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BCFW tilings and cluster adjacency for the amplituhedron

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In 2005, Britto, Cachazo, Feng, and Witten gave a recurrence (now known as the BCFW recurrence) for computing scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills theory. Arkani-Hamed and Trnka subsequently introduced the amplituhedron to give a geometric interpretation of the BCFW recurrence. Arkani-Hamed and Trnka conjectured that each way of iterating the BCFW recurrence gives a "triangulation" or "tiling" of the m=4 amplituhedron. In this article, we prove the BCFW tiling conjecture of Arkani-Hamed and Trnka. We also prove the cluster adjacency conjecture for BCFW tiles of the amplituhedron, which says that facets of tiles are cut out by collections of compatible cluster variables for the Grassmannian Gr4,n. Moreover we show that each BCFW tile is the subset of the Grassmannian where certain cluster variables have particular signs.

amplituhedron | $\mathcal{N} = 4$ super Yang–Mills | positive Grassmannian | scattering amplitudes | cluster algebras

The (tree) amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is the image of the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ under the amplituhedron map $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$. It was introduced by Arkani-Hamed and Trnka (1) in order to give a geometric interpretation of scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills theory (SYM): In particular, they conjectured one can compute $\mathcal{N} = 4$ SYM scattering amplitudes by "tiling" the $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$—that is, decomposing the amplituhedron into "tiles"—and summing the "volumes" of the tiles. While the case $m = 4$ is most important for physics, the amplituhedron is defined for any positive n, k, m with $k + m \leq n$, and has a very rich geometric and combinatorial structure. It generalizes cyclic polytopes (when $k = 1$), cyclic hyperplane arrangements (2) (when $m = 1$), and the positive Grassmannian (when $k = n - m$), and it is connected to the hypersimplex and the positive tropical Grassmanian (3, 4) (when $m = 2$). The present work proves the Britto, Cachazo, Feng, and Witten (BCFW) tiling conjecture and the cluster adjacency conjecture for BCFW tiles, addressing two longstanding problems for the amplituhedron.

The cluster adjacency conjecture says that facets of tiles are cut out by collections of compatible cluster variables. This was motivated by the observation that cluster algebras can be used to describe singularities of scattering amplitudes in $\mathcal{N} = 4$ SYM (5). In particular, Drummond et al. (6, 7) conjectured that the terms in tree-level amplitudes coming from the BCFW recursions are rational functions whose poles correspond to compatible cluster variables of the Grassmannian $\mathrm{Gr}_{4,n}$; see also ref. 8. The cluster adjacency conjecture, formulated for the $m = 2$ and $m = 4$ amplituhedron in refs. 9 and 10, was subsequently proved for all tiles of the $m = 2$ amplituhedron (4), but remained open in the $m = 4$ case, the case of most relevance to physics.

The BCFW tiling conjecture says that any way of iterating the BCFW recurrence (11) gives rise to a collection of cells ("BCFW cells") in the positive Grassmannian whose images tile the $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$ (see *Definition* 6 for a precise definition). This conjecture arose side-by-side with the definition of the amplituhedron—indeed, the goal of Arkani-Hamed and Trnka, realized in ref. 1, was to find a geometric object which could be decomposed into pieces coming from BCFW cells. BCFW-like tilings of the $m = 1$ and $m = 2$ amplituhedron were proved in refs. 2 and 12, building on refs. 13 and 14. A step toward the BCFW tiling conjecture was made in ref. 15, where the authors built on work of Karp et al. (14) to show that the "standard" way of performing the BCFW recursion gives a tiling for the $m = 4$ amplituhedron.

Main Results. In this paper, we extend and generalize the results of refs. 4 and 15 to give a very complete picture of the $m = 4$ amplituhedron. We show that arbitrary BCFW cells give tiles (*Theorem* 20) and that they satisfy the cluster adjacency conjecture (*Theorem* 29). We strengthen the connection with cluster algebras by associating to each BCFW tile a collection of compatible cluster variables for $\mathrm{Gr}_{4,n}$ (*Definition* 25), which

Significance

Scattering amplitudes in a quantum field theory describe probabilities of different outcomes when particles interact. In 2005, Britto, Cachazo, Feng, and Witten gave a recurrence for computing scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills theory. In 2014, Arkani-Hamed and Trnka introduced a geometric object called the amplituhedron, and used it to give a conjectural geometric explanation of the BCFW recurrence: They proposed that each way of iterating the BCFW recurrence corresponds to a "triangulation" or "tiling" of the amplituhedron. In this article, we prove this conjecture. We also give a concrete description of BCFW tiles and their facets using the framework of cluster algebras, a class of commutative rings with a remarkable combinatorial structure.

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we use to describe the tile as a semialgebraic set in $\mathrm{Gr}_{k,k+4}$ (*Theorem* 27). For "standard" BCFW tiles, one can also give a nonrecursive description of these cluster variables (16, theorem 8.4) and the underlying quiver (*Theorem* 34). Finally, we use these results to prove the BCFW tiling conjecture for the $m = 4$ amplituhedron (*Theorem* 31).

Further Motivation. From the point of view of cluster algebras, the study of tiles for the amplituhedron $\mathcal{A}_{n,k,m}$ is useful because it is closely related to the cluster structure on the Grassmannian $\mathrm{Gr}_{m,n}$, as was shown for $m = 2$ in ref. 4 and as this paper demonstrates for $m = 4$. In particular, for $m = 4$, the BCFW product (*Definition* 17) used to recursively build tiles has a cluster quasi-homomorphism counterpart called product promotion (*Definition* 21), that can be used to recursively construct cluster variables and seeds in $\mathrm{Gr}_{4,n}$ (*Theorem* 22). Moreover, this would also have potential implications for loop amplitudes in $\mathcal{N} = 4$ SYM in connection with the cluster bootstrap program (see ref. 17 for a review).

