

Mapping the topological proximity-induced gap in multiterminal Josephson junctions

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Multiterminal Josephson junctions (MTJJs), devices in which a normal metal is in contact with three or more superconducting leads, have been proposed as artificial analogs of topological crystals. The topological nature of MTJJs manifests as a modulation of the quasiparticle density of states (DOS) in the normal metal that may be probed by tunneling measurements. We show that one can reveal this modulation by measuring the resistance of diffusive MTJJs with normal contacts, which shows rich structure as a function of the phase differences $\{\phi_i\}$. Our approach demonstrates a simple yet powerful technique for exploring topological effects in MTJJs.

The superconducting proximity effect, where a normal metal (N) acquires superconducting properties when placed in good electrical contact with a superconductor (S), is characterized by the charge-transfer process at the NS interface. A normal electron incident on the boundary is retro-reflected as a hole producing a correlated pair in S, a process known as Andreev reflection. In the case of a clean normal metal connected to two superconductors, Andreev reflection at the NS interfaces results in discrete quasiparticle energy levels within the normal metal. The energy of these Andreev bound states is periodic in the phase difference between superconducting contacts ϕ similar to the periodicity of the electronic dispersion in a one-dimensional crystal with crystal momentum k . This similarity has led to the suggestion of creating artificial analogs of n dimensional crystals with a normal metal in contact with $n + 1$ superconductors with n distinct phase differences $\{\phi_i\}$, with predictions that the resulting band structure may be topologically non-trivial [1]. The ability to access different topological regions simply by tuning the phase differences $\{\phi_i\}$ is an attractive feature of this system. In the ballistic case, theory predicts topologically distinct gapped regions in the “momentum” space defined by the phases $\{\phi_i\}$ separated from each other by regions where the gap closes, as well as quantized conductance between the superconducting contacts in the gapped regime in direct analogy with real topological materials [1–3]. While the modulation of the gap has been observed in tunneling measurements [4, 5], quantized conductance in transport measurements has not been observed to our knowledge [6–8].

If the normal metal is diffusive, one does not expect well-defined Andreev levels. In this case, the topological behavior is tied to the winding numbers of the phase of the quasiclassical Green’s function describing the proximitized normal metal [9–11]. A modulation of the quasiparticle DOS with the phases $\{\phi_i\}$ is also predicted, with different gapped regions in the quasiparticle DOS defined by unique topological indices separated by regions where

the gap closes. Tunneling measurements on a diffusive device with three superconducting contacts have indeed shown the predicted modulation of the DOS with the phases ϕ_1 and ϕ_2 , albeit only along the line $\phi_2 = -\phi_1$ as the phase modulation was achieved using a uniform external magnetic field [9].

In our experience, fabricating devices for tunneling measurements is challenging; it is easier to fabricate devices for electrical transport measurements. The initial theoretical studies for diffusive systems were for a normal metal connected only to superconducting contacts where the modulation of the DOS is uniform throughout the normal metal. To perform electrical transport measurements on the proximity-coupled normal metal itself, one also needs to attach normal contacts. In this case, the DOS is no longer uniform in the normal metal but varies as a function of position between the normal contacts. Nevertheless, with appropriately designed devices, there can be a region in the diffusive normal metal where the DOS mimics the DOS of a device with only superconducting contacts. More interestingly, measurements of the resistance of the *entire* device reflect this DOS [12]. This is demonstrated in Fig. 1, which shows the DOS and resistance of a diffusive normal metal with three superconducting contacts and two normal contacts (Fig. 1(a)) as a function of the phase differences ϕ_1 and ϕ_2 between the superconducting contacts, calculated using the quasiclassical superconducting Green’s function technique. The parameters used in this simulation are based on Sample 1 of this study: details of the calculations can be found in the *End Matter* and in Ref. 12. The DOS at the juncture of the normal metal wires (Fig. 1(b)) is very similar to the earlier predictions for diffusive normal metals connected to only 3 superconducting contacts [9, 10], with a large modulation of the DOS and gap closings at specific points in the phase space. Calculations of the resistance between the two normal reservoirs (Fig. 1(c)) closely track this DOS, with the resistance achieving its normal state value R_n when the gap in the DOS closes, although there is more structure in the resistance in comparison to the DOS.

