong P(D_{\tau\iota}, \overline{D}_l).$$

Proof. The proof is identical to that of Theorem 2.5, and follows from Proposition 4.3. ∎

4.3 The generic fiber of the Prym map for double covers of genus 2 curves ramified in four points

Consider the Prym map $\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3^{(1,2,2)}$, where $\mathcal{R}_{2,4}$ is the moduli space of double covers of smooth genus 2 curves branched in four distinct points, and $\mathcal{A}_3^{(1,2,2)}$ denotes the moduli space of abelian threefolds with a polarization of type $(1,2,2)$. From [36, Thm. 1.2] or [22, Thm. 5.2], we know that the generic fiber of $\mathcal{P}_{2,4}$ is birational to an elliptic curve. By Theorem 3.1 and Proposition 3.4, a general member of $\mathcal{A}_3^{(1,2,2)}$ can be realized as the invariant part JX^τ for a general cubic threefold X with an involution τ of non-Eckardt type. The goal of this subsection is to give a more concrete description of the generic fiber $\mathcal{P}_{2,4}^{-1}(JX^\tau)$.

Let us keep notation as in § 2.2 and § 4.1. Similarly as in [16, §5], we make the following observation. Let us project a general cubic threefold (X, τ) with a

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non-Eckardt type involution from a general invariant line $l \neq L$. The quotient $b_\tau : D_\tau \to \overline{D}_l$ of the discriminant double cover for the projection $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2_l$ (see Proposition 4.2) is contained in the fiber $\mathcal{P}_{2,4}^{-1}(JX^\tau)$ because of Theorem 4.4. We now show that for τ-invariant lines l and l' contained in the same fiber of $\pi_L : \mathrm{Bl}_L X \to \Pi$ (in other words, there exists a plane $P \subset \mathbb{P}^4$ with $X \cap P = L \cup l \cup l'$) the quotient discriminant double covers $b_\tau : D_\tau \to \overline{D}_l$ and $b'_\tau : D'_\tau \to \overline{D}_{l'}$ are isomorphic and hence correspond to the same element in $\mathcal{R}_{2,4}$. Our main tools are the bigonal and tetragonal constructions; recall that the bigonal construction is also the key ingredient for proving [36, Thm. 1.2] and [22, Thm. 5.2]. We refer the reader to [20, §2] (see also [22, §1]) for the description of the bigonal and tetragonal constructions.

Proposition 4.5. Notation as above. Let l and l' be general τ-invariant lines in X that are contained in the same fiber of $\pi_L : \mathrm{Bl}_L X \to \Pi$. Consider the projections of X from the pointwise fixed line $L \subset X$ (as in § 2.2) and from the invariant lines $l, l' \subset X$ (see § 4.1). Denote the discriminant double covers by $\pi_L : \widetilde{C} \cup \widetilde{Q} \to C \cup Q$, $\pi_l : \widetilde{D}_l \to D_l$ and $\pi_{l'} : \widetilde{D}_{l'} \to D_{l'}$, respectively. Also let $\pi : \widetilde{C} \to C$ be the restriction of π_L to the smooth cubic component, and set $b_\tau : D_\tau \to \overline{D}_l$ (respectively, $b'_\tau : D'_\tau \to \overline{D}_{l'}$) to be the quotient of π_l (respectively, $\pi_{l'}$) by the involution induced from τ (cf. Proposition 4.2).

1. The union of lines $l \cup l'$ corresponds to a point $c_{l\cup l'} \in C \subset \Pi$ and hence determines a degree 4 map $p = p_{l\cup l'} : C \cup Q \to \mathbb{P}^1$ given by $\mathcal{O}_{C\cup Q}(1)(-c_{l\cup l'})$ (i.e. projecting $C \cup Q$ from $c_{l\cup l'} \in C$ to a complementary line in Π). Similarly, D_l (respectively, $D_{l'}$) admits a map $q = q_{L\cup l'} : D_l \to \mathbb{P}^1$ (respectively, $q' = q_{L\cup l} : D_{l'} \to \mathbb{P}^1$) of degree 4. Then

$$\widetilde{C} \cup \widetilde{Q} \xrightarrow{\pi_L} C \cup Q \xrightarrow{p} \mathbb{P}^1;\ \widetilde{D}_l \xrightarrow{\pi_l} D_l \xrightarrow{q} \mathbb{P}^1;\ \widetilde{D}_{l'} \xrightarrow{\pi_{l'}} D_{l'} \xrightarrow{q'} \mathbb{P}^1$$

