Taylor & Francis
Taylor & Francis Group

Optimization

A Journal of Mathematical Programming and Operations Research

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/gopt20

Generalized relative interiors and generalized convexity in infinite-dimensional spaces

Vo Si Trong Long, Boris S. Mordukhovich & Nguyen Mau Nam

OPTIMIZATION
2024, VOL. 73, NO. 12, 3667–3697
https://doi.org/10.1080/02331934.2024.2356205

Generalized relative interiors and generalized convexity in infinite-dimensional spaces

Vo Si Trong Long [a,b], Boris S. Mordukhovich[c] and Nguyen Mau Nam[d]

[a]Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam; [b]Vietnam National University, Ho Chi Minh City, Vietnam; [c]Department of Mathematics, Wayne State University, Detroit, MI, USA; [d]Fariborz Maseeh Department of Mathematics and Statistics, Portland State University, Portland, OR, USA

ABSTRACT

This paper focuses on investigating generalized relative interior notions for sets in locally convex topological vector spaces with particular attentions to graphs of set-valued mappings and epigraphs of extended-real-valued functions. We introduce, study, and utilize a novel notion of *quasi-near convexity* of sets that is an infinite-dimensional extension of the widely acknowledged notion of near convexity. Quasi-near convexity is associated with the quasi-relative interior of sets, which is investigated in the paper together with other generalized relative interior notions for sets, not necessarily convex. In this way, we obtain new results on generalized relative interiors for graphs of set-valued mappings in convexity and generalized convexity settings.

ARTICLE HISTORY
Received 29 July 2023
Accepted 2 May 2024

KEYWORDS
Generalized convexity; generalized relative interiors; quasi-relative interiors; near convexity; quasi-near convexity; locally convex topological vector spaces

AMS SUBJECT CLASSIFICATIONS
49J52; 49J53; 90C31

1. Introduction

The concept of *relative interior* for convex sets is highly important in finite dimensions, as it occupies a pivotal position in convex analysis and its practical applications to, e.g. convex optimization. Recognizing its fundamental importance, there have been significant efforts to explore appropriate notions of generalized relative interior in infinite-dimensional spaces. The notions of *quasi-interior*, *strong quasi-relative interior*, *intrinsic relative interior*, and *quasi-relative interior* for convex sets in infinite dimensions have been well recognized while playing their own notable roles in various aspects of convex analysis and optimization; see [1–15] with the references and discussions therein.

Among the most significant results of finite-dimensional convex geometry, we mention Rockafellar's theorems on nonemptiness of the relative interior of a nonempty convex set in $\mathbb{R}^n$ and the relative interior representation for the graph

CONTACT Nguyen Mau Nam mnn3@pdx.edu
In memory of Diethard Pallaschke, an outstanding mathematician and unique individual

of a convex set-valued mapping; see [16, Theorem 6.8]. Married to convex separation, the latter theorem lies at the core of the *geometric approach* for generalized differentiation in convex analysis developed in [17]. This developed approach provides an easy and unified way to access many important results of convex analysis, optimization, and their applications.

The importance of the aforementioned relative interior representation in finite dimensions calls for its extension to the infinite-dimensional setting by using appropriate generalized relative interior notions. In particular, it is natural to consider representations of the quasi-interior, strong quasi-relative interior, intrinsic relative interior, and quasi-relative interior for graphs of convex set-valued mappings in infinite dimensions. To the best of our knowledge, this question has been addressed only for the case of quasi-relative interior (see [7, 8, 11]), while it remains open for the other listed cases of quasi-interior, strong quasi-relative interior, and intrinsic relative interior of convex graphs.

Another important unsolved issue in this direction is to go *beyond convexity*. Among various notions of generalized convexity for sets, the so-called *near convexity* (known also as 'almost convexity') seems to be the most natural to consider first. This notion actually goes back to Minty [18] in his study of maximal monotone operators in finite dimensions. It can be equivalently formulated as the property that the set in question is situated between a convex set and its closure, with taking into account that any (nonempty) finite-dimensional convex set has nonempty relative interior. In [19], Rockafellar extended Minty's notion and result to smoothly reflexive Banach spaces in terms of the very differently formulated notion of 'virtual convexity' with showing that the latter reduced to [18] in finite dimensions. More recently, the near convexity of sets and associated notion for functions have been consider in [20–24] but only in finite-dimensional spaces.

This paper addresses the aforementioned open questions in convex and nonconvex settings. To proceed with nonconvex sets in the general framework of *locally convex topological vector* (LCTV) spaces, we introduce a new notion of *quasi-near convexity*, which is an infinite-dimensional extension of near convexity with the usage of nonempty quasi-relative interior of convex sets in the definition of the new notion instead of (always nonempty) relative interior of convex sets in the finite-dimensional near convexity. Note that any nonempty convex set has nonempty quasi-relative interior in the case of *separable Banach* spaces; see [3]. Then we establish generalizations of Rockafellar's relative interior representation theorem for set-valued mappings with quasi-nearly convex graphs.

Our paper is structured as follows. Section 2 contains some basic notation and definitions of convex analysis broadly used in the subsequent material. Section 3 focuses on revisiting a number of important notions of generalized relative interior with further clarifications. Section 4 is devoted to the study of the intrinsic relative interior and strong quasi-relative interior of convex graphs. Section 5 introduces and investigates a new notion of quasi-near convexity for nonconvex sets. The final Section 6 provides estimates and representations of the quasi-relative interior for graphs of quasi-nearly convex set-valued mappings.

2. Preliminaries

Throughout the paper, we use standard definitions and notation, which can be founded, e.g. in [11]. Unless otherwise stated, all the spaces under consideration are real LCTV spaces. Recall that the topological dual of X is denoted by X^* with the canonical pairing $\langle x^*, x\rangle := x^*(x)$ for $x \in X$ and $x^* \in X^*$. A nonempty set $\Omega \subset X$ is a *cone* if $\lambda w \in \Omega$ for all $w \in \Omega$ and $\lambda \geq 0$. The *closure, conic hull, affine hull*, and *linear hull* of Ω are denoted by $\overline{\Omega}$, cone(Ω), aff(Ω), and span(Ω), respectively.

Let $f\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be an extended-real-valued function. The *effective domain* and *epigraph* of f are defined, respectively, by

$$\operatorname{dom}(f) := \left\{x \in X \mid f(x) < \infty\right\} \quad \text{and}$$
$$\operatorname{epi}(f) := \left\{(x, \lambda) \in X \times \mathbb{R} \mid f(x) \leq \lambda\right\}.$$

The function f is called *proper* if dom$f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. We also say that f is *convex* if epi(f) is a convex set.

Given a set-valued mapping $F\colon X \rightrightarrows Y$, define the *domain*, *range*, and *graph* of F by

$$\operatorname{dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}, \quad \operatorname{rge}(F) := \bigcup_{x\in X} F(x),$$
$$\operatorname{gph}(F) := \left\{(x, y) \in X \times Y \mid y \in F(x)\right\}.$$

If gph(F) is a convex set in $X \times Y$, then we say that the *mapping* F is *convex*.

For a function $f\colon X \to \overline{\mathbb{R}}$, define the *epigraphical mapping* $E_f\colon X \rightrightarrows \mathbb{R}$ by

$$E_f(x) := \{\alpha \in \mathbb{R} \mid f(x) \leq \alpha\}, \quad x \in X. \tag{1}$$

It is easy to verify the equalities

$$\operatorname{dom}(E_f) = \operatorname{dom}(f) \quad \text{and} \quad \operatorname{gph}(E_f) = \operatorname{epi}(f).$$

We also consider the *epigraphical range* of f given by rge(f) := rge(E_f).

Given a subset Ω of X, define its *polar* by

$$\Omega^\circ := \left\{x^* \in X^* \mid \left\langle x^*, w\right\rangle \leq 1 \text{ for all } w \in \Omega\right\}.$$

Therefore, for a subset Θ of X^* we have

$$\Theta^\circ = \left\{x \in X \mid \left\langle z^*, x\right\rangle \leq 1 \text{ whenever } z^* \in \Theta\right\}.$$

It follows from the definition that if Ω is a cone in X, then

$$\Omega^\circ = \left\{x^* \in X^* \mid \left\langle x^*, w\right\rangle \leq 0 \text{ for all } w \in \Omega\right\},$$

and if Θ is a cone in X^*, then

$$\Theta^\circ = \left\{x \in X \mid \langle z^*, x\rangle \le 0 \text{ for all } z^* \in \Theta\right\}.$$

Finally in this section, we recall the fundamental *separation properties* of sets, which are studied and applied in the subsequent sections.

Definition 2.1: Let Ω_1 and Ω_2 be two nonempty subsets of X. We say that Ω_1 and Ω_2 can be SEPARATED by a closed hyperplane if there exists $x^* \in X^* \setminus \{0\}$ such that

$$\sup\left\{\langle x^*, y\rangle \mid y \in \Omega_2\right\} \le \inf\left\{\langle x^*, x\rangle \mid x \in \Omega_1\right\}. \tag{2}$$

If it holds in addition that

$$\inf\left\{\langle x^*, y\rangle \mid y \in \Omega_2\right\} < \sup\left\{\langle x^*, x\rangle \mid x \in \Omega_1\right\}, \tag{3}$$

then we say that Ω_1 and Ω_2 can be PROPERLY SEPARATED by a closed hyperplane.

Observe that (2) can be rewritten as

$$\langle x^*, y\rangle \le \langle x^*, x\rangle \quad \text{whenever } y \in \Omega_2 \text{ and } x \in \Omega_1,$$

while (3) means that there exist $\bar{y} \in \Omega_2$ and $\bar{x} \in \Omega_1$ satisfying

$$\langle x^*, \bar{y}\rangle < \langle x^*, \bar{x}\rangle.$$

3. Extended relative interiors of sets

In this section, we first revisit the major *generalized relative interior* properties of sets in LCTV spaces that are broadly used in the literature; see, e.g. [1–4, 10–12, 14, 25, 26]. We also present here some refinements of known results in the case of arbitrary (not necessarily convex) sets.

Definition 3.1: Let Ω be a subset of X.

(a) The INTERIOR of Ω *with respect to the affine hull* $\text{aff}(\Omega)$ is the set

$$\text{rint}(\Omega) := \{x \in \Omega \mid \exists \text{ a neighbourhood } V \text{ of the origin such that } (x + V) \cap \text{aff}(\Omega) \subset \Omega\}.$$

(b) The INTERIOR of Ω *with respect to the closed affine hull* $\overline{\text{aff}}(\Omega)$ is the set

$$\text{ri}(\Omega) := \{x \in \Omega \mid \exists \text{ a neighbourhood } V \text{ of the origin such that } (x + V) \cap \overline{\text{aff}}(\Omega) \subset \Omega\}.$$

(c) The QUASI-INTERIOR of Ω is the set

$$\text{qi}(\Omega) := \{x \in \Omega \mid \overline{\text{cone}}(\Omega - x) = X\}.$$

(d) The STRONG QUASI-RELATIVE INTERIOR of Ω is the set

$$\text{sqri}(\Omega) := \{x \in \Omega \mid \text{cone}(\Omega - x) \text{ is a closed subspace of } X\}.$$

(e) The INTRINSIC RELATIVE INTERIOR of Ω is the set

$$\text{iri}(\Omega) := \{x \in \Omega \mid \text{cone}(\Omega - x) \text{ is a subspace of } X\}.$$

(f) The QUASI-RELATIVE INTERIOR of Ω is the set

$$\text{qri}(\Omega) := \{x \in \Omega \mid \overline{\text{cone}}(\Omega - x) \text{ is a subspace of } X\}.$$

If $\text{qri}(\Omega) = \text{iri}(\Omega)$, we say that Ω is QUASI-REGULAR.

The following proposition establishes a relationship between $\text{ri}(\Omega)$ and $\text{rint}(\Omega)$.

Proposition 3.2: *Let Ω be a subset of X, not necessarily convex. Then we have*

$$ri(\Omega) = \begin{cases} rint(\Omega) & \text{if } aff(\Omega) \text{ is closed,} \\ \emptyset & \text{otherwise.} \end{cases} \tag{4}$$

In particular, $ri(\Omega) \subset rint(\Omega)$, where the equality holds if $\text{ri}(\Omega) \neq \emptyset$.