The modulation of the DOS and resistance with $\{\phi_i\}$ reflects the changes in the retarded Green’s function \hat{G}^R

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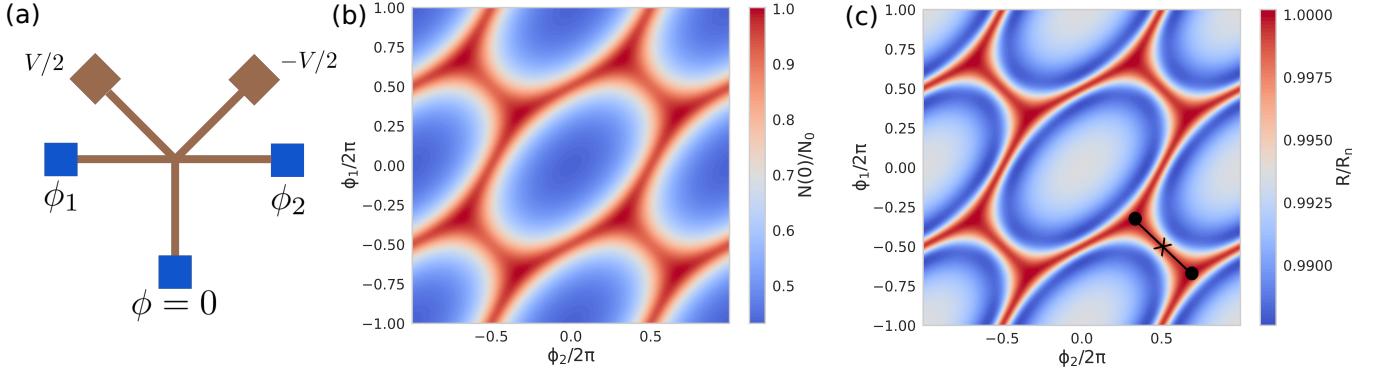


FIG. 1. (a) Schematic of the sample used for the numerical simulations. Gold represents the normal metal, with the squares being the normal contacts, while blue represents the superconducting contacts. A voltage V is applied symmetrically between the two normal contacts. The phases of the superconducting contacts are specified as shown. (b) Density of states at the Fermi energy $N(0)$ at the junction of all the normal wires normalized to the normal state value N_0 as a function of the phase differences ϕ_1 and ϕ_2 (c) Low temperature resistance R between the two normal contacts normalized to the normal state resistance R_n .

that describes the proximitized normal metal. \hat{G}^R can be parameterized by two complex parameters θ and χ [13]

$$\hat{G}^R = \begin{pmatrix} \cos \theta & \sin \theta e^{i\chi} \\ \sin \theta e^{-i\chi} & -\cos \theta \end{pmatrix}. \quad (1)$$

Here, θ and χ characterize the strength of the superconducting pair correlations and their gauge-invariant phase, respectively. $\theta = \pi/2$ at a superconducting contact and vanishes at a normal contact, while χ is set at the superconducting contacts by the imposed phases $\{\phi_i\}$. In the proximitized normal metal, θ and χ will vary between these values. As pointed out by Strambini *et al.* [9], the resulting state in the proximitized normal metal can be mapped on the northern hemisphere of a Bloch sphere, with θ representing the polar and χ representing the azimuthal angle. The equator of the Bloch sphere represents the fully gapped (superconducting) state while the North pole represents the normal state. In terms of transitions between topologically distinct states, Strambini *et al.* define the gapless state for their geometry as whenever the vector defining the Green's function does *not* lie on the equator, i.e., when $\theta < \pi/2$. As our device has normal contacts in addition to superconducting contacts, θ will approach some maximum value $\theta_{max} < \pi/2$ but never equal $\pi/2$ in the proximitized normal metal. Nevertheless, the overall behavior of θ and χ is similar to that expected in a diffusive metal with only superconducting contacts [9, 10], with the gapless state now being defined by $\theta < \theta_{max}$. This is demonstrated in Fig. 2(a) and (b), which shows the real parts of θ and χ in the center of the wire of Fig. 1(a) as a function of ϕ_1 and ϕ_2 (the imaginary parts are effectively zero to numerical accuracy). There are regions with large θ corresponding to the gapped areas in Fig. 1(b). Following the convention of Ref. [9], we label these regions with topological indices $(n_1 n_2)$ corresponding to the winding numbers of the phases ϕ_1 and ϕ_2 , as shown in Fig. 2(a). These regions are separated by narrow sections where $\theta \sim 0$ and

