ON THE IMPROVED CONVERGENCE OF LIFTED DISTRIBUTIONAL GAUSS CURVATURE FROM REGGE ELEMENTS

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Abstract. Although Regge finite element functions are not continuous, useful generalizations of nonlinear derivatives like the curvature, can be defined using them. This paper is devoted to studying the convergence of the finite element lifting of a generalized (distributional) Gauss curvature defined using a metric tensor approximation in the Regge finite element space. Specifically, we investigate the interplay between the polynomial degree of the curvature lifting by Lagrange elements and the degree of the metric tensor in the Regge finite element space. Previously, a superconvergence result, where convergence rate of one order higher than expected, was obtained when the approximate metric is the canonical Regge interpolant of the exact metric. In this work, we show that an even higher order can be obtained if the degree of the curvature lifting is reduced by one polynomial degree and if at least linear Regge elements are used. These improved convergence rates are confirmed by numerical examples.

Keywords: Gauss curvature, finite element method, Regge calculus, differential geometry.

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1. Introduction

Substantial progress has recently been made on computing high-order approximations of Gauss curvature on two-dimensional Riemannian manifolds using non-smooth metrics that are piecewise smooth with respect to a mesh [10, 2, 12]. Perhaps the most well-known example is that of a piecewise constant metric, where the angle defect at mesh vertices yields an approximation of Gauss curvature. Being concentrated at vertices, this angle defect can naturally be lifted into a linear Lagrange finite element space that has one basis function per vertex. Taking this as a point of departure by viewing a piecewise constant metric as a polynomial of degree $k = 0$ and the resulting Lagrange curvature lifting as a polynomial of degree $k + 1 = 1$ within each element, we can generalize to higher degrees k. Namely, in [12], we showed that if a smooth metric is approximated using the canonical interpolant of the Regge finite element space of degree k, then the error in a degree $(k + 1)$-Lagrange finite element approximation of the Gauss curvature converges to zero in the H^{-1}-norm at the rate $\mathcal{O}(h^{k+1})$, where h is the mesh-size. The present work is devoted to answering the following related question. What convergence rates can be expected if we decide to approximate curvature in an even higher degree Lagrange space—or for that matter, a lower degree Lagrange space—while keeping the metric in the degree k Regge space? Since the analysis in [12] used delicate orthogonality properties (such as the orthogonality of the error in Christoffel symbol approximations), the answer is not obvious. In fact, the answer we provide in this paper may even seem counterintuitive at first sight: reducing the degree of curvature approximation to degree k increases the convergence rate, while increasing it to degree $k + 2$ reduces the rate. Specifically, we observe that, under suitable assumptions, the following rates apply:

Curvature approximation		H^{-1} convergence	Source
$k+2$	$k \geqslant 0$	$\mathcal{O}(h^k)$	Section 4 in this paper
$k+1$	$k \geqslant 0$	$\mathcal{O}(h^{k+1})$	[12, Theorem 6.5]
k	$k \geqslant 1$	$\mathcal{O}(h^{k+2})$	Theorem 3.1, 3.8 in this paper
$k-1$	$k \geqslant 2$	$\mathcal{O}(h^{k+1})$	Section 4 in this paper

The remainder of this introduction places these and related prior results into perspective.

By Gauss' Theorema Egregium, Gauss curvature K is an intrinsic quantity. It can be computed considering solely the metric tensor of the manifold, without reference to any embedding. Therefore, it is natural to ask for discrete versions of Gauss curvature that arise when only approximations of the exact metric are given, and how well such discrete versions approximate the exact curvature when the approximated metrics are close to the exact one.

We consider Regge finite elements for discretizing the metric tensor. They originate from Regge calculus, originally developed for solving Einstein field equations in general relativity [21]. Following Regge we consider a simplicial triangulation of the manifold and assign positive numbers to each edge. These numbers are interpreted as squared lengths and determine a piecewise constant metric tensor. Sorkin pointed out in [25, Section II.A] that this piecewise constant metric tensor possesses tangential-tangential continuity, or *tt*-continuity (which we define precisely in Subsection 2.1 below) over element interfaces. Christiansen [4] popularized Regge calculus in the study of finite element methods (FEM) much the same way as finite element exterior calculus (FEEC) popularized the use of Whitney forms [29] in FEM. He defined in [5] the lowest-order Regge finite element space (the $k = 0$ case of the space Reg_h^k defined in (2.3) below) and showed that the linearization of the discretized Einstein–Hilbert action functional around the Euclidean metric equals the distributional incompatibility operator applied to such functions. Further, he proved in [6, 7] that the densitized curvature of a sequence of mollified piecewise constant Regge metrics converges to the angle defect in the sense of measures. Li extended the Regge space to arbitrary polynomial degrees k and to higher-dimensional simplices [18], and Neunteufel defined high-order Regge elements for quadrilaterals, hexahedra, and prisms [19].