Proof: If $\text{aff}(\Omega)$ is closed, then by definition $\text{ri}(\Omega) = \text{rint}(\Omega)$. Consider the case where $\text{aff}(\Omega)$ is not closed and suppose on the contrary that $\text{ri}(\Omega) \neq \emptyset$. Pick $\bar{x} \in \text{ri}(\Omega)$ and find a neighbourhood V of the origin such that

$$(\bar{x} + V) \cap \overline{\text{aff}}(\Omega) \subset \Omega. \tag{5}$$

Fix any $z \in \overline{\text{aff}}(\Omega)$ and select $\lambda > 0$ so small that $\bar{x} + \lambda(z - \bar{x}) \in \bar{x} + V$. Then we get $\bar{y} := \bar{x} + \lambda(z - \bar{x}) = \lambda z + (1 - \lambda)\bar{x} \in (\bar{x} + V) \cap \overline{\text{aff}}(\Omega)$ because $\bar{y}$ is an affine combination of $z, \bar{x} \in \overline{\text{aff}}(\Omega)$. Using (5) gives us $\bar{y} \in \Omega$. Then

$$z = \left(1 - \frac{1}{\lambda}\right)\bar{x} + \frac{1}{\lambda}\bar{y} \in \text{aff}(\Omega)$$

since z is an affine combination of $\bar{x}, \bar{y} \in \Omega$. This yields the closedness of $\text{aff}(\Omega)$, a contradiction completing the proof of (4). The last statement follows from (4). ■

The next result provides relationships between the notions of generalized relative interiors from Definition 3.1.

Theorem 3.3: *Let Ω be a subset of X, not necessarily convex. Then we have the inclusions*

$$ri(\Omega) \subset sqri(\Omega) \subset iri(\Omega) \subset qri(\Omega), \tag{6}$$

$$ri(\Omega) \subset rint(\Omega) \subset iri(\Omega). \tag{7}$$

All the inclusions above become equalities if Ω is convex with $ri(\Omega) \neq \emptyset$.

Proof: Take any $\bar{x} \in \text{ri}(\Omega)$. Then $\bar{x} \in \Omega$ and there exists a neighbourhood V of the origin such that (5) is satisfied. Let us verify that

$$\overline{\text{aff}}(\Omega) - \bar{x} = \text{cone}(\Omega - \bar{x}). \tag{8}$$

Indeed, take any $z \in \overline{\text{aff}}(\Omega) - \bar{x}$ and find $\lambda > 0$ so small that $\lambda z \in V$. Since $\overline{\text{aff}}(\Omega) - \bar{x}$ is a linear subspace, we have $\lambda z \in \overline{\text{aff}}(\Omega) - \bar{x}$ and thus

$$\bar{x} + \lambda z \in (\bar{x} + V) \cap \overline{\text{aff}}(\Omega) \subset \Omega.$$

Then $z \in \lambda^{-1}(\Omega - \bar{x}) \subset \text{cone}(\Omega - \bar{x})$, which justifies the inclusion '$\subset$' in (8). Observe that $\Omega - \bar{x} \subset \overline{\text{aff}}(\Omega) - \bar{x}$. Since the latter set is a linear subspace, we arrive at the reverse inclusion in (8).

By (8), $\text{cone}(\Omega - \bar{x})$ is a closed linear subspace of X and thus $\bar{x} \in \text{sqri}(\Omega)$, which yields $\text{ri}(\Omega) \subset \text{sqri}(\Omega)$. The inclusion $\text{sqri}(\Omega) \subset \text{iri}(\Omega)$ follows from the obvious fact that any closed linear subspace is a linear subspace. Since the closure of a linear subspace is also a linear subspace, we obtain the inclusion $\text{iri}(\Omega) \subset \text{qri}(\Omega)$ and thus complete the proof of (6).

The inclusion $\text{ri}(\Omega) \subset \text{rint}(\Omega)$ is a consequence of Proposition 3.2. Taking now any $\bar{x} \in \text{rint}(\Omega)$ and following the proof of (8) tell us that

$$\text{aff}(\Omega) - \bar{x} = \text{cone}(\Omega - \bar{x}),$$

which implies that $\text{cone}(\Omega - \bar{x})$ is a linear subspace. Hence $\bar{x} \in \text{iri}(\Omega)$ verifying (7).

Finally, assume that Ω is convex with $\text{ri}(\Omega) \neq \emptyset$. Then $\text{ri}(\Omega) = \text{qri}(\Omega)$ (see [3, 11]), which clearly implies that all the inclusions in (6) and (7) become equalities. ∎

Next, we provide an example to demonstrate that the inclusion $\text{rint}(\Omega) \subset \text{iri}(\Omega)$ is strict in general.

Example 3.4: Consider the set $\Omega := \{(x, \lambda) \in \mathbb{R}^2 \mid x^2 \leq \lambda\} \cup (\mathbb{R} \times (-\infty, 0])$. Then we have $\text{cone}(\Omega - (0,0)) = \mathbb{R}^2$, which implies that $(0,0) \in \text{iri}(\Omega)$. However, it is easy to verify that $(0,0) \notin \text{rint}(\Omega)$.

The example below shows that $\text{rint}(\Omega)$ and $\text{sqri}(\Omega)$ can differ for a convex set Ω. More examples that distinguish other notions of generalized relative interiors can be found in [2, 4].

Example 3.5: Let $X := C_{[0,1]}$ (the normed space of real continuous functions on $[0,1]$) with the 'max' norm, and let P be the set of all polynomials with real coefficients on $[0,1]$. It is well known that P is a dense subspace in X. Thus we get

$$\mathrm{rint}(P) = P \neq \emptyset = \mathrm{sqri}(P).$$

To proceed further, take an affine subset M of an LCTV space X. It is well known that there exists a unique linear subspace L of X such that

$$M = x_0 + L \quad \text{for some } x_0 \in M.$$

In this case, L is called the *linear subspace parallel to M*. We have the representation $L = M - M$; see, e.g. [11]. Consider now a nonempty subset Ω of X and get that $M := \mathrm{aff}(\Omega)$ is a nonempty affine set in X. The next result provides a representation of the linear subspace that is parallel to $\mathrm{aff}(\Omega)$.

Proposition 3.6: *Let Ω be a nonempty set in X. Then $L := \mathrm{aff}(\Omega - \Omega)$ is the linear subspace which is parallel to $\mathrm{aff}(\Omega)$.*

Proof: By the above, the linear subspace parallel to $\mathrm{aff}(\Omega)$ is $\mathrm{aff}(\Omega) - \mathrm{aff}(\Omega)$. It is an easy exercise to show that $\mathrm{aff}(\Omega) - \mathrm{aff}(\Omega) = \mathrm{aff}(\Omega - \Omega) = L$. This completes the proof. ■

The following lemma presents a straightforward result involving the convexity of the conic hull of a set Ω given by $\mathrm{cone}(\Omega) := \{\lambda x \mid \lambda \geq 0,\ x \in \Omega\}$.

Lemma 3.7: *If Ω is a convex set in X, then so is cone* (Ω).

Although some results of the next three propositions can be distilled from [14], we prefer for the reader's convenience and completeness to give here their simplified proofs with commentaries.

Proposition 3.8: *If Ω is a nonempty convex set in X, then*

$$span(\Omega - w) = cone(\Omega - \Omega)$$

whenever $w \in \Omega$.

Proof: Since $\Omega - \Omega$ is convex, by Lemma 3.7 the set $\mathrm{cone}(\Omega - \Omega)$ is a convex cone. It is indeed a linear subspace because if $x \in \mathrm{cone}(\Omega - \Omega)$, then $-x \in \mathrm{cone}(\Omega - \Omega)$. For any $w \in \Omega$, we have $\Omega - w \subset \mathrm{cone}(\Omega - \Omega)$ and thus $\mathrm{span}(\Omega - w) \subset \mathrm{cone}(\Omega - \Omega)$.

To verify the reverse inclusion, fix any $x \in \text{cone}(\Omega - \Omega)$ and get the representation

$$x = \lambda(w_1 - w_2) \quad \text{where } \lambda \geq 0, \text{ and } w_1, w_2 \in \Omega.$$

Then $x = \lambda(w_1 - w) - \lambda(w_2 - w) \in \text{span}(\Omega - w)$. This justifies the reverse inclusion and completes the proof. ■

Proposition 3.9: *Let Ω be a nonempty convex set in X, and let $w \in \Omega$. Then we have:*

(a) *$\text{aff}(\Omega) - w = \text{cone}(\Omega - \Omega)$.*
(b) *$\text{cone}(\Omega - \Omega)$ is the linear subspace parallel to $\text{aff}(\Omega)$.*
(c) *$\text{aff}(\Omega) - w = \text{cone}(\Omega - w)$ if and only if $\text{cone}(\Omega - w)$ is a linear subspace of X.*
(d) *$\overline{\text{aff}}(\Omega) - w = \overline{\text{cone}}(\Omega - w)$ if and only if $\overline{\text{cone}}(\Omega - w)$ is a linear subspace of X.*

Proof: (a) It follows from Proposition 3.8 that

$$\text{cone}(\Omega - \Omega) = \text{span}(\Omega - w).$$

Since the set $\text{aff}(\Omega - w)$ is a linear subspace that contains $\Omega - w$, we see that

$$\text{cone}(\Omega - \Omega) = \text{span}(\Omega - w) \subset \text{aff}(\Omega - w) = \text{aff}(\Omega) - w.$$

Observe that $\Omega - w \subset \Omega - \Omega \subset \text{cone}(\Omega - \Omega)$, where the last set is a linear subspace and hence an affine set. Thus we have the inclusion $\text{aff}(\Omega - w) \subset \text{cone}(\Omega - \Omega)$, which completes the proof of assertion (a).

(b) This assertion follows directly from (a).

(c) If $\text{aff}(\Omega) - w = \text{cone}(\Omega - w)$, it is obvious that $\text{cone}(\Omega - w)$ is a linear subspace since $\text{aff}(\Omega) - w$ has this property. To verify the converse implication, suppose that $\text{cone}(\Omega - w)$ is a linear subspace. Observe by (a) that

$$\text{cone}(\Omega - w) \subset \text{cone}(\Omega - \Omega) = \text{aff}(\Omega) - w.$$

Take further any $x \in \text{aff}(\Omega) - w = \text{cone}(\Omega - \Omega)$ and find $\lambda \geq 0$ and $w_1, w_2 \in \Omega$ such that

$$x = \lambda(w_1 - w_2) = \lambda(w_1 - w) + \lambda(w - w_2).$$

Since $w_2 - w \in \text{cone}(\Omega - w)$, where the latter set is a linear subspace, we see that $w - w_2 \in \text{cone}(\Omega - w)$. This shows that $x \in \text{cone}(\Omega - w)$ and hence justifies (c)

(d) Assume that $\overline{\text{cone}}(\Omega - w)$ is a linear subspace of X. It follows from (a) that

$$\overline{\text{cone}}(\Omega - w) \subset \overline{\text{cone}}(\Omega - \Omega) = \overline{\text{aff}(\Omega) - w} = \overline{\text{aff}}(\Omega) - w.$$

To verify the reverse inclusion, it suffices to show that $\text{cone}(\Omega - \Omega) \subset \overline{\text{cone}}(\Omega - w)$. Take any $x \in \text{cone}(\Omega - \Omega)$ and find $\lambda \geq 0$, $w_1, w_2 \in \Omega$ such that

$$x = \lambda(w_1 - w_2) = \lambda(w_1 - w) + \lambda(w - w_2).$$

Similarly to the proof of (c), we see that $w - w_2 \in \overline{\text{cone}}(\Omega - w)$, which implies that x belongs to the set $\overline{\text{cone}}(\Omega - w)$ since the latter is a linear subspace. This completes the proof of (d) by taking into account that the other implication is obvious. ■

The relationship between the strong quasi-relative interior and the intrinsic relative interior of a convex set is established next.