$\chi = 0$, corresponding to the areas of phase space where the system approaches the normal state. One can map trajectories between adjacent minima in this phase space on the Bloch sphere shown in Fig. 2(c). The blue trajectory in Fig. 2(c) corresponds to the blue trace in Fig. 2(a) from the minimum specified by the winding numbers (00) to the minimum specified by the winding numbers (11), while the red trajectories correspond to going from (00) to (-11). In the case of the blue trajectory, both θ and χ vary, while for the red trajectory χ is always 0, and hence the red trajectory on the Bloch sphere appears an arc of a meridian. Note that for both trajectories, as mentioned earlier, the maximum value of θ is $\theta_{max} \sim 0.36\pi$ rather than 0.5π as would be expected for a MTJJ with no normal contacts. In addition, the blue trajectory does not approach the north pole of the Bloch sphere corresponding to the completely normal state, while the red trajectory does (twice, in fact, although this is not discernible as χ is always 0). The reason for this can be seen from Fig. 2(a) where the blue line passes a local maximum in the DOS but not regions corresponding to $\theta = 0$. θ_{max} can be pushed closer towards $\pi/2$ by decreasing the distance between the superconducting contacts and the junction of all the normal wires, but it will never equal $\pi/2$ due to the presence of the normal contacts. However, the main point is that both trajectories shift from the maximum value of $\theta_{max} \sim 0.36\pi$ corresponding to the gapped state in going between minima identified by different winding numbers.

This analysis shows that resistance measurements on proximity-coupled devices are an attractive route to explore topological effects in diffusive MTJJs. To this end, we fabricated and measured three devices based on the geometry of Fig. 1(a) for this study: we discuss the results for two (Sample 1 and Sample 2) for which we have the most complete sets of data. These two devices differed slightly in sample dimensions as well as in the thickness of the normal metal (see the Supplementary