   are tetragonally related (in other words, the tetragonal construction of one produces the other two).
2. Consider the degree 2 map obtained as the restriction of $p = p_{l\cup l'} : C \cup Q \to \mathbb{P}^1$ to the smooth cubic curve C and still use p to denote it. Note that $\overline{D}_l$ and $\overline{D}_{l'}$ are smooth of genus 2 and hence admit degree 2 maps to $\mathbb{P}^1$ defined by the canonical linear systems (which are the unique g^1_2's on $\overline{D}_l$ and $\overline{D}_{l'}$). Then the bigonal construction takes both

$$D_\tau \xrightarrow{b_\tau} \overline{D}_l \xrightarrow{r} \mathbb{P}^1;\ D'_\tau \xrightarrow{b'_\tau} \overline{D}_{l'} \xrightarrow{r'} \mathbb{P}^1$$

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to

$$\widetilde{C} \xrightarrow{\pi} C \xrightarrow{p} \mathbb{P}^1.$$

In particular, $b_\tau : D_\tau \to \overline{D}_l$ and $b'_\tau : D'_\tau \to \overline{D}_{l'}$ are isomorphic.

Proof. The first claim is the content of [20, Ex. 2.15(4)] which we now recall using the notation therein. To apply the tetragonal construction to $\widetilde{D}_l \xrightarrow{\pi_l} D_l \xrightarrow{q} \mathbb{P}^1$, we consider the following commutative diagram:

$$\begin{array}{ccc} q_*\widetilde{D}_l & \hookrightarrow & \widetilde{D}_l^{(4)} \\ \downarrow{\scriptstyle 16:1} & & \downarrow{\scriptstyle \iota^{(4)}} \\ \mathbb{P}^1 & \hookrightarrow & D_l^{(4)} \end{array}$$

where the superscript $^{(n)}$ denotes the n-th symmetric product, the bottom horizontal arrow is defined by sending a point $y \in \mathbb{P}^1$ to the fiber $q^{-1}(y)$, and $q_*\widetilde{D}_l$ is the fiber product which is a curve in $\widetilde{D}_l^{(4)}$. As discussed in [20, §2.1], the covering involution associated with $\pi_l : \widetilde{D}_l \to D_l$ induces an involution ι_l on $q_*\widetilde{D}_l$. Denote the orientation double cover of $\widetilde{D}_l \xrightarrow{\pi_l} D_l \xrightarrow{q} \mathbb{P}^1$ by $\widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ (cf. [20, §2.2]). Then the map $q_*\widetilde{D}_l \xrightarrow{16:1} \mathbb{P}^1$ factors as

$$q_*\widetilde{D}_l \xrightarrow{2:1} q_*\widetilde{D}_l/\iota_l \xrightarrow{4:1} \widetilde{\mathbb{P}}^1 \xrightarrow{2:1} \mathbb{P}^1.$$

Moreover, the orientation double cover $\widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ is trivial; the curves $q_*\widetilde{D}_l$, $q_*\widetilde{D}_l/\iota_l$ and $\widetilde{\mathbb{P}}^1$ are thus reducible and we obtain the other two towers associated with the tower $\widetilde{D}_l \xrightarrow{\pi_l} D_l \xrightarrow{q} \mathbb{P}^1$ via the tetragonal construction (see [20, §2.5] for more details).

Going back to our case, we need to construct injections $\widetilde{C} \cup \widetilde{Q} \hookrightarrow q_*\widetilde{D}_l$ and $\widetilde{D}_{l'} \hookrightarrow q_*\widetilde{D}_l$. Geometrically, there exists a plane $P \subset \mathbb{P}^4$ with $X \cap P = L \cup l \cup l'$ as l and l' are contained in the same fiber of $\pi_L : \mathrm{Bl}_L X \to \Pi$. Consider the conic fibration $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2_l$. The plane P corresponds to a point $d_{L\cup l'} \in D_l \subset \mathbb{P}^2_l$. Projecting the discriminant quintic D_l from $d_{L\cup l'}$, one gets the degree 4 map $q : D_l \to \mathbb{P}^1$. Let us now fix a general point $y \in \mathbb{P}^1 \subset \mathbb{P}^2_l$. Pulling back the line $\langle d_{L\cup l'}, y\rangle \subset \mathbb{P}^2_l$ joining $d_{L\cup l'}$ and y via $\pi_l : \mathrm{Bl}_l X \to \mathbb{P}^2_l$, we obtain a hyperplane $H_y \subset \mathbb{P}^4$ which intersects X along a smooth cubic surface $X \cap H_y$. Note that the lines L, l and l' are contained in $X \cap H_y$. From the configuration of the 27 lines on a smooth cubic surface, we deduce that besides $L \cup l'$ there are four other pairs of coplanar lines $m_y^{(i)} \cup n_y^{(i)}$ (for $1 \le i \le 4$) on $X \cap H_y$ meeting l which corresponds to the