Proposition 3.10: *Let Ω be a nonempty convex set in X. Then we have the representation*

$$sqri(\Omega) = \begin{cases} iri(\Omega) & \text{if } aff(\Omega) \text{ is closed,} \\ \emptyset & \text{otherwise.} \end{cases} \tag{9}$$

Proof: Suppose that $\text{aff}(\Omega)$ is closed. Taking any $\bar{x} \in \text{iri}(\Omega)$, it follows from definition that $\text{cone}(\Omega - \bar{x})$ is a linear subspace of X. By Proposition 3.9(c) we have $\text{cone}(\Omega - \bar{x}) = \text{aff}(\Omega) - \bar{x}$, and so $\text{cone}(\Omega - \bar{x})$ is a closed linear subspace of X. Thus $\bar{x} \in \text{sqri}(\Omega)$, which implies that $\text{iri}(\Omega) \subset \text{sqri}(\Omega)$. Since the reverse inclusion follows from Theorem 3.3, we justify the equality $\text{sqri}(\Omega) = \text{iri}(\Omega)$ in this case.

Assume now that $\text{aff}(\Omega)$ is not closed. Arguing by contradiction, suppose that $\text{sqri}(\Omega)$ is nonempty and pick $\bar{x} \in \text{sqri}(\Omega)$. Then $\text{cone}(\Omega - \bar{x})$ is a closed linear subspace of X. It follows from Proposition 3.9(c) that $\text{aff}(\Omega) - \bar{x} = \text{cone}(\Omega - \bar{x})$ is also a closed linear subspace of X. Hence the affine hull $\text{aff}(\Omega)$ is closed, which is a contradiction verifying that the set $\text{sqri}(\Omega)$ is empty in this case. ■

The next proposition shows that the notions of generalized relative interiors in Definition 3.1 for convex sets agree with those taken from [2] under different names.

Proposition 3.11: *Let Ω be a convex set in X. We have the following assertions:*

(a) $sqri(\Omega) = \{x \in \Omega \mid cone(\Omega - x) = \overline{span}(\Omega - x)\}$.
(b) $iri(\Omega) = \{x \in \Omega \mid cone(\Omega - x) = span(\Omega - x)\}$.
(c) $qri(\Omega) = \{x \in \Omega \mid \overline{cone}(\Omega - x) = \overline{span}(\Omega - x)\}$.

Proof: (a) Suppose that $x \in \text{sqri}(\Omega)$. Then $x \in \Omega$ and $\text{cone}(\Omega - x)$ is a closed linear subspace. By Propositions 3.8 and 3.9(a,c) we have

$$\text{cone}(\Omega - x) = \text{aff}(\Omega) - x = \text{cone}(\Omega - \Omega) = \text{span}(\Omega - x).$$

It follows that

$$\text{cone}(\Omega - x) = \overline{\text{cone}}(\Omega - x) = \overline{\text{span}}(\Omega - x).$$

This justifies the inclusion '$\subset$' of the equality in (a). The reverse inclusion follows directly from the definition of strong quasi relative interior.

(b) Suppose that $x \in \text{iri}(\Omega)$. Then by definition $\text{cone}(\Omega - x)$ is a linear subspace of X. Using Propositions 3.8 and 3.9(a,c) as in the proof of assertion (a), we get

$$\text{cone}(\Omega - x) = \text{span}(\Omega - x).$$

This implies the inclusion '$\subset$' in (b). The reverse inclusion is obvious.

(c) Assume finally that $x \in \text{qri}(\Omega)$ and get that $\overline{\text{cone}}(\Omega - x)$ is a linear subspace. Using again Propositions 3.8 and 3.9(a,d) gives us

$$\overline{\text{cone}}(\Omega - x) = \overline{\text{aff}}(\Omega) - x = \overline{\text{cone}}(\Omega - \Omega) = \overline{\text{span}}(\Omega - x).$$

This justifies the inclusion '$\subset$' in (c). The reverse inclusion follows from the definition of quasi-relative interior and the fact that $\overline{\text{span}}(\Omega - x)$ is a linear subspace. ■

The following result, which extends [3, Lemma 2.3] from convex sets to arbitrary sets in LCTV spaces, plays an important role in representations of generalized relative interiors for graphs of set-valued mappings.

Proposition 3.12: *Let Ω be a subset of X, and let $\bar{x} \in \Omega$. Then $\bar{x} \in iri(\Omega)$ if and only if for each $x \in \Omega$ there exists $x_0 \in \Omega$ such that $\bar{x} = (1 - t_0)x + t_0 x_0$ with some $t_0 \in (0, 1)$.*

Proof: Suppose $\bar{x} \in \text{iri}(\Omega)$. Fix any $x \in \Omega$. If $x = \bar{x}$, we immediately get the conclusion. Otherwise, since $\text{cone}(\Omega - \bar{x})$ is a linear subspace, $\bar{x} - x \in \text{cone}(\Omega - \bar{x})$. Thus there exist $\lambda_0 > 0$ and $x_0 \in \Omega$ such that $\bar{x} - x = \lambda_0(x_0 - \bar{x})$, which reads as $\bar{x} = \frac{1}{1+\lambda_0}x + \frac{\lambda_0}{1+\lambda_0}x_0$. Setting $t_0 := \frac{\lambda_0}{1+\lambda_0} \in (0, 1)$, we obtain $\bar{x} = (1 - t_0)x + t_0 x_0$.

To verify the converse implication, pick any nonzero vector $v \in \text{cone}(\Omega - \bar{x})$. Then we find $\lambda_0 > 0$ and $x \in \Omega$ such that $v = \lambda_0(x - \bar{x})$. By the assumption, there exists $x_0 \in \Omega$ with $\bar{x} = (1 - t_0)x + t_0 x_0$ for some $t_0 \in (0, 1)$. It follows that

$$-v = \lambda_0 \left(\bar{x} - \frac{\bar{x} - t_0 x_0}{1 - t_0} \right) = \frac{\lambda_0 t_0}{1 - t_0} (x_0 - \bar{x}).$$

Consequently, $-v \in \text{cone}(\Omega - \bar{x})$, which completes the proof. ■

Proposition 3.13: *Let Ω be a nonempty convex set. If $\bar{x} \in sqri(\Omega)$ ($ri(\Omega)$, $iri(\Omega)$, $qri(\Omega)$, respectively) and $x_0 \in \Omega$, then for every $t \in [0, 1)$ we have the inclusions $(1-t)\bar{x} + tx_0 \in sqri(\Omega)$ ($ri(\Omega)$, $iri(\Omega)$, $qri(\Omega)$, respectively).*

Proof: We only provide the proof for the case of sqri(Ω). The proofs for the other cases can be found in [3, 11]. Let $w_t = (1-t)\bar{x} + tx_0$ for $0 \le t < 1$. Then

$$\Omega - \Omega \supset \Omega - w_t = \Omega - (1-t)\bar{x} - tx_0 = (1-t)(\Omega - \bar{x}) + t(\Omega - x_0)$$
$$\supset (1-t)(\Omega - \bar{x}).$$

It follows therefore that

$$\overline{\text{cone}}(\Omega - \Omega) \supset \text{cone}(\Omega - w_t) \supset \text{cone}(\Omega - \bar{x}).$$

As seen in the proof of Proposition 3.11(a), we have

$$\overline{\text{cone}}(\Omega - \Omega) = \text{cone}(\Omega - w_t) = \text{cone}(\Omega - \bar{x}),$$

which implies by definition that $w_t \in$ sqri (Ω) and thus completes the proof of the proposition. ■

Proposition 3.14: *Let $T\colon X \to Y$ be a linear mapping, and let Ω be a subset of X. Then*

$$T(iri(\Omega)) \subset iri(T(\Omega)),$$

where the equality holds if Ω is convex and $iri(\Omega) \neq \emptyset$.

Proof: Fixing any $\bar{x} \in$ iri(Ω), we get that cone$(\Omega - \bar{x})$ is a linear subspace of X. It follows from the linearity of T that the set cone$(T(\Omega) - T(\bar{x})) = T($cone$(\Omega - \bar{x}))$ is a linear subspace of Y. Thus $T(\bar{x}) \in$ iri$(T(\Omega))$, which justifies the claimed inclusion.

Now assume that iri$(\Omega) \neq \emptyset$ and fix $\bar{x} \in$ iri(Ω). Letting $\bar{y} = T(\bar{x})$, observe that $\bar{y} \in$ iri$(T(\Omega))$. Picking any $y \in$ iri$(T(\Omega)$, we deduce from Proposition 3.12 that there exists $y_0 \in T(\Omega)$ such that $y = (1-t_0)\bar{y} + t_0 y_0$ for some $t_0 \in (0, 1)$. Choose $x_0 \in \Omega$ with $y_0 = T(x_0)$. Then

$$y = (1-t_0)\bar{y} + t_0 y_0 = (1-t_0)T(\bar{x}) + t_0 T(x_0) = T((1-t_0)\bar{x} + t_0 x_0).$$

Letting further $x := (1-t_0)\bar{x} + t_0 x_0$ and taking into account the assumed convexity of Ω allow us to apply Proposition 3.13 and conclude that $x \in$ iri(Ω). Thus $y \in T($iri$(\Omega))$, which yields the reverse inclusion and hence completes the proof of the proposition. ■

To study below images of strong quasi-relative interiors of sets under linear mappings, we need the following definition.

Definition 3.15: A linear mapping $T\colon X \to Y$ is said to be SUBSPACE CLOSED if it maps any closed subspace of X to a closed subspace of Y.

First, we present an easy verifiable sufficient condition for subspace closedness.

Proposition 3.16: *Let X be a Banach space, and let Y be a normed space. Consider a continuous linear mapping $T\colon X \to Y$. Suppose that there exists $\gamma > 0$ such that*

$$\gamma \|x\| \le \|T(x)\| \quad \text{for all } x \in X.$$

Then the mapping T is subspace closed.

Proof: Take any convergent sequence $\{y_n\} \subset T(Z)$, where Z is a closed subspace of X. Then $y_n = T(x_n)$ for $x_n \in Z$, $n \in \mathbb{N}$. We have

$$\gamma \|x_m - x_n\| \le \|T(x_m) - T(x_n)\| = \|y_m - y_n\| \to 0 \quad \text{as } m, n \to \infty.$$

This yields $\{x_n\}$ is a Cauchy sequence, and hence it converges to $x_0 \in Z$ due to the completeness of X. Then $y_n = T(x_n) \to T(x_0) \in T(Z)$ as $n \to \infty$, which verifies that $T(Z)$ is closed. ■

In the next proposition, we use the notion of *strong quasi-regularity* for a nonempty set $\Omega \subset X$ meaning that $\text{sqri}(\Omega) = \text{iri}(\Omega)$.

Proposition 3.17: *Let $T\colon X \to Y$ be a continuous linear mapping which is subspace closed, and let Ω be a nonempty convex set in X. Then we have*

$$T(sqri(\Omega)) \subset sqri(T(\Omega)).$$

If in addition $sqri(\Omega) \neq \emptyset$ and Ω is strongly quasi-regular, then the reverse inclusion holds.

Proof: Suppose that $w \in \text{sqri}(\Omega)$. Then by Proposition 3.11(a) we have

$$\text{cone}(\Omega - w) = \overline{\text{span}}\,(\Omega - w).$$

Therefore, we obtain the equalities

$$\begin{aligned}\text{cone}(T(\Omega) - T(w)) &= T(\text{cone}(\Omega - w)) = T(\overline{\text{span}}\,(\Omega - w))\\ &= \overline{\text{span}}(T(\Omega) - T(w)),\end{aligned}$$

which tell us that $T(w) \in \text{sqri}(T(\Omega))$. To verify the reverse inclusion, assume that $\text{sqri}(\Omega) \neq \emptyset$ and Ω is strongly quasi-regular. Then Proposition 3.14 ensures that

$$\text{sqri}(T(\Omega)) \subset \text{iri}(T(\Omega)) = T(\text{iri}(\Omega)) = T(\text{sqri}(\Omega)),$$

which thus completes the proof of the claimed result. ■

4. Generalized relative interiors of graphical sets

This section is devoted to extending Rockafellar's relative interior representation theorem in $\mathbb{R}^n$ to generalized relative interiors of graphs of set-valued mappings and epigraphs of extended-real-valued functions.

