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Apportionable matrices and gracefully labelled graphs

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To apportion a complex matrix means to apply a similarity so that all entries of the resulting matrix have the same magnitude. We initiate the study of apportionment, both by unitary matrix similarity and general matrix similarity. There are connections between apportionment and classical graph decomposition problems, including graceful labellings of graphs, Hadamard matrices, and equiangular lines, and potential applications to instantaneous uniform mixing in quantum walks. The connection between apportionment and graceful labellings allows the construction of apportionable matrices from trees. A generalization of the well-known Eigenvalue Interlacing Inequalities using graceful labellings is also presented. It is shown that every rank one matrix can be apportioned by a unitary similarity, but there are 2×2 matrices that cannot be apportioned. A necessary condition for a matrix to be apportioned by unitary matrix is established. This condition is used to construct a set

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of matrices with nonzero Lebesgue measure that are not apportionable by a unitary matrix.

1. Introduction

There has been extensive study of diagonalization of matrices, and finding the Jordan canonical form for matrices that are not diagonalizable. Diagonalization can be viewed as using a similarity to concentrate the magnitude of all the entries within a small subset of entries. Here we study what can be viewed as reversing this process, spreading out the magnitudes as uniformly as possible. A square complex matrix is *uniform* if all entries have the same absolute value. Hadamard matrices and discrete Fourier transforms are important examples of uniform matrices. Uniform matrices play the role of the target (like diagonal matrices) in this process. A square complex matrix is *apportionable* if it is similar (by a specified type of matrix) to a uniform matrix (formal definitions of various types of apportionability and other terms are given below). There are interesting connections between apportionability and classical combinatorial problems, such as graceful labelling of graphs (see Section 5), construction of equiangular lines (see Section 3), and construction of Hadamard matrices (see discussion of these matrices later in this section). There are also connections with the relatively new study of instantaneous uniform mixing in continuous-time quantum walks. Specifically, the continuous-time quantum walk on a simple graph G has the transition operator e^{-itA_G}, where A_G is the adjacency matrix of G, and has instantaneous uniform mixing at time t_0 if and only if $e^{-it_0A_G}$ is uniform [1,5].

We index the entries of $A = [a_{kj}] \in \mathbb{C}^{n\times n}$ from 0 to $n-1$. The set of unitary $n \times n$ matrices is denoted by $\mathcal{U}(n)$, the set of $n \times n$ matrices of determinant one is the *special linear group* and is denoted by $\mathrm{SL}(n)$, and the set of invertible $n \times n$ matrices is denoted by $\mathrm{GL}(n)$. Obviously $\mathcal{U}(n)$ and $\mathrm{SL}(n)$ are subgroups of $\mathrm{GL}(n)$, and we will sometimes have occasion to consider additional subgroups, such as the set of real orthogonal $n \times n$ matrices, which is denoted by $\mathcal{O}(n)$. The *max-norm* of A is $\|A\|_{\max} = \max_{0\le k,j<n} |a_{kj}|$ and the *Frobenius norm* of A is $\|A\|_{\mathrm{F}} = \sqrt{\mathrm{tr}\,(A^*A)} = \sqrt{\sum_{0\le j,k<n} |a_{kj}|^2}$.

Definition 1.1. A complex square matrix $A = [a_{kj}]$ is *uniform* if there exists a nonnegative real number c such that $|a_{kj}| = c$ for all k and j. A matrix $A \in \mathbb{C}^{n\times n}$ is *unitarily apportionable* or *$\mathcal{U}$-apportionable* if there exists a matrix $U \in \mathcal{U}(n)$ such that UAU^* is uniform. In this case, $\|UAU^*\|_{\max}$ is called a *unitary apportionment constant* and U is called an *apportioning matrix*. A matrix $A \in \mathbb{C}^{n\times n}$ is *GL-apportionable* or *generally apportionable* if there exists a matrix $M \in \mathrm{GL}(n)$ such that MAM^{-1} is uniform. In this case, $\|MAM^{-1}\|_{\max}$ is called a *general apportionment constant* and M is called an *apportioning matrix*.

An apportionment constant is usually denoted by κ. If $A = [a_{kj}] \in \mathbb{C}^{n\times n}$ is uniform, then $\|A\|_{\max} = |a_{kj}| = \frac{\|A\|_F}{n}$. Since the Frobenius norm is unitarily invariant, the unitary apportionment constant is unique.

We have not defined *specially apportionable*, because every generally apportionable matrix can be apportioned by a matrix in $\mathrm{SL}(n)$: If $MAM^{-1} = B$, then

$$(\det(M)^{-1/n}M)A(\det(M)^{1/n}M^{-1}) = B$$

and $\det(M)^{-1/n}M \in \mathrm{SL}(n)$. Similarly, we may consider only special unitary matrices when studying $\mathcal{U}$-apportionment.

Example 2.4 shows that a matrix may be apportionable but not $\mathcal{U}$-apportionable. Just as unitarily apportionable matrices were defined to measure apportionability relative to $\mathcal{U}(n)$, the apportionability of A can be assessed relative to any subgroup of $\mathrm{GL}(n)$.

We first study unitary apportionability and then consider general apportionability. Determining whether a matrix is $\mathcal{U}$- or GL-apportionable can be challenging but every rank one matrix is $\mathcal{U}$-apportionable and we present an algorithm for finding a unitary matrix to apportion a rank one matrix in Section 2. We also show there that a positive semidefinite matrix H is $\mathcal{U}$-apportionable if and only if $\operatorname{rank} H \le 1$. In Section 3 we present a condition on a matrix $A \in \mathbb{C}^{n\times n}$ that is sufficient to show that A is not $\mathcal{U}$-apportionable; however, this condition is not necessary. In Section 4 we define a function that measures how far away from $\mathcal{U}$-apportionable a matrix is and establish bounds on this function. Connections with Rosa's ρ-labellings of graphs, which include graceful labellings, are studied in Section 5. There we show that a loop-graph has a ρ-labelling if and only if a specific expansion of its adjacency matrix can be apportioned by a unitary matrix of a specific form. In Section 6 we use gracefully labelled loop-graphs to generalize the well-known Eigenvalue Interlacing Inequalities. General apportionment is studied in Section 7, where it is shown that most pairs of real numbers are not realizable as the spectrum of a 2×2 apportionable matrix, and Section 8, where the problem of finding an apportioning matrix is studied. Section 9 contains concluding remarks. Some of the work in Section 5 relies on the Composition Lemma proved in [3], and in Appendix A we provide a proof of the Recovery Lemma, which is used in [3].

The remainder of this introduction contains additional examples, terminology, and notation. We use the notation $\mathbb{i}$ for the imaginary unit and we work over the field of complex numbers unless otherwise indicated. The $n \times n$ identity matrix is denoted by I_n (or I if n is clear), the $n \times n$ all zeros matrix is denoted by O_n (or O) and the $n \times n$ all ones matrix is denoted by J_n (or J). Let E_{kj} be the $n \times n$ matrix with (k, j)-entry equal to 1 and all other entries 0 (note that n must be specified when using E_{kj}). Let C_n denote the circulant matrix with first row $[0, 1, 0, \ldots, 0]$, so C_n is the adjacency matrix of a directed cycle spanning all n vertices. Let $\mathbb{1} = [1, \ldots, 1]^\top$, ω be a primitive nth

Table 1.1
Multiplicities of the eigenvalues $\pm 1, \pm\,\mathbb{i}$ for F_n [8].

n	1	-1	$-\,\mathbb{i}$	$\mathbb{i}$
$4k$	$k+1$	k	k	$k-1$
$4k+1$	$k+1$	k	k	k
$4k+2$	$k+1$	$k+1$	k	k
$4k+3$	$k+1$	$k+1$	$k+1$	k

root of unity, $\mathbf{w} = [1, \omega, \omega^2, \dots, \omega^{n-1}]^\top$, and $F_n = \frac{1}{\sqrt{n}}[\mathbf{w}_0 = \mathbb{1}, \mathbf{w}_1 = \mathbf{w}, \mathbf{w}_2, \dots, \mathbf{w}_{n-1}]$ where $\mathbf{w}_j$ is the entrywise product of j copies of $\mathbf{w}$ for $j = 2, \dots, n-1$. The matrix F_n is called a *discrete Fourier transform* or *DFT* matrix. Since we index matrix entries by $0, \dots, n-1$, the (k, j)-entry of F_n is $\frac{\omega^{kj}}{\sqrt{n}}$. DFT matrices are very useful for the study of apportionment and also provide a nice example of a family of uniform matrices for which the eigenvalues are completely known.

Example 1.2. It is known that every eigenvalue of the $n \times n$ DFT matrix F_n is one of $1, -1, \mathbb{i}, -\,\mathbb{i}$. Furthermore the multiplicities are given in Table 1.1.

A *Hadamard matrix* is an $n \times n$ matrix H with every entry equal to 1 or -1 and such that $H^\top H = nI_n$. A Hadamard matrix of order n is necessarily uniform and every eigenvalue has magnitude $\sqrt{n}$. If there exists an $n \times n$ Hadamard matrix, then $n = 1, 2$ or $n \equiv 0 \mod 4$. It is not known whether there exist Hadamard matrices of all orders of the form $n = 4k$, but it is known that there is a Hadamard matrix for each $n = 2^k$. If H_m and H_n are $m \times m$ and $n \times n$ Hadamard matrices, then $H_{mn} = H_m \otimes H_n$ is an $mn \times mn$ Hadamard matrix, where $\otimes$ denotes the Kronecker product. Thus, Hadamard matrices of order $n = 2^k$ can be constructed as in the next example.

Example 1.3. Let $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and define $H_{2^{k+1}} = H_2 \otimes H_{2^k}$. For a symmetric Hadamard matrix H_n, every eigenvalue is $\pm\sqrt{n}$. Observe that $\operatorname{tr}(H_{2^k}) = 0$ and H_{2^k} is symmetric, so $\operatorname{spec}(H_{2^k}) = \left\{ \left(-\sqrt{2^k}\right)^{(2^{k-1})}, \left(\sqrt{2^k}\right)^{(2^{k-1})} \right\}$. By scaling H_{2^k}, any spectrum of the form $\left\{ (-\lambda)^{(2^{k-1})}, \lambda^{(2^{k-1})} \right\}$ can be realized by a $2^k \times 2^k$ uniform matrix.

There is a close connection between the matrix apportionment problem and two classical graph decomposition problems. One of these problems (graceful labelling) is discussed in Section 5. The other problem consists in determining the existence of a decomposition of K_n (where n is even) into $n-1$ overlapping copies of $K_{n/2,n/2}$ such that each edge of K_n occurs as an edge in exactly $\frac{n}{2}$ distinct copies of the given $n-1$ copies of $K_{n/2,n/2}$. This graph decomposition problem is well known to be equivalent to the problem of establishing the existence of a $n \times n$ Hadamard matrix. Its relation to apportioning follows from the observation that a symmetric $n \times n$ Hadamard matrix exists if and only if one of the $\frac{n}{2}$ diagonal matrices in the set

$$\left\{ \mathrm{diag}(\underbrace{-1,\ldots,-1}_{k \text{ times}}, \underbrace{1,\ldots,1}_{n-k \text{ times}}) : 0 < k \le \frac{n}{2} \right\}$$

is $\mathcal{O}$-apportionable. Thus determining whether or not one of the diagonal matrices above is $\mathcal{O}$-apportionable must be at least as hard as establishing the existence of symmetric $n \times n$ Hadamard matrices.

A *complex Hadamard matrix* is an $n \times n$ matrix H with complex entries of modulus 1 such that $HH^* = nI$. An $n \times n$ complex Hadamard matrix is $\sqrt{n}U$ where $U \in \mathbb{C}^{n\times n}$ is both uniform and unitary. The appropriate scalar multiple of the DFT matrix is a complex Hadamard matrix. Thus, unlike real Hadamard matrices, complex Hadamard matrices exist for every order.

We use spec(A) to denote the spectrum of A, i.e., the multiset of n eigenvalues of A. The *spectral radius* of A is $\rho(A) = \max\{|\lambda| : \lambda \in \mathrm{spec}(A)\}$; ρ can be used when A is clear. The eigenvalues of a Hermitian matrix H are real and denoted by $\lambda_{\max}(H) = \lambda_0(H) \ge \lambda_1(H) \ge \cdots \ge \lambda_{n-1}(H) = \lambda_{\min}(H)$ (or $\lambda_{\max} = \lambda_0 \ge \lambda_1 \ge \cdots \ge \lambda_{n-1} = \lambda_{\min}$ if H is clear).