In the closely related field of total positivity, one prototypical problem is to give an efficient characterization of the "positive part" of a space as the subset where a collection of functions take on positive values (18). A minimal collection of functions whose positivity cuts out the positive part is called a "positivity test." For example, for any cluster $\mathbf{x}$ for $\mathrm{Gr}_{k,n}$ (19), the positive Grassmannian $\mathrm{Gr}_{k,n}^{>0}$ can be described as the region in $\mathrm{Gr}_{k,n}$, where all the cluster variables of $\mathbf{x}$ are positive, so each $\mathbf{x}$ provides a positivity test.

We think of *Theorem* 27 as a "positivity test" for membership in a BCFW tile of the amplituhedron. See (4, theorem 6.8) for an analogous result for $m = 2$, and (16, conjecture 7.17) for some conjectures for general m.

From the point of view of discrete geometry, one can think about tiles as a generalization of polytopes in the Grassmannian. In particular, the positivity tests for the positive Grassmannian and BCFW tiles can be thought of as analogues of the hyperplane description of polytopes. Finally, it is expected that tiles are positive geometries (20).

Background

Let $[n]$ denote $\{1, \ldots, n\}$, and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$.

The (positive) Grassmannian. The Grassmannian $\mathrm{Gr}_{k,n}$ is the space of all k-dimensional subspaces of $\mathbb{C}^n$. We denote its real points by $\mathrm{Gr}_{k,n}(\mathbb{R})$. We can represent a point $V \in \mathrm{Gr}_{k,n}$ as the row-span of a full-rank $k \times n$ matrix C. For $I = \{i_1 < \cdots < i_k\} \in \binom{[n]}{k}$, we let $\langle I \rangle_V = \langle i_1\, i_2 \ldots i_k \rangle_V$ be the $k \times k$ minor of C using the columns I. The $\langle I \rangle_V$ are called the *Plücker coordinates* of V and are independent of the choice of matrix representative C (up to common rescaling). The Plücker embedding $V \mapsto \{\langle I \rangle_V\}_{I \in \binom{[n]}{k}}$ embeds $\mathrm{Gr}_{k,n}$ into projective space.[*] If C has columns $v_1, \ldots, v_n$, we may also identify $\langle i_1\, i_2 \ldots i_k \rangle$ with $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$, hence e.g. $\langle i_1\, i_2 \ldots i_k \rangle = -\langle i_2\, i_1 \ldots i_k \rangle$. We occasionally will also work with $\mathrm{Gr}_{k,N}$, the Grassmannian of k-planes in a vector space with basis indexed by $N \subset [n]$.

Definition 1 [Positive Grassmannian (21, 22)]. We say that $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$ is totally nonnegative if (up to a global change of sign) $\langle I \rangle_V \geq 0$ for all $I \in \binom{[n]}{k}$. Similarly, V is totally positive if $\langle I \rangle_V > 0$ for all $I \in \binom{[n]}{k}$. We let $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$ denote the set of totally nonnegative and totally positive elements of $\mathrm{Gr}_{k,n}$, respectively. $\mathrm{Gr}_{k,n}^{\geq 0}$ is called the totally nonnegative Grassmannian, or sometimes just the positive Grassmannian.

If we partition $\mathrm{Gr}_{k,n}^{\geq 0}$ into strata based on which Plücker coordinates are strictly positive and which are 0, we obtain a cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ into positroid cells (22). Each positroid cell S gives rise to a matroid $\mathcal{M}$, whose bases are precisely the k-element subsets I such that the Plücker coordinate $\langle I \rangle$ does not vanish on S; $\mathcal{M}$ is called a positroid. There are many ways to index positroid cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ (22), such as plabic graphs.

Definition 2: Let G be a plabic graph,[†] i.e. a planar graph embedded in a disk, with boundary vertices $1, 2, \ldots, n$ on the boundary of the disk and with internal vertices colored black or white. A perfect orientation $\mathcal{O}$ of G is an orientation of the edges so that each black vertex has a unique outgoing edge and each white vertex has a unique incoming edge. Each perfect orientation has the same number of sources, which are boundary vertices. The positroid associated to G is the collection $\mathcal{M} := \{I : I \text{ the source set of a perfect orientation of } G\}$.

Both $\mathrm{Gr}_{k,n}$ and $\mathrm{Gr}_{k,n}^{\geq 0}$ admit the following set of operations, which will be useful to us.

Definition 3 (Operations on the Grassmannian). We define the following maps on $\mathrm{Mat}_{k,n}$, which descend to maps (denoted in the same way) on $\mathrm{Gr}_{k,n}$ and $\mathrm{Gr}_{k,n}^{\geq 0}$:

- (cyclic shift) The map $\mathrm{cyc} : \mathrm{Mat}_{k,n} \to \mathrm{Mat}_{k,n}$ sends $v_1 \mapsto (-1)^{k-1} v_n$ and $v_i \mapsto v_{i-1}$, $2 \leq i \leq n$.
- (reflection) The map $\mathrm{refl} : \mathrm{Mat}_{k,n} \to \mathrm{Mat}_{k,n}$ sends $v_i \mapsto v_{n+1-i}$ and rescales the top row by $(-1)^{\binom{k}{2}}$.
- (zero column) The map $\mathrm{pre}_i : \mathrm{Mat}_{k,[n]\setminus\{i\}} \to \mathrm{Mat}_{k,n}$ adds a zero column at i.