We start with considering *intrinsic relative interiors* for graphs of set-valued mappings.

Theorem 4.1: *Let $F\colon X \rightrightarrows Y$ be a set-valued mapping. Then*

$$iri(gph(F)) \subset \left\{(x,y) \in X \times Y \mid x \in iri(dom(F)), y \in iri(F(x))\right\}. \tag{10}$$

Assuming in addition that F is convex, we have the equality

$$iri(gph(F)) = \left\{(x,y) \in X \times Y \mid x \in iri(dom(F)), y \in iri(F(x))\right\}.$$

Proof: To verify the inclusion '$\subset$', take any $(\bar{x},\bar{y}) \in \text{iri}(\text{gph}(F))$. Considering the *projection mapping $\mathcal{P}\colon X \times Y \to X$ given by*

$$\mathcal{P}(x,y) = x,\ (x,y) \in X \times Y,$$

gives us $\mathcal{P}(gph(F)) = dom(F)$. Using Proposition 3.14, we have

$$\mathcal{P}(\text{iri}(\text{gph}(F))) \subset \text{iri}(\text{dom}(F)), \tag{11}$$

which implies that $\bar{x} \in iri(dom(F))$. To show next that $\bar{y} \in \text{iri}(F(\bar{x}))$, fix any $y \in F(\bar{x})$ telling us that $(\bar{x},y) \in \text{gph}(F)$. By Proposition 3.12, find $(x_0,y_0) \in \text{gph}(F)$ such that

$$(\bar{x},\bar{y}) = (1-t_0)(\bar{x},y) + t_0(x_0,y_0) \quad \text{for some } t_0 \in (0,1).$$

This yields $x_0 = \bar{x}$ and $\bar{y} = (1-t_0)y + t_0y_0$ with $y_0 \in F(\bar{x})$. It follows therefore by Proposition 3.12 that $\bar{y} \in \text{iri}(F(\bar{x}))$, which completes the proof of the inclusion '$\subset$'.

To verify now the reverse inclusion under the convexity of $\text{gph}(F)$, fix $\bar{x} \in \text{iri}(\text{dom}(F))$ and $\bar{y} \in \text{iri}(F(\bar{x}))$. Arguing by contradiction, suppose that $(\bar{x},\bar{y}) \notin \text{iri}(\text{gph}(F))$. Then Proposition 3.12 yields the existence of $(x',y') \in \text{gph}(F)$ such that

$$(\bar{x},\bar{y}) \neq (1-t)(x',y') + t(x,y) \quad \text{for all } (x,y) \in \text{gph}(F) \text{ and } t \in (0,1). \tag{12}$$

Consider the following two cases:

(A) $\bar{x} \neq (1-t)x' + tx$ for all $(x,t) \in \text{dom}(F) \times (0,1)$,
(B) $\bar{x} = (1-t_0)x' + t_0x_0$ for some $(x_0,t_0) \in \text{dom}(F) \times (0,1)$.

Let us show below that in each of these cases we arrive at a contradiction. In case (A), we clearly have a contradiction due to $\bar{x} \in \text{iri}(\text{dom}(F))$.

In case (B), we distinguish two subcases:

(B_1) $x' = \bar{x}$. Since $\bar{x} = (1-t_0)x' + t_0x_0$, we get $\bar{x} = x' = x_0$ and hence $F(\bar{x}) = F(x') = F(x_0)$. Then it follows from (12) that $\bar{y} \neq (1-t)y' + ty$ for all $(y,t) \in F(\bar{x}) \times (0,1)$. By Proposition 3.12 we have $\bar{y} \notin \text{iri}(F(\bar{x}))$, which gives us a contradiction.

(B_2) $x' \neq \bar{x}$. Note first that since $x_0 \in \text{dom}(F)$, there exists $y_0 \in Y$ such that $(x_0, y_0) \in \text{gph}(F)$. Using the convexity of gph(F) ensures that

$$(1-t_0)(x',y') + t_0(x_0,y_0) \in \text{gph}(F).$$

Define further the vector

$$\bar{y}' := (1-t_0)y' + t_0y_0.$$

Since $\bar{x} = (1-t_0)x' + t_0x_0$, we get $\bar{y}' \in F(\bar{x})$ and

$$(\bar{x},\bar{y}') = (1-t_0)(x',y') + t_0(x_0,y_0). \tag{13}$$

On the other hand, it follows from $\bar{y} \in \text{iri}(F(\bar{x}))$ and Proposition 3.12 that there exists $y_0' \in F(\bar{x})$ such that $\bar{y} = (1-t_0')\bar{y}' + t_0'y_0'$ for some $t_0' \in (0,1)$. This together with (13) ensures the equalities

$$\begin{aligned}(\bar{x},\bar{y}) &= (1-t_0')(\bar{x},\bar{y}') + t_0'(\bar{x},y_0') \\ &= (1-t_0')[(1-t_0)(x',y') + t_0(x_0,y_0)] + t_0'(\bar{x},y_0') \\ &= (1-t_0')(1-t_0)(x',y') + (1-t_0')t_0(x_0,y_0) + t_0'(\bar{x},y_0') \\ &= (1-t_0')(1-t_0)(x',y') + (t_0 - t_0't_0 + t_0') \\ &\quad \times \left[\frac{(t_0 - t_0't_0)}{(t_0 - t_0't_0 + t_0')}(x_0,y_0) + \frac{t_0'}{(t_0 - t_0't_0 + t_0')}(\bar{x},y_0')\right] \\ &= (1-s)(x',y') + s(x'',y''),\end{aligned}$$

where $s := t_0 - t_0't_0 + t_0' \in (0,1)$ and

$$(x'',y'') := \frac{(t_0 - t_0't_0)}{(t_0 - t_0't_0 + t_0')}(x_0,y_0) + \frac{t_0'}{(t_0 - t_0't_0 + t_0')}(\bar{x},y_0') \in \text{gph}(F).$$

This also gives us a contradiction due to (12), which therefore completes the proof of the theorem. ■

Remark 4.2: If in the setting of Theorem 4.1 an additional assumption $\text{iri}(\text{gph}(F)) \neq \emptyset$ is imposed, then we have the following *alternative simple proof* for the reverse inclusion of (10). Fix $\bar{x} \in \text{iri}(\text{dom}(F))$ and $\bar{y} \in \text{iri}(F(\bar{x}))$. By (11) which holds as an equality under the convexity of gph(F), we find $y_0 \in F(\bar{x})$ with $(\bar{x}, y_0) \in \text{iri}(\text{gph}(F))$. If $y_0 = \bar{y}$, then $(\bar{x},\bar{y}) \in \text{iri}(\text{gph}(F))$ and we are done. Otherwise, since $\bar{y} \in \text{iri}(F(\bar{x}))$, we deduce from [3, Lemma 3.1] that there exists $\gamma > 0$

such that $(1+\gamma)\bar{y}-\gamma y_0 \in F(\bar{x})$. Then it follows from the corresponding result of Proposition 3.13 that

$$(1-t)(\bar{x}, y_0) + t(\bar{x}, (1+\gamma)\bar{y}-\gamma y_0) \in \text{iri}(\text{gph}(F)) \quad \text{for all } t \in [0,1).$$

Choosing $t := \frac{1}{1+\gamma} \in (0,1)$, we obtain

$$(\bar{x}, \bar{y}) = \left(1 - \frac{1}{1+\gamma}\right)(\bar{x}, y_0) + \frac{1}{1+\gamma}\left(\bar{x}, (1+\gamma)\bar{y}-\gamma y_0\right) \in \text{iri}(\text{gph}(F)),$$

which completes the alternative proof of Theorem 4.1.

To proceed further, we need one more useful result.

Lemma 4.3: *Let $F: X \rightrightarrows Y$ be a set-valued mapping. Suppose that $int(F(x)) \neq \emptyset$ for all $x \in dom(F)$. Then we have the equality*

$$\mathit{aff}(\mathit{gph}(F)) = \mathit{aff}(\mathit{dom}(F)) \times Y.$$

Proof: The inclusion '$\subset$' is obvious. Let us verify the reverse one. Take any $(x_0, y_0) \in \text{aff}(\text{dom}(F)) \times Y$, which yields $x_0 \in \text{aff}(\text{dom}(F))$ and $y_0 \in Y$. Choose $\lambda_i \in \mathbb{R}$ and $x_i \in \text{dom}(F)$ for $i = 1, \ldots, m$ such that $\sum_{i=1}^m \lambda_i = 1$ and $x_0 = \sum_{i=1}^m \lambda_i x_i$. Then select $y_i \in \text{int}(F(x_i))$ and define $\hat{y} := \sum_{i=1}^m \lambda_i y_i$ while having in this way that $(x_0, \hat{y}) \in \text{aff}(\text{gph}(F))$. We also choose $\epsilon > 0$ so small that

$$y_i + \epsilon(y_0 - \hat{y}) \in F(x_i) \quad \text{for all } i = 1, \ldots, m,$$

which yields $(x_0, \hat{y} + \epsilon(y_0 - \hat{y})) \in \text{aff}(\text{gph}(F))$, or equivalently

$$(0, \epsilon(y_0 - \hat{y})) \in \text{aff}(\text{gph}(F)) - (x_0, \hat{y}).$$

Note that since $(x_0, \hat{y}) \in \text{aff}(\text{gph}(F))$, the affine hull $\text{aff}(\text{gph}(F)) - (x_0, \hat{y})$ is a linear subspace. This gives us the inclusion

$$\begin{aligned}(0, y_0 - \hat{y}) = \frac{1}{\epsilon}(0, \epsilon(y_0 - \hat{y})) &\in \frac{1}{\epsilon}\left(\text{aff}(\text{gph}(F)) - (x_0, \hat{y})\right) \\ &= \text{aff}(\text{gph}(F)) - (x_0, \hat{y}),\end{aligned}$$

and tells us therefore that

$$(x_0, y_0) = (x_0, \hat{y}) + (0, y_0 - \hat{y}) \in \text{aff}(\text{gph}(F)) - (x_0, \hat{y}) + (x_0, \hat{y}) = \text{aff}(\text{gph}(F))$$

and thus completes the proof of the lemma. ■

The next theorem provides a new representation of *strong quasi-relative interiors* for graphs of convex set-valued mappings.

Theorem 4.4: *Let $F\colon X \rightrightarrows Y$ be a convex set-valued mapping. Then we have*

$$sqri(gph(F)) = \{(x,y) \mid x \in sqri(dom(F)),\ y \in sqri(F(x)\}, \tag{14}$$

provided that $int(F(x)) \neq \emptyset$ for all $x \in dom(F)$.

Proof: Take any $(x,y) \in \text{sqri}(\text{gph}(F))$. Then it follows from Proposition 3.10 and Lemma 4.3 that the affine hull $\text{aff}(\text{gph}(F)) = \text{aff}(\text{dom}(F)) \times Y$ is closed and that $(x,y) \in \text{iri}(\text{gph}(F))$. By Theorem 4.1, we have $x \in \text{iri}(\text{dom}(F))$ and $y \in \text{iri}(F(x))$. Since $\text{aff}(\text{dom}(F))$ is closed, using Proposition 3.10 again gives us $x \in \text{sqri}(\text{dom}F)$. In addition, $y \in \text{iri}(F(x)) = \text{sqri}(F(x))$ since $\text{int}(F(x)) \neq \emptyset$. This justifies the inclusion '$\subset$' in (14).

Conversely, take $x \in \text{sqri}(\text{dom}(F))$ and $y \in \text{sqri}(F(x))$. Then $\text{aff}(\text{dom}(F))$ is closed and $x \in \text{iri}(\text{dom}(F))$. Since $\text{int}(F(x)) \neq \emptyset$, we see that $y \in \text{iri}(F(x))$. It follows from Theorem 4.1 and Lemma 4.3 that $(x,y) \in \text{iri}(\text{gph}F)$ and that $\text{aff}(\text{gph}(F))$ is closed. This yields $(x,y) \in \text{sqri}(\text{gph}(F))$, which completes the proof. ■

The following important statement is a consequence of both Theorems 4.1 and 4.4 applying to *epigraphs* of extended-real-valued convex functions.