Note that $\|\cdot\|_{\max}$ is a vector norm but not a matrix norm. We use several matrix norms in addition to the Frobenius norm, many of which are defined in terms of the singular values of A, which we denote by $\sigma_0(A) \ge \cdots \ge \sigma_{n-1}(A)$. If A is a normal matrix, then $\sigma_0(A) = \rho(A)$ and $\{\sigma_0(A), \ldots, \sigma_{n-1}(A)\} = \{|\lambda| : \lambda \in \mathrm{spec}(A)\}$. For a positive integer p (or $p = \infty$), the Schatten-p norm of $A \in \mathbb{C}^{n\times n}$ is

$$\|A\|_{\mathrm{S},p} = \left(\sum_{0\le k<n} \sigma_k(A)^p \right)^{1/p};$$

$\|A\|_{\mathrm{S},1} = \sum_{0\le k<n} \sigma_k(A)$ is called the *nuclear norm* and denoted by $\|A\|_*$. $\|A\|_{\mathrm{S},\infty} = \sigma_0(A)$ is called the *spectral norm* and denoted by $\|A\|_2$ (because it is the matrix norm induced by the vector 2-norm). Since the Frobenius norm of A is invariant under multiplication of A by a unitary matrix, $\|A\|_{\mathrm{F}} = \|A\|_{\mathrm{S},2}$; note that the Frobenius norm of A is the vector 2-norm of A viewed as an n^2-vector.

2. Rank one matrices are $\mathcal{U}$-apportionable

The only rank zero matrix in $\mathbb{C}^{n\times n}$ is O_n, and it is uniform. In this section we show that every rank one matrix is $\mathcal{U}$-apportionable and provide an algorithm for finding a unitary apportioning matrix and similar uniform matrix. The situation changes dramatically when the rank is of two or more. In this section we also show that positive semidefinite matrices of rank two or more are not $\mathcal{U}$-apportionable. Section 7 provides examples of spectra that cannot be realized by apportionable matrices.

Lemma 2.1 and Theorem 2.2 establish that Algorithm 2.1 produces the claimed results; to assist in making connections between the proofs and the algorithm, we identify algorithm steps by number within the proofs of Lemma 2.1 and Theorem 2.2.

Recall that for any vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ with $\|\mathbf{v}\|_2 = \|\mathbf{w}\|_2$, there exists a unitary matrix U such that $U\mathbf{v} = \mathbf{w}$. This can be accomplished, for example, by the *Householder matrix defined by v and w*, $I_n - \frac{\mathbf{u}\mathbf{u}^*}{\mathbf{u}^*\mathbf{v}}$, where $\mathbf{u} = \mathbf{v} - \mathbf{w}$ (cf. [2]). The (complex) sign function is $\operatorname{sgn}(z) = \frac{z}{|z|}$ for all nonzero $z \in \mathbb{C}$ and $\operatorname{sgn}(0) = 1$. Let $\mathbf{e}_i$ denote the i-th standard basis vector.

Lemma 2.1. *Let $n \geq 2$ and $A \in \mathbb{C}^{n\times n}$. If $\operatorname{rank} A = 1$, then A is unitarily similar to $\gamma\mathbf{e}_0(\alpha\mathbf{e}_0^\top + \beta\mathbf{e}_1^\top)$, where $|\gamma| = 1$ and $\alpha, \beta \in \mathbb{R}$ are nonnegative.*

Proof. Suppose that $\operatorname{rank} A = 1$. This implies that $\operatorname{spec}(A) = \{\lambda, 0^{(n-1)}\}$ where $\lambda = \operatorname{tr} A$ (Step 1 in Algorithm 2.1); λ may be nonzero or zero. Furthermore, $A = \mathbf{x}\mathbf{y}^*$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ with $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ and $\|\mathbf{x}\|_2 = 1$ (Step 2). Let H_1 be a Householder matrix such that $H_1\mathbf{x} = \mathbf{e}_0$ and let $\mathbf{z} = H_1\mathbf{y}$ (Steps 3 and 4). This implies $(\mathbf{z})_0 = \overline{\lambda}$. Define $\hat{\mathbf{z}} = \mathbf{z} - \overline{\lambda}\mathbf{e}_0$, so $(\hat{\mathbf{z}})_0 = 0$. If $\hat{\mathbf{z}} = \mathbf{0}$, then let $H_2 = I_n$. Otherwise, let H_2 be the Householder matrix defined by $\hat{\mathbf{z}}$ and $\operatorname{sgn}(\overline{\lambda})\|\hat{\mathbf{z}}\|_2\mathbf{e}_1$ (Step 5). Thus $H_2\hat{\mathbf{z}} = \operatorname{sgn}(\overline{\lambda})\|\hat{\mathbf{z}}\|_2\mathbf{e}_1$, $H_2\mathbf{e}_0 = \mathbf{e}_0$, and $H_2H_1\mathbf{y} = \operatorname{sgn}(\overline{\lambda})(|\lambda|\mathbf{e}_0 + \|\hat{\mathbf{z}}\|_2\mathbf{e}_1)$. Hence, $H_2H_1AH_1^*H_2^* = \operatorname{sgn}(\lambda)\mathbf{e}_0(|\lambda|\mathbf{e}_0 + \|\hat{\mathbf{z}}\|_2\mathbf{e}_1)^\top$ has the required form. □

Note that this algorithm is intended to summarize the constructive method of proof. For accurate implementation in decimal arithmetic, it is important to apply well-known numerical techniques to minimize errors.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$. If $\operatorname{rank} A = 1$, then A is $\mathcal{U}$-apportionable.*

Proof. The claim clearly holds for $n = 1$, so suppose $n \geq 2$. Assume that $\operatorname{rank} A = 1$. By Lemma 2.1 we may assume that A is of the form $\gamma\mathbf{e}_0(\alpha\mathbf{e}_0^\top + \beta\mathbf{e}_1^\top)$, where $|\gamma| = 1$ and $\alpha, \beta \in \mathbb{R}$ are nonnegative. Observe that if $\alpha = 0$ or $\beta = 0$, then $F_nAF_n^*$ is uniform (Steps 6-13 of Algorithm 2.1). So suppose that α and β are positive real numbers. Since $\mathcal{U}$-apportionability is invariant under scaling we may assume that $\gamma = 1$ and $\alpha = 1$.

Let U be a unitary matrix whose first two columns are $\mathbf{u}_0$ and $\mathbf{u}_1$. Then

$$UAU^* = (U\mathbf{e}_0\,(\mathbf{e}_0^\top + \beta\mathbf{e}_1^\top)U^*) = \mathbf{u}_0\,(\mathbf{u}_0 + \beta\mathbf{u}_1)^*.$$

Thus UAU^* is uniform if and only if $\mathbf{u}_0$ and $\mathbf{u}_0 + \beta\mathbf{u}_1$ are uniform. If n is even and U is a unitary matrix whose first two columns are $\mathbf{u}_0 = \frac{1}{\sqrt{n}}\mathbb{1}$ and $\mathbf{u}_1 = \frac{1}{\sqrt{n}}[\,\mathbb{i}, -\mathbb{i}, \mathbb{i}, \ldots, -\mathbb{i}]^\top$, then $\mathbf{u}_0$ and $\mathbf{u}_0 + \beta\mathbf{u}_1$ are uniform (Steps 14-17).

Now consider the case where n is odd. Let U be a unitary matrix whose first two columns are $\mathbf{u}_0 = \frac{1}{\sqrt{n}}\mathbb{1}$ and

$$\mathbf{u}_1 = [(1-n)a, a + b\,\mathbb{i}, a - b\,\mathbb{i}, a + b\,\mathbb{i}, \ldots, a - b\,\mathbb{i}]^\top,$$

Algorithm 2.1 Given $A \in \mathbb{C}^{n\times n}$ with $\operatorname{rank} A = 1$, construct uniform matrix B and unitary matrix V such that $B = VAV^*$.

1: $\lambda = \operatorname{tr} A$.
2: Factor A as $A = \mathbf{x}\mathbf{y}^*$ where $\|\mathbf{x}\|_2 = 1$.
3: $H_1 = I_n - \frac{\mathbf{u}\mathbf{u}^*}{\mathbf{u}^*\mathbf{x}}$ where $\mathbf{u} = \mathbf{x} - \mathbf{e}_0$ (assuming $\mathbf{x} \neq \mathbf{e}_0$, else $H_1 = I_n$).
4: $\mathbf{z} = H_1\mathbf{y}$.
5: $H_2 = I_n - \frac{\hat{\mathbf{u}}\hat{\mathbf{u}}^*}{\hat{\mathbf{u}}^*\hat{\mathbf{z}}}$ where $\hat{\mathbf{z}} = [0, z_1, \ldots, z_{n-1}]^\top$ and $\hat{\mathbf{u}} = \hat{\mathbf{z}} - \operatorname{sgn}(z_0)\|\hat{\mathbf{z}}\|_2\,\mathbf{e}_1$ (assuming $\hat{\mathbf{z}} \neq \mathbf{0}$ and $\hat{\mathbf{z}} \neq \operatorname{sgn}(z_0)\|\hat{\mathbf{z}}\|_2\,\mathbf{e}_1$, else $H_2 = I_n$).
6: **if** $\lambda = 0$ **then**
7: $V = F_nH_2H_1$.
8: $(B)_{kj} = VAV^* = \frac{\|A\|_{\mathrm{F}}\,\omega^{-j}}{n}$.
9: **end if**
10: **if** $\lambda \neq 0$ **and** $\|A\|_{\mathrm{F}} = |\operatorname{tr} A|$ **then**
11: $V = F_nH_2H_1$.
12: $B = VAV^* = \frac{\|A\|_{\mathrm{F}}}{n}J_n$.
13: **end if**
14: **if** $\lambda \neq 0$ **and** $\|A\|_{\mathrm{F}} \neq |\operatorname{tr} A|$ **then**
15: $\mathbf{u}_0 = \frac{1}{\sqrt{n}}\mathbb{1}$.
16: **if** n is even **then**
17: $\mathbf{u}_1 = \frac{1}{\sqrt{n}}[\,\mathbb{i}, -\mathbb{i}, \ldots, \mathbb{i}, -\mathbb{i}]^\top$.
18: **else**
19: $\mathbf{u}_1 = [(1-n)a, a + b\,\mathbb{i}, a - b\,\mathbb{i}, a + b\,\mathbb{i}, \ldots, a - b\,\mathbb{i}]^\top$ where

$$a = \frac{1-\sqrt{\beta^2+1}}{(n-1)\sqrt{n}\beta} \qquad \text{and} \qquad b = \sqrt{(n-1)^{-1} - na^2}.$$

20: **end if**
21: Construct a unitary matrix $U = [\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}]$ (e.g., by extending $\{\mathbf{u}_0, \mathbf{u}_1\}$ to a basis and applying the standard Gram-Schmidt process).
22: $V = UH_2H_1$
23: $B = VAV^*$
24: **end if**

where $b = \sqrt{(n-1)^{-1} - na^2}$ and $a = \frac{1-\sqrt{\beta^2+1}}{(n-1)\sqrt{n}\beta}$ (Steps 14-15 and 18-19). Then it is not difficult, albeit rather tedious, to verify algebraically that $\mathbf{u}_0 + \beta\mathbf{u}_1$ is uniform, i.e.,

$$\begin{aligned}\left|\frac{1}{\sqrt{n}} + \beta(1-n)a\right| = \sqrt{\frac{\beta^2+1}{n}} &= \left|\frac{1}{\sqrt{n}} + \beta\left(a + \sqrt{(n-1)^{-1} - na^2}\,\mathbb{i}\right)\right| \\ &= \left|\frac{1}{\sqrt{n}} + \beta(a + b\,\mathbb{i})\right|. \quad \square\end{aligned}$$

Recall that a matrix H is *positive semidefinite (PSD)* if and only if H is Hermitian and $\lambda \geq 0$ for every eigenvalue λ of H. An $n \times n$ matrix of the form $H = X^*X$ for some $X \in \mathbb{C}^{m\times n}$ is called a *Gram matrix*. It is well known that a matrix is a Gram matrix if and only if it is positive semidefinite. In fact, the least d such that a PSD matrix H can be expressed as X^*X with $X \in \mathbb{C}^{d\times n}$ is the rank of H.