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fiber $q^{-1}(y) \in D_l^{(4)}$. Now we choose a line on $X \cap H_y$ meeting L which is different from l or l'; such a line is parametrized by a point of $\widetilde{C} \cup \widetilde{Q}$. Observe that this line intersects four of the eight lines $m_y^{(1)}, n_y^{(1)}, \ldots, m_y^{(4)}, n_y^{(4)}$, one in each of the four coplanar pairs, and hence defines an element of $q_*\widetilde{D}_l$. Letting y vary and using continuity, we obtain a map $\widetilde{C} \cup \widetilde{Q} \to q_*\widetilde{D}_l$ which is clearly injective. The definition of $\widetilde{D}_{l'} \hookrightarrow q_*\widetilde{D}_l$ is similar. (Note also that the local pictures of this tetragonal construction are given in [20, 2.14(3)].)

Let us now move to the proof of the second assertion. Following [20, §2.3], we recall that the bigonal construction associates with the tower of double covers $D_\tau \xrightarrow{b_\tau} \overline{D}_l \xrightarrow{r} \mathbb{P}^1$ another tower of double covers $r_*D_\tau \to r_*D_\tau/\iota_\tau \to \mathbb{P}^1$, where r_*D_τ is defined via the following fiber product diagram (where the bottom horizontal arrow is defined by sending $y \in \mathbb{P}^1$ to $r^{-1}(y) \in \overline{D}_l^{(2)}$):

$$\begin{array}{ccc} r_*D & \hookrightarrow & D^{\cdot 2/} \\ \downarrow{\scriptstyle 4:1} & & \downarrow{\scriptstyle b^{\cdot 2/}} \\ \mathbb{P}^1 & \hookrightarrow & \overline{D}_l^{\cdot 2/} \end{array}$$

and ι_τ denotes the involution on r_*D_τ induced by the covering involution of $b_\tau : D_\tau \to \overline{D}_l$. The proof of the second claim is similar to that of the first one, but we need to take the non-Eckardt type involution into consideration. Specifically, we project X from the τ-invariant line l to $\mathbb{P}_l^2$ which is also τ-invariant and admits an involution $\tau_{\mathbb{P}_l^2}$. Note that the point $d_{L\cup l'} \subset \mathbb{P}_l^2$ corresponding to the plane $P = \langle L, l, l' \rangle$ is fixed by $\tau_{\mathbb{P}_l^2}$. Note also that the discriminant double cover $\pi_l : \widetilde{D}_l \to D_l$ is equivariant with respect to the involutions τ on $\widetilde{D}_l$ and τ_{D_l} on D_l (cf. Proposition 4.1); the double cover $b_\tau : D_\tau \to \overline{D}_l$ is obtained as the quotient of π_l by the involutions. Now let us project D_l from $d_{L\cup l'}$ to an invariant complementary line $\mathbb{P}^1 \subset \mathbb{P}_l^2$ and fix a general point $y \in \mathbb{P}^1$. Again let $H_y \subset \mathbb{P}^4$ be the τ-invariant hyperplane corresponding to the line $\langle d_{L\cup l'}, y \rangle \subset \mathbb{P}_l^2$. The key observation is that the smooth cubic surface $X \cap H_y$ admits an involution τ_y whose fix locus consists of a line and three distinct points (e.g. [19, §9.5.1]). To verify this, we choose coordinates as in § 4.1, noting that $d_{L\cup l'} = [0, 0, 1] \in \mathbb{P}_l^2 \cong \mathbb{P}^2_{x_0,x_1,z}$. We then suppose that $y = [a_0, a_1, 0]$ and plug $a_0x_1 = a_1x_0$ into Equation (5). As in the proof of the first assertion, let $m_y^{(i)} \cup n_y^{(i)}$ (with $1 \leq i \leq 4$) denote the four coplanar pairs of lines on $X \cap H_y$ meeting l which are different from $L \cup l'$. It is not difficult to see that these four pairs of lines are related by the involution τ_{D_l} on D_l and hence give an element of $\overline{D}_l^{(2)}$. Without loss of generality, we assume that $\tau(m_y^{(1)}) = m_y^{(3)}$ (respectively, $\tau(n_y^{(1)}) = n_y^{(3)}$) and $\tau(m_y^{(2)}) = m_y^{(4)}$ (respectively, $\tau(n_y^{(2)}) = n_y^{(4)}$). Now choose a τ_y-invariant line on $X \cap H_y$ meeting the pointwise fixed line L;