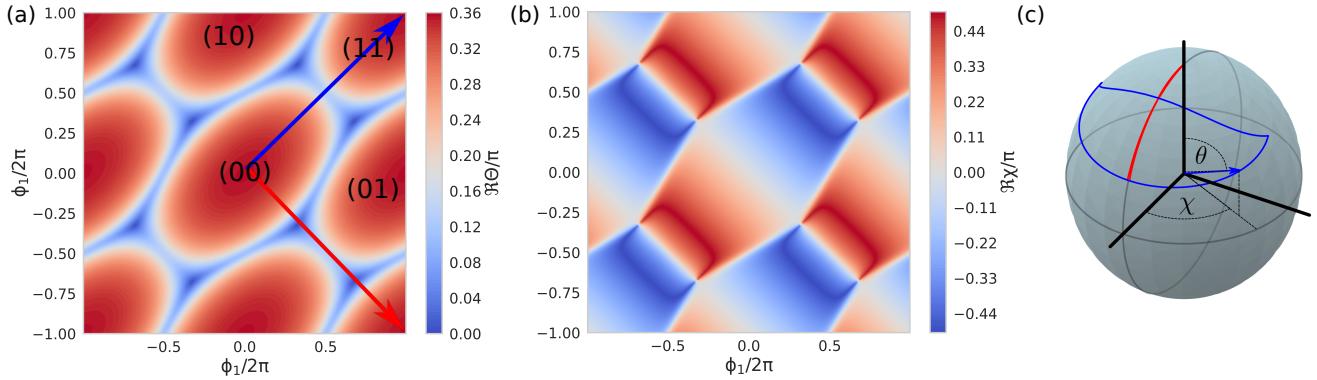


FIG. 2. (a) (b) Real part of θ (a) and χ (b) as the junction of all the normal wires (see Fig. 1(a)) as a function of ϕ_1 and ϕ_2 . ($n_1 n_2$) in (a) identify the topological indices that characterize different gapped regions. (c) Mapping of the blue and red trajectories shown in (a) on the Bloch sphere where the polar angle represents $\Re \theta$ and the azimuthal angle represents $\Re \chi$, and where the north pole denotes the completely normal state.

Materials for details; details of the fabrication can be found in the End Matter). Figure 3(a) shows a SEM image of Sample 1. In order to independently and statically vary the phase between the superconducting contacts and thus map out the entire phase space $\{\phi_1, \phi_2\}$, the superconducting contacts were connected to form two loops through which a magnetic field could be threaded by on-chip field coils generating a flux, as shown schematically in Fig. 3(a). The area of each flux loop was $\sim 5 \mu\text{m} \times 50 \mu\text{m}$. Measurements on a device in which one flux loop was broken showed that each field coil coupled only to its respective flux loop, with no measurable flux coupled to the other loop within the range of field coil drive currents used.

The resistance of the device was measured with standard 4-terminal low-frequency ac lock-in techniques with a modified Adler-Jackson resistance bridge [14] using low-frequency (< 120 Hz) ac excitation currents of amplitudes of $\sim 1 \mu\text{A}$. The ac excitation current was chosen experimentally to optimize the signal to noise without modifying the measured response. Field coil bias was provided by two home-built current sources driven by separate Agilent synthesizers. To avoid interference from line-frequency noise, the battery-operated first-stage pre-amplifiers were placed in a μ -metal shielded enclosure attached to the cryostat.

Two length scales in the normal metal determine the overall dimensions of the devices: the electron phase coherence length L_ϕ and the superconducting coherence length $\xi_N = \sqrt{\hbar D/kT}$ [13]. Here T is the temperature and $D = (1/3)v_F\ell$ the electronic diffusion coefficient, v_F being the Fermi velocity and ℓ the elastic scattering length of the electrons in the normal metal. Fits to the weak localization magnetoresistance (MR) [15] of co-deposited Au meander wires gave L_ϕ in excess of $7 \mu\text{m}$ at base temperature (~ 20 mK), so that the devices were entirely phase coherent in the measurement temperature range. Sheet resistance measurements of the devices gave $\xi_N \sim 0.5\mu\text{m}/\sqrt{T}$, with T in Kelvin. Thus $\xi_N \sim 3 \mu\text{m}$

at base temperature, also larger than the lengths of the normal metal wires in the samples.

Ideally, one would like the length of the normal metal wires between the superconducting contacts and the node at which these wires meet to be as short as possible in order to maximize the proximity effect and push the value of θ close to $\pi/2$, corresponding to the fully gapped state as discussed above. However, this leads to hysteresis in the response as a function of the flux threaded through each flux loop, as the large critical current of the proximitized region coupled with the self-inductance of the flux loops results in screening of the externally applied flux. In the first device we measured (Sample 1), the length between superconducting contacts L was $\sim 1 \mu\text{m}$, and the resistance traces of this device at base temperature were indeed hysteretic (see Supplementary Materials). The critical current of the proximitized region depends exponentially on the ratio L/ξ_N [16]. Since ξ_N decreases with increasing temperature, we can reach a non-hysteretic regime by raising the temperature. The lowest temperature at which this should occur is when $L \sim \xi_N$. Although we cannot directly measure the critical current between the superconducting contacts, from a comparison of the period of the oscillations seen at base temperature to the range over which no oscillations are observed on switching the sweep direction of the field coil, one can roughly estimate the temperature at which $L \sim \xi_N$. From the data shown in the Supplementary, we estimated this temperature to be ~ 600 mK for Sample 1.

