Discrete Mathematics 347 (2024) 114071

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Rainbow solutions to the Sidon equation in cyclic groups and the interval

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A R T I C L E I N F O

Article history:
Received 10 October 2022
Received in revised form 16 January 2024
Accepted 30 April 2024
Available online 17 May 2024

Keywords:
Rainbow numbers
Sidon equation

A B S T R A C T

Given a coloring of group elements, a rainbow solution to an equation is a solution whose every element is assigned a different color. The rainbow number of $X \in \{\mathbb{Z}_n, [n]\}$ for an equation eq, denoted $rb(X, eq)$, is the smallest number of colors r such that every exact r-coloring of X admits a rainbow solution to the equation eq. We prove that for every exact 4-coloring of $\mathbb{Z}_p$, where $p \geq 3$ is prime, there exists a rainbow solution to the Sidon equation $x_1 + x_2 = x_3 + x_4$. Furthermore, we determine the rainbow number of $\mathbb{Z}_n$ and $[n]$ for the Sidon equation.

1. Introduction

An *r-coloring* of a set X is a function $c : X \to [r]$, where $[r] = \{1, 2, \dots, r\}$. An r-coloring is *exact* if the function c is surjective. In this paper we focus on r-colorings of the cyclic group $\mathbb{Z}_n$ and the interval $[n]$, and all r-colorings are assumed to be exact. A subset $A \subseteq \mathbb{Z}_n$ or $A \subseteq [n]$ is called *rainbow* (under the coloring c) if each element of A is colored distinctly. Given an equation eq, the *rainbow number* of $X \in \{\mathbb{Z}_n, [n]\}$ for an equation eq, denoted $rb(X, eq)$, is the smallest number of colors r such that every exact r-coloring of X admits a rainbow solution to the equation eq. By convention, if no such integer r exists we set $\mathrm{rb}(X, eq) = |X| + 1$. We say that a coloring c is *rainbow eq-free* if there is no rainbow solution to eq under c.

Rainbow numbers of $\mathbb{Z}_n$ and $[n]$ for the equation $x_1 + x_2 = 2x_3$, for which the solutions are 3-term arithmetic progressions, are known as anti-van der Waerden numbers. Rainbow arithmetic progressions have been studied extensively in [1,3,5,8,14]. Generalizing the equation $x_1 + x_2 = 2x_3$, Bevilacqua et al. in [4] determined the rainbow number of $\mathbb{Z}_n$ for the equation $x_1 + x_2 = kx_3$ for every integer n when $k = 1$ (which is known as the Schur equation). Ansaldi et al. determined the rainbow number of $\mathbb{Z}_p$ for the equation $a_1x_1 + a_2x_2 + a_3x_3 = b$, and established the rainbow number of $\mathbb{Z}_n$ for this equation under certain conditions on the coefficients in [2]. Structures of rainbow-free colorings for various equations have been studied in [7,9,10].

In this paper, we establish the rainbow number of $\mathbb{Z}_n$ and $[n]$ for the Sidon equation $x_1 + x_2 = x_3 + x_4$. The Sidon equation is a classical object in additive number theory and is used to measure the additive energy of a set (see [11,13]). Fox, Mahdian, and Radoiĉić showed in [6] that for every 4-coloring of $[n]$, where each color class has cardinality more than $\frac{n+1}{6}$, there exists a rainbow solution to the Sidon equation. The lower bound on a color class cardinality is tight. Taranchuk and Timmons in [12] studied the maximum number of rainbow solutions to the Sidon equation for a fixed number of colors.

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1 Research in part supported by NSF grants DMS-1839918 and DMS-2136228.

https://doi.org/10.1016/j.disc.2024.114071

In this paper we denote the Sidon equation by S. We determine $\mathrm{rb}(\mathbb{Z}_p, S)$, where p is prime in Section 2. Furthermore, we determine $\mathrm{rb}(\mathbb{Z}_n, S)$ in Section 3. Finally, we determine $\mathrm{rb}([n], S)$ in Section 4. Notice that $\mathrm{rb}(\mathbb{Z}_2, S) = 3$ and $\mathrm{rb}(\mathbb{Z}_3, S) = 4$ by convention. Our main results are as follows:

Theorem 1.1. *Let $p \geq 3$ be a prime. Then $rb(\mathbb{Z}_p, S) = 4$.*

Theorem 1.2. *Let $n = p_1 \cdots p_k$ be a prime factorization such that $p_i \leq p_j$ whenever $i < j$. Let m be an arbitrary index such that $p_m \geq 3$ (or $m = k$ if this index does not exist), $f_1 = |\{i : p_i \leq 3, i \neq m\}|$, and $f_2 = |\{i : p_i \geq 5, i \neq m\}|$. Then*

$$rb(\mathbb{Z}_{p_m}, S) + f_1 + 2f_2 = rb(\mathbb{Z}_n, S).$$

Theorem 1.3. *For any integer $n \geq 2$,*

$$rb([n], S) = \lfloor \log_2(n-1) \rfloor + 3.$$

2. Rainbow numbers for $\mathbb{Z}_p$

Suppose c is a coloring of $\mathbb{Z}_p$. We say that a color X is *dominant* if for any pair of elements $x, x+1 \in \mathbb{Z}_p$, either $c(x) = c(x+1)$ or $X \in \{c(x), c(x+1)\}$. More generally, we say a color X is *i-dominant* if for any pair of elements $x, x+i \in \mathbb{Z}_p$, either $c(x) = c(x+i)$ or $X \in \{c(x), c(x+i)\}$. We should note that the idea of dominant colors has appeared in [6]. Lemma 2.2 shows that i-dominant colors must exist for all $i \in \mathbb{Z}_p^*$ and Lemma 2.3 lets us translate and scale colorings, where $\mathbb{Z}_p^*$ is the cyclic multiplicative group.

The remainder of the lemmas, we assume for the sake of contradiction that a rainbow Sidon-free 4-coloring c on $\mathbb{Z}_p$ exists. The assumptions that c uses 4 colors and that c is rainbow Sidon-free imply that certain structures must exist in the coloring. Ultimately, these structures will allow us to find a rainbow solution to the Sidon equation. In this sense, the entire section is to be read as a proof by contradiction, in which the contradiction is found in the proof of Theorem 1.1.

As noted in the introduction, $\mathrm{rb}(\mathbb{Z}_2, S) = 3$ and $\mathrm{rb}(\mathbb{Z}_3, S) = 4$ by convention. Since any rainbow solution to the Sidon equation requires at least four distinct colors, we have $\mathrm{rb}(\mathbb{Z}_p, S) \geq 4$ when $p \geq 5$.

Observation 2.1. Let $p \geq 3$ be a prime. Then $\mathrm{rb}(\mathbb{Z}_p, S) > 3$.