Proposition 2.3. *Let H be a PSD matrix. If C is nonsingular and C^*HC is uniform, then* $\operatorname{rank} H \leq 1$. *Furthermore, H is $\mathcal{U}$-apportionable if and only if* $\operatorname{rank} H \leq 1$.

Proof. If $H = O$ then $\operatorname{rank} H = 0$. So assume $H \neq O$. Let C be nonsingular and suppose $B = C^*HC$ is uniform. Since H is PSD, B is also PSD and there exists a matrix R such that $B = R^*R$. Let $\mathbf{r}_k$ denote column k of R. Since B is uniform, $\|\mathbf{r}_k\|_2 = \|\mathbf{r}_j\|_2$ and

$|\mathbf{r}_k^*\mathbf{r}_j| = \|\mathbf{r}_k\|_2\|\mathbf{r}_j\|_2$ for any row indices k and j. Thus equality holds for the Cauchy-Schwarz inequality applied to any pair of columns of R. This implies $\operatorname{rank} R = 1$, and thus $\operatorname{rank} H = 1$.

If $\operatorname{rank} H \leq 1$, then H is $\mathcal{U}$-apportionable by Theorem 2.2. Now suppose that H is $\mathcal{U}$-apportionable, so there exists a unitary matrix U such that U^*HU is uniform. Then $\operatorname{rank} H \leq 1$. □

The next example shows that a PSD matrix H with $\operatorname{rank} H = 2$ may be GL-apportionable, demonstrating the existence of a matrix that is apportionable but not $\mathcal{U}$-apportionable.

Example 2.4. Observe that

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

is a uniform matrix with distinct nonnegative eigenvalues, $H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is PSD, and $\operatorname{rank} H = 2$.

It is interesting to note that while the previous example shows that $\{1, 2, 0\}$ is realizable as the spectrum of a uniform matrix, Theorem 7.6 implies $\{1, 2\}$ is not realizable.

3. Necessary condition for $\mathcal{U}$-apportionability

In this section we establish a necessary condition on $|c|$ for a translation $A + cI$ to be $\mathcal{U}$-apportionable. This condition is used to show that for a given $n \geq 2$, a positive fraction of $n \times n$ matrices are not $\mathcal{U}$-apportionable.

Theorem 3.1. *Let $n \geq 2$, $A = [a_{kj}] \in \mathbb{C}^{n\times n}$, and $c \in \mathbb{C}$. Suppose $A + cI_n$ is $\mathcal{U}$-apportionable. Then*

$$|c| \leq \frac{\sum_{k=0}^{n-1} |a_{kk}|}{n} + \sqrt{\left(\frac{\sum_{k=0}^{n-1} |a_{kk}|}{n}\right)^2 + \frac{\|A\|_{\mathrm{F}}^2}{n(n-1)}}$$

Proof. Let $B = A - A \circ I_n$. Since $A + cI_n$ is $\mathcal{U}$-apportionable, the $\mathcal{U}$-apportionment constant for $A + cI_n$ is

$$\kappa = \frac{\|A + cI\|_{\mathrm{F}}}{n} = \frac{\|A \circ I_n + cI + B\|_{\mathrm{F}}}{n} = \frac{1}{n}\sqrt{\sum_{k=0}^{n-1} |a_{kk} + c|^2 + \|B\|_{\mathrm{F}}^2}.$$

Since $A+cI$ is $\mathcal{U}$-apportionable, there exists a unitary matrix U such that $U(A+cI)U^*$ is uniform. Observe that $U(A+cI)U^* = UAU^* + cI$. Thus $|\,(UAU^*)_{kj}\,| = \kappa$ for all $k \neq j$. So

$$\|A\|_{\mathrm{F}}^2 = \|UAU^*\|_{\mathrm{F}}^2 \geq \sum_{k=0}^{n-1}\sum_{j\neq k} |\,(UAU^*)_{kj}\,|^2 = n(n-1)\kappa^2.$$

Replacing $\|A\|_{\mathrm{F}}^2$ and κ^2 and rearranging yields the following.

$$\begin{aligned}
\sum_{k=0}^{n-1}|a_{kk}|^2 + \|B\|_{\mathrm{F}}^2 &\geq \frac{n-1}{n}\left(\sum_{k=0}^{n-1}|a_{kk}+c|^2 + \|B\|_{\mathrm{F}}^2\right).\\
\sum_{k=0}^{n-1}|a_{kk}|^2 + \frac{1}{n}\|B\|_{\mathrm{F}}^2 &\geq \frac{n-1}{n}\sum_{k=0}^{n-1}(|a_{kk}|-|c|)^2\\
&= \frac{n-1}{n}\left(\sum_{k=0}^{n-1}|a_{kk}|^2 + \sum_{k=0}^{n-1}|c|^2 - 2\sum_{k=0}^{n-1}|a_{kk}||c|\right).\\
\frac{1}{n}\sum_{k=0}^{n-1}|a_{kk}|^2 + \frac{1}{n}\|B\|_{\mathrm{F}}^2 &\geq (n-1)|c|^2 - \left(\frac{2(n-1)}{n}\sum_{k=0}^{n-1}|a_{kk}|\right)|c|.\\
0 &\geq (n-1)|c|^2 - \left(\frac{2(n-1)}{n}\sum_{k=0}^{n-1}|a_{kk}|\right)|c| - \frac{1}{n}\|A\|_{\mathrm{F}}.
\end{aligned}$$

Since $n \geq 2$, we can apply the quadratic formula to obtain

$$|c| \leq \frac{\sum_{k=0}^{n-1}|a_{kk}|}{n} + \sqrt{\left(\frac{\sum_{k=0}^{n-1}|a_{kk}|}{n}\right)^2 + \frac{\|A\|_{\mathrm{F}}^2}{n(n-1)}}. \qquad \square$$

Note that it follows from Theorem 7.4 below that translation also does not preserve GL-apportionability.

An equivalent form of Theorem 3.1,

$$\begin{aligned}
\exists c, |c| > &\frac{\sum_{k=0}^{n-1}|a_{kk}-c|}{n}\\
&+ \sqrt{\left(\frac{\sum_{k=0}^{n-1}|a_{kk}-c|}{n}\right)^2 + \frac{\|A-cI\|_{\mathrm{F}}^2}{n(n-1)}} \text{ implies } A \text{ is not } \mathcal{U}\text{-apportionable,}
\end{aligned} \tag{3.1}$$

gives a condition that can be used to show a matrix is not $\mathcal{U}$-apportionable. We apply this to show that it is not the case that almost all $n\times n$ matrices are $\mathcal{U}$-apportionable. Consider the complex vector space $\mathbb{C}^{n\times n}$ of $n \times n$ matrices with max-norm (or equivalently, $\mathbb{C}^{n^2}$ with ∞-norm where a matrix represented by the vector of its entries). Define the region $T_n = \{A \in \mathbb{C}^{n\times n} : \|A - \frac{3}{4}I_n\|_{\max} \le \frac{1}{4}\}$.

Proposition 3.2. *For $n \ge 2$, no matrix in T_n is $\mathcal{U}$-apportionable.*

Proof. For $c = \frac{3}{4}$,

$$\begin{aligned}
&\frac{\sum_{k=0}^{n-1}|a_{kk}-c|}{n} + \sqrt{\left(\frac{\sum_{k=0}^{n-1}|a_{kk}-c|}{n}\right)^2 + \frac{\|A-cI\|_{\mathrm{F}}^2}{n(n-1)}} \\
&\le \frac{1}{4} + \sqrt{\left(\frac{1}{4}\right)^2 + \frac{n\left(\frac{1}{4}\right)^2 + n(n-1)\left(\frac{1}{4}\right)^2}{n(n-1)}} \\
&\le \frac{1}{4} + \sqrt{\frac{1}{16} + \frac{1}{16(n-1)} + \frac{1}{16}} < \frac{3}{4}.
\end{aligned}$$

The result now follows by applying condition (3.1). $\square$

Remark 3.3. Since T_n has nonzero Lebesgue measure and is scalable, it is not the case that almost all $n \times n$ matrices are $\mathcal{U}$-apportionable.

Although (3.1) is a sufficient condition, it is not a necessary condition, as we can see by examining a connection between $\mathcal{U}$-apportionability and equiangular lines. The unit vectors $\mathbf{x}_0, \ldots, \mathbf{x}_{d-1} \in \mathbb{C}^n$ are *equiangular with angle θ* if $|\mathbf{x}_i^* \mathbf{x}_j| = \cos(\theta)$ for all $0 \le i < j \le d-1$; note that $\mathbf{x}_j{}^* \mathbf{x}_j = 1$ for $j = 0, \ldots, d-1$. Given $B \in \mathbb{C}^{d\times n}$ with unit length columns, consider determining the existence of a unitary matrix U whose right action on B results in (unit length) equiangular columns (with angle θ). Note that for any $c \in \mathbb{C}$, $U^*(B^*B + cI_n)U = (BU)^*(BU) + cI_n$. Thus BU has equiangular columns with angle θ if and only if $|\left((BU)^*(BU)\right)_{ij}| = \cos(\theta)$ and $\left((BU)^*(BU)\right)_{jj} = 1$ if and only if

$$U^*\left(B^*B - I_n + \cos(\theta)I_n\right)U \quad \text{is uniform}$$

So the matrix $B^*B + (\cos(\theta) - 1)\, I_n$ is $\mathcal{U}$-apportionable if and only if there exists a unitary matrix U such that the columns of BU are equiangular with angle θ.

Observe that $B^*B - I$ has zeros on the diagonal and $\|B^*B - I\|_{\mathrm{F}} = \|U(B^*B - I)U\|_{\mathrm{F}}$ for any unitary matrix U. So if $B^*B - I$ is $\mathcal{U}$-apportionable, the $\mathcal{U}$-apportionment constant is $\kappa = \sqrt{\frac{\|B^*B-I\|_{\mathrm{F}}^2}{n(n-1)}}$ and we are trying to apportion $A = B^*B + (\kappa - 1)I$. However, we cannot use Theorem 3.1 to show this is not $\mathcal{U}$-apportionable, because we see that for any $c \in \mathbb{C}$:

$$
\begin{aligned}
&\frac{\sum_{k=0}^{n-1}|a_{kk}-c|}{n}+\sqrt{\left(\frac{\sum_{k=0}^{n-1}|a_{kk}-c|}{n}\right)^2+\frac{\|A-cI\|_{\mathrm{F}}^2}{n(n-1)}}\\
&=|\kappa-c|+\sqrt{|\kappa-c|^2+\frac{\|A-cI\|_{\mathrm{F}}^2}{n(n-1)}}\\
&\geq|\kappa-c|+\sqrt{|\kappa-c|^2+\kappa^2}\\
&\geq(|c|-\kappa)+\kappa=|c|
\end{aligned}
$$

It is well known that there is a limit to the number of equiangular lines in a given dimension d. In the case $d = 2$, no set of 4 lines is equiangular. Thus for any 2×4 matrix B, $A = B^*B + (\sqrt{\frac{\|B^*B-I\|_{\mathrm{F}}^2}{n(n-1)}} - 1)I$ is not $\mathcal{U}$-apportionable, but no c satisfies the hypothesis of (3.1).

Question 3.4. *Can we efficiently certify that a matrix is not $\mathcal{U}$-apportionable?*

4. Measure of closeness to uniformity

In this section we define a function that measures how far away from $\mathcal{U}$-apportionable a matrix is and establish bounds on this function.

Definition 4.1. For $A \in \mathbb{C}^{n\times n}$, define

$$u(A) = \inf_{U\in\mathcal{U}(n)} \|UAU^*\|_{\max}.$$

The *unitary apportionment gap* of an arbitrary matrix $A \in \mathbb{C}^{n\times n}$ is $\left|u(A) - \frac{\|A\|_{\mathrm{F}}}{n}\right|$

Observe that A is $\mathcal{U}$-apportionable if and only if $u(A) = \frac{\|A\|_{\mathrm{F}}}{n}$ and the unitary apportionment gap is zero. Note that in defining the unitary apportionment gap, we are using the fact that the unitary apportionment constant is unique.

Remark 4.2. Since the set of $n \times n$ unitary matrices is compact, the infimum is actually the minimum:

$$u(A) = \min_{U\in\mathcal{U}(n)} \|UAU^*\|_{\max}.$$

Next we establish bounds on $\|UAU^*\|_{\max}$ and thus on $u(A)$. Recall that the nuclear norm of A is $\|A\|_* = \sum_{k=1}^{n} \sigma_k(A)$ and the spectral norm of A is $\|A\|_2 = \sigma_0(A)$ (where $\sigma_k(A)$ is the kth singular value of A).