The Amplituhedron. Building on refs. 23 and 24, Arkani-Hamed and Trnka (1) introduced the (tree) amplituhedron, which is the image of the positive Grassmannian under a positive linear map. Let $\mathrm{Mat}_{n,p}^{>0}$ denote the set of $n \times p$ matrices whose maximal minors are positive.

Definition 4 (Amplituhedron). Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$, where $k + m \leq n$. The amplituhedron map $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ is defined by $\tilde{Z}(C) := CZ$, where as usual we identify the matrices C, CZ with their rowspans. The amplituhedron $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ is the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$.

The amplituhedron is km-dimensional, so it is full-dimensional in $\mathrm{Gr}_{k,k+m}$. We will be concerned with decompositions of $\mathcal{A}_{n,k,m}(Z)$ into full-dimensional images of particular positroid cells.

Definition 5 (Tiles). Fix k, n, m with $k + m \leq n$ and choose $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Given a positroid cell S of $\mathrm{Gr}_{k,n}^{\geq 0}$, we let $Z_S^{\circ} := \tilde{Z}(S)$ and $Z_S := \overline{\tilde{Z}(S)} = \tilde{Z}(\overline{S})$. We call Z_S and Z_S° a *tile* and an *open tile* for $\mathcal{A}_{n,k,m}(Z)$ if $\dim(S) = km$ and $\tilde{Z}$ is injective on S.

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[*] We will sometimes abuse notation and identify C with its row-span; we will also drop the subscript V on Plücker coordinates when it does not cause confusion.

[†] We will always assume that plabic graphs are reduced (22, definition 12.5).

Definition 6 (Tilings). A tiling of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{Z_S \mid S \in \mathcal{C}\}$ of tiles, such that their union equals $\mathcal{A}_{n,k,m}(Z)$ and all open tiles in $\{Z_S^\circ \mid S \in \mathcal{C}\}$ are pairwise disjoint.

There is a natural notion of facet of a tile, generalizing the notion of facet of a polytope.

Definition 7 (Facet of a cell and a tile). Given two positroid cells S' and S, we say that S' is a facet of S if $S' \subset \partial S$ and S' has codimension 1 in $\overline{S}$. If S' is a facet of S and Z_S is a tile of $\mathcal{A}_{n,k,m}(Z)$, we say that $Z_{S'}$ is a facet of Z_S if $Z_{S'} \subset \partial Z_S$ and has codimension 1 in Z_S.

In order to describe tiles and their facets, we will use functions that are adapted to the amplituhedron map.

Definition 8 (Twistor coordinates). Fix $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ with rows $Z_1, \ldots, Z_n \in \mathbb{R}^{k+m}$. Given $Y \in \mathrm{Gr}_{k,k+m}$ with rows $y_1, \ldots, y_k$, and $\{i_1, \ldots, i_m\} \subset [n]$, we define the twistor coordinate $\langle\langle i_1 i_2 \cdots i_m \rangle\rangle$ to be the determinant of the matrix with rows $y_1, \ldots, y_k, Z_{i_1}, \ldots, Z_{i_m}$.

Note that the twistor coordinates are defined only up to a common scalar multiple. An element of $\mathrm{Gr}_{k,k+m}$ is uniquely determined by its twistor coordinates (2). Moreover, $\mathrm{Gr}_{k,k+m}$ can be embedded into $\mathrm{Gr}_{m,n}$ so that the twistor coordinate $\langle\langle i_1 \ldots i_m \rangle\rangle$ is the pullback of the Plücker coordinate $\langle i_1, \ldots, i_m \rangle$ in $\mathrm{Gr}_{m,n}$.

Definition 9: We refer to a homogeneous polynomial in twistor coordinates as a functionary. For $S \subseteq \mathrm{Gr}_{k,n}^{\geq 0}$, we say a functionary F has a definite sign $s \in \{\pm 1\}$ on Z_S° if for all $Z \in \mathrm{Mat}_{n,k+4}^{>0}$ and for all $Y \in Z_S^\circ$, $F(Y)$ has sign s.

We will use functionaries to describe BCFW tiles.

Cluster Algebras. Cluster algebras were introduced by Fomin and Zelevinsky in ref. 25, motivated by the study of total positivity; see ref. 26 for an introduction. We quickly recall notation for cluster algebras of geometric type from quivers.

A quiver Q is a finite directed graph. We require that quivers do not have any oriented cycles of length 1 or 2.

Definition 10: Choose $s \geq r$ positive integers. Let $\mathcal{F}$ be a field of rational functions in r independent variables over $\mathbb{C}(x_{r+1}, \ldots, x_s)$. A labeled seed in $\mathcal{F}$ is a pair $(\mathbf{x}, Q)$, where $\mathbf{x} = (x_1, \ldots, x_s)$ forms a free generating set for $\mathcal{F}$ and Q is a quiver on $[s]$, where vertices $[r]$ are called mutable and vertices $r+1, \ldots, s$ are called frozen. The tuple $\mathbf{x}$ is a cluster. The functions $\{x_1, \ldots, x_r\}$ are mutable cluster variables and the functions $c = \{x_{r+1}, \ldots, x_s\}$ are frozen cluster variables.

An operation called mutation produces new seeds (25).

Definition 11 (Quiver Mutation). Let Q be a quiver and let k be a vertex. The mutated quiver $\mu_k(Q)$ has the same vertex set as Q and its set of arrows is obtained as follows:

1. for each subquiver $i \to k \to j$, add a new arrow $i \to j$;
2. reverse all arrows with source or target k;
3. remove all 2-cycles.