Proposition 4.5: *Let $f\colon X \to \overline{\mathbb{R}}$ be an extended-real-valued convex function. Then we have the generalized relative interior representations*

$$iri(epif) = \left\{(x,\alpha) \in X \times \mathbb{R} \mid x \in iri(domf), f(x) < \alpha\right\},$$
$$sqri(epi(f)) = \left\{(x,\alpha) \in X \times \mathbb{R} \mid x \in sqri(dom(f)), f(x) < \alpha\right\}.$$

Proof: Consider the epigraphical mapping E_f associated with f given in (1). Then the first formula follows from Theorem 4.1. Since $\text{int}(E_f(x))$ is nonempty for all $x \in \text{dom}(f) = \text{dom}(E_f)$, the second formula follows from Theorem 4.4. ■

To conclude this section, we provide a result on the representation of *strong quasi-relative interiors* for *ideally convex* graphs of set-valued mappings, the notion taken from [27].

Definition 4.6: A subset Ω of X is called IDEALLY CONVEX if for any bounded sequence $\{x_n\} \subset \Omega$ and sequence $\{\lambda_n\}$ of nonnegative numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$, the series $\sum_{n=1}^{\infty} \lambda_n x_n$ either converges to an element of Ω, or does not converge at all.

As observed in [27], any convex subset of X, which is either open or closed, is ideally convex. Moreover, every finite-dimensional convex set is ideally convex.

Proposition 4.7: *Let $F\colon X \rightrightarrows Y$ be a convex set-valued mapping. Assume that $gph(F)$ is an ideally convex subset of the product space $X \times Y$ and that $sqri(gph(F)) \neq \emptyset$. Then*

$$sqri(gph(F)) = \{(x, y) \in X \times Y \mid x \in sqri(dom(F)), y \in sqri(F(x))\}.$$

Proof: Combining Theorem 4.1 with [28, Proposition 3.2] verifies the result. ■

5. Generalized relative interiors of quasi-nearly convex sets

In this section, we introduce a new notion of generalized convexity called *quasi-near convexity*, investigate its basic properties, and establish its relationship with the near convexity or almost convexity notions known in the literature; see, e.g. [20–22, 24]).

Definition 5.1: Let Ω be a nonempty subset of X. We say that Ω is QUASI-NEARLY CONVEX if there exists a convex subset $C \subset X$ such that $\text{qri}(C) \neq \emptyset$ and

$$C \subset \Omega \subset \overline{C}.$$

Remark 5.2: If Ω is a nonempty convex subset of X with $\text{qri}(\Omega) \neq \emptyset$, then Ω QUASI-NEARLY CONVEX.

The next proposition shows that the notion of quasi-near convexity agrees with the near convexity in finite dimensions; see, e.g. [16, 18–22].

Proposition 5.3: *Let Ω be a nonempty set in $\mathbb{R}^n$. The following properties are equivalent:*

(a) *Ω is quasi-nearly convex.*
(b) *There exists a convex set $C \subset \mathbb{R}^n$ such that*

$$ri(C) \subset \Omega \subset \overline{C}.$$

(c) *There exists a convex set $D \subset \mathbb{R}^n$ such that*

$$D \subset \Omega \subset \overline{D}.$$

(d) *$\overline{\Omega}$ is convex and $ri(\overline{\Omega}) \subset \Omega$.*

Proof: (a)$\Longrightarrow$(b): Obvious.

(b)$\Longrightarrow$(c): Suppose that (b) is satisfied. Define the convex set $D := \text{ri}(C)$. Then $\overline{D} = \overline{\text{ri}}(C) = \overline{C}$, and thus

$$D \subset \Omega \subset \overline{D}.$$

(c)$\Longrightarrow$(d): Suppose that (c) is satisfied. It can be easily checked that $\overline{\Omega} = \overline{D}$, and hence $\overline{\Omega}$ is a convex set. In addition, we have

$$\text{ri}(\overline{\Omega}) = \text{ri}(\overline{D}) = \text{ri}(D) \subset D \subset \Omega,$$

which ensures that (d) is satisfied.

(d)$\Longrightarrow$(a): Let (d) hold. Defining $C := \text{ri}(\overline{\Omega})$, which is nonempty convex set, we get

$$C = \text{ri}(\overline{\Omega}) \subset \Omega \subset \overline{\Omega} = \overline{\text{ri}}(\overline{\Omega}) = \overline{C}.$$

Since $\text{qri}(C) = \text{ri}(C) = \text{ri}(\text{ri}(\overline{\Omega}) = \text{ri}(\overline{\Omega}) \neq \emptyset$, it follows that (a) is satisfied. ■

Some basic properties of quasi-nearly convex sets are established below.

Proposition 5.4: *Let Ω be a quasi-nearly convex set with $C \subset \Omega \subset \overline{C}$, where C is a convex subset of X satisfying the condition $qri(C) \neq \emptyset$. Then we have the following:*

(a) $\overline{\Omega} = \overline{C}$.
(b) $qri(C) \subset qri(\Omega) \subset qri(\overline{C})$.

Consequently, the set $qri(\Omega)$ is quasi-nearly convex. Moreover, the set $qri(\Omega)$ is convex if $qri(\overline{\Omega}) \subset \Omega$, which holds when $X = \mathbb{R}^n$.

Proof: (a) The conclusion is an immediate consequence of the definition.

(b) For each $x \in X$, we have

$$\text{cone}(C - x) \subset \text{cone}(\Omega - x) \subset \text{cone}(\overline{C} - x) \subset \overline{\text{cone}}(C - x),$$

which implies in turn the equalities

$$\overline{\text{cone}}(C - x) = \overline{\text{cone}}(\Omega - x) = \overline{\text{cone}}(\overline{C} - x).$$

The latter brings us to the inclusions

$$\text{qri}(C) \subset \text{qri}(\Omega) \subset \text{qri}(\overline{C}),$$

which completes the proof of (b).

We therefore have

$$\text{qri}(C) \subset \text{qri}(\Omega) \subset \text{qri}(\overline{C}) \subset \overline{\text{qri}}(\overline{C}) = \overline{C} = \overline{\text{qri}}(C),$$

where the equalities hold by Borwein and Lewis [4, Proposition 2.12]. Moreover, we get that $\text{qri}(C)$ is convex by Borwein and Lewis [4, Lemma 2.9] and that

$\text{qri}(\text{qri}(C)) = \text{qri}(C) \neq \emptyset$ by Boț et al. [6, Proposition 2.5(vii)]. Consequently, the set $\text{qri}(\Omega)$ is quasi-nearly convex. If $\text{qri}(\overline{\Omega}) \subset \Omega$, we can easily check that $\text{qri}(\overline{C}) = \text{qri}(\overline{\Omega}) \subset \text{qri}(\Omega)$ by definition. This implies that $\text{qri}(\Omega) = \text{qri}(\overline{\Omega}) = \text{qri}(\overline{C})$, which is a convex set. This completes the proof of the proposition. ∎

The following example shows that each inclusion in Proposition 5.4(b) can be strict.

Example 5.5: Let X and P be the given as in Example 3.5. Consider an element $\bar{x} \in X \setminus P$ and let $\Omega := P \cup \{\bar{x}\}$. Then Ω is *not convex* but *quasi-nearly convex*, and we have

$$\text{qri}(P) = P \subset \text{qri}(\Omega) = \Omega \subset \text{qri}(\overline{P}) = X.$$

Some other useful properties of quasi-nearly convex sets are collected in the next proposition.

Proposition 5.6: *Let Ω be a quasi-nearly convex set in X, and let $\bar{x} \in X$. Then we have the following properties:*

(a) $qi(\Omega) = \Omega \cap qi(\overline{\Omega})$ and $qri(\Omega) = \Omega \cap qri(\overline{\Omega})$.
(b) $\overline{qri}(\Omega) = \overline{\Omega}$.
(c) $qri(\Omega) = qri(qri(\Omega))$.
(d) $qri(\Omega + \bar{x}) = qri(\Omega) + \bar{x}$.

Proof: (a) It is easy to see that $\text{qi}(\Omega) \subset \Omega \cap \text{qi}(\overline{\Omega})$. Conversely, take any $x \in \Omega \cap \text{qi}(\overline{\Omega})$. Then $\overline{\text{cone}}(\Omega - x) = \overline{\text{cone}}(\overline{\Omega} - x) = X$, which means that $x \in \text{qi}(\Omega)$. Similarly, we have $\text{qri}(\Omega) = \Omega \cap \text{qri}(\overline{\Omega})$.

(b) Since Ω is quasi-nearly convex, there exists a convex set $C \subset X$ satisfying $\text{qri}(C) \neq \emptyset$ and $C \subset \Omega \subset \overline{C}$. Then it follows from Proposition 5.4 and [4, Proposition 2.12] that

$$\overline{C} = \overline{\text{qri}}(C) = \overline{\text{qri}}(\Omega) \subset \overline{\Omega} = \overline{C}.$$

This justifies assertion (b).

(c) It follows directly from (a) and (b) that

$$\text{qri}\left(\text{qri}(\Omega)\right) = \text{qri}(\Omega) \cap \text{qri}(\overline{\text{qri}}(\Omega)) = \text{qri}(\Omega) \cap \text{qri}(\overline{\Omega}) = \text{qri}(\Omega).$$

(d) Take any $x \in \text{qri}(\Omega) + \bar{x}$. Then we have $x - \bar{x} \in \text{qri}(\Omega)$ and hence $\overline{\text{cone}}(\Omega + \bar{x} - x)$ is subspace, which ensures that $x \in \text{qri}(\Omega + \bar{x})$.

Conversely, take any $x \in \text{qri}(\Omega + \bar{x})$. Then the set $\overline{\text{cone}}(\Omega + \bar{x} - x)$ is a subspace, and hence $-(\bar{x} - x) \in \text{qri}(\Omega)$. This tells us that $x \in \text{qri}(\Omega) + \bar{x}$ and thus completes the proof. ∎

To proceed further, we define the *normal cone* to a quasi-nearly convex set in the same way as in the case of pure convexity.

Definition 5.7: Let $\Omega \subset X$ be a quasi-nearly convex set, and let $\bar{x} \in \Omega$. The NORMAL CONE to Ω at $\bar{x}$ is

$$N(\bar{x};\Omega) := \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \leq 0 \text{ for all } x \in \Omega\}.$$

The next result provides a *characterization of quasi-relative interiors* of quasi-nearly convex sets in LCTV spaces.

Proposition 5.8: *Let $\Omega \subset X$ be a quasi-nearly convex set, and let $\bar{x} \in \Omega$. Then $\bar{x} \in qri(\Omega)$ if and only if the normal cone $N(\bar{x};\Omega)$ is a subspace.*

Proof: By the continuity of $x^* \in X^*$ on X, we have that $\langle x^*, x - \bar{x}\rangle \leq 0$ for all $x \in \Omega$ if and only if $\langle x^*, u\rangle \leq 0$ whenever $u \in \overline{\text{cone}}(\Omega - \bar{x})$. It follows that $N(\bar{x};\Omega) = \overline{\text{cone}}(\Omega - \bar{x})^\circ$. This tells us that if $\bar{x} \in \text{qri}(\Omega)$, then $\overline{\text{cone}}(\Omega - \bar{x})$ is a subspace, and so is $N(\bar{x};\Omega)$.

Conversely, we have that the set $\overline{\text{cone}}(\Omega - \bar{x})$ is closed and $0 \in \overline{\text{cone}}(\Omega - \bar{x})$. Furthermore, it follows from Proposition 5.4 that $\overline{\Omega}$ is convex, and so is $\overline{\text{cone}}(\overline{\Omega} - \bar{x}) = \overline{\text{cone}}(\Omega - \bar{x})$. Employing the classical bipolar theorem (see, e.g. [14, Theorem 1.1.9]) gives us the representation

$$N(\bar{x};\Omega)^\circ = (\overline{\text{cone}}(\Omega - \bar{x})^\circ)^\circ = \overline{\text{cone}}(\Omega - \bar{x}).$$

This implies that if $N(\bar{x};\Omega)$ is a subspace, then so is $\overline{\text{cone}}(\Omega - \bar{x})$, and thus $\bar{x} \in \text{qri}(\Omega)$. ■

The next result, which is often employed in the subsequent sections, presents an equivalent description of quasi-relative interiors via *proper separation* of *quasi-nearly convex sets* in LCTV spaces.