Proposition 4.3. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then*

$$\frac{\|A\|_{\mathrm{F}}}{n} \leq \|UAU^*\|_{\max} \leq \|A\|_2$$

The lower bound is realized by a uniform matrix unitarily similar to A if A is $\mathcal{U}$-apportionable. The upper bound is realized by a diagonal matrix unitarily similar to A when A is normal.

Proof. It is known that for any matrix $A \in \mathbb{C}^{n\times n}$, $\|A\|_{\max} \leq \|A\|_2$ (e.g., this is immediate from [6, Theorem 5.6.2(d)]). Thus $\|UAU^*\|_{\max} \leq \|UAU^*\|_2 = \|A\|_2$. For any matrix $B \in \mathbb{C}^{n\times n}$, $\|B\|_{\mathrm{F}} \leq \sqrt{n^2 \left(\max_{k,j} |b_{kj}|\right)^2} = n\|B\|_{\max}$ with equality if and only if B is uniform. Thus $\|A\|_{\mathrm{F}} \leq n\|UAU^*\|_{\max}$ since $\|UAU^*\|_{\mathrm{F}} = \|A\|_{\mathrm{F}}$ for any unitary matrix U. The last two statements are now immediate. □

Proposition 4.4. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then*

$$\frac{\|A\|_{\mathrm{F}}}{n} \leq u(A) \leq \|A\|_2 \tag{4.1}$$

and $\frac{\|A\|_{\mathrm{F}}}{n} = u(A)$ if and only if A is $\mathcal{U}$-apportionable. If A is normal, then $u(A) \leq \frac{\|A\|_*}{n}$. All bounds are sharp.

Proof. Equation (4.1) is immediate from Proposition 4.3, where it is also shown that equality in the first inequality requires A to be $\mathcal{U}$-apportionable.

Now assume A is normal and recall F_n is the DFT. Since A is normal, there exists a unitary matrix V such that $V^*AV = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where $\operatorname{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$. The singular values of A are the absolute values of the eigenvalues of A because A is normal, so

$$\left|(F_n V^* AV F_n^*)_{\ell j}\right| = \Big|\sum_k \frac{\omega^{\ell k}}{\sqrt{n}} \lambda_k \frac{\overline{\omega^{kj}}}{\sqrt{n}}\Big| \leq \frac{1}{n}\sum_k |\omega^{\ell k}||\lambda_k||\overline{\omega^{jk}}| = \frac{1}{n}\sum_k \sigma_k = \frac{\|A\|_*}{n}.$$

Thus $u(A) \leq \frac{\|A\|_*}{n}$. The upper bounds are equality when A is the identity matrix, which is normal. □

Question 4.5. *Is upper bound on $u(A) \leq \|A\|_2$ sharp for any A with $\frac{\|A\|_*}{n} < \|A\|_2$?*

5. Graph labellings and $\mathcal{U}$-apportionment

One of the inspirations for writing this paper is a connection between apportionment and graph labellings introduced by Rosa in [10]. Rosa demonstrated that some of these labellings produce cyclic decompositions of the complete graph. We show that these cyclic decompositions can naturally be described as an apportionment problem using a reasonable choice of unitary matrix and adjacency matrices.

We first provide the necessary background information. The term *graph* will always refer to a simple graph G, i.e., G does not have any loops or multi-edges. A *loop-graph* $\mathfrak{G}$ is a graph that allows loops but not multi-edges. We will always label the vertices of a graph on n vertices with the set $\{0, \ldots, n-1\}$. The adjacency matrix of a graph G or loop-graph $\mathfrak{G}$ is denoted A_G or $A_{\mathfrak{G}}$, respectively. The edge set of a graph G or loop-graph $\mathfrak{G}$ is denoted by E_G or $E_{\mathfrak{G}}$.

We denote the complete graph on n vertices by K_n. The complete loop-graph $\mathfrak{K}_n$ is the loop-graph obtained from K_n by adding a loop to each vertex. Define $\phi : V(K_n) \to V(K_n)$ to be the graph isomorphism that maps $i \mapsto i+1 \mod n$. Note that ϕ is also a graph isomorphism of $\mathfrak{K}_n$. Recall that C_n denotes the $n \times n$ (cyclic) permutation matrix corresponding to ϕ, i.e., $C_n \mathbf{e}_i = \mathbf{e}_{i+1 \mod n}$. The *length* of an edge $\{i, j\}$ in $\mathfrak{K}_n$ is $\min\{|i-j|, n-|i-j|\}$. Observe that ϕ does not change the length of an edge and when n is odd, $\mathfrak{K}_n$ consists of n edges of length i for $i = 0, \ldots, \frac{n-1}{2}$.

Given a loop-graph $\mathfrak{G}$, a $\mathfrak{G}$-*decomposition* of $\mathfrak{K}_n$ is a set $\Delta = \{\mathfrak{G}_1, \ldots, \mathfrak{G}_t\}$ of subgraphs of $\mathfrak{K}_n$ such that each $\mathfrak{G}_i$ is isomorphic to $\mathfrak{G}$ and the edge sets of the loop-graphs in Δ partition the edge set of $\mathfrak{K}_n$. The $\mathfrak{G}$-decomposition Δ is *cyclic* if $\phi(\mathfrak{G}_i) = \mathfrak{G}_{(i+1) \mod n}$ for each $\mathfrak{G}_i$. Let $\mathfrak{G}$ be a loop-graph with m edges. An injective function $f : V(\mathfrak{G}) \to \{0, \ldots, 2m-2\}$ is a ρ-*labelling* of $\mathfrak{G}$ provided

$$\left\{ \min\{|f(u) - f(v)|,\, 2m+1 - |f(u) - f(v)|\} : \{u, v\} \in E(\mathfrak{G}) \right\} = \{0, \ldots, m-1\}.$$

Thus a ρ-labelling is an embedding from $\mathfrak{G}$ to $\mathfrak{K}_2 m - 1$ such that the image of $\mathfrak{G}$ has exactly one edge of length i for $i = 0, \ldots, m-1$ (an *embedding* of $\mathfrak{G}$ in $\mathfrak{G}'$ is an injective mapping of vertices that maps edges of $\mathfrak{G}$ to edges of $\mathfrak{G}'$).

The definitions in the preceding two paragraphs were originally given for simple graphs. We have modified the original definitions to fit the setting of loop-graphs. This small change simplifies the connection to apportionment. The following theorem is a loop-graph version of Theorem 7 in [10].

Theorem 5.1. [10] *Let $\mathfrak{G}$ be a loop-graph with m edges, consisting of $m-1$ non-loop edges and one loop. Then there exists a cyclic $\mathfrak{G}$-decomposition of the complete loop-graph $\mathfrak{K}_{2m-1}$ if and only if $\mathfrak{G}$ has a ρ-labelling.*

Let $\mathfrak{G}$ be a loop-graph with n vertices, $n-1$ non-loop edges and one loop edge. Let $f : V(\mathfrak{G}) \to V(\mathfrak{K}_{2n-1})$ be an embedding and define $\widehat{\mathfrak{G}}$ to be the loop-graph with vertex set $\{0, \ldots, 2n-1\}$ and edge set $f(E_{\mathfrak{G}})$. In the special case that $f(i) = i$ for all i, the $(2n-1) \times (2n-1)$ adjacency matrix of $\widehat{\mathfrak{G}}$ would be $A_{\mathfrak{G}} \oplus O$. In the more general setting, a permutation similarity corresponds to a relabelling of vertices, so there exists a permutation matrix P such that the adjacency matrix of $\widehat{\mathfrak{G}}$ is $A = P(A_{\mathfrak{G}} \oplus O)P^T$. Observe that there exists a cyclic $\mathfrak{G}$-decomposition of $\mathfrak{K}_{2n-1}$ if and only if

$$\sum_{0\le i<2n-1} C_{2n-1}^{i} A C_{2n-1}^{-i} = J_{2n-1}. \tag{5.1}$$

This observation can be thought of as a matrix version of Theorem 5.1. With some minor modifications, we can directly connect these ideas to apportionment.

Let ω denote a primitive $(2n-1)$-th root of unity and let $\mathbf{w} = [1, \omega, \omega^2, \ldots, \omega^{2n-2}]^\top$ (for convenience we may assume $\mathbf{w}$ is the second column of $\sqrt{2n-1}\,F_{2n-1}$). Define $U_n \in \mathbb{C}^{(2n-1)^2\times(2n-1)^2}$ to be the $(2n-1)\times(2n-1)$ block matrix whose (i,j)-block is the $(2n-1)\times(2n-1)$ matrix

$$U_n(i,j) = \frac{C_{2n-1}^{j}\,\mathrm{diag}(\mathbf{w})^i}{\sqrt{2n-1}}, \qquad \text{for } 0 \le i,\, j < 2n-1. \tag{5.2}$$

Observe that the (i,k)-block of $U_nU_n^*$ is

$$\begin{aligned}(U_nU_n^*)\,(i,k) &= \sum_{j=0}^{2n-2} U_n(i,j)U_n(k,j)^* \\ &= \frac{1}{2n-1}\sum_{j=0}^{2n-2} C_{2n-1}^{j}\,\mathrm{diag}\left(1, \omega^{i-k}, \omega^{2(i-k)}, \ldots, \omega^{(2n-2)(i-k)}\right) C_{2n-1}^{-j}.\end{aligned}$$

If $i = k$, then $\mathrm{diag}\left(1, \omega^{i-k}, \omega^{2(i-k)}, \ldots, \omega^{(2n-2)(i-k)}\right) = I_{2n-1}$, and so $(U_nU_n^*)\,(i,i) = I_{2n-1}$. If $i \ne k$, then

$$\begin{aligned}(U_nU_n^*)\,(i,k) &= \sum_{j=0}^{2n-2} \mathrm{diag}\left(\omega^{j(i-k)}, \omega^{(j+1)(i-k)}, \ldots, \omega^{(2n-2)(i-k)}, 1, \ldots \omega^{(j-1))(i-k)}\right) \\ &= O_{2n-1}.\end{aligned}$$

Thus U_n is unitary. Using U_n we define the following matrix representation of the symmetric group S_{2n-1} on $2n-1$ elements:

$$\mathfrak{U}_n := \{U_n(I_{2n-1}\otimes P)U_n^* : P \text{ is a } (2n-1)\times(2n-1) \text{ permutation matrix}\}.$$

Note that $\mathfrak{U}_n$ is a subgroup of the unitary group $\mathcal{U}((2n-1)^2)$. Define the *cyclic blowup matrix* of $\mathfrak{G}$ to be the $(2n-1)^2 \times (2n-1)^2$ Hermitian matrix

$$H_{\mathfrak{G}} = U_n(I_{2n-1}\otimes(A_{\mathfrak{G}}\oplus O_{n-1}))U_n^*.$$

With these definitions in hand, we are now ready for the main result of this section.