Definition 12 (Seed mutation). Let $(\mathbf{x}, Q)$ be a seed in $\mathcal{F}$ and let k be a mutable vertex of Q. The mutated seed $\mu_k(\mathbf{x}, Q) = (\mathbf{x}', Q')$ is another seed in $\mathcal{F}$, with $Q' = \mu_k(Q)$ and $\mathbf{x}' = \mathbf{x} \setminus \{x_k\} \cup \{x_k'\}$, where the new cluster variable x_k' is determined by the exchange relation

$$x_k' \, x_k = \prod_{i \to k \text{ in } Q} x_i + \prod_{i \leftarrow k \text{ in } Q} x_i.$$

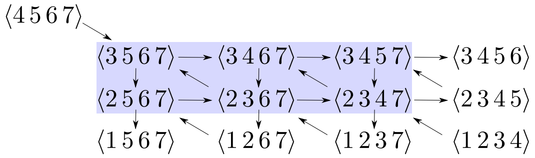


Fig. 1. The rectangle seed $\Sigma_{4,7}$. Mutable variables are in the colored box.

Definition 13 (Cluster algebra). Given a seed $(\mathbf{x}, Q)$ in $\mathcal{F}$, we denote as $\mathcal{X}$ the union of all mutable variables of all the seeds obtained from $(\mathbf{x}, Q)$ by any sequence of mutations. Let $\mathbb{C}[c^{\pm 1}]$ be the ground ring consisting of Laurent polynomials in the frozen variables. The cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is the $\mathbb{C}[c^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by all mutable variables, with coefficients which are Laurent polynomials in the frozen variables: $\mathcal{A}(\mathbf{x}, Q) = \mathbb{C}[c^{\pm 1}][\mathcal{X}]$. A subset C of cluster variables in $\mathcal{X}$ are compatible if there exists a seed $(\mathbf{x}', Q')$ obtained from $(\mathbf{x}, Q)$ by a sequence of mutations such that C is a subset of the cluster $\mathbf{x}'$.

The coordinate ring of the Grassmannian $\mathrm{Gr}_{k,n}$ is a cluster algebra (19). One may take the rectangles seed $\Sigma_{k,n}$ as the initial seed (27); see Fig. 1.

Theorem 14 (19). *Let* $\mathrm{Gr}_{k,n}^\circ$ *be the open subset of the Grassmannian where the frozen variables do not vanish. Then the coordinate ring* $\mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}^\circ]$ *of the affine cone over* $\mathrm{Gr}_{k,n}^\circ$ *is the cluster algebra* $\mathcal{A}(\Sigma_{k,n})$.

Moreover, the operations on the Grassmannian cyc, refl, pre in Definition 3 induce maps on $\mathbb{C}[\widehat{\mathrm{Gr}}_{k,N}^\circ]$ which are compatible with the cluster structure.

Proposition 15. *The pullbacks* $\mathrm{cyc}^*, \mathrm{refl}^* : \mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}^\circ] \to \mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}^\circ]$, $\mathrm{pre}_i^* : \mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}^\circ] \to \mathbb{C}[\widehat{\mathrm{Gr}}_{k,[n]\setminus\{i\}}^\circ]$ *take cluster variables to cluster variables and preserve compatibility and exchange relations.*

Results

Our first main result is that a class of cells called BCFW cells give rise to tiles for $\mathcal{A}_{n,k,4}(Z)$. We will build BCFW cells recursively using the BCFW product. Let us first introduce some notation we will use throughout this section.

Notation 16. *Choose integers* $1 \leq a < b < c < d < n$ *with* a, b *and* c, d, n *consecutive. Let*[‡] $N_L = \{1, 2, \ldots, a, b, n\}$, $N_R = \{b, \ldots, c, d, n\}$ *and* $B = (a, b, c, d, n)$. *Also fix* $k \leq n$ *and two nonnegative integers* $k_L \leq |N_L|$ *and* $k_R \leq |N_R|$ *such that* $k_L + k_R + 1 = k$. *Note that, for any set of indices* $N \subset [n]$, *our results hold with* N *instead of* $[n]$, *by replacing* 1 *and* n *in the definition with the smallest and largest elements of* N, *respectively.*

Definition 17 (BCFW product). Using Notation 16, let $S_L \subseteq \mathrm{Gr}_{k_L,N_L}^{\geq 0}$, $S_R \subseteq \mathrm{Gr}_{k_R,N_R}^{\geq 0}$ be positroid cells and G_L, G_R be the respective plabic graphs. The BCFW product of S_L and S_R is the positroid cell $S_L \bowtie S_R \subseteq \mathrm{Gr}_{k,n}$ corresponding to the plabic graph in the Right-hand side of Fig. 2.

We now define the family of BCFW cells to be the set of positroid cells which is closed under the operations in *Definitions* 3 and 17.

[‡] Note that we will overload the notation and let n index an element of a vector space basis for different vector spaces; however, in what follows, the meaning should be clear from context.

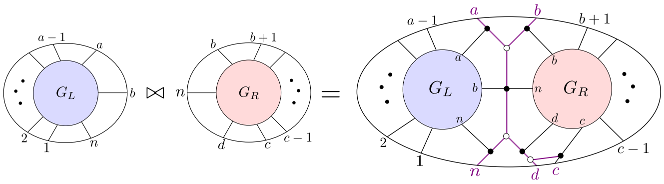


Fig. 2. The BCFW product $S_L \bowtie S_R$ of S_L and S_R in terms of their plabic graphs.

Definition 18 (BCFW cells). The set of BCFW cells is defined recursively. For $k = 0$, the trivial cell $\mathrm{Gr}^{\geq 0}_{0,n}$ is a BCFW cell. If S is a BCFW cell, so are $\mathrm{cyc}\, S$, $\mathrm{refl}\, S$ and $\mathrm{pre}_i\, S$. If S_L, S_R are BCFW cells, so is their BCFW product $S_L \bowtie S_R$.