To prove the corresponding upper bound, we show that any 4-coloring of $\mathbb{Z}_p$ with $p \geq 5$ prime admits a rainbow solution to the Sidon equation. Suppose $c : \mathbb{Z}_p \to \{R, Y, G, B\}$ is a rainbow Sidon-free 4-coloring. For a color X, an interval of integers $[i, i+j]$ is an *X-string* if $c([i, i+j]) = \{X\}$. Similarly, given two colors X_1, X_2, an interval $[i, i+j]$ is an *X_1X_2-string* if $c([i, i+j]) = \{X_1, X_2\}$ (these strings are also called bichromatic). An X or X_1X_2-string $[i, i+j]$ is *maximal* if $c(i-1), c(i+j+1) \notin \{X\}$ or $c(i-1), c(i+j+1) \notin \{X_1, X_2\}$, respectively. A pattern (sequence) of colors $X_0X_1X_2 \cdots X_k$ *appears at position j* if $c(j+i) = X_i$ for $0 \leq i \leq k$; if such a j does not exist, then $X_0X_1X_2 \cdots X_k$ *does not appear*. A string A is *i-periodic* if for all $x, x+i \in A$ we have $c(x) = c(x+i)$. Often we will abuse notation and identify a string A with its induced pattern of colors.

Lemma 2.2. *Every rainbow Sidon-free 4-coloring c of $\mathbb{Z}_p$ has an i-dominant color for any $i \in \mathbb{Z}_p \setminus \{0\}$.*

Proof. Let $i \in \mathbb{Z}_p \setminus \{0\}$. Form a graph H on $V(H) = \{R, Y, G, B\}$ where $X_1X_2 \in E(H)$ if and only if there exists $x \in \mathbb{Z}_p$ such that $\{c(x), c(x+i)\} = \{X_1, X_2\}$. For the sake of contradiction, suppose that $d(X) = 0$ in H. Since c is a exact, we know there exists $x \in \mathbb{Z}_p$ such that $c(x) = X$. By induction on k and the fact that $d(X) = 0$ in H, we have that $c(x+ik) = X$ for all $k \geq 0$. Since p is prime, $i \neq 0$ is an additive generator of $\mathbb{Z}_p$. Thus,

$$c(\mathbb{Z}_p) = c(\{x + ik : k \geq 0\}) = \{X\},$$

which contradicts the fact that c is a 4-coloring.

By construction, if H contains a $2K_2$ subgraph, then c admits a rainbow solution to the Sidon equation. Therefore, H is $2K_2$-free. Since $\delta(H) \geq 1$ and H is $2K_2$-free, H must be isomorphic to $K_{1,3}$. Let $X \in V(H)$ such that $d(X) = 3$. Notice that X is an i-dominant color. $\square$

The next lemma shows that the rainbow Sidon-free property of colorings is preserved by translating and scaling colorings.

Lemma 2.3. *Let c be a coloring of $\mathbb{Z}_p$, $i \in \mathbb{Z}_p^*$, and $j \in \mathbb{Z}_p$. Let $c_{i,j}$ be given by $c_{i,j}(x) = c(ix + j)$. The coloring c is rainbow Sidon-free if and only if $c_{i,j}$ is rainbow Sidon-free.*

Proof. Let $A = \{x_1, x_2, x_3, x_4\} \subset \mathbb{Z}_p$. Notice that A is rainbow under $c_{i,j}$ if and only if $A_{i,j} = \{ix_1 + j, ix_2 + j, ix_3 + j, ix_4 + j\}$ is rainbow under c. Furthermore, $x_1 + x_2 = x_3 + x_4$ if and only if $ix_1 + j + ix_2 + j = ix_3 + j + ix_4 + j$. □

It should be noted that Proposition 3.5 in [5] and Theorem 3.5 in [8] together determine when $\mathrm{rb}(\mathbb{Z}_p, eq) = 3$ and when $\mathrm{rb}(\mathbb{Z}_p, eq) = 4$, where eq is the equation $x_1 + x_2 = 2x_3$. Since we use Proposition 3.5 from [5], we have stated it below.

Proposition 2.4 *(Proposition 3.5 in [5]). Let eq be the equation $x_1 + x_2 = 2x_3$. For every prime number p, $3 \le rb(\mathbb{Z}_p, eq) \le 4$.*

Let $x, x+i, x+2i$ be a rainbow 3-term arithmetic progression in $\mathbb{Z}_p$ under the coloring c with colors Y, R, B, respectively. This implies that $i^{-1}x, i^{-1}x+1, i^{-1}x+2$ are colored Y, R, B under the coloring $c_{i,0}$. Since a 1-dominant color must exist in $c_{i,0}$ by Lemma 2.2 and the one dominant color must appear in $\{c_{i,0}(i^{-1}x), c_{i,0}(i^{-1}x+1)\} \cap \{c_{i,0}(i^{-1}x+1), c_{i,0}(i^{-1}x+2)\} = \{R\}$, it follows that R must be dominant. Furthermore, R is not 2-dominant given $c_{i,0}$. Since c and $c_{i,0}$ have the same behavior with respect to rainbow Sidon solutions, we will always assume that R is dominant, and R is not 2-dominant. In particular, we will assume that the pattern YRB appears in c (otherwise, we can find a rainbow 3-term arithmetic progression, and scale/translate the coloring to put ourselves in this position). Furthermore, since a 2-dominant color must exist (and it is either Y or B), we can assume that Y is 2-dominant (otherwise consider $c_{-i,0}$). From this point forward in this section, we will use these assumptions to prove structural results about c. Ultimately, these structures will lead to a contradiction, in the sense that we will find a rainbow solution to the Sidon equation in the proof of Theorem 1.1.

Lemma 2.5. *Let $X \in \{B, G\}$. Every maximal RX-string which uses two colors is 2-periodic with exactly one element colored by X within every substring of length 2. In particular, BB and GG do not appear.*

Proof. Let A be a maximal RG-string. Since R is dominant, A cannot start or end with G. Recall that we assume the pattern YRB appears under the coloring c. Therefore, if A contains patterns RRG, GRR, RGG, or GGR, then c admits a rainbow solution to the Sidon equation. Furthermore, if GGG appears, then GGR appears since R is dominant. Similarly, if RRR appears as a sub-pattern of A, then either RRG or GRR must appear as a sub-pattern of A since we assumed that A uses both G and R. Thus, A must be of the form $RGR \cdots RGR$.