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there are four such lines besides l and l' all of which are parameterized by points on $\widetilde{C}$. This τ_Y-invariant line intersects two of the four lines $m_Y^{(1)}, n_Y^{(1)}, m_Y^{(2)}, n_Y^{(2)}$, one in each of the two coplanar pairs. Similarly as in [20, Ex. 2.15(4)], we get an element in r_*D_τ and thus define an injection $\widetilde{C} \hookrightarrow r_*D_\tau$. Since (X, τ, l) is general, r_*D_τ is smooth and irreducible (i.e. the situation in [20, p. 69 (v)] does not happen) and therefore $\widetilde{C} = r_*D_\tau$. Summing it up, $D_\tau \xrightarrow{b_\tau} \overline{D}_l \xrightarrow{r} \mathbb{P}^1$ and $\widetilde{C} \xrightarrow{\pi} C \xrightarrow{p} \mathbb{P}^1$ are related by the bigonal construction.

Similarly, $D'_\tau \xrightarrow{b'_\tau} \overline{D}_{l'} \xrightarrow{r'} \mathbb{P}^1$ and $\widetilde{C} \xrightarrow{\pi} C \xrightarrow{p} \mathbb{P}^1$ are also related by the bigonal construction, noting that l' is contained in the same fiber of $\pi_L : \mathrm{Bl}_L X \to \Pi$ as l and that the bigonal structure $p = p_{l \cup l'} : C \to \mathbb{P}^1$ on C is determined by $l \cup l'$. Since the bigonal construction is symmetric (cf. [20, Lem. 2.7]), $b_\tau : D_\tau \to \overline{D}_l$ and $b'_\tau : D'_\tau \to \overline{D}_{l'}$ are isomorphic. ■

As a consequence of [37, §3] and Proposition 4.5, the Prym varieties $P(\widetilde{C}, C)$ and $P(D_\tau, \overline{D}_l)$ are dual to each other; this matches our results in Theorems 2.5 and 4.4. We conclude the discussion using the following proposition.

Proposition 4.6. Consider the Prym map $\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3^{(1,2,2)}$. A general member $A \in \mathcal{A}_3^{(1,2,2)}$ can be realized as the invariant part JX^τ of the intermediate Jacobian of a general cubic threefold (X, τ) with a non-Eckardt type involution. Furthermore, set $C \subset X$ to be the pointwise fixed plane section as in Lemma 1.3; note that C is smooth and of genus 1. Then the generic fiber of $\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3^{(1,2,2)}$ over $JX^\tau \in \mathcal{A}_3^{(1,2,2)}$ is birational to C.

Proof. The first assertion is a corollary of Theorem 3.1 and Proposition 3.4. The second claim follows from Proposition 4.5 and the argument in the proof of [36, Thm. 1.2] or [22, Thm. 5.2]. Specifically, notation remains the same as in § 2.2 and § 4.1. From Theorem 4.4, we know that the invariant part JX^τ is isomorphic to the Prym variety $P(D_\tau, \overline{D}_l)$, where $l \subset X$ is a general invariant line and $b_\tau : D_\tau \to \overline{D}_l$ is the quotient discriminant double cover (see Proposition 4.2). As argued in the proof of [36, Thm. 1.2] or [22, Thm. 5.2], the generic fiber of the Prym map $\mathcal{P}_{2,4} : \mathcal{R}_{2,4} \to \mathcal{A}_3^{(1,2,2)}$ over $P(D_\tau, \overline{D}_l)$ is birational to $\mathrm{Pic}^2(E) \cong E$, where E is an elliptic curve obtained via the bigonal construction for $D_\tau \xrightarrow{b_\tau} \overline{D}_l \to \mathbb{P}^1$. By Proposition 4.5, we have that $E \cong C$ which completes the proof. ■