Example 19: For $k = 1$, the BCFW cells in $\mathrm{Gr}^{\geq 0}_{1,n}$ have plabic graphs as in Fig. 3 (*Left*). The associated positroid has bases $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}$. In Fig. 3 (*Right*), $S_{ex} \subset \mathrm{Gr}^{\geq 0}_{2,7}$ is obtained as $S_L \bowtie S_R$, with S_L, S_R BCFW cells in $\mathrm{Gr}^{\geq 0}_{1,N_L}$, $\mathrm{Gr}^{\geq 0}_{0,N_R}$ respectively, with $N_L = \{1, 2, 3, 4, 7\}$, $N_R = \{4, 5, 6, 7\}$ and $B = (3, 4, 5, 6, 7)$.

Our first main result shows that images of BCFW cells are in fact tiles in $\mathcal{A}_{n,k,4}(Z)$.

Theorem 20 (BCFW tiles). *The amplituhedron map is injective on each BCFW cell. That is, the closure $Z_S := \overline{\tilde{Z}(S)}$ of the image of a BCFW cell S is a tile, which we refer to as a BCFW tile.*

A key ingredient to prove *Theorem* 20 is inverting the amplituhedron map on BCFW tiles (16, theorem 7.7) by using product promotion—an operation which interacts nicely both with the cluster structure on the Grassmannian and with the BCFW product.

Definition 21: Given 5 vectors v_i, v_j, v_r, v_s, v_q, we set $(ij) \cap (rsq) := v_i\langle j\,r\,s\,q\rangle - v_j\langle i\,r\,s\,q\rangle = -v_r\langle i\,j\,s\,q\rangle + v_s\langle i\,j\,r\,q\rangle - v_q\langle i\,j\,r\,s\rangle$, which is in the intersection of the subspace spanned by v_i, v_j and that spanned by v_r, v_s, v_q. Using Notation 16, product promotion is the homomorphism

$$\Psi_B = \Psi : \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_L}) \times \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_R}) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,n}),$$

induced by the following substitution:

$$\text{on } \widehat{\mathrm{Gr}}_{4,N_L}\text{: } v_b \;\mapsto\; \frac{(ba) \cap (cdn)}{\langle a\,c\,d\,n\rangle},$$

$$\text{on } \widehat{\mathrm{Gr}}_{4,N_R}\text{: } v_n \;\mapsto\; \frac{(ba) \cap (cdn)}{\langle a\,b\,c\,d\rangle},\; v_d \;\mapsto\; \frac{(dc) \cap (abn)}{\langle a\,b\,c\,n\rangle}.$$

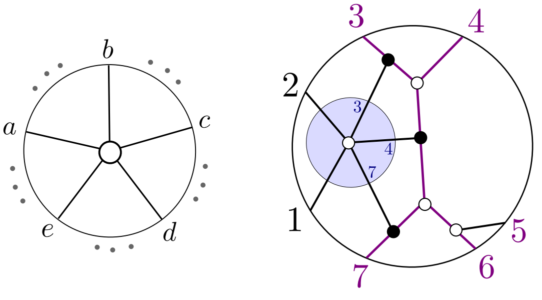


Fig. 3. Plabic graphs of a BCFW cell in $\mathrm{Gr}^{\geq 0}_{1,n}$ (*Left*) and in $\mathrm{Gr}^{\geq 0}_{2,7}$ (*Right*).

We show that Ψ is a quasi-homomorphism from the cluster algebra[§] $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$ to a subcluster algebra of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,n}]$. See (28, definition 3.1, Proposition 3.2) for the precise definition of a quasi-homomorphism.

Theorem 22. *Product promotion Ψ is a quasi-homomorphism of cluster algebras. In particular, Ψ maps a cluster variable (respectively, cluster) of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$, to a cluster variable (respectively, subcluster) of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,n}]$, up to multiplication by Laurent monomials in $\mathcal{T}' := \{\langle a\,b\,c\,n\rangle, \langle a\,b\,c\,d\rangle, \langle b\,c\,d\,n\rangle, \langle a\,c\,d\,n\rangle\}$.*

Product promotion sends cluster variables to cluster variables up to scaling by a Laurent monomial factor. It will also be convenient to strip off this factor.

Definition 23: Let x be a cluster variable of $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}]$ or $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$. We define the rescaled product promotion $\overline{\Psi}(x)$ of x to be the cluster variable of $\mathrm{Gr}_{4,n}$ obtained from $\Psi(x)$ by removing the Laurent monomial[¶] in $\mathcal{T}'$ (c.f. *Theorem* 22).

Example 24: For N_L and N_R as in *Example* 19, we have

$$\Psi(\langle 1\,2\,4\,7\rangle) = \frac{\langle 1\,2\,4\,7\rangle\langle 3\,5\,6\,7\rangle - \langle 1\,2\,3\,7\rangle\langle 4\,5\,6\,7\rangle}{\langle 3\,4\,6\,7\rangle}.$$

The remaining Plückers $\langle 1\,2\,3\,4\rangle$, $\langle 1\,2\,3\,7\rangle$, $\langle 1\,3\,4\,7\rangle$, $\langle 2\,3\,4\,7\rangle$ are fixed by Ψ. We have $\overline{\Psi}(\langle 1\,2\,4\,7\rangle) = \langle 1\,2\,4\,7\rangle\langle 3\,5\,6\,7\rangle - \langle 1\,2\,3\,7\rangle\langle 4\,5\,6\,7\rangle$. Note that this is a quadratic cluster variable in $\mathrm{Gr}_{4,7}$, obtained by mutating $\langle 2367\rangle$ in $\Sigma_{4,7}$ of Fig. 1.