Now let A be a maximal RB-string. Since R is dominant, A cannot start or end with B. Since Y is 2-dominant, we know that pattern GRY must appear under c. This implies that A cannot contain RRB, BRR, BBR, RBB. Furthermore, if BBB appears, then BBR appears since R is dominant. Similarly, if RRR appears as a sub-pattern of A, then either RRB or BRR must appear as a sub-pattern of A since we assumed that A uses both G and R. Thus, A must be of the form $RBR \cdots RBR$. □

The next lemma extends Lemma 2.5 to shows that the pattern YY cannot appear under the coloring c. It will be very useful since it implies that if $c(x) = Y$, then $c(x-1) = c(x+1) = R$.

Lemma 2.6. *The pattern YY does not appear under the coloring c.*

Proof. Let $c(\{i, i+m\}) = \{G, B\}$. Suppose the pattern RYY exists at position j, and consider colors of $j+m, j+m+1, j+m+2$. There are three cases: $c(j+m)$ in $\{Y\}$, $\{B, G\}$, or $\{R\}$. If $c(j+m) = Y$, then $\{i, j+m, i+m, j\}$ is a rainbow Sidon solution. If $c(j+m) \in \{B, G\}$, then $c(j+m+1) = R$ by Lemma 2.5 and $\{i, j+m+1, i+m, j+1\}$ is a rainbow Sidon solution. Therefore, $c(j+m) = R$.

Next consider the color of $j+m+2$. There are three cases: $c(j+m+2)$ is in $\{R\}$, $\{B, G\}$, or $\{Y\}$. If $c(j+m+2) = R$, then $\{i, j+m+2, i+m, j+2\}$ is a rainbow Sidon solution. If $c(j+m+2) \in \{B, G\}$, then $c(j+m+1) = R$ by Lemma 2.5 and $\{i, j+m+1, i+m, j+1\}$ a is rainbow Sidon solution. Therefore, $c(j+m+2) = Y$.

It follows that $c(j+m+1) \notin \{B, G\}$ since $c(j+m+2) = Y$ and R is dominant. Furthermore, $c(j+m+1) \neq R$, else $\{i, j+m+1, i+m, j+1\}$ is a rainbow Sidon solution. Therefore, $c(j+m+1) = Y$.

Thus if there is an RYY-string starting at j, then there is an RYY at $j+m$. Since m is a generator of $\mathbb{Z}_p$, which implies that $c(x) = R$ for all $x \in \mathbb{Z}_p$; this is a contradiction. □

The following lemma is a technical detail that will be used in the proof of Lemma 2.8.

Lemma 2.7. *Suppose that X_1RY appears at i and YRX_2 appears at $i+j-2$ where j is odd and $\{X_1, X_2\} = \{B, G\}$. Then $c(i + \frac{j-1}{2}) = c(i + \frac{j+1}{2}) = R$.*

Proof. Let $y = i + \frac{j-1}{2}$. For the sake of contradiction, suppose that $\{c(y), c(y+1)\} \neq \{R\}$. By the symmetry of the problem, there are three cases: $c(y) = Y$, $c(y) = X_1$, and $c(y) = X_2$.

If $c(y) = Y$, then by the dominance of R and Lemma 2.6, we have $c(y+1) = R$; hence $\{i, i+j, y, y+1\}$ is a rainbow Sidon solution via $i + (i+j) = y + (y+1)$. If $c(y) = X_1$, then $c(y+1) = R$ and $c(y+2) \in \{X_1, Y\}$ since Y is 2-dominant.

Depending on the value of $c(y+2)$, either $\{i+1, i+j, y, y+2\}$ or $\{i+2, i+j, y+1, y+2\}$ is a rainbow Sidon solution. If $c(y) = X_2$, then $c(y-1) = R$ and $\{i, i+j-2, y, y-1\}$ is a rainbow Sidon solution.

In each case, we get a contradiction. □

Lemma 2.8. *The patterns BRB and GRG do not appear under the coloring c.*

Proof. Without loss of generality, suppose that BRB exists. Let $i_b, \ldots, i_b + j_b$ be a maximal BR-string that starts and ends with B (we are truncating the R colored element at the start and end of a maximal BR-string). Note that this string alternates colors between B and R. This implies that $j_b \geq 2$ and j_b is even. Let $i_g, \ldots, i_g + j_g$ be a maximal GR-string that starts and ends with G. This implies that j_g is even.

Since p is odd, either $i_g - i_b - j_b$ or $i_b - i_g - j_g$ is odd. Without loss of generality, suppose that $j = i_g - i_b - j_b$ is odd. Let $i = i_b + j_b$ so that $i + j = i_g$. In particular, using the fact that Y is 2-dominant, we have chosen i and j such that $BRBRY$ appears at $i-2$ and YRG appears at $i+j-2$, where j is odd.

Let $y = i + \frac{j-1}{2}$. Notice that $c(y) = c(y+1) = R$ by Lemma 2.7. Let k be the smallest integer such that either $c(y-k) = Y$ or $c(y+1+k) = Y$. Notice that if $\{c(y-k), c(y+1+k)\} = \{R, Y\}$, then $\{i, i+j, y-k, y+1+k\}$ is a rainbow Sidon solution. Therefore, $c(y-k) = c(y+1+k) = Y$.

Notice that $c(i-2) = B$; and we claim that $c(y+1+k-2) = R$. We will prove the claim by showing that $c(y+1+\ell) = R$ for $0 \leq \ell \leq k-1$ using induction. As a base case, recall that $c(y+1+0) = R$. Now we assume that $c(y+1+\ell') = R$ for $0 \leq \ell' < \ell$. If $c(y+1+\ell-2) = R$ by the induction hypothesis and the fact that $c(y) = R$ in the case when $\ell = 1$. Since Y is 2-dominant and $c(y+1+\ell-2) = R$, we know that $c(y+1+\ell) \in \{R, Y\}$. If $c(y+1+\ell) = Y$, then we contradict the choice of k. Thus, $c(y+1+\ell) = R$.

However,

$$(y-k) + (y+1+k-2) = 2i + j - 2 = (i-2) + (i+j).$$

Therefore, $\{i-2, i+j, y-k, y+1+k-2\}$ is a rainbow Sidon solution, a contradiction. Therefore, neither BRB nor GRG can appear under the coloring c. □

As an interesting note, we would like to point out that Lemmas 2.5 and 2.8 have analogous results in the context of the interval $[n]$ as shown in [6] by Fox et al. In these cases, our proofs are similar, but expedited by the assumptions Lemmas 2.2 and 2.3 afford us. Curiously, Lemma 2.6 contrasts with Lemma 3 in [6], where it is shown that a YY-string must exist.

Lemma 2.9. *All R-strings have length* 1 *or* 3.

Proof. Let elements i and $i+m$ be colored with B and G, respectively. If every R-string has length 1, then we are done. Therefore, we will assume that YRR appears at position x. Our goal is to show that $YRRRY$ must appear at x.