Figure 3(b) shows the resistance of Sample 1 as a function of the current I_{dc1} through one field coil at a few different values over a narrow range of the current I_{dc2} through the other field coil at a temperature of 600 mK. Clear periodic oscillations as a function of I_{dc1} are observed that evolve systematically with I_{dc2} , with no hysteresis. The oscillations in general show an asymmetric, double-peaked structure, with the asymmetry evolving as I_{dc2} is varied over a narrow range. The peaks appear

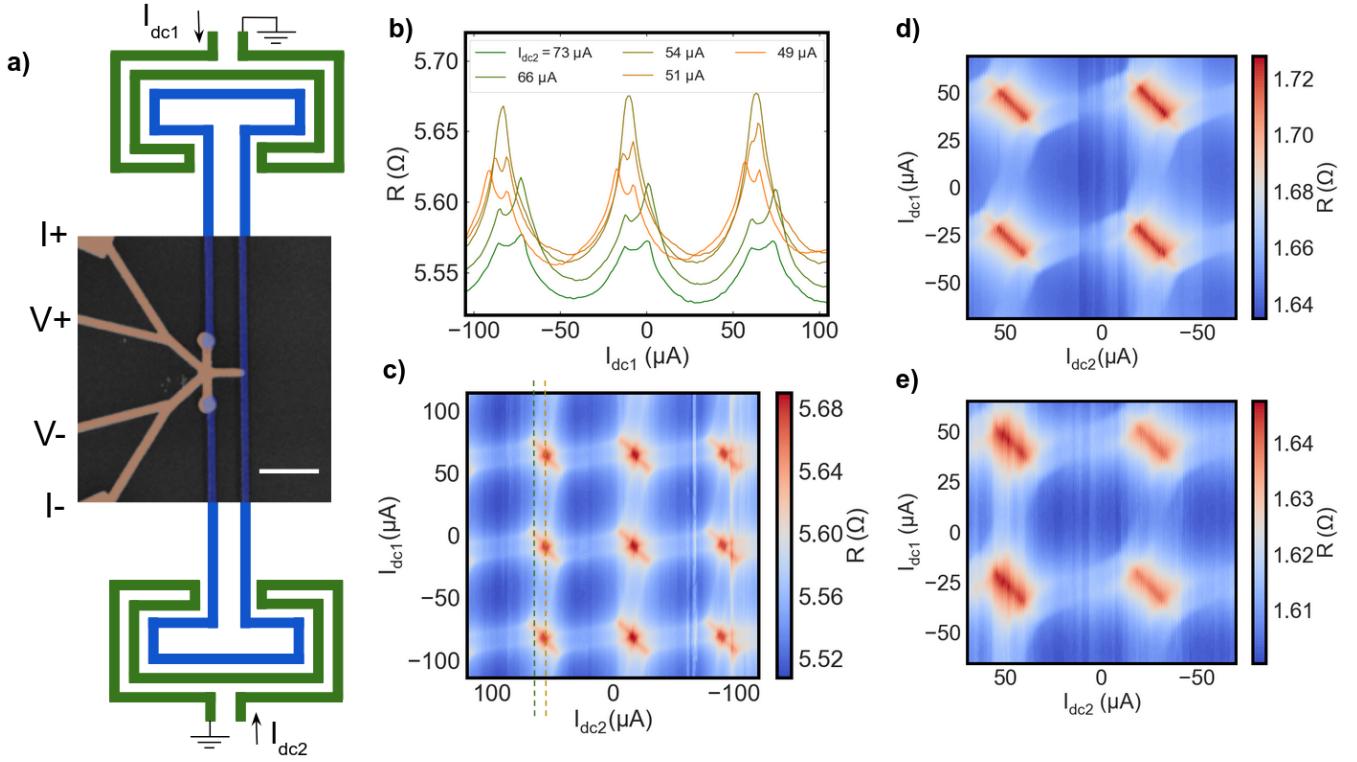


FIG. 3. (a) False-color SEM image of Sample 1. Gold (blue) corresponds to Au (Al). The scale bar is $1 \mu\text{m}$. The schematic shows the superconducting (Al) flux loops attached to the superconducting contacts, as well as the on-chip Al field-coils (green). The leads used for the 4-terminal resistance measurement are labeled. (b) Resistance of Sample 1 as a function of coil bias I_{dc1} at a few different values of I_{dc2} at 600 mK . (c) Contour map of the resistance in ohms of Sample 1 at 600 mK as a function of I_{dc1} and I_{dc2} . Vertical dotted lines show cuts corresponding to $I_{dc2} = 73 \mu\text{A}$ and $I_{dc2} = 54 \mu\text{A}$ in (b). (d), (e) Contour map of the resistance of Sample 2 at 300 mK and 30 mK , respectively.

to merge into a single maximum for $I_{dc2} = 54 \mu\text{A}$. To obtain the full phase diagram, we swept I_{dc1} over the range shown in Fig. 3(b) while stepping I_{dc2} sequentially to obtain the contour map shown in Fig. 3(c). Small current offsets likely arise from the remanent field of the superconducting solenoid of the dilution refrigerator. The contour map shares similar features with the results of the simulation shown in Fig. 1(c), with structure that is periodic in both I_{dc1} and I_{dc2} . Each period corresponds to one superconducting flux quantum $\Phi_0 = h/2e$ through an individual flux loop. As with the simulations, there are central areas of low resistance separated by regions of higher resistance. In particular, there are localized, periodic regions of high resistance; these correspond to the diagonal line shown in Fig. 1(c), and are responsible for the evolution of the traces with I_{dc2} seen in Fig. 3(b).

While the structure is similar, there are also significant differences. First, while the unit cell of the structure in the simulations of Fig. 1(c) is roughly elliptical in shape, it is rhomboidal in the experimental data seen in Fig. 3(c). Second, a key feature of the simulations is that the maxima in resistance (and correspondingly the maxima in $N(0)$) occur *not* at the points defined by $\phi_1, \phi_2 = (2n + 1)\pi$ (marked with the cross on the diagonal line in Fig. 1(c)), but at pairs of points in phase