Proposition 5.9: *Let Ω be a quasi-nearly convex set in X, and let $\bar{x} \in \Omega$. Then $\bar{x} \notin qri(\Omega)$ if and only if the sets $\{\bar{x}\}$ and Ω can be properly separated by a closed hyperplane.*

Proof: As seen in Proposition 5.8, $\bar{x} \in \text{qri}(\Omega)$ if and only if the normal cone $N(\bar{x};\Omega)$ is a linear subspace of X^*. It follows that $\bar{x} \notin \text{qri}(\Omega)$ if and only if there is $x^* \in N(\bar{x};\Omega)$ such that $-x^* \notin N(\bar{x};\Omega)$. By the definition of the normal cone, we have

$$\langle x^*, x\rangle \leq \langle x^*, \bar{x}\rangle \quad \text{for all } x \in \Omega,$$

while $-x^* \notin N(\bar{x};\Omega)$ means that there exists $\hat{x} \in \Omega$ satisfying

$$\langle x^*, \hat{x}\rangle < \langle x^*, \bar{x}\rangle,$$

which completes the proof. ■

The example below shows that even if Ω is *convex*, the assumption $\bar{x} \in \Omega$ in Proposition 5.9 is *essential*.

Example 5.10: Let X, P, and $\bar{x}$ are given as in Example 5.5. We have seen that $\text{qri}(P) = P$ and $\bar{x} \notin \text{qri}(P)$. Let us now show that the sets $\{\bar{x}\}$ and P can not be properly separated by a closed hyperplane. Suppose on the contrary that there exists $x^* \in X^* \setminus \{0\}$ such that

$$\langle x^*, x\rangle \leq \langle x^*, \bar{x}\rangle \text{ for all } x \in P, \quad \text{ and } \langle x^*, \bar{y}\rangle < \langle x^*, \bar{x}\rangle \text{ for some } \bar{y} \in P.$$

Since P is dense in X and x^* is continuous, we get that

$$\langle x^*, z\rangle \leq \langle x^*, \bar{x}\rangle \quad \text{for all } z \in X,$$

which is impossible. Therefore, the sets $\{\bar{x}\}$ and P cannot be properly separated by a closed hyperplane because the assumption $\bar{x} \in P$ is violated.

The next theorem establishes *proper separation* of two quasi-nearly convex sets.

Theorem 5.11: *Let Ω_1 and Ω_2 be quasi-nearly convex subsets of an LCTV space X. Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and that*

$$qri(\Omega_1 - \Omega_2) = qri(\Omega_1) - qri(\Omega_2). \tag{15}$$

Then the sets Ω_1 and Ω_2 can be properly separated by a closed hyperplane if and only if we have

$$qri(\Omega_1) \cap qri(\Omega_2) = \emptyset. \tag{16}$$

Proof: The imposed assumptions ensure that $0 \in \Omega$ and $\text{qri}(\Omega) = \text{qri}(\Omega_1) - \text{qri}(\Omega_2)$, where $\Omega := \Omega_1 - \Omega_2$. Therefore, if relation (16) holds, then

$$0 \notin \text{qri}(\Omega) = \text{qri}(\Omega_1) - \text{qri}(\Omega_2).$$

According to Proposition 5.9, the sets $\Omega_1 - \Omega_2$ and $\{0\}$ can be properly separated by a closed hyperplane, which clearly ensures the proper separation of the sets Ω_1 and Ω_2.

Conversely, suppose that Ω_1 and Ω_2 can be properly separated by a closed hyperplane. Then the sets $\Omega_1 - \Omega_2$ and $\{0\}$ can be properly separated by a closed hyperplane as well. Using Proposition 5.9 and (15) yields

$$0 \notin \text{qri}(\Omega) = \text{qri}(\Omega_1) - \text{qri}(\Omega_2),$$

and hence $\text{qri}(\Omega_1) \cap \text{qri}(\Omega_2) = \emptyset$, which completes the proof. ■

Based on the sufficient conditions for the fulfilment of (15) from [11, Theorem 2.183(c)], we obtain the following result for the convex case as a direct consequence of Theorem 5.11; see [7, Theorem 4.2] and [11, Theorem 2.184].

Corollary 5.12: *Let Ω_1 and Ω_2 be convex subsets of X such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Assume that $qri(\Omega_1) \neq \emptyset$, $qri(\Omega_2) \neq \emptyset$, and the set difference $\Omega_1 - \Omega_2$ is quasi-regular. Then Ω_1 and Ω_2 can be properly separated if and only if*

$$qri(\Omega_1) \cap qri(\Omega_2) = \emptyset.$$

To continue our study of quasi-nearly convex sets and their generalized relative interiors, we need the following proposition.

Proposition 5.13: *Let Ω be a quasi-nearly convex set in X. If $\bar{x} \in qri(\Omega)$, $x_0 \in \Omega$, and $(1-t_0)\bar{x} + t_0 x_0 \in \Omega$ for some $t_0 \in (0,1]$, then $(1-t_0)\bar{x} + t_0 x_0 \in qri(\Omega)$.*

Proof: Suppose on the contrary that $\hat{x} := (1-t_0)\bar{x} + t_0 x_0 \notin \text{qri}(\Omega)$. Then Proposition 5.9 ensures the existence of $x^* \in X^* \setminus \{0\}$ with

$$\langle x^*, x - \hat{x} \rangle \leq 0 \quad \text{for all } x \in \Omega, \tag{17}$$

$$\langle x^*, \bar{y} - \hat{x} \rangle < 0 \quad \text{for some } \bar{y} \in \Omega. \tag{18}$$

By the continuity of x^* on X, we deduce from (17) that

$$\langle x^*, v \rangle \leq 0 \quad \text{for all } v \in \overline{\text{cone}}(\Omega - \hat{x}). \tag{19}$$

On the other hand, the quasi-near convexity of Ω yields the existence of a convex subset $C \subset X$ such that $\text{qri}(C) \neq \emptyset$ and $C \subset \Omega \subset \overline{C}$. Thus $x_0 \in \overline{C}$ and by Proposition 5.4 we have $\bar{x} \in \text{qri}(\overline{C})$. Using Proposition 3.13 gives us $\hat{x} \in \text{qri}(\overline{C})$, i.e. $\overline{\text{cone}}(\overline{C} - \hat{x})$ is a subspace. Since $\overline{\text{cone}}(\overline{C} - \hat{x}) = \overline{\text{cone}}(\Omega - \hat{x})$, it follows from (19) that

$$\langle x^*, v \rangle = 0 \quad \text{for all } v \in \overline{\text{cone}}(\Omega - \hat{x}).$$

This contradicts (18) and completes the proof. ∎

Now we give a characterization of the *quasi-interior* of quasi-nearly convex sets.

Proposition 5.14: *Let Ω be a quasi-nearly convex set in X, and let $\bar{x} \in \Omega$. Then $\bar{x} \in qi(\Omega)$ if and only if $N(\bar{x}; \Omega) = \{0\}$.*

Proof: Assume that $\bar{x} \in \text{qi}(\Omega)$ and take any $x^* \in N(\bar{x}; \Omega)$. Then the continuity of x^* on X ensures that $\langle x^*, v \rangle \leq 0$ for all $v \in \overline{\text{cone}}(\Omega - \bar{x}) = X$, which yields $x^* = 0$.

Conversely, assume that $N(\bar{x}; \Omega) = \{0\}$. Arguing by contradiction, consider an arbitrary element $\bar{v} \in X$, and suppose that $\bar{v} \notin \overline{\text{cone}}(\Omega - \bar{x})$. Note that $\overline{\text{cone}}(\Omega - \bar{x})$ is convex as seen in the proof of Proposition 5.4(b). By Zălinescu [14, Theorem 1.1.5], the sets $\{\bar{v}\}$ and $\overline{\text{cone}}(\Omega - \bar{x})$ can be *strictly separated* by a closed hyperplane, i.e. there exists $x^* \in X^* \setminus \{0\}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\langle x^*, v\rangle \le \alpha_1 < \alpha_2 \le \langle x^*, \bar{v}\rangle \quad \text{for all } v \in \overline{\text{cone}}(\Omega - \bar{x}).$$

Fix any $y \in \Omega$ and get $\langle x^*, y - \bar{x}\rangle \le \alpha_1/\lambda$ as $\lambda > 0$, which yields $\langle x^*, y - \bar{x}\rangle \le 0$. Since y was chosen arbitrarily in Ω, we arrive at $x^* \in N(\bar{x}; \Omega)$, a contradiction completing the proof. ■

The next proposition describes the separation of quasi-nearly convex sets via quasi-interiors.

Proposition 5.15: *Let Ω be a quasi-nearly convex set in X, and let $\bar{x} \in \Omega$. Then $\bar{x} \notin qi(\Omega)$ if and only if the sets $\{\bar{x}\}$ and Ω can be separated by a closed hyperplane.*

Proof: By Proposition 5.14, we have $\bar{x} \notin \text{qi}(\Omega)$ if and only if there is a nonzero element $x^* \in N(\bar{x}; \Omega)$. It follows from the normal cone definition that $x^* \in N(\bar{x}; \Omega)$ if and only if

$$\langle x^*, y\rangle \le \langle x^*, \bar{x}\rangle \quad \text{for all } y \in \Omega,$$

which completes the proof of the proposition. ■

By Definition 3.1(c,f) we have $\text{qi}(\Omega) \subset \text{qri}(\Omega)$. The next proposition shows that the equality holds therein if the quasi-interior of Ω is nonempty.

Proposition 5.16: *Let Ω be a quasi-nearly convex set in X. If $qi(\Omega) \neq \emptyset$, then*

$$qi(\Omega) = qri(\Omega).$$

Proof: Observe first that if $\text{qi}(\Omega) \neq \emptyset$, then $N(0; \Omega - \Omega) = \{0\}$. Indeed, take any $x \in \text{qi}(\Omega)$. Then $0 \in \text{qi}(\Omega - x)$, and since $\Omega - x \subset \Omega - \Omega$, we get $0 \in \text{qi}(\Omega - \Omega)$. Therefore, it follows from Proposition 5.14 that

$$N(0; \Omega - \Omega) = \{0\}. \tag{20}$$

To show that $\text{qri}(\Omega) \subset \text{qi}(\Omega)$, pick any $x \in \text{qri}(\Omega)$ and $x^* \in N(0; \Omega - x)$. Then we get

$$\langle x^*, w - x\rangle \le 0 \quad \text{for all } w \in \Omega, \tag{21}$$

which implies that $x^* \in N(x; \Omega)$. Since $x \in \text{qri}(\Omega)$, it follows from Proposition 5.8 that $N(x, \Omega)$ is a linear subspace. Thus we arrive at $-x^* \in N(x; \Omega)$, i.e.

$$\langle x^*, x - w\rangle \le 0 \quad \text{for all } w \in \Omega. \tag{22}$$

Summing up the two inequalities (21) and (22) gives us

$$\langle x^*, w_1 - w_2\rangle \le 0 \quad \text{whenever } w_1, w_2 \in \Omega,$$

and so $x^* \in N(0; \Omega - \Omega)$. Then it follows from (20) that $x^* = 0$. Since $x^* \in N(0; \Omega - x)$ was chosen arbitrarily, we get $N(0; \Omega - x) = \{0\}$. Applying finally Proposition 5.14 tells us that $x \in \text{qi}(\Omega)$ and thus completes the proof of the proposition. ■

Now we ready to derive some *calculus rules* for quasi-relative interiors of quasi-nearly convex sets via continuous linear mappings.