Theorem 5.2. *Let $\mathfrak{G}$ be a loop-graph with $n-1$ edges and one loop. Then $\mathfrak{G}$ has a ρ-labelling if and only if $H_{\mathfrak{G}}$ is $\mathfrak{U}_n$-apportionable. In this case, the $\mathcal{U}$-apportionment constant of $H_{\mathfrak{G}}$ is $\kappa = (2n-1)^{-1}$.*

Proof. Assume that $\mathfrak{G}$ has a ρ-labelling $f : V(\mathfrak{G}) \to V(\mathfrak{K}_{2n-1})$. Let P be the permutation matrix such that the $(2n-1) \times (2n-1)$ adjacency matrix of the image of $\mathfrak{G}$ under f is $A = P(A_{\mathfrak{G}} \oplus O)P^*$. Let $Q = U_n(I_{2n-1} \otimes P)U_n^*$. Note that $Q \in \mathfrak{U}_n$. Direct calculation gives

$$QH_{\mathfrak{G}}Q^* = U_n(I_{2n-1} \otimes A)U_n^*$$

and so the (i,k)-block of $QH_{\mathfrak{G}}Q^*$ is

$$\sum_{j=0}^{2n-2} U(i,j)AU(k,j)^* = \frac{1}{2n-1} \sum_{j=0}^{2n-2} C_{2n-1}^{j} \operatorname{diag}(\mathbf{w})^i A \operatorname{diag}(\mathbf{w})^{-k} C_{2n-1}^{-j}.$$

Since $\mathfrak{G}$ has $n-1$ non-loop edges and one loop, Theorem 5.1 implies that each term in this sum has disjoint support if and only if f is a ρ-labelling. This holds if and only if the (i,k)-block of $QH_{\mathfrak{G}}Q^*$ is uniform with apportionment constant $(2n-1)^{-1}$ since the zero-nonzero pattern of each term in this sum matches that of the corresponding term in (5.1). □

Rosa also introduced other labellings in [10], in particular graceful labellings. Similar versions of Theorem 5.2 can be made for each these labellings since they all imply the existence of ρ-labellings. A graceful labelling of an n-vertex graph is a ρ-labelling where the previously mentioned injective function $f : V(\mathfrak{G}) \to \{0, \dots, 2n-2\}$ is such that its image is $\mathbb{Z}_n := \{0, \dots, n-1\}$. In a graceful labelling, the subset of vertices being acted upon by the permutation group is reduced from $2n-1$ to n, thereby further reducing the apportioning unitary subgroup from a representation of S_{2n-1} as

$$\mathfrak{U}_n := \{U_n(I_{2n-1} \otimes P)U_n^* : P \text{ is a } (2n-1) \times (2n-1) \text{ permutation matrix}\}$$

to a representation of S_n as

$$\mathfrak{U}_n' := \{U_n\left(I_{2n-1} \otimes (P \oplus I_{n-1})\right)U_n^* : P \text{ is a } n \times n \text{ permutation matrix}\}.$$

In a *directed graph* or *digraph* $\Gamma = (V, E)$, the edge set E is a set of ordered pairs (i,j) with $i, j \in V$. Thus a digraph allows loops (arcs of the form (i,i)) and allows both (i,j) and (j,i) when $i \neq j$, but not multiple identical arcs between a pair of vertices. To an arbitrary function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ we associate a *functional directed graph*, denoted by Γ_f, whose vertex set and arc set are respectively

$$V\left(\Gamma_f\right) := \mathbb{Z}_n, \quad E\left(\Gamma_f\right) := \{(i, f(i)) \,:\, i \in \mathbb{Z}_n\}.$$

Of course not every digraph arises from a function, i.e., not every digraph is functional, but the mapping $f \to \Gamma_f$ is injective. A digraph Γ can be mapped to its *underlying simple graph* G that has the edge $\{i,j\}$ exactly when Γ has one or both of the arcs (i,j) and

(j, i) and $i \neq j$. Similarly, Γ can be mapped to its *underlying loop-graph* $\mathfrak{G}$ that has the edge $\{i, j\}$ exactly when Γ has one or both of the arcs (i, j) and (j, i). When Γ_f is a functional digraph, we denote the underlying simple graph and loop-graph by G_f and $\mathfrak{G}_f$, respectively.

We define the sets of *contracting functions* and *non-increasing functions* as follows.

$$\mathrm{Con}(n) := \{h : \mathbb{Z}_n \to \mathbb{Z}_n \text{ such that } h(0) = 0 \text{ and } h(i) < i \text{ for } i \neq 0\}.$$
$$\mathrm{NIF}(n) := \{f : \mathbb{Z}_n \to \mathbb{Z}_n \text{ such that } f(i) \leq i, \forall\, i \in \mathbb{Z}_n\}.$$

Note that $\mathrm{Con}(n) \subset \mathrm{NIF}(n)$. Observe that while in general the mapping of a functional digraph to its underlying loop-graph is not injective, with the restriction that $f \in \mathrm{NIF}(n)$ the mapping $\Gamma_f \to \mathfrak{G}_f$ is injective, so there is a unique association of $\mathfrak{G}_f$ and Γ_f.

In Proposition 5.3 we show that $\mathrm{NIF}(n)$ is in one-to-one correspondence with the set of gracefully labelled loop-graphs on n vertices by associating each $f \in \mathrm{NIF}(n)$ with the loop-graph having edges $\{f(i) + n - 1 - i, f(i)\}$ for $i = 0, \ldots, n-2$ and loop $\{f(n-1), f(n-1)\}$. Note that $\{f(i)+n-1-i, f(i)\}$ is never a loop for $i = 0, \ldots, n-2$.

Proposition 5.3. *The set of gracefully labelled loop-graphs on n vertices are in one-to-one correspondence with* $\mathrm{NIF}(n)$.

Proof. Each $f \in \mathrm{NIF}(n)$ is defined by the set of ordered pairs $\{(i, f(i)) : i = 0, \ldots, n-1\}$. Observe that the map sending $(i, f(i))$ to the multiset $\{f(i)+n-1-i, f(i)\}$ is bijective since it is represented by

$$\begin{pmatrix} -1 & 1 & n-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ f(i) \\ 1 \end{pmatrix} = \begin{pmatrix} f(i) + n - 1 - i \\ f(i) \\ 1 \end{pmatrix}.$$

Thus the mapping which sends $f \in \mathrm{NIF}(n)$ to the loop-graph having edges $\{n - 1 + f(i) - i, f(i)\}$ for $i = 0, \ldots, n-2$ and loop $\{f(n-1), f(n-1)\}$ is bijective. $\square$

If the loop graph $\mathfrak{G}_f$ is gracefully labelled, it follows that the blowup construction H_f is uniform and thus the Hermitian matrix $I_{2n-1} \otimes \left(A_{\mathfrak{G}_f} \oplus O_{n-1}\right)$ is $\mathfrak{U}'_n$-apportionable. We now establish a result which shows that the spectra of $\mathcal{U}$-apportionable matrices varies quite widely. It was recently shown in [9] for large n that every n-vertex functional tree admits a ρ-labelling. It is known that the spectra of trees are many and varied [4,7]. Consequently, there must be a wide ranging family of spectra of matrices devised from our blowup construction of trees that are $\mathcal{U}$-apportionable, albeit with many repeated eigenvalues and additional zero eigenvalues. We can also invoke the stronger result from [3] that every tree admits a graceful labelling. We show that for all $f : \mathbb{Z}_n \to \mathbb{Z}_n$ subject to the iterative fixed point condition $\left|f^{(n-1)}(\mathbb{Z}_n)\right| = 1$, the matrix $I_{2n-1} \otimes \left(\mathrm{diag}\left(\Lambda_{\mathfrak{G}_f}\right) \oplus O_{n-1}\right)$ is $\mathcal{U}$-apportionable (where $\Lambda_{\mathfrak{G}_f}$ is an ordering of the spectrum of $A_{\mathfrak{G}_f}$).

Observe that any loop-tree can be relabelled to be the loop graph of a function in Con(n). We have restated the next lemma from [3] in the language of loop-graphs (this is permitted because for $f \in \mathrm{Con}(n)$ the mapping $\Gamma_f \to \mathfrak{G}_f$ is injective).

Lemma 5.4 *(Composition Lemma). [3] Let $f \in \mathrm{Con}(n)$ be such that the length of the path whose endpoints in $\mathfrak{G}_f$ are 0 and $n-1$ is equal to the diameter of $\mathfrak{G}_f$ and the set $f^{-1}(\{f(n-1)\})$ consists of consecutive integers including $n-1$, i.e., $f^{-1}(\{f(n-1)\}) = \left\{n-1, n-2, ..., n - \left|f^{-1}(\{f(n-1)\})\right|\right\}$. If $g \in \mathrm{Con}(n)$ is defined from f by*

$$g(i) = \begin{cases} f^{(2)}(i) & \text{if } i \in f^{-1}(\{f(n-1)\}) \\ f(i) & \text{otherwise} \end{cases},$$

then

$$\max_{\pi \in S_n} \left|\left\{\left|\pi g \pi^{-1}(i) - i\right| : i \in \mathbb{Z}_n\right\}\right| \leq \max_{\pi \in S_n} \left|\left\{\left|\pi f \pi^{-1}(i) - i\right| : i \in \mathbb{Z}_n\right\}\right|.$$

Theorem 5.5. *If $f : \mathbb{Z}_n \to \mathbb{Z}_n$ subject to the iterative fixed point condition $\left|f^{(n-1)}(\mathbb{Z}_n)\right| = 1$, then*

$$A = I_{2n-1} \otimes \left(A_{\mathfrak{G}(f)} \oplus O_{n-1}\right)$$

is $\mathcal{U}$-apportionable.

Proof. By hypothesis, there is a permutation $\pi \in S_n$ such that $\pi^{-1} f \pi \in \mathrm{Con}(n)$. Starting from any member of Con(n), if we repeatedly perform the local iteration described in the statement of the Composition Lemma 5.4, the resulting sequence of functions converges to the identically zero function whose loop-graph (a star with zero at the centre) is gracefully labelled. The composition lemma asserts that the local iteration transformation never increases the maximum number of induced edge labels. Therefore the loop-graph of all members of the said sequence have graceful loop-graphs. Thus, for all functions $f \in \mathrm{Con}(n)$ we have

$$n = \max_{\pi \in \mathrm{S}_n} \left|\left\{\left|\pi f \pi^{-1}(i) - i\right| : i \in \mathbb{Z}_n\right\}\right|.$$

Consider the map that associates with an arbitrary $f \in \mathrm{Con}(n)$ the $(2n-1)^2 \times (2n-1)^2$ cyclic blowup matrix

$$H_f = U_n \left(I_{2n-1} \otimes \left(A_{\mathfrak{G}(f)} \oplus O_{n-1}\right)\right) U_n^*.$$

By Theorem 5.2 there exists a unitary matrix in the subgroup $\mathfrak{U}_n'$ which apportions H_f. $\square$

When symmetry is removed, the situation is very different. For our next result, consider the map which associates with an arbitrary function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ a matrix of size $(2n-1)^2 \times (2n-1)^2$ as follows:

$$f \mapsto T_f = U_n \left(I_{2n-1} \otimes (A_f \oplus O_{n-1}) \right) U_n^*,$$

where $A_f \in \{0,1\}^{n\times n}$ denotes the adjacency matrix of the functional directed graph Γ_f of f with entries given by

$$(A_f)_{i,j} = \begin{cases} 1 & \text{if } j = f(i) \\ 0 & \text{otherwise} \end{cases}, \quad \forall\, 0 \le i, j < n.$$

This matrix construction results in a non-Hermitian matrix unless f is an involution. Observe that the subset of matrices $\{T_f : f : \mathbb{Z}_n \to \mathbb{Z}_n\}$ yields a matrix representation of the transformation monoid of functions from $\mathbb{Z}_n$ to $\mathbb{Z}_n$ prescribed by the antihomomorphism identity

$$T_f\, T_g = T_{g\circ f}, \text{ for all } f, g : \mathbb{Z}_n \to \mathbb{Z}_n$$

(because the adjacency matrix of a digraph acts on the right to identify the adjacencies of a vertex).

Definition 5.6. For $A \in \mathbb{C}^{n\times n}$, define

$$\mathfrak{u}(A) = \inf_{U \in \mathfrak{U}_n} \|UAU^*\|_{\max}.$$

The $\mathfrak{U}_n$-*apportionment gap* of an arbitrary matrix $A \in \mathbb{C}^{n\times n}$ is $\left|\mathfrak{u}(A) - \frac{\|A\|_{\mathrm{F}}}{n}\right|$.

Remark 5.7. Since the set $\mathfrak{U}_n$ is compact, the infimum is actually the minimum:

$$\mathfrak{u}(A) = \min_{U \in \mathfrak{U}_n} \|UAU^*\|_{\max}.$$

Theorem 5.8. *If $f \in \mathrm{Con}(n)$, then T_f is not $\mathfrak{U}_n$-apportionable. Furthermore, the $\mathfrak{U}_n$-apportionment gap for every T_f is*

$$\mathfrak{u}(T_f) - \frac{\|T_f\|_F}{(2n-1)^2} = \frac{1}{2n-1} - \frac{\sqrt{(2n-1)\,n}}{(2n-1)^2}.$$

Proof. By Theorem 5.2, the matrix

$$U_n \left(I_{2n-1} \otimes (A_f \oplus O_{n-1}) \right) U_n^* + U_n \left(I_{2n-1} \otimes \left((A_f - E_{0,0})^\top \oplus O_{n-1} \right) \right) U_n^*$$

is $\mathfrak{U}_n$-apportionable with apportion constant $(2n-1)^{-1}$. For any $V \in \mathfrak{U}_n$, the set of indices of the nonzero entries of each $(2n-1) \times (2n-1)$ block indexed by (i,j) of VT_fV^* are identical (except possibly for some zeros created by cancellation). Because V has at its core a permutation, this set of indices is disjoint from the set of indices of the non-zero entries in the corresponding $(2n-1) \times (2n-1)$ block indexed by (i,j) of

$$VU_n\left(I_{2n-1} \otimes \left((A_f - E_{0,0})^\top \oplus O_{n-1}\right)\right)U_n^*V^*.$$

This implies that

$$(VT_fV^*) \circ \left(VU_n\left(I_{2n-1} \otimes \left((A_f - E_{0,0})^\top \oplus O_{n-1}\right)\right)U_n^*V^*\right) = O_{(2n-1)^2} \tag{5.3}$$

where $\circ$ denotes the entrywise product. Therefore the action on T_f by similarity transformation mediated by members of $\mathfrak{U}_n$ can not result in a uniform matrix. We conclude that T_f is not $\mathfrak{U}_n$-apportionable as claimed. This also implies $\mathfrak{u}(T_f) = (2n-1)^{-1}$ and

$$\mathfrak{u}(T_f) - \frac{\|T_f\|_{\mathrm{F}}}{(2n-1)^2} = \frac{1}{2n-1} - \frac{\sqrt{(2n-1)\,n}}{(2n-1)^2}. \quad \square$$

As a corollary of Theorem 5.8, for all f, g lying in the semigroup $\mathrm{Con}(n)$ the following property holds: $\mathfrak{u}(T_g) = \mathfrak{u}(T_f)$.