space a short distance away along a diagonal (marked with filled circles in Fig. 1(c)). In the experimental data, these diagonals of high resistance can be observed, but the maxima in resistance do appear to occur at field coil currents corresponding to $\phi_1, \phi_2 = (2n + 1)\pi$. In the simulations, the regions of high resistance are connected to each other by narrow strips in phase space where the resistance almost reaches its maximum value. The experimental data also shows similar strips, but the resistance does not approach the maximum resistance seen near $\phi_1, \phi_2 = (2n + 1)\pi$. These regions correspond to the (almost) symmetrical double peaked oscillations shown in Fig. 3(b) for $I_{dc2} = 49 \mu\text{A}$. Indeed, it should be noted that the experimental data are perhaps more similar the results for an alternate geometry discussed in Ref. [12] which differs from Fig. 1(a) in that there are regions along the path in the normal metal between the normal reservoirs where the injected quasiparticle current overlaps with the induced supercurrent between the superconducting contacts. The wires in the experimental sample have finite width, unlike the idealized geometry of Fig. 1(a), so that there is some overlap between the quasiparticle current and the induced supercurrent. Finally, we note the absence of the local resistance maximum in the experimental data present at the center of

Fig. 1(c). We believe this might be due to the measurement averaging over the small decrease in resistance immediately prior to the peaks.

To confirm the reproducibility of these results, we fabricated and measured a second set of devices of the same design. In order to make measurements at the base temperature of our refrigerator without encountering hysteresis as noted above, these devices were fabricated with a length $L \sim 2 \mu\text{m}$ between the superconducting contacts. Figures 3(d) and (e) show the phase diagrams for one of these devices (Sample 2) at 300 mK and 30 mK respectively. These data are similar to those of Sample 1 shown in Fig. 3(c), except that the distinct maxima seen at $\phi_1, \phi_2 = (2n + 1)\pi$ are no longer visible. Sample 2 shows only diagonal lines of high resistance, more consistent with the simulations. There is no significant difference in the structure of the phase diagram between 30 mK and 300 mK, although the overall resistance is higher as expected, and the modulation of the resistance is larger at higher temperatures (5.2% compared to 2.7%). This is consistent with the numerical simulations (see Supplementary Materials).