The fact that product promotion is a cluster quasi-homomorphism may be of independent interest in the study of the cluster structure on $\mathrm{Gr}_{4,n}$. Much of the work thus far on the cluster structure of the Grassmannian has focused on cluster variables which are polynomials in Plücker coordinates with low degree; by contrast, the cluster variables obtained by repeated application of $\overline{\Psi}$ can have arbitrarily high degree in Plücker coordinates [e.g. see the chain polynomials in (16, theorem 8.3)].

Using rescaled product promotion and the operations in *Proposition* 15, we associate to each BCFW tile Z_S a collection of compatible cluster variables $\mathbf{x}(S)$ for $\mathrm{Gr}_{4,n}$. We ultimately use these cluster variables to invert $\tilde{Z}$ on the tile and to give a semialgebraic description of each tile.

Definition 25 (Cluster variables for BCFW tiles). Let $S \subset \mathrm{Gr}^{\geq 0}_{k,n}$ be a BCFW cell. We define the set of coordinate cluster variables $\mathbf{x}(S)$ for S recursively as follows:

- If $S = S_L \bowtie S_R$ with indices $B = (a, b, c, d, n)$, then
$$\mathbf{x}(S) = \overline{\Psi}_B(\mathbf{x}(S_L) \cup \mathbf{x}(S_R)) \cup \left\{\langle I\rangle, I \in \binom{B}{4}\right\}.$$
- If $S = \begin{cases} \mathrm{pre}_i(S') \\ \mathrm{cyc}(S') \\ \mathrm{refl}(S') \end{cases}$ then $\mathbf{x}(S) = \begin{cases} \mathbf{x}(S') \\ (\mathrm{cyc}^{-1})^*(\mathbf{x}(S')) \\ \mathrm{refl}^*(\mathbf{x}(S')) \end{cases}$.

For the base case $k = 0$, we set $\mathbf{x}(S) = \emptyset$.

[§] $\mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_L}] \times \mathbb{C}[\widehat{\mathrm{Gr}}^{\circ}_{4,N_R}]$ is a cluster algebra where each seed is the disjoint union of a seed of each factor.

[¶] If $x = \langle bcdn\rangle$, then $\overline{\Psi}(x) = \Psi(x) = x$.

For a BCFW cell S, $\mathbf{x}(S)$ depends on the sequence of operations in *Definition* 18 used to build S, but we will drop this dependence for brevity. Note that $\mathbf{x}(S)$ is a collection of compatible cluster variables for $\mathrm{Gr}_{4,n}$ (16, lemma 7.6).

Example 26: From *Example* 19, $S_{ex} = S_L \bowtie S_R$ with $B = (3,4,5,6,7)$. We have $\mathbf{x}(S_L) = \{\langle I\rangle, I \in \binom{B_L}{4}\}$, where $B_L = (1,2,3,4,7)$, and $\mathbf{x}(S_R) = \emptyset$. Then by *Example* 24 the coordinate cluster variables $\mathbf{x}(S_{ex})$ are $\langle 1234\rangle$, $\langle 1237\rangle$, $\langle 1247\rangle\langle 3567\rangle - \langle 1237\rangle\langle 4567\rangle$, $\langle 1347\rangle$, $\langle 2347\rangle$ from $\overline{\Psi}(\mathbf{x}(S_L))$ together with $\langle 3456\rangle$, $\langle 3457\rangle$, $\langle 3467\rangle$, $\langle 3567\rangle$, $\langle 4567\rangle$.

Recall from *Definitions* 8 and 9 the notion of twistor coordinates $\langle\langle I\rangle\rangle$ and functionaries. Given a cluster variable x in $\mathrm{Gr}_{4,n}$, we let $x(Y)$ denote the functionary on $\mathrm{Gr}_{k,k+4}$ obtained by identifying Plücker coordinates $\langle I\rangle$ in $\mathrm{Gr}_{4,n}$ with twistor coordinates $\langle\langle I\rangle\rangle$ in $\mathrm{Gr}_{k,k+4}$. Interpreting each cluster variable as a functionary in this way, we describe each BCFW tile as the semialgebraic subset of $\mathrm{Gr}_{k,k+4}$, where the coordinate cluster variables take on particular signs.

Theorem 27 (Sign description of BCFW tiles). *Let Z_S be a BCFW tile. For each element x of $\mathbf{x}(S)$, the functionary $x(Y)$ has a definite sign s_x on the open tile Z_S° and*

$$Z_S^\circ = \{Y \in \mathrm{Gr}_{k,k+4} : s_x\, x(Y) > 0 \text{ for all } x \in \mathbf{x}(S)\}.$$

Example 28: The open tile $Z^\circ_{S_{ex}} = \tilde{Z}(S_{ex})$, with S_{ex} from *Example* 19, is the semialgebraic set in $\mathrm{Gr}_{2,6}$ described by the following: $x(Y)$ is negative for $x \in \{\langle 3567\rangle, \langle 3457\rangle, \langle 2347\rangle, \langle 3567\rangle\}$ and $x(Y)$ is positive for all other $x \in \mathbf{x}(S_{ex})$ (computed in *Example* 26).

Theorem 27 is particularly relevant to *Theorem* 20 as the ability to describe tiles using cluster variables follows from the fact that we can invert the amplituhedron map using cluster variables (16, theorem 7.7).

We now turn to facets of BCFW tiles (cf. *Definition* 7); we will give functionaries which vanish on facets. The next result shows that each facet lies in the vanishing locus of a coordinate cluster variable (interpreting coordinate cluster variables as functionaries as above).