Since $m \geq 1$ is a generator of $\mathbb{Z}_p$, it follows that there exists a smallest non-negative integer ℓ such that YRR does not appear at $x + m(\ell+1)$. Let $j = x + m\ell$. We will prove that $YRRRY$ appears at $j - m\ell'$ for $0 \leq \ell' \leq \ell$. Notice that by our choice of ℓ, we immediately have that YRR appears at $j - m - \ell'$ for $0 \leq \ell' \leq \ell$.

To prove the base case, we will consider the colors of $j+m$, $j+m+1$, $j+m+2$. First we will conclude that $c(j+m) = Y$ by ruling out the other three options.

If $c(j+m) = R$, then $\{i, j+m, i+m, j\}$ is a rainbow Sidon solution. If $c(j+m) = B$, then $\{i+2, j+m, i+m, j+2\}$ is a rainbow Sidon solution because $c(i+2) = Y$ by Lemma 2.8. If $c(j+m) = G$, then $\{i, j+m, i+m-2, j+2\}$ is a rainbow Sidon solution because $c(i+m-2) = Y$ by Lemma 2.8. Therefore, $c(j+m) = Y$. Furthermore, by Lemma 2.6, $c(j+m+1) = R$ since R is dominant.

Next consider the color of $j+m+2$. Notice that if $c(j+m+2) = Y$, then $\{i, j+m+2, i+m, j+2\}$ is a rainbow Sidon solution. Furthermore, recall that j was selected so that YRR does not appear at $j+m$. Therefore, $c(j+m+2) \neq R$. In particular, $c(j+m+2) \in \{B, G\}$.

Now $c(j+4) \neq R$, else either $\{i, j+m+2, i+m-2, j+4\}$ or $\{i+2, j+m+2, i+m, j+4\}$ is a rainbow Sidon solution depending on the color of $j+m+2$. Therefore, $c(j+4) = Y$ and $c(j+3) = R$. This concludes the proof of the base case $\ell' = 0$.

For our induction hypothesis, we assume that $YRRRY$ appears at $j - m(\ell'-1)$. Recall that we already know that YRR appears that $j - m\ell'$, so we only need to determine the colors of $j - m\ell' + 3$ and $j - m\ell' + 4$. Since Y is 2-dominant and YY cannot appear (by Lemma 2.6), it follows that $c(j - m\ell' + 3), c(j - m\ell' + 4) \in \{R, Y\}$. If $c(j - m\ell' + 4) = R$, then $\{i+m, j - m\ell', i, j - m(\ell'-1)\}$ is a rainbow Sidon Solution. Therefore, $c(j - m\ell' + 4) = Y$, and so by Lemma 2.6 we have $c(j - m\ell' + 3) = R$. Hence, $YRRRY$ appears at position $j - m\ell'$.

Thus, by induction, $YRRRY$ appears at $j - m\ell'$ for $0 \leq \ell' \leq \ell$. In particular, $YRRRY$ appears at $j - m\ell = x$ and any R-string with length greater than 1 must have length 3. □

With no further ado:

Theorem 1.1. *The coloring c does not exist. In particular, $rb(\mathbb{Z}_p, S) = 4$ for all prime $p \geq 3$.*

Proof. Since c is a 4-coloring, there must exist i and j such that $c(i) = B$ and $c(j) = G$. Furthermore, since p is odd, there are distinct $m_1, m_2 \in \mathbb{Z}_p$ such that $i+m_1 = j$ and $i-m_2 = j$ where either m_1 is odd or m_2 is odd. Without loss of generality, suppose that m_1 is odd. By Lemma 2.5 and Lemma 2.8, we have that $c(i+1) = c(j-1) = R$ and $c(i+2) = c(j-2) = Y$. By Lemma 2.7, let $y = i + \frac{m_1-1}{2}$ and $c(y) = c(y+1) = R$. Notice that Lemma 2.9 guarantees that the R-string containing y has length 3. Therefore, either $c(y-1) = Y$ and $c(y+2) = R$ or $c(y-1) = R$ and $c(y+2) = Y$. In either case, $\{i, y-1, y+2, j\}$ is a rainbow Sidon solution. This contradicts the assumption of the section, that c is a rainbow Sidon-free 4-coloring of $\mathbb{Z}_p$. $\square$

3. Rainbow numbers for $\mathbb{Z}_n$

This section is divided into two subsections. The lower bound for $\mathrm{rb}(\mathbb{Z}_n, S)$ is shown in the first subsection, which will provide insight into where (and how) to look for rainbow solutions to the Sidon equation in the upper bound argument. The upper bound is shown in the second subsection.

3.1. Lower bound

We construct a lower bound coloring for $\mathbb{Z}_n$ by expanding a coloring for $\mathbb{Z}_{n/p}$. Essentially, we insert $p-1$ elements between each pair of elements in $\mathbb{Z}_{n/p}$ (taken in their natural cyclic ordering), and color the elements appropriately. The method for coloring the new inserted elements is specifically chosen to maintain i-dominant colors.

Lemma 3.1. *Let c be a rainbow Sidon-free r-coloring of $\mathbb{Z}_n$ for some natural number n and p be prime. If $p \leq 3$, then there exists a rainbow Sidon-free $(r+1)$-coloring of $\mathbb{Z}_{pn}$. If $p \geq 5$, then there exists a rainbow Sidon-free $(r+2)$-coloring of $\mathbb{Z}_{pn}$.*

Proof. Assume that $p \leq 3$. Let

$$\hat{c}(x) = \begin{cases} c(x/p) & \text{if } x \equiv 0 \mod p \\ r+1 & \text{otherwise} \end{cases}$$

be an $(r+1)$-coloring of $\mathbb{Z}_{pn}$. Let $X = \{x_1, x_2, x_3, x_4\} \subseteq \mathbb{Z}_{pn}$ such that $x_1 + x_2 = x_3 + x_4$.

Without loss of generality, if $x_1, x_2, x_3 \in \langle p \rangle$, then $x_4 \in \langle p \rangle$ where $\langle p \rangle$ is the subgroup of $\mathbb{Z}_{pn}$ generated by p. In this case, X is not rainbow, since c is a rainbow Sidon-free coloring of $\langle p \rangle \cong \mathbb{Z}_n$. Furthermore, if $x_i, x_j \notin \langle p \rangle$ for $1 \leq i < j \leq 4$, then $\hat{c}(x_i) = \hat{c}(x_j)$ and X is not rainbow. Thus, $\hat{c}$ is a rainbow Sidon-free $(r+1)$-coloring of $\mathbb{Z}_{pn}$.