The results of the simulations shown in Fig. 1 are obtained by imposing the phases $\{\phi_1, \phi_2\}$ at the superconducting contacts and the voltage at normal contacts before calculating the currents through all normal metal wires. In general, the phases ϕ_i need to be determined self-consistently by applying the usual flux quantization condition for each flux loop [17]

$$\phi_i + 2\pi \frac{\Phi_i}{\Phi_0} = \oint \nabla \phi \cdot d\mathbf{l} = 2\pi n_i \quad (2)$$

where n_i is the topological index introduced by Strambini *et al.* [9], ϕ_i is the (gauge-invariant) phase difference between the superconducting contact and the reference contact ($\phi = 0$) and Φ_i is the *net* flux enclosed by the corresponding loop. If the self-inductance L_i of the flux loop is not negligible, Φ_i is not equal to the flux Φ_{ext} generated by the on-chip field coils, but is given by $\Phi_i = \Phi_{ext} - L_i I_{si}$, I_{si} being the supercurrent flowing through the flux loop. In the experiments, we estimate the self-inductance of each flux loop to be approximately 0.1 nH and the screening supercurrents to be on the order of μA based on prior experience with similar proximity coupled devices [18], so that the flux generated can be an appreciable fraction of Φ_0 . We have clearly seen this screening effect in Sample 1 at low temperatures (see Supplementary Materials). In order to take the effects of the screening current into account, we have simulated the DOS and resistance of the device of Fig. 1(a) by self-consistently calculating the phases ϕ_1 and ϕ_2 on each superconducting contact using Eqn. (2). Theoretically, the maximum value of the critical supercurrent in a SNS junction in the long junction limit is $I_c \sim 10.82 E_c / e R_n$ assuming perfectly transparent normal-metal/superconductor interfaces [16]. Here $E_c = \hbar D / L^2$ is the Thouless energy, L being the length

of the normal metal between the two superconductors. In previous experiments, we have found that the critical current is typically a factor of ~ 20 smaller than this, likely due to the fact that the interface transparency is finite [18]. Since the numerical simulations, which assume perfect normal-metal/superconductor interfaces, overestimate the supercurrents in our device, we use a correspondingly smaller value of the self-inductances L_i in our simulation. The results of these simulations are shown in Fig. 4, where now the x and y axes represent the normalized external flux coupled to each flux loop. The overall structure is similar to that seen in Fig. 1, with an elliptical region in the center of each cell where the DOS is at a minimum with correspondingly smaller resistance, but the detailed structure is modified. In particular, the transitions between these minima are broader, in line with our experimental observations. While a more accurate agreement between theory and experiment would likely require taking into account the inevitable asymmetries in the flux loops as well as the different transparencies of each normal-superconducting interface, it is clear that it is important to take into account the self-inductance of the flux loops in such devices. Regardless, we note that the overall topological structure is similar to that of Fig. 1 and is thus robust against such perturbations.

In conclusion, we have measured the resistance of diffusive three-terminal Josephson junctions as a function of the two independent superconducting phase differences imposed by on-chip field coils. The resistance shows a rich structure as a function of these phase differences, reflective of the topological nature of the underlying anomalous superconducting Green's function. The overall structure of the phase diagram is consistent with numerical predictions based on the quasiclassical theory of superconductivity, although the detailed structure is different, likely due to the finite linewidths in the device and screening effects arising from the finite self-inductance of the flux loops used to generate the superconducting phases. Our results demonstrate the power of using electrical transport measurements with normal contacts to explore the physics of MTJJs, including potentially in ballistic devices where a different class of topological effects is predicted.