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that of [16, Prop. 3.10], which was due to her. We also thank the referee for helpful suggestions. Research of the first named author is supported in part by a grant from the Simons Foundation [grant number 581058]. Research of the second named author is supported in part by a National Science Foundation grant [grant number DMS-2101640] (PI Laza). Research of the third named author is supported in part by the National Natural Science Foundation of China (grant number: 12201406).

References

[1] Alexeev, V., C. Birkenhake, and K. Hulek. "Degenerations of Prym varieties." *J. Reine Angew. Math.* 553 (2002): 73–116.

[2] Allcock, D., J. A. Carlson, and D. Toledo. "The complex hyperbolic geometry of the moduli space of cubic surfaces." *J. Algebraic Geom.* 11, no. 4 (2002): 659–724.

[3] Allcock, D., J. A. Carlson, and D. Toledo. "The moduli space of cubic threefolds as a ball quotient." *Mem. Amer. Math. Soc.* 209, no. 985 (2011) xii+70.

[4] Andreotti, A. "On a theorem of Torelli." *Amer. J. Math.* 80 (1958): 801–28.

[5] Ballico, E., C. Casagrande, and C. Fontanari. "Moduli of Prym curves." *Doc. Math.* 9 (2004): 265–81.

[6] Beauville, A. "Prym varieties and the Schottky problem." *Invent. Math.* 41, no. 2 (1977): 149–96.

[7] Beauville, A. "Variétés de Prym et jacobiennes intermédiaires." *Ann. Sci. École Norm. Sup. (4)* 10, no. 3 (1977): 309–91.

[8] Beauville, A. Les singularités du diviseur Θ de la jacobienne intermédiaire de l'hypersurface cubique dans $\mathbf{P}^4$. In *Algebraic threefolds (Varenna, 1981), volume 947 of Lecture Notes in Math.*, pages 190–208. Springer, Berlin-New York, 1982.

[9] Beauville, A. "Determinantal hypersurfaces." *Michigan Math. J.* 48 (2000): 39–64 Dedicated to William Fulton on the occasion of his 60th birthday.

[10] Birkenhake, C. and H. Lange. "An isomorphism between moduli spaces of abelian varieties." *Math. Nachr.* 253 (2003): 3–7.

[11] Birkenhake, C. and H. Lange. *Complex abelian varieties, volume 302 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004, http://10.1007/978-3-662-06307-1.

[12] Catanese, F. M. E. "Infinitesimal Torelli theorems and counterexamples to Torelli problems." *Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), volume 106 of Ann. of Math. Stud.* 143–56. Princeton, NJ: Princeton Univ. Press, 1984.

[13] Clemens, C. H. and P. A. Griffiths. "The intermediate Jacobian of the cubic threefold." *Annals of Mathematics. Second Series* 95 (1972): 281–356.

[14] Casalaina-Martin, S. and R. Friedman. "Cubic threefolds and abelian varieties of dimension five." *J. Algebraic Geom.* 14, no. 2 (2005): 295–326.

[15] Carlson, J., S. Müller-Stach, and C. Peters. *Period mappings and period domains, volume 168 of Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press, 2017 Second edition of [MR2012297].

Downloaded from https://academic.oup.com/imrn/article/2023/18/16104/7190408 by Columbia University Libraries user on 29 March 2025

[16] Casalaina-Martin, S. and Z. Zheng. "The moduli space of cubic surface pairs via the intermediate Jacobians of Eckardt cubic threefolds." *J. Lond. Math. Soc. (2)* 104, no. 1 (2021): 1–34.

[17] Cornalba, M. "Moduli of curves and theta-characteristics." *Lectures on Riemann surfaces (Trieste, 1987)*. 560–89. Teaneck, NJ: World Sci. Publ., 1989.

[18] Debarre, O. and A. Kuznetsov. "Double covers of quadratic degeneracy and Lagrangian intersection loci." *Math. Ann.* 378, no. 3-4 (2020): 1435–69.