Theorem 5.17: *Let $T\colon X \to Y$ be a continuous linear mapping between LCTV spaces, and let Ω be a quasi-nearly convex set in X. Then the following assertions hold:*

(a) $T(qri(\Omega)) \subset qri(T((\Omega))$.
(b) *If T is injective and $T(\Omega)$ is quasi-regular, then $T(qri(\Omega)) = qri(T(\Omega))$. The injectivity of T is not required if Ω is convex with $qri(\Omega) \neq \emptyset$.*
(c) *$T(\Omega)$ and $T(qri(\Omega))$ are quasi-nearly convex sets in Y.*

Proof: (a) Pick any $\bar{x} \in \text{qri}(\Omega)$ and deduce from Proposition 5.8 that $N(\bar{x}; \Omega)$ is a linear subspace. Take $y^* \in N(T(\bar{x}); T(\Omega))$ and get $\langle y^*, T(x) - T(\bar{x})\rangle \le 0$ for all $x \in \Omega$, which implies that $T^*y^* \in N(\bar{x}; \Omega)$, where $T^*\colon Y^* \to X^*$ is the classical adjoint operator. Since $N(\bar{x}; \Omega)$ is a subspace, we have that $-T^*y^* \in N(\bar{x}; \Omega)$. Equivalently, it holds that $-y^* \in N(T(\bar{x}); T(\Omega))$, which implies that $N(T(\bar{x}); T(\Omega))$ is a linear subspace of Y^*. Hence Proposition 5.8 tells us that $T(\bar{x}) \in \text{qri}(T(\Omega))$.

(b) We have by the assumption that there is $\bar{x} \in \text{qri}(\Omega)$, and thus

$$\bar{y} := T(\bar{x}) \in \text{qri}(T(\Omega)).$$

Fix any $\hat{y} \in \text{qri}(T(\Omega)) = \text{iri}(T(\Omega))$ and deduce from Proposition 3.12 that there exists $y_0 \in T(\Omega)$ such that $\hat{y} = (1 - t_0)\bar{y} + t_0 y_0 \in T(\Omega)$ for some $t_0 \in (0, 1)$. Then we find $x_0, \hat{x} \in \Omega$ satisfying $y_0 = T(x_0)$ and $\hat{y} = T(\hat{x})$. Since T is injective and

$$T(\hat{x}) = \hat{y} = T((1 - t_0)\bar{x} + t_0 x_0),$$

we obtain $(1 - t_0)\bar{x} + t_0 x_0 = \hat{x} \in \Omega$. Moreover, since $\bar{x} \in \text{qri}(\Omega)$, Proposition 5.13 tells us that $\hat{x} \in \text{qri}(\Omega)$, and hence $\hat{y} \in T(\text{qri}(\Omega))$. The imposed assumptions and the proved result in (a) yield

$$\text{qri}(T(\Omega)) \subset T(\text{qri}(\Omega)) \subset \text{qri}(T(\Omega)).$$

If Ω is convex with $\mathrm{qri}(\Omega) \neq \emptyset$, then the conclusion follows directly from [11, Theorem 2.183(c)]. (c) Since Ω is quasi-nearly convex, we get from Definition 5.1 that there exists a convex set $C \subset X$ such that $\mathrm{qri}(C) \neq \emptyset$ and

$$C \subset \Omega \subset \overline{C}.$$

Then Proposition 5.4(a) together with [4, Proposition 2.12] tells us that $\overline{\mathrm{qri}}(C) = \overline{C} = \overline{\Omega}$. Thus

$$T(\mathrm{qri}(C)) \subset T(\mathrm{qri}(\Omega)) \subset T(\Omega) \subset T(\overline{\Omega}) = T(\overline{\mathrm{qri}}(C)) \subset \overline{T(\mathrm{qri}(C))}, \quad (23)$$

where the first inclusion is satisfied by Proposition 5.4(b) and the last inclusion holds because T is continuous. Note that $\mathrm{qri}(C)$ is convex by Borwein and Lewis [4, Lemma 2.9], and we deduce from the linearity of T that $T(\mathrm{qri}(C))$ is convex. Combining this with (23) implies that $T(\Omega)$ and $T(\mathrm{qri}(\Omega))$ are quasi-nearly convex. ■

6. Generalized relative interiors of quasi-nearly convex graphs

The last section of the paper is devoted to deriving new results on quasi-relative interiors and quasi-interiors of graphs of set-valued mappings in the quasi-near convexity framework.

Given a *function* $f: X \to \overline{\mathbb{R}}$, we say that f is *quasi-nearly convex* if $\mathrm{epi}(f)$ is a quasi-nearly convex set. We also say that a *set-valued mapping* $F: X \rightrightarrows Y$ between LCTV spaces is *quasi-nearly convex* convex if its graph $\mathrm{gph}(F)$ is a quasi-nearly convex set in $X \times Y$.

The following proposition is useful to establish the major results of this section.

Proposition 6.1: *Let $F: X \rightrightarrows Y$ is quasi-nearly convex set-valued mapping between LCTV spaces. Then both sets $\mathrm{dom}(F)$ and $\mathrm{rge}(F)$ are quasi-nearly convex sets. Consequently, if $f: X \to \overline{\mathbb{R}}$ is an extended-real-valued proper quasi-nearly convex function, then the sets $\mathrm{dom}(f)$ and $\mathrm{rge}(f)$ are quasi-nearly convex.*

Proof: We first see that $\mathrm{dom}(F) = \mathcal{P}(\mathrm{gph}(F))$, where $\mathcal{P}$ is the continuous linear mapping given by

$$\mathcal{P}(x, y) := x, \quad (x, y) \in X \times Y.$$

Then it follows from Theorem 5.17(c) that $\mathrm{dom}(F)$ is quasi-nearly convex. Similarly, $\mathrm{rge}(F)$ is quasi-convex because $\mathrm{rge}(F) = \mathcal{P}_1(\mathrm{gph}(F))$, where $\mathcal{P}_1$ is the continuous linear mapping defined by

$$\mathcal{P}_1(x, y) := y, \quad (x, y) \in X \times Y.$$

Finally, suppose that f is quasi-nearly convex. Then the epigraphical mapping E_f defined in (1) is quasi-nearly convex. Since $\mathrm{dom}(E_f) = \mathrm{dom}(f)$ and $\mathrm{rge}(E_f) =$

rge(f), the sets dom(f) and rge(f) are both quasi-nearly convex, and we are done. ■

Now we are ready for the first main result.

Theorem 6.2: *Let $F\colon X \rightrightarrows Y$ be a quasi-nearly convex set-valued mapping. The following assertions are valid:*

(a) *If gph(F) is quasi-regular, then we have*

$$qri(gph(F)) \subset \{(x, y) \in X \times Y \mid x \in qri(dom(F)), y \in qri(F(x))\}.$$

(b) *If $F(x)$ is quasi-nearly convex and $qi(F(x)) \neq \emptyset$ for every $x \in dom(F)$, then*

$$qri(gph(F)) \supset \{(x, y) \in X \times Y \mid x \in qri(dom(F)), y \in qri(F(x))\}.$$

Consequently, if gph(F) is quasi-regular and $qi(F(x)) \neq \emptyset$ for every $x \in dom(F)$, then

$$qri(gph(F)) = \{(x, y) \in X \times Y \mid x \in qri(dom(F)), y \in qri(F(x))\}.$$

Proof: (a) Observe first that dom(F) is quasi-nearly convex by Proposition 6.1. We now fix any $(\bar{x}, \bar{y}) \in \text{qri}(\text{gph}(F))$ and suppose on the contrary that $\bar{x} \notin \text{qri}(\text{dom}(F))$. Then it follows from Proposition 5.9 that the sets $\{\bar{x}\}$ and dom(F) can be properly separated by a closed hyperplane, i.e. there exist $x^* \in X^* \setminus \{0\}$ and $\hat{x} \in \text{dom}(F)$ such that

$$\langle x^*, x\rangle \leq \langle x^*, \bar{x}\rangle \quad \text{for all } x \in \text{dom}(F)$$

and

$$\langle x^*, \hat{x}\rangle < \langle x^*, \bar{x}\rangle .$$

Thus, we have

$$\langle (x^*, 0), (x, y)\rangle = \langle x^*, x\rangle \leq \langle x^*, \bar{x}\rangle = \langle (x^*, 0), (\bar{x}, \bar{y})\rangle \quad \text{for all } (x, y) \in \text{gph}(F)$$

and for each $\hat{y} \in F(\hat{x})$,

$$\langle (x^*, 0), (\hat{x}, \hat{y})\rangle = \langle x^*, \hat{x}\rangle < \langle x^*, \bar{x}\rangle = \langle (x^*, 0), (\bar{x}, \bar{y})\rangle .$$

This implies that the sets gph(F) and $\{(\bar{x}, \bar{y})\}$ can be properly separated. It follows from Proposition 5.9 that $(\bar{x}, \bar{y}) \notin \text{qri}(\text{gph}(F))$, a contradiction, which $\bar{x} \in \text{qri}(\text{dom}(F))$. It remains to show that $\bar{y} \in \text{qri}(F(\bar{x}))$. Fix any $y \in F(\bar{x})$. Then by the quasi-regularity of gph(F) and Proposition 3.12, there exist $(x_0, y_0) \in \text{gph}(F)$ and $t_0 \in (0,1)$ such that

$$(\bar{x}, \bar{y}) = (1 - t_0)(\bar{x}, y) + t_0(x_0, y_0).$$

This yields $x_0 = \bar{x}$ and $\bar{y} = (1 - t_0)y + t_0 y_0$ with $y_0 \in F(\bar{x})$. It follows therefore by Proposition 3.12 and Theorem 3.3 that $\bar{y} \in \text{iri}(F(\bar{x})) \subset \text{qri}(F(\bar{x}))$, which completes the proof of (a).

(b) To verify the reverse inclusion in this assertion under the imposed assumptions that $F(x)$ is quasi-nearly convex and $\text{qi}(F(x)) \neq \emptyset$ for every $x \in \text{dom}(F)$, we fix $\bar{x} \in \text{qri}(\text{dom}(F))$ and $\bar{y} \in \text{qri}(F(\bar{x}))$. Arguing by contradiction, suppose that $(\bar{x}, \bar{y}) \notin \text{qri}(\text{gph}(F))$. Then it follows from Proposition 5.9 that there exist $(x^*, y^*) \in X^* \times Y^* \setminus \{(0,0)\}$ and $(\hat{x}, \hat{y}) \in X \times Y$ such that

$$\langle x^*, x\rangle + \langle y^*, y\rangle \leq \langle x^*, \bar{x}\rangle + \langle y^*, \bar{y}\rangle \quad \text{whenever } x \in \text{dom}(F) \text{ and } y \in F(x) \tag{24}$$

together with the strict inequality

$$\langle x^*, \hat{x}\rangle + \langle y^*, \hat{y}\rangle < \langle x^*, \bar{x}\rangle + \langle y^*, \bar{y}\rangle. \tag{25}$$

We distinguish the two possible cases: (A) $y^* = 0$ and (B) $y^* \neq 0$.

In case (A), we get from (24) that

$$\langle x^*, x\rangle \leq \langle x^*, \bar{x}\rangle \quad \text{whenever } x \in \text{dom}(F)$$

and from (25) that

$$\langle x^*, \hat{x}\rangle < \langle x^*, \bar{x}\rangle.$$

Then it follows from Proposition 5.9 that $\bar{x} \notin \text{qri}(\text{dom}(F))$, a contradiction.

In case (B), letting $x = \bar{x}$ in (24) gives us $\langle y^*, y\rangle \leq \langle y^*, \bar{y}\rangle$ for all $y \in F(\bar{x})$. Then Propositions 5.15 and 5.16 tell us that

$$\bar{y} \notin \text{qi}(F(\bar{x})) = \text{qri}(F(\bar{x})).$$

This contradiction shows that $(\bar{x}, \bar{y}) \in \text{qri}(\text{gph}(F))$ and hence completes the proof of (B) and of the whole theorem. ∎

Another description of the quasi-relative interiors of graphs for quasi-nearly convex set-valued mappings is formulated as follows.