Assume that $p \geq 5$. Let

$$\hat{c}(x) = \begin{cases} c(x/p) & \text{if } x \equiv 0 \mod p \\ r+1 & \text{if } x \equiv 1 \mod p \text{ or } x \equiv p-1 \mod p \\ r+2 & \text{otherwise} \end{cases}$$

be an $(r+2)$-coloring of $\mathbb{Z}_{pn}$. Let $X = \{x_1, x_2, x_3, x_4\} \subseteq \mathbb{Z}_{pn}$ such that $x_1 + x_2 = x_3 + x_4$.

As in the previous case, c is a rainbow Sidon-free coloring of $\langle p \rangle \cong \mathbb{Z}_n$. Therefore, if any three elements in X are in $\langle p \rangle$, then X is not rainbow. Thus, assume that exactly two elements in X are in $\langle p \rangle$. In particular, assume that $x_i, x_j \in \langle p \rangle$.

Without loss of generality, if $i = 1$ and $j = 2$, then $x_3 + x_4 = sp$ for some integer s. Therefore, $x_3 \equiv -x_4 \mod p$ and $\hat{c}(x_3) = \hat{c}(x_4)$.

Without loss of generality, if $i = 1$ and $j = 3$, then $x_2 \equiv x_4 \mod p$. Therefore, $\hat{c}(x_2) = \hat{c}(x_4)$.

In either case, X is not rainbow. Thus, $\hat{c}$ is a rainbow Sidon-free $(r+2)$-coloring of $\mathbb{Z}_{pn}$. $\square$

Repeatedly applying Lemma 3.1 gives a lower bound for $\mathrm{rb}(\mathbb{Z}_n, S)$. Notice that the statement of Proposition 3.2 selects p_m to maximize the number of colors used in the construction.

Proposition 3.2. *Let $n = p_1 \cdots p_k$ be a prime factorization such that $p_i \leq p_j$ whenever $i < j$. Let m be an arbitrary index such that $p_m \geq 3$ (or $m = k$ if this index does not exist), $f_1 = |\{i : p_i \leq 3, i \neq m\}|$, and $f_2 = |\{i : p_i \geq 5, i \neq m\}|$. Then*

$$rb(\mathbb{Z}_{p_m}, S) + f_1 + 2f_2 \leq rb(\mathbb{Z}_n, S).$$

Proof. By starting with a rainbow Sidon-free $(\mathrm{rb}(\mathbb{Z}_{p_m}, S) - 1)$-coloring of $\mathbb{Z}_{p_m}$, we can construct a rainbow Sidon-free r-coloring of $\mathbb{Z}_n$ with

$$r = \mathrm{rb}(\mathbb{Z}_{p_m}, S) - 1 + f_1 + 2f_2$$

by repeatedly applying Lemma 3.1 for primes p_i, $i \neq m$. $\square$

3.2. Upper bound

Since the lower bound construction expands colorings depending on primes p, it is intuitive that an upper bound argument would reduce colorings modulo p until a rainbow Sidon solution can be found. Let t be a positive integer that divides n. Let $R_i = i + \langle t \rangle$ be the i^{th} coset of the subgroup generated by t in $\mathbb{Z}_n$. This is notation that is consistent with work in [4]. Lemma 3.3 identifies which coset of t we want to narrow in on.

Lemma 3.3. *Let $t = n/p$ (p prime) divide n and c be a rainbow Sidon-free coloring of $\mathbb{Z}_n$. Consider the cosets of $\langle t \rangle$, R_i with $0 \le i < t$. There exists an index j such that $|c(R_i) \setminus c(R_j)| \le 1$ for all $0 \le i < t$.*

Proof. Let j be the index that maximizes $|c(R_j)|$. For the sake of contradiction, assume that $|c(R_i) \setminus c(R_j)| \ge 2$ for index i. This implies that there exists $x_1, x_2 \in R_i$ such that $c(x_1) \ne c(x_2)$ and $c(x_1), c(x_2) \notin c(R_j)$. Let $x_3 \in R_j$. Notice that there exist integers s_1, s_2, s_3 such that

$$x_1 = s_1 t + i$$

$$x_2 = s_2 t + i$$

$$x_3 = s_3 t + j.$$

We claim that $c(x_3 + kt(s_1 - s_2)) = c(x_3)$ for all $k \ge 0$. The base case holds when $k = 0$. As an induction hypothesis, we will assume that $c(x_3 + (k-1)t(s_1 - s_2)) = c(x_3)$. Toward the induction step, observe that

$$x_3 + kt(s_1 - s_2) = x_1 + x_3 + (k-1)t(s_1 - s_2) - x_2.$$

Since c is rainbow Sidon-free and $x_3 + kt(s_1 - s_2) \in R_j$,

$$c(x_3 + kt(s_1 - s_2)) = x_3 + (k-1)t(s_1 - s_2)) = c(x_3).$$

Thus, the claim holds by induction. Since $s_1 - s_2 \ne 0$ and $s_1 - s_2$ is relatively prime to p, it follows that $R_j = \{x_3 + kt(s_1 - s_2) : k \ge 0\}$. Therefore, we can conclude that $|c(R_j)| = 1$. This contradicts our choice of index j. □

The next lemma focuses our attention on the coset with the most number of colors under a coloring c. Furthermore, it controls the number of colors lost in the process.

Lemma 3.4. *Let p be a prime divisor of a natural number n and $pt = n$. Then*

$$rb(\mathbb{Z}_n, S) \le rb(\mathbb{Z}_p, S) + rb(\mathbb{Z}_t, S) - 2.$$

Proof. Let c be a rainbow Sidon-free $(\mathrm{rb}(\mathbb{Z}_n, S) - 1)$-coloring of $\mathbb{Z}_n$. Let R_i be the i^{th} coset of $\langle t \rangle$ in $\mathbb{Z}_n$. By Lemma 3.3, there exists index j such that $|c(R_i) \setminus c(R_j)| \le 1$ for $0 \le i < t$. Notice that c is rainbow Sidon-free if and only if c' given by $c'(x) = c(x + j)$ is rainbow Sidon-free. Since $R_0 \cong \mathbb{Z}_p$ is a subgroup of $\mathbb{Z}_n$, it follows c' must be a rainbow Sidon-free coloring of $\mathbb{Z}_p$. Furthermore, $|c'(R_i) \setminus c'(R_0)| \le 1$ for $0 \le i < t$.