Remark 5.9. Observe that the three matrix summands

$$\begin{aligned} &I_{2n-1} \otimes (E_{0,0} \oplus O_{n-1}),\ I_{2n-1} \otimes ((A_f - E_{0,0}) \oplus O_{n-1}) \\ &\text{and } I_{2n-1} \otimes \left((A_f - E_{0,0})^\top \oplus O_{n-1}\right) \end{aligned}$$

are pair-wise orthogonal when viewed as members the standard (vector) inner-product space of matrices and remain so after any unitary similarity transform.

6. Application of gracefully labelled graphs to spectral inequalities

In this section we generalize the well-known Eigenvalue Interlacing Inequalities (involving deletion of one or more rows and columns) to a new eigenvalue interlacing inequality involving matrices obtained by zeroing the entries associated with edges of a gracefully labelled loop-graph. Of course there are complications for interlacing when zeroing entries is done instead of deletion even with a star (replacing deletion of row and column j by zeroing row and column j). Our results describe how a combination of n permutations of $n-1$ edge zeroings in a dense arbitrarily weighted undirected graph give an interlacing bound, and the proof uses the original Eigenvalue Interlacing Inequality.

Definition 6.1. Let $\mathfrak{G}$ be a loop-graph and recall that $A_{\mathfrak{G}}$ denotes its adjacency matrix. Recall that C_n denotes the $n\times n$ cyclic shift matrix. For $n \geq 3$ and any matrix $M \in \mathbb{C}^{n\times n}$, define

$$M_k = M \circ \left((C_n)^k \left(J_n - A_{\mathfrak{G}} \circ (J_n + I_n)\right) (C_n)^{-k} \right)$$

and denote $\operatorname{spec}(M_k) = \{\lambda_{k,0}, \ldots, \lambda_{k,n-1}\}$.

We illustrate this definition in the next example.

Example 6.2. Let $\mathfrak{G}$ be the gracefully labelled loop-graph with edge set

$$E(\mathfrak{G}) = \{\{0,2\}, \{0,3\}, \{1,2\}, \{2,2\}\}$$

i.e., $\mathfrak{G}$ is a path on 4 vertices with path order 3,0,2,1 and a loop at 2. Then

$$A_{\mathfrak{G}} = \begin{pmatrix} 0&0&1&1\\ 0&0&1&0\\ 1&1&1&0\\ 1&0&0&0 \end{pmatrix}, \quad A_{\mathfrak{G}} \circ (J_n + I_n) = \begin{pmatrix} 0&0&1&1\\ 0&0&1&0\\ 1&1&2&0\\ 1&0&0&0 \end{pmatrix},$$

$$J_n - (\, A_{\mathfrak{G}} \circ (J_n + I_n)) = \begin{pmatrix} 1&1&0&0\\ 1&1&0&1\\ 0&0&-1&1\\ 0&1&1&1 \end{pmatrix}$$

Observe that if we take

$$M = \begin{pmatrix} m_{00}&m_{01}&m_{02}&m_{03}\\ m_{10}&m_{11}&m_{12}&m_{13}\\ m_{20}&m_{21}&m_{22}&m_{23}\\ m_{30}&m_{31}&m_{32}&m_{33} \end{pmatrix},$$

then

$$M_0 = \begin{pmatrix} m_{00}&m_{01}&0&0\\ m_{10}&m_{11}&0&m_{13}\\ 0&0&-m_{22}&m_{23}\\ 0&m_{31}&m_{32}&m_{33} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} m_{00}&0&m_{02}&m_{03}\\ 0&-m_{11}&m_{12}&0\\ m_{20}&m_{21}&m_{22}&0\\ m_{30}&0&0&m_{33} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -m_{00}&m_{01}&0&0\\ m_{10}&m_{11}&0&m_{13}\\ 0&0&m_{22}&m_{23}\\ 0&m_{31}&m_{32}&m_{33} \end{pmatrix}, \qquad M_3 = \begin{pmatrix} m_{00}&0&m_{02}&m_{03}\\ 0&m_{11}&m_{12}&0\\ m_{20}&m_{21}&m_{22}&0\\ m_{30}&0&0&-m_{33} \end{pmatrix}.$$

Observe that

$$M_0 + M_1 + M_2 + M_3 = \begin{pmatrix} 2m_{00} & 2m_{01} & 2m_{02} & 2m_{03} \\ 2m_{10} & 2m_{11} & 2m_{12} & 2m_{13} \\ 2m_{20} & 2m_{21} & 2m_{22} & 2m_{23} \\ 2m_{30} & 2m_{31} & 2m_{32} & 2m_{33} \end{pmatrix} = (4-2)M. \tag{6.1}$$

We note that with M_k as defined in Definition 6.1, the property illustrated in (6.1) is true in general (provided $\mathfrak{G}$ is gracefully labelled). The effects of the permutation similarity $(C_n)^k$ described in the proof can be observed in Example 6.2.

Proposition 6.3. *Let $\mathfrak{G}$ be a gracefully labelled loop-graph of order $n \geq 4$ and let M_k be as defined in Definition 6.1. Then*

$$(n-2)\, M = \sum_{0 \leq k < n} M_k.$$

Proof. Consider the effect of the permutation similarity $(C_n)^k$: For $M = [m_{ij}]$, $\left((C_n)^k\, M\, (C_n)^{-k}\right)_{ij} = m_{i-k,j-k}$ (with arithmetic done modulo n). Because $\mathfrak{G}$ is gracefully labelled, the $n-1$ zeros above the diagonal in M_0 will land in each off-diagonal position exactly once as k ranges over $0, \ldots, n-1$, and similarly for the zeros below the diagonal. Thus $\sum_{k=0}^{n-1} (M_k)_{ij} = (n-2)m_{ij}$. For the diagonal, observe that $\mathfrak{G}$ has a unique loop, so $A_{\mathfrak{G}}$ has exactly one nonzero diagonal entry $m_{\ell\ell}$. The effect of the cyclic permutation is that this one nonzero entry, which transforms via $J_n - (\, A_{\mathfrak{G}} \circ (J_n + I_n))$ from positive to negative, hits every index once. Thus for each $j = 0, \ldots, n-1$, $\sum_{k=0}^{n-1} (M_k)_{jj} = (n-1)m_{jj} - m_{jj} = (n-2)m_{jj}$. $\square$

Definition 6.4. For M_k and $\lambda_{k,t}$ as defined in Definition 6.1, order the multiset $\{\lambda_{k,t} : t = 0, \ldots, n-1, k = 0, \ldots, n-1\}$ in nonincreasing order and denote these values by $\theta_j, j = 0, \ldots, n^2 - 1$, so that

$$\{\theta_j, j = 0, \ldots, n^2 - 1\} = \{\lambda_{k,t} : t = 0, \ldots, n-1, k = 0, \ldots, n-1\}$$
$$\text{and } \theta_0 \geq \theta_1 \geq \cdots \geq \theta_{n^2-1}.$$

Recall that all gracefully labelled loop graphs can be constructed from nonincreasing functions (Proposition 5.3).

Theorem 6.5. *Let $\mathfrak{G}$ be a gracefully labelled loop-graph of order $n \geq 4$, let M be an $n \times n$ Hermitian matrix, and let θ_j be as defined in Definition 6.4. For $\ell = 0, \ldots, n-1$,*

$$\frac{n}{n-2}\theta_{\ell+(n^2-n)} \leq \lambda_\ell(M) \leq \frac{n}{n-2}\theta_\ell.$$

Proof. Since M is Hermitian, each M_k is a Hermitian matrix. Thus by the Spectral Theorem, each matrix M_k admits a spectral decomposition of the form

$$M_k = U_k \operatorname{diag}(\Lambda_k) U_k^*, \tag{6.2}$$

where U_k is a real unitary matrix, $\Lambda_k = (\lambda_{k,0}, \ldots, \lambda_{k,n-1})$ and $\operatorname{spec}(M_k) = \{\lambda_{k,0}, \ldots, \lambda_{k,n-1}\} \subset \mathbb{R}$. Thus

$$(M_k)_{ij} = \sum_{0 \le t < n} \left((U_k)_{i,t} \sqrt{\lambda_{k,t}} \right) \left(\sqrt{\lambda_{k,t}} \, \left(\overline{U_k}\right)_{j,t} \right). \tag{6.3}$$

From Proposition 6.3,

$$m_{ij} = \sum_{0 \le k < n} \sum_{0 \le t < n} \left(\frac{(U_k)_{i,t}}{\sqrt{n}} \sqrt{\frac{n}{n-2} \lambda_{k,t}} \right) \left(\sqrt{\frac{n}{n-2} \lambda_{k,t}} \, \frac{\left(\overline{U_k}\right)_{j,t}}{\sqrt{n}} \right), \ \forall \, 0 \le i, j < n. \tag{6.4}$$

Reversing the process used to go from (6.2) to (6.3), we view the entry-wise equality in (6.4) as expressing the product of three matrices. The first matrix is $n \times n^2$ matrix $\hat{U}$ defined as follows: The i-th row of $\hat{U}$ is obtained by concatenating row i of the n matrices U_k for $k = 0, \ldots, n-1$. The second matrix is the $n^2 \times n^2$ diagonal matrix

$$\Lambda = \bigoplus_{k=0}^{n-1} \operatorname{diag}\left(\frac{n}{n-2} \Lambda_k \right).$$

Finally the third matrix is the Hermitian adjoint of the first matrix $\hat{U}$. Observe that the rows of $\hat{U}$ are orthonormal. By extending the rows of $\hat{U}$ to an orthonormal basis for $\mathbb{R}^{n^2}$ and applying Gram-Schmidt, we can expand $\hat{U}$ to a unitary matrix U of size $n^2 \times n^2$.

Then M is the $\{0, \ldots, n-1\}$ principal submatrix of the $n^2 \times n^2$ matrix $U \Lambda U^*$, i.e.,

$$U \Lambda U^* = \begin{bmatrix} M & B_{0,1} \\ {B_{0,1}}^* & B_{1,1} \end{bmatrix}$$

for some matrices $B_{0,1}$ and $B_{1,1}$. Then by the Eigenvalue Interlacing Theorem [11, Theorem 8.10],

$$\lambda_{\ell + (n^2 - n)} \left(\begin{bmatrix} M & B_{0,1} \\ B_{0,1}^* & B_{1,1} \end{bmatrix} \right) \le \lambda_\ell (M) \le \lambda_\ell \left(\begin{bmatrix} M & B_{0,1} \\ B_{0,1}^* & B_{1,1} \end{bmatrix} \right)$$

for $\ell = 1, \ldots, n$. Observe that

$$\frac{n}{n-2} \theta_\ell = \lambda_\ell (\Lambda) = \lambda_\ell \left(\begin{bmatrix} M & B_{0,1} \\ B_{0,1}^* & B_{1,1} \end{bmatrix} \right). \quad \square$$

7. Spectra and Jordan canonical forms of GL-apportionable matrices

We now shift focus from unitary apportionability to general apportionability. In this section we study the question of what multisets of n complex numbers can and cannot be realized as spectra or Jordan canonical forms of uniform $n \times n$ matrices. Being the spectrum or Jordan canonical form of an apportionable matrix is equivalent to being the spectrum or Jordan canonical form of a uniform matrix since similarity by matrices in $\mathrm{GL}(n)$ is allowed for GL-apportionability. We begin with some elementary observations and then focus on the case of 2×2 matrices.