Theorem 29 (Cluster adjacency for BCFW tiles). *Let Z_S be a BCFW tile of $\mathcal{A}_{n,k,4}(Z)$. Each facet $Z_{S'}$ of Z_S lies on a hypersurface cut out by a functionary $F_{S'}(\langle\langle I\rangle\rangle)$ such that $F_{S'}(\langle I\rangle)$ is in $\mathbf{x}(S)$. Thus $\{F_{S'}(\langle I\rangle) : Z_{S'}$ a facet of $Z_S\}$ is a collection of compatible cluster variables of $\mathrm{Gr}_{4,n}$.*

Finally, we explain how to construct BCFW tilings of $\mathcal{A}_{n,k,4}(Z)$ (cf. *Definition* 6). This result shows that all ways of running the BCFW recurrence correspond to tilings of $\mathcal{A}_{n,k,4}(Z)$. *Theorems* 20 and 27 are important ingredients to prove *Theorem* 31.

In what follows, we use Notation 16, fix $n \geq k+4$, and define $b_{min} := 2$ if $k_L = 0$ and otherwise $b_{min} := k_L + 3$.

Definition 30 (BCFW collections).

We say a collection $\mathcal{T}$ of $4k$-dimensional BCFW cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ is a BCFW collection of cells for $\mathcal{A}_{n,k,4}$ if it has the following recursive form:

- If $k = 0$ or $k = n-4$, $\mathcal{T}$ is the single BCFW cell $\mathrm{Gr}_{k,n}^{>0}$.
- If $\mathcal{T} = \{S\}$ is a BCFW collection of cells, so is $\{\mathrm{refl}\, S\}_{S\in\mathcal{T}}$ and $\{\mathrm{cyc}\, S\}_{S\in\mathcal{T}}$.

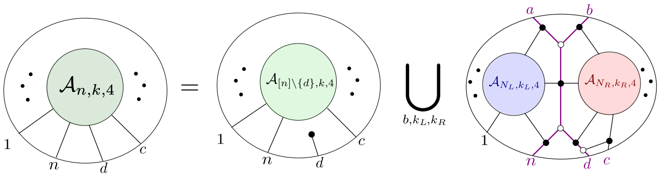


Fig. 4. BCFW tiling for $\mathcal{A}_{n,k,4}$. On the *Right*: The first term is obtained by tiling $\mathcal{A}_{[n]\setminus\{d\},k,4}$ (from $\mathcal{T}_{pre}$); the second term is the union over b, k_L, k_R as in *Definition* 30 of the collections of tiles obtained by tiling $\mathcal{A}_{N_L,k_L,4}$ and $\mathcal{A}_{N_R,k_R,4}$ (from $\mathcal{T}_{k_L,k_R,b}$).

- Otherwise $\mathcal{T} = \mathcal{T}_{pre} \bigsqcup_{b,k_L,k_R} \mathcal{T}_{k_L,k_R,b}$, where k_L, k_R are as in Notation 16, b ranges from b_{min} to $n-3-k_R$, and
  - $\mathcal{T}_{pre} = \{\mathrm{pre}_d(S)\}_{S\in\mathcal{C}}$, where $\mathcal{C}$ is a BCFW collection of cells for $\mathcal{A}_{[n]\setminus\{d\},k,4}$;
  - $\mathcal{T}_{k_L,k_R,b} = \{S_L \bowtie S_R\}_{(S_L,S_R)\in\mathcal{C}_L\times\mathcal{C}_R}$ where $\mathcal{C}_L$ and $\mathcal{C}_R$ are BCFW collections of cells for $\mathcal{A}_{N_L,k_L,4}$ and $\mathcal{A}_{N_R,k_R,4}$.

The definition of BCFW collections comes directly from the BCFW recursion. Written graphically (Fig. 4), the BCFW recursion expresses the scattering amplitude as a sum of terms indexed by cells in a BCFW collection.

Theorem 31 (BCFW tilings). *Every BCFW collection of cells $\mathcal{T} = \{S\}$ as in Definition 30 gives rise to a tiling $\{Z_S\}_{S\in\mathcal{T}}$ of the amplituhedron $\mathcal{A}_{n,k,4}(Z)$, which we refer to as a BCFW tiling.*

Theorem 31 generalizes the main result of ref. 15, which proved the same result for the standard BCFW cells. *Theorem* 31 also proves the main conjecture of ref. 1.

Tiles which do not come from the BCFW recurrence are also expected to satisfy cluster adjacency, have a sign description in terms of cluster variables, and appear in tilings of $\mathcal{A}_{n,k,4}(Z)$. By leveraging the methods presented in this paper, we provided an example in (29, section 5).

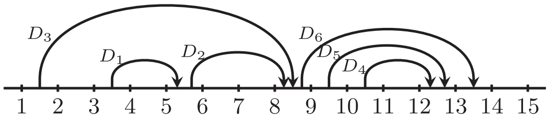
Standard BCFW Tiles. A BCFW cell is called standard if it is obtained from the trivial cell $\mathrm{Gr}_{0,n}^{>0}$ by a series of BCFW products and applying pre_i at the penultimate index. The standard BCFW tiles are the images of standard BCFW cells under the amplituhedron map. The collection of standard BCFW tiles for a fixed n, k forms one tiling of the amplituhedron (15). We can sharpen a number of our results in the standard setting. Standard BCFW tiles are in bijection with chord diagrams.