Let

$$\hat{c}(x) = \begin{cases} i & \text{if } \{i\} = c'(R_x) \setminus c'(R_0) \\ \alpha & \text{if } c'(R_x) \subseteq c'(R_0) \end{cases}$$

be a coloring of $\mathbb{Z}_t$ (so that α is a color not used by c'). For the sake of contradiction, let $\{x_1, x_2, x_3, x_4\} \subseteq \mathbb{Z}_t$ be rainbow given $\hat{c}$ such that $x_1 + x_2 = x_3 + x_4$. Without loss of generality, assume that $\hat{c}(x_1), \hat{c}(x_2), \hat{c}(x_3) \ne \alpha$. Therefore, there exists $y_i \in R_{x_i}$ such that $c'(y_i) = \hat{c}(x_i)$ for $1 \le i \le 3$. Notice that $y_4 = y_1 + y_2 - y_3 \in R_{x_4}$. There are two cases: either $\hat{c}(x_4) = \alpha$, or $\hat{c}c(x_4) \ne \alpha$. In the case that $\hat{c}(x_4) = \alpha$, we have that $c'(y_4) \subseteq c'(R_0)$ which implies that $c'(y_4) \ne c'(y_1), c'(y_2), c'(y_3)$. Now consider the case when $\hat{c}(x_4) \ne \alpha$. Notice that $c'(R_{x_4})$ does not contain any of $c'(y_1), c'(y_2), c'(y_3)$, since otherwise $\hat{c}(x_4)$ would not be distinct from $\hat{c}(x_1), \hat{c}(x_2)$, and $\hat{c}(x_3)$. Therefore, $c'(y_4) \ne c'(y_1), c'(y_2), c'(y_3)$.

In any case, $\{y_1, y_2, y_3, y_4\}$ is a rainbow Sidon solution in $\mathbb{Z}_n$ given c'; this is a contradiction. Thus, $\hat{c}$ is a rainbow Sidon-free coloring of $\mathbb{Z}_t$.

We can combine all this information to bound the number of colors of used by c. In particular,

$$c(\mathbb{Z}_n) = c'(\mathbb{Z}_n) = (c'(R_0) \cup \hat{c}(\mathbb{Z}_t)) \setminus \{\alpha\}.$$

This implies that

$$\mathrm{rb}(\mathbb{Z}_n, S) - 1 = |c(\mathbb{Z}_n)| = |c'(R_0)| + |\hat{c}(\mathbb{Z}_t)| - 1 \le \mathrm{rb}(\mathbb{Z}_p, S) - 1 + \mathrm{rb}(\mathbb{Z}_t, S) - 2,$$

completing the proof. □

Proposition 3.5 is the result of repeatedly applying Lemma 3.4. In this sense, it reverses the process used in the construction of a rainbow Sidon-free coloring in the proof of Proposition 3.2. It is helpful to recall that $\mathrm{rb}(\mathbb{Z}_p, S) = 4$ for all primes $p \geq 3$ while comparing these two propositions.

Proposition 3.5. *Let $n = p_1 \cdots p_k$ be a prime factorization of n. Then*

$$rb(\mathbb{Z}_n, S) \leq 2(1-k) + \sum_{i=1}^{k} rb(\mathbb{Z}_{p_i}, S).$$

Proof. Recursively apply Lemma 3.4. □

Notice that the upper bound given in Proposition 3.5 does not meet the lower bound in Proposition 3.2. In particular, if $3^k | n$ for $k \geq 2$, then the upper bound exceeds the lower bound by at least $k - 1$. This suggests that Lemma 3.4 is too generous when $p = 3$.

To improve the upper bound, we want to focus on the case when $n = 3t$. Suppose that c is an r-coloring of $\mathbb{Z}_n$. Since the goal is to show that c admits a rainbow solution to the Sidon equation, we will assume that c is rainbow Sidon-free and pursue a contradiction. Partition $\mathbb{Z}_n$ into cosets of $\langle t \rangle$ denoted R_i, $1 \leq i \leq t$. By Lemma 3.3, there exists an index j such that $|c(R_i) \setminus c(R_j)| \leq 1$ for all $0 \leq i < t$. By shifting the coloring, we can assume that $j = 0$.

Lemma 3.6. *If $|c(R_i) \setminus c(R_0)| = 1$ and $|c(R_0)| = 3$, then $|c(R_i)| = 1$.*

Proof. Notice that $R_0 = \{0, t, 2t\}$, while $R_i = \{i, t+i, 2t+i\}$. Without loss of generality, suppose that $c(st + i) \notin c(R_0)$ for some $0 \leq s \leq 2$. For the sake of contradiction, suppose that $c((s \pm 1)t + i) \neq c(st + i)$ (where $(s \pm 1)t + i$ is taken modulo n). By assumption, there exists $s't, (s'+1)t \in R_0$ such that $c((s \pm 1)t + i) \notin \{c(s't), c((s' \pm 1)t)\}$. However, $(s' \pm 1)t + st + i = s't + (s \pm 1)t + i$. Since $\{s't, (s \pm 1)t + i, (s' \pm 1)t, st + i\}$ is rainbow under c, we have a contradiction. □

Rainbow numbers for the Schur equation $x_1 + x_2 = x_3$ will be useful in analyzing the rainbow number for the Sidon equation when 9 divides n. For convenience, we state the relevant results below. Let $\mathrm{rb}(\mathbb{Z}_n, 1)$ denote the fewest number of colors that guarantee a rainbow solution to $x_1 + x_2 = x_3$ in $\mathbb{Z}_n$.

Theorem 3.7 *(Theorem 1 in [4]). For a prime $p \geq 5$, $rb(\mathbb{Z}_p, 1) = 4$.*

Remark 3.8. It can be deduced through inspection that $\mathrm{rb}(\mathbb{Z}_2, 1) = \mathrm{rb}(\mathbb{Z}_3, 1) = 3$.

This result is important for us because $\mathrm{rb}(\mathbb{Z}_p, S) = \mathrm{rb}(\mathbb{Z}_p, 1)$ except when $p = 3$. In the case that $p = 3$, $\mathrm{rb}(\mathbb{Z}_p, 1) = \mathrm{rb}(\mathbb{Z}_p, S) - 1$.

Theorem 3.9 *(Theorem 2 in [4]). For a positive integer n with prime factorization $n = p_1 \cdot p_2 \cdots p_k$,*

$$rb(\mathbb{Z}_n, 1) = 2(1-k) + \sum_{i=1}^{k} rb(\mathbb{Z}_{p_i}, 1).$$

The following fact is immediate from Theorems 3.7 and 3.9, and Proposition 3.2:

Observation 3.10. If 3 divides n, then $rb(\mathbb{Z}_n, S) \geq 1 + \mathrm{rb}(\mathbb{Z}_n, 1)$.

We use this fact to find solutions to $x_1 + x_2 = x_3$ in the proof of Lemma 3.11. To our knowledge, this is the first time in the literature on rainbow solutions to equations in $\mathbb{Z}_n$ where previously known rainbow numbers for different equation are employed in the proof. This method may be useful in proving rainbow numbers for more general 4-term equations.