For every $\lambda \in \mathbb{C}$, there is a uniform matrix $B \in \mathbb{C}^{n\times n}$ with $\operatorname{rank} B = 1$ and $\operatorname{spec}(B) = \{\lambda, 0, \dots, 0\}$ by Theorem 2.2. We can scale the spectrum of a uniform matrix: If $\Lambda = \operatorname{spec}(B)$ for a uniform matrix B, then for $b \in \mathbb{C}$, $b\Lambda = \operatorname{spec}(bB)$ and bB is uniform.

Kronecker products can be used to construct bigger uniform matrices from smaller uniform matrices, and thus expand the set of spectra that we are able to realize with uniform matrices. For two multisets $S = \{s_0, \dots, s_{n-1}\}$ and $T = \{t_0, \dots, t_{m-1}\}$, define $ST = \{s_k t_j : k = 0, \dots, n-1, j = 0, \dots, m-1\}$.

Remark 7.1. Suppose B_n and B_m are uniform $n \times n$ and $m \times m$ matrices, with spectra Λ_n and Λ_m. Then $B_n \otimes B_m$ is uniform. From known properties of Kronecker products [11, Theorem 4.8], $\operatorname{spec}(B_n \otimes B_m) = \Lambda_n \Lambda_m$. By using $B_n = F_n$ (recall that F_n denotes the $n \times n$ DFT matrix) and B_m as any uniform matrix with spectrum Λ, we conclude that if Λ is the spectrum of a uniform matrix, then so is $\cup_{j=1}^{n} \omega^j \Lambda$ where ω is a primitive nth root of unity.

Proposition 7.2. *Let B be a uniform $n \times n$ matrix. Then there is a uniform matrix with spectrum* $\operatorname{spec}(B) \cup \{0^{((r-1)n)}\}$ *for any positive integer r.*

Proof. Let $E_{0,0}$ be the $r \times r$ matrix with $(0, 0)$-entry equal to 1 and all other entries 0. Consider $B' = E_{00} \otimes B = B \oplus O_{(r-1)n}$. Then $\operatorname{spec}(B') = \operatorname{spec}(B) \cup \{0^{((r-1)n)}\}$. Define $U = F_r \otimes I_n$ and observe that U is unitary. Furthermore,

$$UB'U^* = (F_r \otimes I_n)(E_{00} \otimes B)(F_r^* \otimes I_n) = (F_r E_{00} F_r^*) \otimes B = J_r \otimes B$$

is uniform. $\square$

Since extending the spectrum with blocks of zeros preserves apportionability, it is natural to ask whether we can add just one zero and preserve apportionability.

Question 7.3. *If B is uniform, is* $\operatorname{spec}(B) \cup \{0\}$ *the spectrum of a uniform matrix?*

Next we study the question of what multisets of 2 complex numbers cannot be realized as spectra of uniform 2×2 matrices, and thus of apportionable matrices. Very few real spectra are realizable by uniform matrices.

Theorem 7.4. *For any $n \geq 2$, the spectrum $\{0^{(n)}\}$ can be realized by a nonzero uniform matrix. For any nonzero $\lambda \in \mathbb{C}$, the spectrum $\{\lambda^{(2)}\}$ cannot be realized by a uniform matrix.*

Proof. Since rank $E_{0,1} = 1$ where $E_{0,1}$ is $n \times n$, Theorem 2.2 shows $\{0^{(n)}\}$ can be realized as the spectrum of a nonzero uniform matrix.

Suppose that $B \in \mathbb{C}^{2\times 2}$ is a uniform matrix with spectrum $\{\lambda^{(2)}\}$. We show that $\lambda = 0$. There exists a matrix M such that $M^{-1}BM$ is in Jordan canonical form. Since B is uniform and $M\lambda I_2 M^{-1} = \lambda I_2$, if the Jordan canonical form of B is λI_2, then $\lambda = 0$. So assume the Jordan canonical form of B is $\lambda I_2 + N$ where $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $M(\lambda I_2 + N)M^{-1} = \lambda I_2 + MNM^{-1}$. Without loss of generality, $\det M{=}1$ and $M = \begin{bmatrix} x & y \\ z & \frac{1+yz}{x} \end{bmatrix}$. Then

$$B = M(\lambda I_2 + N)M^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} -xz & x^2 \\ -z^2 & xz \end{bmatrix} = \begin{bmatrix} -xz+\lambda & x^2 \\ -z^2 & xz+\lambda \end{bmatrix}$$

is uniform, so $|x| = |z|$. If $x = 0$, then $\lambda = 0$, so assume $x \neq 0$. Now uniformity implies $|xz+\lambda| = |-xz+\lambda| = |x^2| = |xz|$. Let $\delta = xz$. Then $|xz+\lambda|^2 = \delta\overline{\delta} + \delta\overline{\lambda} + \lambda\overline{\delta} + \lambda\overline{\lambda}$, $|-xz+\lambda|^2 = \delta\overline{\delta} - \delta\overline{\lambda} - \lambda\overline{\delta} + \lambda\overline{\lambda}$, and $|xz|^2 = \delta\overline{\delta}$. Thus $\delta\overline{\lambda} + \lambda\overline{\delta} = 0$. So $\delta\overline{\delta} = \delta\overline{\delta} + \lambda\overline{\lambda}$, which implies $\lambda = 0$. $\square$

The next result is established by computation.

Lemma 7.5. *Let $c \in \mathbb{C}$, $D = \operatorname{diag}(1, c)$ and $M = \begin{bmatrix} x & y \\ z & \frac{1+yz}{x} \end{bmatrix}$ Then*

$$B = MDM^{-1} = \begin{bmatrix} 1+(1-c)yz & -(1-c)xy \\ (1-c)(1+yz)\frac{z}{x} & c-(1-c)yz \end{bmatrix}.$$

Theorem 7.6. *For a real number r, the spectrum $\{1, r\}$ can be realized by a uniform matrix if and only if $r = 0$ or $r = -1$.*

Proof. The spectra $\{1, 0\}$ and $\{1, -1\}$ are realized by $\frac{1}{2}J_2$ and $\frac{1}{\sqrt{2}}H_2$, respectively (recall H_2 is a 2×2 Hadamard matrix).

The eigenvalues of a uniform 2×2 matrix are distinct unless both are zero by Theorem 7.4. Let $r \in \mathbb{R}$ and $D = \operatorname{diag}(1, r)$. Assume D is apportionable. Then we may assume the apportioning matrix M has the form in Lemma 7.5, so $B = \begin{bmatrix} 1+(1-r)yz & -(1-r)xy \\ (1-r)(1+yz)\frac{z}{x} & r-(1-r)yz \end{bmatrix}$ is uniform. Let $yz = a + b\,\mathbb{i}$ with $a, b \in \mathbb{R}$. Compare the absolute values of the $(0,0)$ and $(1,1)$ entries of B:

$$|1 + (1-r)(a + b\,\mathbb{i})| = |r - (1-r)(a + b\,\mathbb{i})|.$$

$$|(1+(1-r)a)+((1-r)b)\,\mathbb{i}| = |(r-(1-r)a)+((1-r)b)\,\mathbb{i}|.$$
$$|1+(1-r)a| = |r-(1-r)a|.$$

Thus $1+(1-r)a = r-(1-r)a$ or $-(1+(1-r)a) = r-(1-r)a$. Since $-(1+(1-r)a) = r-(1-r)a$ implies $r=-1$ and we have seen that $r=-1$ can be realized, assume $1+(1-r)a = r-(1-r)a$. Thus $a=-\frac{1}{2}$. Since B is uniform, the absolute value of product of the off-diagonal entries equals the absolute value of product of the diagonal entries. Since $1+yz = -\overline{yz}$, the product of the off diagonal entries is $-(1-r)^2yz(1+yz) = (1-r)^2|yz|^2 = (1-r)^2(\frac{1}{4}+b^2)$. The square of the absolute value of each entry must be $|1+(1-r)yz|^2 = |1+(1-r)(-\frac{1}{2}+b\,\mathbb{i})|^2 = |\frac{1+r}{2}+(1-r)b\,\mathbb{i}| = \frac{(1+r)^2}{4}+(1-r)^2b^2$. Thus

$$\begin{aligned}(1-r)^2(\frac{1}{4}+b^2) &= \frac{(1+r)^2}{4}+(1-r)^2b^2\\ \frac{(1-r)^2}{4} &= \frac{(1+r)^2}{4}\\ 0 &= r.\end{aligned}$$

Thus the uniformity of B implies $r=-1$ or $r=0$. □

It is immediate from the previous theorem that two nonzero eigenvalues of a 2×2 uniform matrix may or may not have the same magnitude. This is also illustrated in the next two examples.

Example 7.7. For $B = \frac{1}{2}\begin{bmatrix} 1+\mathbb{i} & -1+\mathbb{i} \\ -1+\mathbb{i} & 1+\mathbb{i}\end{bmatrix}$, $\operatorname{spec}(B) = \{1,\ \mathbb{i}\}$.

Example 7.8. For $A = \begin{bmatrix} 1 & 1 \\ 1 & \frac{1+\sqrt{3}\,\mathbb{i}}{2}\end{bmatrix}$, the (approximate) decimal values of the eigenvalues of A are $1.69244+0.318148\,\mathbb{i}$ and $-0.19244+0.547877\,\mathbb{i}$.

8. Finding an apportioning matrix M and constant κ

In this section we discuss how to find GL-apportioning matrices. We begin with a simple 2×2 example that illustrates a matrix can have infinitely many GL-apportionment constants each of which can be obtained from infinitely many apportioning matrices.

Example 8.1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0\end{bmatrix}$ and let $M = \begin{bmatrix} w & x \\ y & z\end{bmatrix}$ be nonsingular (so $wz \neq xy$). Then

$$MAM^{-1} = \frac{2}{wz-xy}\begin{bmatrix} wz & -wx \\ yz & -xy\end{bmatrix}.$$

Thus the matrix MAM^{-1} is uniform if and only if $|x| = |z|$ and $|w| = |y|$. Observe that $|wz - xy| \leq 2|wz|$ and so each apportionment constant for A must be at least 1. Let $a, \theta \in \mathbb{R}$ such that $a \neq 0$ and $0 < \theta < 2\pi$, and let

$$M = \begin{bmatrix} a & a^{-1} \\ a & a^{-1}e^{i\theta} \end{bmatrix}.$$

Then M is nonsingular and apportions A with apportionment constant $\kappa = \left|\sin\left(\frac{\theta}{2}\right)\right|^{-1}$. Notice that $\kappa = 1$ for $\theta = \pi$, and that κ can be made arbitrarily large for a sufficiently small choice of θ. Thus $[1, \infty)$ is the set of apportionment constants for A.

Example 8.1 utilizes ad hoc methods to solve for apportioning matrices of a small and curated matrix. It may seem rather hopeless to find apportioning matrices in a more general setting. The search for apportioning matrices can be simplified with the following proposition. Let $\operatorname{vec}(A)$ denote the vectorization of the matrix A. Recall that $\circ$ denotes the entrywise product.

Proposition 8.2. *Let $A \in \mathbb{C}^{n\times n}$ and $M \in \mathrm{GL}(n)$. Let $\mathbf{v} = \operatorname{vec}\left((MAM^{-1}) \circ (\overline{MAM^{-1}})\right)$ and let F be the $n^2 \times n^2$ DFT matrix. Then M apportions A if and only if $F\mathbf{v} \in \operatorname{span}(\mathbf{e}_0)$.*

Proof. Suppose that M apportions A and let κ be the apportioning constant for M. Then $(MAM^{-1}) \circ (\overline{MAM^{-1}}) = \kappa^2 J$ and so $F\mathbf{v} = n\kappa^2\mathbf{e}_0$.

Now suppose that $F\mathbf{v} \in \operatorname{span}(\mathbf{e}_0)$. Then $\mathbf{v} = c\mathbb{1}$ for some $c \in \mathbb{R}$ (by construction $\mathbf{v} \in \mathbb{R}^{n^2}$) and hence $(MAM^{-1}) \circ (\overline{MAM^{-1}}) = cJ$. Thus MAM^{-1} is uniform and so M apportions A. $\square$

Proposition 8.2 can be used to solve for apportioning matrices by generating a system of $n^2 - 1$ equations in the entries of M. Note that F can be replaced with any unitary matrix whose first row is a multiple of $\mathbb{1}$.