Theorem 6.3: *Let $F\colon X \rightrightarrows Y$ be a quasi-nearly convex set-valued mapping. Then we have*

$$\text{qri}(\text{gph}(F)) \supset \{(x, y) \in X \times Y \mid x \in \text{qri}(\text{dom}(F)), y \in \text{int}(F(x))\}.$$

If in addition gph(F) is quasi-regular, $F(x)$ is convex and $\text{int}(F(x)) \neq \emptyset$ *for every $x \in$ dom(F), then*

$$\text{qri}(\text{gph}(F)) = \{(x, y) \in X \times Y \mid x \in \text{qri}(\text{dom}(F)), y \in \text{int}(F(x))\}.$$

Proof: Pick any $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} \in \text{qri}(\text{dom}(F))$ and $\bar{y} \in \text{int}(F(\bar{x}))$. We prove by contradiction, suppose that $(\bar{x}, \bar{y}) \notin \text{qri}(\text{gph}(F))$. By Proposition 5.9, there exists $(x^*, y^*) \in X^* \times Y^* \setminus \{(0,0)\}$ such that

$$\langle x^*, x\rangle + \langle y^*, y\rangle \le \langle x^*, \bar{x}\rangle + \langle y^*, \bar{y}\rangle \quad \text{for all } x \in \text{dom}(F) \text{ and } y \in F(x), \tag{26}$$

and there exists $(\hat{x}, \hat{y}) \in \text{gph}(F)$ for which

$$\langle x^*, \hat{x}\rangle + \langle y^*, \hat{y}\rangle < \langle x^*, \bar{x}\rangle + \langle y^*, \bar{y}\rangle. \tag{27}$$

Letting $x = \bar{x}$ in (26), we obtain that

$$\langle y^*, y\rangle \le \langle y^*, \bar{y}\rangle \quad \text{whenever } y \in F(\bar{x}). \tag{28}$$

Since $\bar{y} \in \text{int}(F(\bar{x}))$, there exists a symmetric neighbourhood V of the origin satisfying $V \subset F(\bar{x}) - \{\bar{y}\}$. It follows from (28) that $\langle y^*, v\rangle \le 0$ and $\langle y^*, -v\rangle \le 0$ for all $0 \ne v \in V$, and hence $y^* = 0$ on V. Moreover, since V is a symmetric neighbourhood of $0 \in X$, for every $0 \ne y \in Y$ we find $0 \ne t \in \mathbb{R}$ such that $ty = v \in V$, and therefore $\langle y^*, y\rangle = \frac{1}{t}\langle y^*, v\rangle = 0$. This implies that $y^* = 0$ on Y, which implies by (26) and (27) that the sets $\{\bar{x}\}$ and $\text{dom}(F)$ can be properly separated by a closed hyperplane. Then Proposition 5.9 tells us that $\bar{x} \notin \text{qri}(\text{dom}(F))$, a contradiction that justifies the inclusion

$$\text{qri}(\text{gph}(F)) \supset \{(x, y) \in X \times Y \mid x \in \text{qri}(\text{dom}(F)), y \in \text{int}(F(x))\}.$$

To check the inclusion '$\subset$' in the theorem under the additional assumptions made, we take any $(\bar{x}, \bar{y}) \in \text{qri}(\text{gph}(F))$ and suppose on the contrary that $\bar{x} \notin \text{qri}(\text{dom}(F))$. By Proposition 5.9, there exist a nonzero function $x^* \in X^*$ and $\hat{x} \in \text{dom}(F)$ such that

$$\langle x^*, x\rangle \le \langle x^*, \bar{x}\rangle \quad \text{for all } x \in \text{dom}(F)$$

and

$$\langle x^*, \hat{x}\rangle < \langle x^*, \bar{x}\rangle.$$

Then for all $(x, y) \in \text{gph}(F)$, we get

$$\big\langle (x^*, 0), (x, y)\big\rangle = \langle x^*, x\rangle \le \langle x^*, \bar{x}\rangle = \big\langle (x^*, 0), (\bar{x}, \bar{y})\big\rangle.$$

Taking any fixed $\hat{y} \in F(\hat{x})$ gives us

$$\big\langle (x^*, 0), (\hat{x}, \hat{y})\big\rangle = \langle x^*, \hat{x}\rangle < \langle x^*, \bar{x}\rangle = \big\langle (x^*, 0), (\bar{x}, \bar{y})\big\rangle.$$

Thus, the sets $\{(\bar{x}, \bar{y})\}$ and $\text{gph}(F)$ can be properly separated. Applying Proposition 5.9 again, we arrive at the condition $(\bar{x}, \bar{y}) \notin \text{qri}(\text{gph}(F))$, which is a contradiction telling us that $\bar{x} \in \text{qri}(\text{dom}(F))$.

To proceed further, let us verify that $\bar{y} \in \text{int}(F(\bar{x}))$. Under the assumptions of the convexity of $F(\bar{x})$ and the nonemptiness of $\text{int}(F(\bar{x}))$, we have by Borwein and

Goebel [3, Theorem 2.12(b)] that $\text{int}(F(\bar{x})) = \text{iri}(F(\bar{x}))$. Therefore, it suffices to show $\bar{y} \in \text{iri}(F(\bar{x}))$. To justify this, fix any $y \in F(\bar{x})$. Then the quasi-regularity of $\text{gph}(F)$ and Proposition 3.12 give us $(x_0, y_0) \in \text{gph}(F)$ such that

$$(\bar{x}, \bar{y}) = (1 - t_0)(\bar{x}, y) + t_0(x_0, y_0) \quad \text{for some } t_0 \in (0, 1).$$

This yields $x_0 = \bar{x}$ and $\bar{y} = (1 - t_0)y + t_0 y_0$ with $y_0 \in F(\bar{x})$. Applying Proposition 3.12 again tells that $\bar{y} \in \text{iri}(F(\bar{x}))$, which completes the proof of the theorem. ■

Next, we employ the above result to deduce representations of the quasi-relative interiors of *epigraphs* for extended-real-valued functions.

Corollary 6.4: *Let $f: X \to \overline{\mathbb{R}}$ be a proper quasi-nearly convex function. If epi(f) is quasi-regular, then we have*

$$qri(epi(f)) = \left\{ (x, \alpha) \in X \times \mathbb{R} \mid x \in qri(dom(f)), f(x) < \alpha \right\}.$$

Proof: Consider the epigraphical mapping E_f associated with f. Since $E_f(x) = [f(x), \infty)$ is convex and $\text{int}(E_f(x))$ is nonempty every $x \in \text{dom}(E_f) = \text{dom}(f)$, the conclusion follows from Theorem 6.3. ■

We end this section with the following inclusion of quasi-interiors for graphs of quasi-nearly convex set-valued mappings.

Theorem 6.5: *Let $F: X \rightrightarrows Y$ be a quasi-nearly convex set-valued mapping between LCTV spaces. Then we have the inclusion*

$$\text{qi}(\text{gph}(F)) \supset \{(x, y) \in X \times Y \mid x \in \text{qi}(\text{dom}(F)), \; y \in \text{qi}(F(x))\}.$$

Proof: We proceed as in the proof of Theorem 6.2(b) with using now Proposition 5.15 instead of Proposition 5.9. ■

Acknowledgments

The authors are very grateful to the anonymous referees for their invaluable remarks and suggestions, which significantly contributed to enhancing both the content and the presentation of our paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

Research of Vo Si Trong Long is funded by University of Science, VNU-HCM, under grant number T2023-97. Research of Boris S. Mordukhovich was partly supported by the USA National Science Foundation under grants DMS-1808978 and DMS-2204519, by the Australian Research Council under grant DP-190100555, and by Project 111 of China under grant D21024. Research of Nguyen Mau Nam was partly supported by the USA National Science Foundation under grant DMS-2136228.

ORCID

Vo Si Trong Long http://orcid.org/0000-0002-4781-1674

References

[1] Bao TQ, Mordukhovich BS. Relative pareto minimizers for multiobjective problems: existence and optimality conditions. Math Program. 2010;122:301–347. doi: 10.1007/s10107-008-0249-2
[2] Bauschke HH, Combettes PL. Convex analysis and monotone operator theory in Hilbert spaces. 2nd ed. New York: Springer; 2017.
[3] Borwein JM, Goebel R. Notions of relative interior in Banach spaces. J Math Sci. 2003;115:2542–2553. doi: 10.1023/A:1022988116044
[4] Borwein JM, Lewis AS. Partially finite convex programming, part 1: quasi-relative interiors and duality theory. Math Program. 1992;57:15–48. doi: 10.1007/BF01581072
[5] Boț RI, Csetnek ER. Regularity conditions via generalized interiority notions in convex optimization: new achievements and their relation to some classical statements. Optimization. 2012;61:35–65. doi: 10.1080/02331934.2010.505649
[6] Boț RI, Csetnek ER, Wanka G. Regularity conditions via quasi-relative interior in convex programming. SIAM J Optim. 2008;19:217–233. doi: 10.1137/07068432X
[7] Cuong DV, Mordukhovich B, Nam NM, et al. Fenchel–Rockafellar theorem in infinite dimensions via generalized relative interiors. Optimization. 2023;72:135–162. doi: 10.1080/02331934.2022.2048383
[8] Cuong DV, Mordukhovich BS, Nam NM. Quasi-relative interiors for graphs of convex set-valued mappings. Optim Lett. 2021;15:933–952. doi: 10.1007/s11590-019-01447-4
[9] Khan A, Tammer C, Zălinescu C. Set-valued optimization: an introduction and applications. Berlin: Springer; 2015.
[10] Mordukhovich BS. Variational analysis and applications. Cham, Switzerland: Springer; 2018.
[11] Mordukhovich BS, Nam NM. Convex analysis and beyond: volume I: basic theory. Cham, Switzerland: Springer; 2022.
[12] Ng KF, Song W. Fenchel duality in infinite-dimensional setting and its applications. Nonlinear Anal. 2003;55:845–858. doi: 10.1016/j.na.2003.07.008
[13] Pallaschke D, Rolewiecz S. Foundations of mathematical optimization: convexc analysis without linearity. Dordrecht, The Netherlands: Kluwer; 1998.
[14] Zălinescu C. Convex analysis in general vector spaces. Singapore: World Scientific; 2002.
[15] Zhou ZA, Yang XM. Optimality conditions of generalized subconvexlike set-valued optimization problems based on the quasi-relative interior. J Optim Theory Appl. 2011;150:327–340. doi: 10.1007/s10957-011-9844-0
[16] Rockafellar RT. Convex analysis. Princeton (NJ): Princeton University Press; 1970.

[17] Mordukhovich BS, Nam NM. An easy path to convex analysis and applications. 2nd ed. Cham, Switzerland: Springer; 2023.
[18] Minty GJ. On the maximal domain of a 'monotone' function. Michigan Math J. 1961;8:135–137. doi: 10.1307/mmj/1028998564
[19] Rockafellar RT. On the virtual convexity of the domain and range of a nonlinear maximal monotone operator. Math Ann. 1970;185:81–90. doi: 10.1007/BF01359698
[20] Bauschke HH, Moffat SM, Wang X. Near equality, near convexity, sums of maximally monotone operators, and averages of firmly nonexpansive mappings. Math Program. 2013;139:55–70. doi: 10.1007/s10107-013-0659-7
[21] Boţ RI, Kassay G, Wanka G. Duality for almost convex optimization problems via the perturbation approach. J Global Optim. 2008;42:385–399. doi: 10.1007/s10898-008-9300-3
[22] Li J, Mastroeni G. Near equality and almost convexity of functions with applications to optimization and error bounds. J Convex Anal. 2019;26:785–822.
[23] Moffat SM, Moursi WM, Wang X. Nearly convex sets: fine properties and domains or ranges of subdifferentials of convex functions. Math Program. 2016;160:193–223. doi: 10.1007/s10107-016-0980-z
[24] Nam NM, Thieu NN, Yen ND. Near convexity and generalized differentiation; 2023. arXiv:2302.08986.
[25] Cammaroto F, Bella BD. Separation theorem based on the quasirelative interior and application to duality theory. J Optim Theory Appl. 2005;125:223–229. doi: 10.1007/s10957-004-1724-4
[26] Zălinescu C. On the use of the quasi-relative interior in optimization. Optimization. 2015;64:1795–1823. doi: 10.1080/02331934.2015.1032284
[27] Holmes RB. Geometric functional analysis and its applications. New York: Springer; 1975.
[28] Jeyakumar V, Wolkowicz H. Generalizations of Slater's constraint qualification for infinite convex programs. Math Program. 1992;57:85–101. doi: 10.1007/BF01581074