Lemma 3.11. *Let 9 be a divisor of n and $3t = n$. Then*

$$rb(\mathbb{Z}_n, S) \leq rb(\mathbb{Z}_3, S) + rb(\mathbb{Z}_t, S) - 3.$$

Proof. Suppose that c is a rainbow Sidon-free r-coloring of $\mathbb{Z}_n$ where $r = \mathrm{rb}(\mathbb{Z}_3, S) + \mathrm{rb}(\mathbb{Z}_t, S) - 3$. We choose not to evaluate $\mathrm{rb}(\mathbb{Z}_3, S)$ to 4 for conceptual clarity. It is easier to keep track of and compare the number of colors used in the unevaluated form. Without loss of generality, we assume that $|c(R_i) \setminus c(R_0)| \leq 1$ and that $|c(R_0)| \geq |c(R_i)|$ for all i (otherwise, we can shift the coloring to put ourselves in this position). Let

$$\hat{c}(x) = \begin{cases} i & \text{if } \{i\} = c(R_x) \setminus c(R_0) \\ \alpha & \text{if } c(R_x) \subseteq c(R_0) \end{cases}$$

be a coloring of $\mathbb{Z}_t$ (so that α is a color not used by c). Notice that $\hat{c}$ is a rainbow Sidon-free $(\mathrm{rb}(\mathbb{Z}_t, S) - 1)$-coloring of $\mathbb{Z}_t$. This implies that $|c(R_0)| = 3$. By Observation 3.10, $\hat{c}$ admits a rainbow solution to $x_1 + x_2 = x_3$ in $\mathbb{Z}_t$. In particular, suppose that $i + j = k$ such that $\{i, j, k\}$ is rainbow under $\hat{c}$. Without loss of generality, $\hat{c}(i) \neq \alpha$. By construction, if $\hat{c}(j), \hat{c}(k) \neq \alpha$, then there exists $s_j t + j$ and $s_k t + k$ such that $c(s_j t + j) = \hat{c}(j)$ and $c(s_k t + k) = \hat{c}(k)$. If either $\hat{c}(j) = \alpha$ or $\hat{c}(k) = \alpha$, then let $s_j t + j$ (resp. $s_k t + k$) be some element in R_j (resp. R_k). Furthermore, there exists $s_0 t \in R_0$ such that $c(s_0 t) \notin \{c(s_j t + j), c(s_k t + k), \hat{c}(i)\}$. By construction, there exists s_i such that

$$s_k t + k + s_0 t - s_j t - j = s_i t + i \in R_i.$$

Furthermore, $|c(R_i)| = 1$ by Lemma 3.6. In particular, $c(s_i t + i) = \hat{c}(i)$ and $\{s_i t + i, s_j t + j, s_k t + k, s_0 t\}$ is a rainbow solution to the Sidon equation under c. This is a contradiction; therefore, any r-coloring of $\mathbb{Z}_n$ is not rainbow Sidon-free. □

Notice that Lemma 3.11 is a refinement of Lemma 3.4. This refinement lets us improve the upper bound in Proposition 3.5 enough to meet the lower bound in Proposition 3.2.

Theorem 1.2. *Let $n = p_1 \cdots p_k$ be a prime factorization such that $p_i \leq p_j$ whenever $i < j$. Let m be the smallest index such that $p_m \geq 3$ (or $m = k$ if this index does not exist), $f_1 = |\{p_i : p_i \leq 3, i \neq m\}|$, and $f_2 = |\{p_i : p_i \geq 5, i \neq m\}|$. Then*

$$rb(\mathbb{Z}_{p_m}, S) + f_1 + 2f_2 = rb(\mathbb{Z}_n, S).$$

Proof. The lower bound is provided by Proposition 3.2. Notice that if 9 does not divide n, then the upper bound in Proposition 3.5 meets the lower bound. Therefore, assume that 9 divides n.

Let α be the largest integer such that 3^α divides n. To prove the upper bound, iteratively apply Lemma 3.11 $\alpha - 1$ times to conclude that

$$\mathrm{rb}(\mathbb{Z}_n, S) \leq (\alpha - 1)(\mathrm{rb}(\mathbb{Z}_3, S) - 3) + \mathrm{rb}(\mathbb{Z}_{n/3^{\alpha-1}}, S).$$

By Proposition 3.5,

$$\mathrm{rb}(\mathbb{Z}_n, S) \leq (\alpha - 1)(\mathrm{rb}(\mathbb{Z}_3, S) - 3) + \mathrm{rb}(\mathbb{Z}_{p_m}, S) + 2(-k + \alpha) + \sum_{\substack{p_i \neq 3 \\ i \neq m}} \mathrm{rb}(\mathbb{Z}_{p_i}, S).$$

Let β be the largest integer such that 2^β divides n. Notice that $\alpha + \beta - 1 = f_1$. By regrouping terms and evaluating $\mathrm{rb}(\mathbb{Z}_{p_i}, S)$,

$$\begin{aligned} \mathrm{rb}(\mathbb{Z}_n, S) &\leq \mathrm{rb}(\mathbb{Z}_{p_m}, S) + (\alpha - 1)(\mathrm{rb}(\mathbb{Z}_3, S) - 3) + \beta(\mathrm{rb}(\mathbb{Z}_2, S) - 2) + \sum_{\substack{p_i \neq 2,3 \\ i \neq m}} (\mathrm{rb}(\mathbb{Z}_{p_i}, S) - 2) \\ &= \mathrm{rb}(\mathbb{Z}_{p_m}, S) + f_1 + 2f_2. \end{aligned}$$

This concludes the proof. □

4. Rainbow numbers for $[n]$

First, we use the lower bound construction of $\mathrm{rb}(\mathbb{Z}_n, S)$ to prove our lower bound. Then we prove lemmas that give structure of a rainbow Sidon-free coloring of $[n]$, leading to the proof of the upper bound.

Given a rainbow Sidon-free r-coloring $c : [n] \to [r]$, we say that a color X is *dominant* if for any pair of elements $x, x+1 \in [n]$, either $c(x) = c(x+1)$ or $X \in \{c(x), c(x+1)\}$. More generally, we say a color X is *i-dominant* if for any pair of elements $x, x+i \in [n]$, either $c(x) = c(x+i)$ or $X \in \{c(x), c(x+i)\}$. This definition of i-dominance is analogous to the definition previously given for cyclic groups.

Lemma 4.1. *There is a rainbow Sidon-free k-coloring of $[2^{k-1}]$ where every color appears on elements in $[2^{k-2} + 1]$.*

Proof. The idea is to start coloring [2] with two distinct colors, say R and B, and iteratively at every stage insert a new color between every two previously colored integers.