We revisit Example 8.1 to illustrate how to apply Proposition 8.2.

Example 8.3. Let A and M be the same as in Example 8.1. We may assume, without loss of generality, that $\det(M) = wz - xy = 2$. Then

$$(MAM^{-1}) \circ (\overline{MAM^{-1}}) = \begin{bmatrix} |wz|^2 & |wx|^2 \\ |yz|^2 & |xy|^2 \end{bmatrix}.$$

By Proposition 8.2

$$\begin{aligned} |wz|^2 + \mathbb{i}|yz|^2 - |wx|^2 - \mathbb{i}|xy|^2 &= 0, \\ |wz|^2 - |yz|^2 + |wx|^2 - |xy|^2 &= 0, \\ |wz|^2 - \mathbb{i}|yz|^2 - |wx|^2 + \mathbb{i}|xy|^2 &= 0. \end{aligned}$$

This system of equations can be reduced to $|x| = |z|$ and $|w| = |y|$.

Remark 8.4. Suppose that $A \in \mathbb{C}^{n\times n}$ is $\mathcal{U}$-apportionable. Then the entries of a unitary matrix U that apportions A can be determined by Proposition 8.2 along with the system of equations resulting from $UU^* = I$.

Note that $A^{\circ^{-1}}$ means the entrywise inverse of a matrix A because $\circ$ is the entrywise product.

Theorem 8.5. *Let $A \in \mathbb{C}^{n\times n}$ be nonzero and apportionable with an apportionment constant $\kappa \geq 0$. Then there exists an $M \in \mathrm{GL}(n)$ such that*

$$A = \kappa^2 M^{-1} \left(\overline{MAM^{-1}}\right)^{\circ^{-1}} M.$$

Proof. Since A is apportionable with apportionment constant κ, there exists an $M \in \mathrm{GL}(n)$ such that $B = MAM^{-1}$ is uniform and $\kappa = \|B\|_{\max}$. Observe that $B \circ \overline{B} = \kappa^2 J$. Since $\overline{B}$ has no zero entries, $A = \kappa^2 M^{-1} \overline{B}^{\circ^{-1}} M$, as claimed. □

Question 8.6. *When $A \in \mathbb{C}^{n\times n}$ is not apportionable how do we find and certify the matrix M that achieves the infimum, $\inf_{M\in\mathrm{GL}(n)} \left\|MAM^{-1}\right\|_{max}$?*

9. Concluding remarks

We have included open questions throughout when relevant to the material discussed. In this section we list some additional open questions.

We begin with questions related to how 'common' apportionable matrices are. For context, recall that set of matrices that cannot be diagonalized is of measure zero (because an eigenvalue must be repeated). What about apportionability? It was shown in Proposition 3.2 that the set of matrices that are not $\mathcal{U}$-apportionable has positive measure.

Question 9.1. *Is the set of $\mathcal{U}$-apportionable matrices of measure zero or positive measure?*

Question 9.2. *Is the set of matrices that are not GL-apportionable of measure zero or positive measure? Is the set of GL-apportionable matrices of measure zero or positive measure?*

There are numerous ways to measure closeness to apportionability. Section 4 contains results about one such measure for $\mathcal{U}$-apportionability, $u(A) = \min_{U\in\mathcal{U}(n)} \|UAU^*\|_{\max}$. Here we mention other possibilities.

Definition 9.3. For $A \in \mathbb{C}^{n\times n}$ with no zero entries, define the *uniformity ratio* to be $ur(A) = \frac{\max\{|a_{ij}|\}}{\min\{|a_{ij}|\}}$; if there are both zero and nonzero entries in A, then $ur(A) = \infty$. Define the *unitary apportionability ratio* of $A \neq O$ to be

$$uar(A) = \inf_{U\in\mathcal{U}(n)} ur(UAU^*).$$

Let $A \in \mathbb{C}^{n\times n}$ and $A \neq O$. Observe that A is $\mathcal{U}$-apportionable if and only if $uar(A) = 1$. A unitary matrix U obtained from a random $n \times n$ matrix via orthonormalization of the columns will have the property that UAU^* has no zero entries and thus $ur(A) < \infty$.

Declaration of competing interest

All authors have no competing interests.

Data availability

No data was used for the research described in the article.

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Appendix A. Recovery lemma

The Composition Lemma, which is proved in [3], is applied in Section 5. It relies on the Recovery Lemma. Here we provide a proof of the Recovery Lemma.

Remark A.1. Let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ be a function. Recall that the functional digraph Γ_f associated with F has $V(\Gamma_f) = \mathbb{Z}_n$ and $E(\Gamma_f) = \{(i, f(i)) : i \in \mathbb{Z}_n\}$. Each vertex in Γ_f has out degree one. A fixed point of f corresponds to a loop in Γ_f. Note that f can be determined from Γ_f (but not always from the underlying simple graph G_f). If G_f is connected, then f has at most one fixed point, because $n - 1$ non-loop arcs are needed. If G_f is connected and f has a fixed point, then Γ_f does not have any cycles except the loop at the fixed point. If f has a fixed point and G_f is connected, then the fixed point and the edges of G_f uniquely determine f and Γ_f: Let u be the unique fixed point and let initially define $X = \{u\}$; X is the set of vertices x for which $f(x)$ is determined. If v

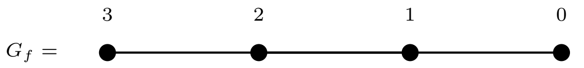

Fig. A.1. The graph G_f.

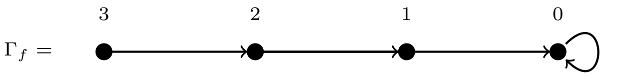

Fig. A.2. The graph Γ_f.

is a neighbour of $x \in X$, then $f(v) = x$ and the arc is (v, x), since each vertex of Γ_f has out degree one. So now $X := N[X]$. Repeat this neighbourhood step until $X = \mathbb{Z}_n$.

Definition A.2. For a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$, define the *edge-labelling* polynomial of f to be

$$p_f\left(x_0, \ldots, x_{n-1}\right) = \prod_{0 \le i < j < n} \left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right),$$

We illustrate why this is called the edge-labelling polynomial in the next example.

Example A.3. Let $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ and suppose that $G_f = P_4$ as depicted in Fig. A.1.

Assume 0 is the unique fixed point of f. Then $f(0) = 0$ and $f(i) = i - 1$ for $0 < i \le 3$. The functional directed graph Γ_f is shown in Fig. A.2 above.

In order to determine p_f note that $f(0) = 0$, and so $(x_{f(j)} - x_j)^2 - (x_{f(0)} - x_0)^2 = (x_{f(j)} - x_j)^2$ for $j > 0$. Thus

$$\begin{aligned} p_f(x_0, x_1, x_2, x_3) &= \prod_{0 \le i < j < 4} \left((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2\right) \\ &= \left(\prod_{j=1}^{4} (x_{j-1} - x_j)^2\right) \left((x_1 - x_2)^2 - (x_0 - x_1)^2\right) \\ &\quad \times \left((x_2 - x_3)^2 - (x_0 - x_1)^2\right) \left((x_2 - x_3)^2 - (x_1 - x_2)^2\right) \end{aligned}$$

Observation A.4. *The edge-labelling polynomial $p_f\left(x_0, \ldots, x_{n-1}\right)$ is not identically zero if and only if f has at most one fixed point and Γ_f has no 2-cycles.*

The next result gives an algorithm for recovering G_f from p_f when p_f is not identically zero and f has a fixed point.

Lemma A.5 *(Recovery Lemma). Suppose the edge-labelling polynomial $p_f\left(x_0, \ldots, x_{n-1}\right)$ is defined from some function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ and p_f is not identically zero. It can be determined from p_f whether or not f has a (necessarily unique) fixed point. If f has* *a fixed point, then G_f can be determined from p_f. If f has a fixed point and G_f is connected, then f and Γ_f can be determined from p_f and the fixed point. Let S denote the set of functions $f : \mathbb{Z}_n \to \mathbb{Z}_n$ such that f has a unique fixed point 0 and G_f is connected. The function from S to $\mathbb{Q}[x_0, \cdots, x_{n-1}]$ that assigns p_f to f is injective.*

Proof. We show that each factor in a factorization of p_f is a quadrinomial (a linear combination of exactly four distinct variables), a trinomial (a linear combination of exactly three distinct variables), or a binomial (a linear combination of exactly two distinct variables), and analyze how each can occur.

We factor

$$\left(\left(x_{f(j)} - x_j\right)^2 - \left(x_{f(i)} - x_i\right)^2\right) = \left(x_{f(j)} - x_j + x_{f(i)} - x_i\right)\left(x_{f(j)} - x_j - x_{f(i)} + x_i\right)$$

A factor $x_{f(j)} - x_j - x_{f(i)} + x_i$ or $x_{f(j)} - x_j + x_{f(i)} - x_i$ has the form $a+b-c-d$; it is a quadrinomial with a, b, c, d distinct if and only if $|\{a, b, c, d\}| = 4$, i.e., $\left|\left\{x_{f(j)}, x_j, x_{f(i)}, x_i\right\}\right| = 4$. In this case both $x_{f(j)} - x_j - x_{f(i)} + x_i$ and $x_{f(j)} - x_j + x_{f(i)} - x_i$ are quadrinomials.

The expression $a + b - c - d$ collapses to a binomial if $|\{a, b\} \cap \{c, d\}| = 1$ (note that $|\{a, b\} \cap \{c, d\}| = 2$ is impossible since p_f is not identically zero). Notice that $a + b - c - d$ occurs in two forms in p_f: $\{a, b\} = \{x_{f(j)}, x_{f(i)}\}, \{c, d\} = \{x_j, x_i\}$ or $\{a, b\} = \{x_{f(j)}, x_i\}, \{c, d\} = \{x_j, x_{f(i)}\}$. First consider the case that f has a (unique) fixed point u. Then for each $j \neq u$ we obtain two copies of the binomial $f(x_j) - x_j$ from $\pm\left((f(x_j) - x_j)^2 - (f(x_u) - x_u)^2\right) = \pm (f(x_j) - x_j)^2$ with $+$ if $j > u$ and $-$ otherwise.

Now assume neither i nor j is a fixed point. A binomial-trinomial pair of factors arises from $\left(x_{f(j)} - x_j + x_{f(i)} - x_i\right)\left(x_{f(j)} - x_j - x_{f(i)} + x_i\right)$ when $\{a, b\} = \{x_{f(j)}, x_{f(i)}\}$, $\{c, d\} = \{x_j, x_i\}$, and $j = f(i)$ or $i = f(j)$. Without loss of generality, we choose $j = f(i)$. This produces

$$\pm\left(x_{f(f(i))} - x_i\right)\left(x_{f(f(i))} + x_i - 2x_{f(i)}\right).$$

Similarly, a binomial-trinomial pair of factors arises when $\{a, b\} = \{x_{f(j)}, x_i\}, \{c, d\} = \{x_j, x_{f(i)}\}$, which implies $f(i) = f(j)$. Setting $i < j$, this produces

$$\left(2x_{f(j)} - x_j - x_i\right)(x_i - x_j).$$

We have now described all possible ways binomial factors can occur in p_f. Furthermore, a trinomial factor of p_f can only occur in a binomial-trinomial pair. Observe that in each binomial-trinomial pair, the trinomial has the form $\pm(2r - s - t)$ and the associated binomial is of the form $(s - t)$.

We now take a given polynomial p_f that is not identically zero, with no information about f except that $f : \mathbb{Z}_n \to \mathbb{Z}_n$ is a function. Define $h(x_0, \ldots, x_{n-1})$ to be the product of all the binomials that occur in binomial-trinomial pairs. That is, $s - t$ is a factor of h if and only if $2r - s - t$ is a factor of p_f for some r. Now define

$$q\left(x_0, \ldots, x_{n-1}\right)=\frac{p_f\left(x_0, \ldots, x_{n-1}\right)}{h\left(x_0, \ldots, x_{n-1}\right)},$$

which is a polynomial. Then q has no binomial factors if and only if f does not have a fixed point. Otherwise, q has $2(n-1)$ binomial factors, which occur in pairs: $(x_k - x_\ell)^2$. Then $E(G_f) = \{k\ell : (x_k - x_\ell)^2$ is a factor of $q\}$. The remaining two statements now follow from knowing G_f by Remark A.1. $\square$

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