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End Matter

Numerical Simulations

The simulations for this paper were performed by solving the quasiclassical equations of superconductivity in the Keldysh formulation, which allows one to obtain solu-

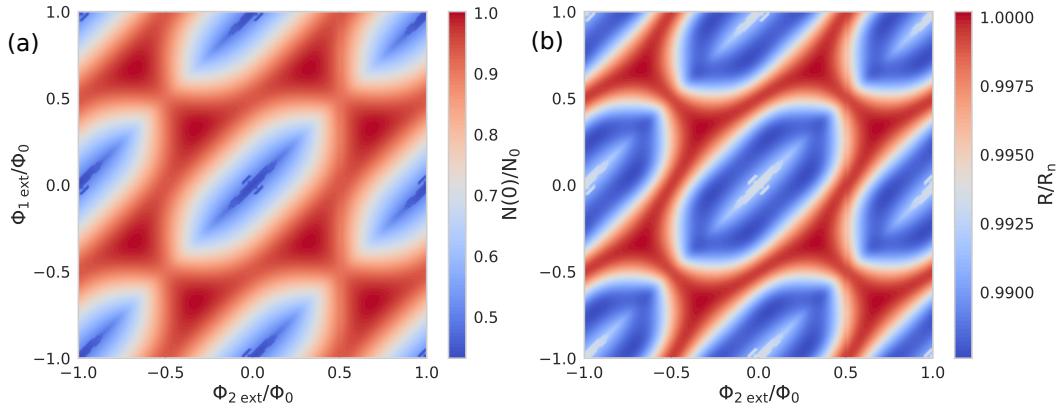


FIG. 4. (a), (b) Normalized DOS and resistance, respectively, for the geometry of Fig. 1(a) determined by calculating the phase on the superconducting contacts self-consistently using Eqn. (2).

tions for the superconducting Green's functions (the Usadel equation) as well as for the quasiparticle distribution functions (the kinetic equations) [13]. The equations are conveniently solved using the software package developed by Pauli Virtanen [19]. Details of these numerical calculations can be found in Ref. [12]. In brief, for the simulations shown in Figs. 1 and 2, referring to the schematic in Fig. 1(a), we set the phase of one superconducting contact to 0 without loss of generality, and assign phases ϕ_1 and ϕ_2 to the remaining two superconducting contacts. A small voltage bias V is applied symmetrically between the two normal contacts. The normal-superconducting contacts were assumed to be perfectly transparent. Solution of the Usadel equation enables one to determine the normalized density of states at any point along the normal wire, and in particular at the junction of all the normal wires, which is what is plotted in Fig. 1(b). Solution of the kinetic equations allows one to calculate the quasiparticle current flowing into the normal contact and from this the normalized resistance between the normal contacts plotted in Fig. 1(c). The parameters used in the simulations are based on the dimensions and film parameters of Sample 1. In particular, the Thouless energy $E_c = \hbar D/L^2$ which sets the energy scale of the proximity

effect is determined by the length L of the normal wire between the voltage probes (see Fig. 3(a)).

For the self-consistent calculation whose results are shown in Fig. 4, the Usadel ad kinetic equations were solved iteratively until the condition given by Eqn. (2) was satisfied.

Device Fabrication

Devices included in this study were fabricated using e-beam lithography on Si substrates with a 1 μm SiO_2 insulating layer. All features were patterned in MMA/PMMA bilayers using a Tescan MIRA 4 electron microscope onto which ultra pure Au and Al films were deposited in an Edwards thermal evaporator used exclusively for 99.999% (5N) Au and Al source material. Separate lithographic steps were used for the Au and Al films. Prior to the Au deposition, an *in situ* O_2^+ plasma etch was used to clean the substrate. Additionally, to ensure high NS interface transparency, an *in situ* Ar^+ plasma etch was performed immediately prior to a deposition of Al, leaving only enough time to pump back to a base pressure of 8×10^{-7} torr. The devices were cooled in an MX100 Oxford dilution refrigerator within hours of the Al deposition to prevent degradation of the NS interfaces.

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