Let c be a rainbow Sidon-free k-coloring of $[2^{k-1}]$, and define an $(k+1)$-coloring of $[2^k]$ as follows:

$$\hat{c}(x) = \begin{cases} c(\frac{x+1}{2}) & \text{if } x \text{ is odd} \\ k+1 & \text{otherwise.} \end{cases}$$

Notice that $\hat{c}$ is rainbow Sidon-free. Let $\{x_1, x_2, x_3, x_4\}$ be a Sidon solution in $[2^k]$. If all x_i's are odd, then this solution is not rainbow since c is rainbow Sidon-free. If some x_i is even, then there are at least two integers of the solution are even and are colored $k+1$; hence the solution is not rainbow.

By construction, $c(1) = R$ and $c(2^{k-1}+1) = B$ and these colors appear uniquely on these two integers. Furthermore, every other color must appear between 1 and $2^{k-1}+1$. Therefore, claim is proven by induction. □

Lemma 4.1 is a slight adjustment of Proposition 3.2 and gives a rainbow Sidon-free coloring of $\mathbb{Z}_{2^{k-1}}$ with k colors such that every color appears on the elements $2^{k-2}+1$. This gives the following proposition.

Proposition 4.2. *For all $n > 0$, $\lfloor \log_2(n-1) \rfloor + 2 < rb([n], S)$.*

Proof. Choose an integer k such that $2^{k-2}+1 \le n \le 2^{k-1}$. This k represents the most colors that Lemma 4.1 lets us use to color n elements. Solving for k gives the lower bound. □

In order to prove an upper bound on $\mathrm{rb}([n], S)$, suppose that c is a rainbow Sidon-free k-coloring of $[n]$. Notice that we can restrict ourselves to the smallest sub-interval of $[n]$ that contains all k colors, since any rainbow solution to the Sidon equation contained in a sub-interval of $[n]$ will exist in $[n]$. Furthermore, if $[a, b]$ is the smallest sub-interval of $[n]$ that contains all k colors, then $c(a)$ and $c(b)$ are unique within $[a, b]$. Therefore, without loss of generality, we will assume that $c(1)$ and $c(n)$ are uniquely colored (if this is not the case, we can use $[b-a+1]$ colored by $\hat{c} := c(x+a-1)$). This has a very important consequence given by the next proposition.

Proposition 4.3. *Suppose that c is a rainbow Sidon-free k-coloring such that $c(1)$ and $c(n)$ are uniquely colored. Then there is a d-dominant color for every $1 \le d \le n-1$. Furthermore, a d-dominant color is uniquely determined by d for $1 \le d \le n-2$.*

Proof. Let $1 \le d < n-1$. Notice that $c(1+d) = c(n-d) = R$. For the sake of contradiction, suppose that there exists $x \in [n]$ such that $c(x) \ne c(x+d)$ and $c(x), c(x+d) \ne R$. In this implies that either $\{x, 1+d, 1, x+d\}$ or $\{x, n, x+d, n-d\}$ is a rainbow solution to the Sidon equation. Therefore, R is d-dominant by definition. Notice that R also uniquely d-dominant.

If $d = n-1$, then both $c(1)$ and $c(n)$ are trivially $(n-1)$-dominant. □

We will maintain the assumption that $c(1)$ and $c(n)$ are uniquely colored. In order to proceed, we will construct two sequences: one of distances $\{d_i\}$ and one of colors $\{X_i\}$. The idea is to keep track of the smallest distance such that a "new" color becomes dominant, and to order colors according to the smallest distance at which they are dominant. Then we will analyze the sequence $\{d_i\}$ to get a lower bound on n depending on the number of colors k. Equivalently, this will give an upper bound on k depending only on n.

Let X_0 be a 1-dominant color and $d_0 = 1$. Let

$$d_i = \min\{|x-y| : c(x) \ne c(y) \text{ and } c(x), c(y) \notin \{X_0, \dots, X_{i-1}\}\}.$$

Let X_i be a d_i-dominant color. Notice that X_i and d_i are defined for $0 \le i \le k-2$. Furthermore, X_i is uniquely determined by d_i for $0 \le i \le k-3$, and $d_i < d_j$ whenever $i < j$.

Lemma 4.4. *Suppose that c is a rainbow Sidon-free k-coloring such that $c(1)$ and $c(n)$ are uniquely colored. For $0 \le i \le k-2$, we have $d_i \ge 2^i$.*

Proof. We will proceed by induction on i. The base case is true by inspection since $d_0 = 1 = 2^0$. By the induction hypothesis, suppose that $d_{i-1} \ge 2^{i-1}$. Notice that by definition of d_i, there exists x such that, without loss of generality, $c(x) = X_i$ and $c(x+d_i) = B \notin \{X_0, \dots, X_i\}$. Furthermore, $c((x, x+d_i)) \subseteq \{X_0, \dots, X_{i-1}\}$.

Let $y = x + d_{i-1}$. Since X_{i-1} is d_{i-1}-dominant and $d_{i-1} < d_i$, it follows that $c(y) = X_{i-1}$. By definition of d_{i-1}, every element in $(y - d_{i-1}, y + d_{i-1})$ receives a color in $\{X_0, \dots, X_{i-1}\}$. In particular, $y - d_{i-1} = x$ and $x + d_i \ge y + d_{i-1}$. Using these facts, we obtain the bound:

$$\begin{aligned} d_i &= x - x + d_i \\ &= y - x + x + d_i - y \\ &\ge d_{i-1} + d_{i-1} \\ &\ge 2^i. \end{aligned}$$

Therefore, the claim is proven. □

Lemma 4.4 suggests that the "first" $k-1$ colors require 2^{k-2} spaces. In particular, $2^{k-2}+1 \le n$.

Theorem 1.3. *Any rainbow Sidon-free coloring of* $[n]$ *uses at most* $\lfloor \log_2(n-1) \rfloor + 2$ *colors. In particular,*

$$rb([n], S) = \lfloor \log_2(n-1) \rfloor + 3.$$

Proof. Let c be a rainbow Sidon-free coloring of $[n]$ with k colors. Then there exists a minimum interval $[a, b]$ with $b - a + 1 = n' \leq n$ that contains all k colors. By Lemma 4.4,

$$2^{k-2} + 1 \leq d_{k-2} + 1 \leq n' \leq n.$$

Therefore,

$$\begin{aligned} 2^{k-2} + 1 &\leq n \\ 2^{k-2} &\leq n - 1 \\ k - 2 &\leq \lfloor \log_2(n-1) \rfloor \\ k &\leq \lfloor \log_2(n-1) \rfloor + 2. \end{aligned}$$

This gives a matching upper bound for Proposition 4.2, and completes the proof. □

Declaration of competing interest

The authors attest that they do not have a conflict of interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

The second author would like to thank Michael Young for his mentorship and helpful discussions while working on this project. This material is based upon work supported by the National Science Foundation under Grant Numbers DMS-1839918 and DMS-2136228. Finally, we thank the anonymous referees for reading our manuscript carefully and providing feedback which improved the paper.

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