Definition 32 [Chord diagram (15)]. Let $k, n \in \mathbb{N}$. A chord diagram $D \in \mathcal{CD}_{n,k}$ is a set of k quadruples, called chords, in the marker set $\{1,\ldots,n\}$

$$D = \{(a_1,b_1,c_1,d_1),\ldots,(a_k,b_k,c_k,d_k)\}$$

such that every chord $D_i = (a_i,b_i,c_i,d_i) \in D$ satisfies $1 \leq a_i < b_i = a_i + 1 < c_i < d_i = c_i + 1 \leq n-1$ and no two chords $D_i, D_j \in D$ satisfy $a_i = a_j$ or $a_i < a_j < c_i < c_j$.

See Fig. 5 for an example. Notice that the chords are noncrossing and no two chords start in the same place. For



$\{(3,4,5,6),(5,6,8,9),(1,2,8,9),(10,11,12,13),(9,10,12,13),(8,9,13,14)\}$

Fig. 5. A chord diagram D with $k = 6$ chords and $n = 15$ markers.

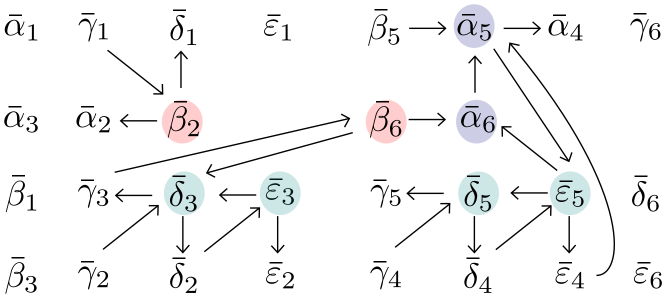


Fig. 6. The quiver Q_D for the chord diagram in Fig. 5. The highlighted variables are the mutable variables, which do not cut out facets of the tile Z_D.

the details of the correspondence between standard BCFW tiles and chord diagrams, see (16, definition 6.12); each chord D_i corresponds to a BCFW product with $B_i = (a_i, b_i, c_i, d_i, n)$. Recall that the corresponding product promotion adds $\{\langle I\rangle : I \in \binom{B_i}{4}\}$ to the coordinate cluster variables. In what follows, we denote by $\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{\delta}_i, \bar{\varepsilon}_i$ the coordinate cluster variables obtained from $\langle b_ic_id_in\rangle$, $\langle a_ic_id_in\rangle$, $\langle a_ib_id_in\rangle$, $\langle a_ib_ic_in\rangle$, $\langle a_ib_ic_id_i\rangle$, respectively, by applying some product promotions.

A chord D_i is a child of the chord immediately above it, which is its parent. For example, D_4 has parent D_5 in Fig. 5. Two chords are same-end if their ends occur in a common segment $(e, e+1)$ (e.g. D_2, D_3); are head-to-tail if the first ends in the segment where the second starts (e.g. D_1, D_3); and are sticky if their starts lie in consecutive segments $(s, s+1)$ and $(s+1, s+2)$ (e.g. D_5, D_6).

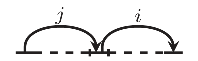
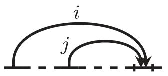
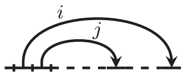
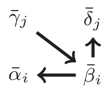
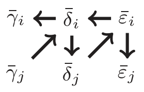
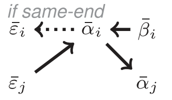
Given a chord diagram D corresponding to a standard BCFW cell S_D, we give explicit formulas for the coordinate cluster variables $\mathbf{x}(S_D)$ and their signs on the BCFW tile Z_D in (16, theorem 8.4, proposition 8.10). We also determine which coordinate cluster variables cut out facets of Z_D.

Theorem 33. *Let $\bar{\zeta}_i$ be a coordinate cluster variable for Z_D, considered as a functionary. A facet of Z_D lies on the hypersurface $\{\bar{\zeta}_i = 0\}$ if and only if:*

- *$\bar{\zeta}_i = \bar{\alpha}_i$ and D_i does not have a sticky child, or*
- *$\bar{\zeta}_i = \bar{\beta}_i$ and D_i does not start where another chord ends or have a sticky same-end parent, or*
- *$\bar{\zeta}_i = \bar{\gamma}_i$, or*
- *$\bar{\zeta}_i \in \{\bar{\delta}_i, \bar{\varepsilon}_i\}$ and D_i does not have a same-end child.*

We call $\bar{\zeta}_i \in \mathbf{x}(S_D)$ frozen if a facet of Z_D lies in its zero locus; otherwise we call ζ_i mutable. Recall that the coordinate cluster variables $\mathbf{x}(S_D)$ are compatible cluster variables for $\mathrm{Gr}_{4,n}$, and so appear together in some seed $\Sigma(D)$. We additionally determine all arrows in the quiver for $\Sigma(D)$ which involve the mutable variables of $\mathbf{x}(S_D)$ (i.e., Fig. 6).

Theorem 34. *Let $D \in \mathcal{CD}_{n,k}$, let S_D be the standard BCFW cell, and let $\Sigma_D = (\mathbf{x}, Q)$ be a seed for $\mathrm{Gr}_{4,n}$ whose cluster contains $\mathbf{x}(S_D)$. Then the arrows of Q which involve the mutable variables of $\mathbf{x}(S_D)$ are as follows: For each chord D_i, check if it forms a configuration in the table below, and if so, draw the indicated arrows.*[#]

 *head-to-tail left sibling* D_j	 *same-end child* D_j	 *sticky child* D_j
		

Data, Materials, and Software Availability. There are no data underlying this work.

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[#] In the third configuration, if D_i and D_j end in the same place, then the dotted arrow from $\bar{\alpha}_i$ to $\bar{\varepsilon}_i$ appears, along with the arrows from the second and third configuration.

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