[19] Dolgachev, I. V. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012. A modern view., http://10.1017/CBO9781139084437

[20] Donagi, R. "The fibers of the Prym map." *Curves, Jacobians, and abelian varieties (Amherst, MA, 1990), volume 136 of Contemp. Math.* 55–125. Providence, RI: Amer. Math. Soc., 1992.

[21] Donagi, R. and R. C. Smith. "The structure of the Prym map." *Acta Math.* 146, no. 1-2 (1981): 25–102.

[22] Frediani, P., J. C. Naranjo, and I. Spelta. "The fibers of the ramified Prym map." *Commun. Contemp. Math.* 24, no. 5 (2022) Paper No. 2150030, 30.

[23] González-Aguilera, V. and A. Liendo. "Automorphisms of prime order of smooth cubic n-folds." *Arch. Math. (Basel)* 97, no. 1 (2011): 25–37.

[24] Ikeda, A. "Global Prym-Torelli theorem for double coverings of elliptic curves." *Algebraic Geometry* 7, no. 5 (2020): 544–60.

[25] Kudla, S. and M. Rapoport. "On occult period maps." *Pacific J. Math.* 260, no. 2 (2012): 565–81.

[26] Laza, R., G. Pearlstein, and Z. Zheng. "On the moduli space of pairs consisting of a cubic threefold and a hyperplane." *Adv. Math.* 340 (2018): 684–722.

[27] Looijenga, E. and R. Swierstra. "The period map for cubic threefolds." *Compositio Math.* 143, no. 4 (2007): 1037–49.

[28] Laza, R., G. Saccà, and C. Voisin. "A hyper-Kähler compactification of the intermediate Jacobian fibration associated with a cubic 4-fold." *Acta Math.* 218, no. 1 (2017): 55–135.

[29] Laza, R. and Z. Zheng. "Automorphisms and periods of cubic fourfolds." *Math. Z.* 300, no. 2 (2022): 1455–507.

[30] Martens, H. H. "An extended Torelli theorem." *Amer. J. Math.* 87 (1965): 257–61. http://10.2307/2373002.

[31] Marquand, L. "Cubic fourfolds with an involution." *Trans. Amer. Math. Soc.* 376, no. 2 (2023): 1373–406.

[32] Marquand, L. and S. Muller. *Classification of symplectic birational involutions of manifolds of OG10 type*, 2022 arXiv:2206.13814.

[33] Matsumura, H. and P. Monsky. "On the automorphisms of hypersurfaces." *J. Math. Kyoto Univ.* 3 (1963/64): 347–61.

[34] Mumford, D. "Prym varieties. I." *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*. 325–50. New York: Academic Press, 1974.

[35] Naranjo, J. C. and A. Ortega. "Generic injectivity of the Prym map for double ramified coverings." *Trans. Amer. Math. Soc.* 371, no. 5 (2019): 3627–46 With an appendix by Alessandro Verra.

Downloaded from https://academic.oup.com/imrn/article/2023/18/16104/7190408 by Columbia University Libraries user on 29 March 2025

[36] Naranjo, J. C. and A. Ortega. *Global Prym-Torelli for double coverings ramified in at least 6 points*, 2020 arXiv:2005.11108, to appear in J. Algebraic Geom.

[37] Pantazis, S. "Prym varieties and the geodesic flow on $SO(n)$." *Math. Ann.* 273, no. 2 (1986): 297–315.

[38] Recillas, S. and R. E. Rodríguez. "Prym varieties and fourfold covers. II. The dihedral case." *The geometry of Riemann surfaces and abelian varieties, volume 397 of Contemp. Math.* 177–91. Providence, RI: Amer. Math. Soc., 2006.

[39] Saccà, G. *Birational geometry of the intermediate Jacobian fibration of a cubic fourfold*, 2021 arXiv: 2002.01420, to appear in Geom. Topol.

[40] Usui, S. "Effect of automorphisms on variation of Hodge structures." *J. Math. Kyoto Univ.* 21, no. 4 (1981): 645–72.

[41] Yu, C. and Z. Zheng. "Moduli spaces of symmetric cubic fourfolds and locally symmetric varieties." *Algebra & Number Theory* 14, no. 10 (2020): 2647–83. http://10.2140/ant.2020.14.2647.

[42] Zheng, Z. "Orbifold aspects of certain occult period maps." *Nagoya Math. J.* 243 (2021): 137–56.

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