Due to the non-smoothness of the approximated metric $g_h \in \mathrm{Reg}_h^k$ (which only has tangential-tangential continuity) and the nonlinearity of curvature, a definition of consistent and convergent notion of discrete Gauss curvature is not obvious. We refer to Sullivan [27, Section 4.1] for a historical discussion and to Strichartz [26, Corollary 3.1] for a definition of curvature as a measure on singular surfaces, where the curvature quantities, being multiplied by the corresponding volume forms, are handled as densities. In [2], Berchenko-Kogan and Gawlik defined a distributional version of the densitized Gauss curvature, namely $K\omega$, a generalization of the product of Gauss curvature K with the volume form ω (and we analyzed their curvature generalization further in [12]). In their work, in addition to the elementwise Gauss curvature $K|_T = K(g_h)|_T$, they consider the jump of the geodesic curvature κ over edges and the angle defect at vertices as sources of Gauss curvature, i.e., for any u in a space $\mathring{\mathcal{V}}(\mathscr{T})$ based on a mesh $\mathscr{T}$ defined below,

$$\widetilde{K\omega}(g_h)(u) = \sum_{T\in\mathscr{T}} \int_T K|_T \, u \, \omega_T + \sum_{E\in\mathscr{E}} \int_E [\![\kappa]\!]_E \, u \, \omega_E + \sum_{V\in\mathscr{V}} \Theta_V \, u(V), \tag{1.1}$$

where ω_T and ω_E denote the volume forms of the respective (sub-)domains. This allows for putting the well-established Gauss curvature approximation by angle deficit Θ_V (2π minus the sum of the interior angles of triangles attached to the vertex) into a finite element context and to extend it to higher polynomial order. In fact, considering piecewise constant metrics, $g_h \in \mathrm{Reg}_h^0$, the angle deficit is recovered, since then $\widetilde{K\omega}(g_h)(u) = \sum_{V\in\mathscr{V}} \Theta_V\, u(V)$. The distributional Gauss curvature (1.1) acts on piecewise smooth and globally continuous 0-forms defined by

$$\begin{aligned} \mathcal{V}(\mathscr{T}) &= \{u \in \Lambda^0(\mathscr{T}) \,:\, u \text{ is continuous}\}, \\ \mathring{\mathcal{V}}_\Gamma(\mathscr{T}) &= \{u \in \mathcal{V}(\mathscr{T}) \,:\, u|_\Gamma = 0\}, \quad \mathring{\mathcal{V}}(\mathscr{T}) = \mathring{\mathcal{V}}_{\partial\Omega}(\mathscr{T}). \end{aligned} \tag{1.2}$$

The meaning of "piecewise smooth" with respect to a "mesh" $\mathscr{T}$ and definition of piecewise smooth k-form fields $\Lambda^k(\mathscr{T})$ appear in Subsection 2.1 below. The standard degree k Lagrange finite element subspaces of the spaces in (1.2) are denoted by $\mathcal{V}_h^k$, $\mathcal{V}_{h,\Gamma}^k$, and $\mathring{\mathcal{V}}_h^k$, respectively, Berchenko-Kogan and Gawlik proved [2] error estimates in the H^{-1}-norm by using an integral representation of (1.1). Indeed, let δ denote the Euclidean metric, whose coordinate components coincide with the classical Kronecker delta, $[\delta]_{ij} = \delta_{ij}$ (not to be confused with the Dirac delta, which is never used in this work). Then there holds

$$\widetilde{K\omega}(g)(u) = \frac{1}{2}\int_0^1 b(\delta + t(g-\delta); g-\delta, u)\, dt, \tag{1.3}$$

where the bilinear form $b(g;\sigma,u)$ is the covariant version of the Hellan–Herrmann–Johnson (HHJ) method [14, 15, 16] extending the covariant differential operator $\mathrm{div}_g\mathrm{div}_g(\mathbb{S}_g\sigma)$ in the sense of distributions, where $\mathbb{S}_g\sigma = \sigma - \mathrm{tr}_g(\sigma)g$. Recently, Gawlik and Neunteufel extended the analysis to the H^{-2}-norm for the Gauss curvature, see [10]. Further, they considered an integral representation for the error $\big(\widetilde{K\omega}(g_h) - \widetilde{K\omega}(g)\big)(u)$.

It is often useful (or even necessary) to consider Gauss curvature as a function instead of as a functional or a distribution. In [9], Gawlik computed a discrete Riesz representative $K_h \in \mathring{\mathcal{V}}_h^r$ in the Lagrange finite element space $\mathring{\mathcal{V}}_h^r$ as a lifting of the distributional Gauss curvature (1.1) via

$$\int_\Omega K_h\, u_h\, \omega_h = \widetilde{K\omega}(g_h)(u_h) \qquad \text{for all } u_h \in \mathring{\mathcal{V}}_h^r,$$

where $\omega_h = \sqrt{\det g_h}\, dx^1 \wedge dx^2$ denotes the approximated volume form.

He proved error estimates of this lifting also for Sobolev norms under the assumption that $g_h \in \mathrm{Reg}_h^k$ is an optimal-order approximation of the exact metric $\bar g$, with exact Gauss curvature $\bar K = K(\bar g)$,

$$\|K_h - \bar K\|_{H_h^l} \leqslant C\, h^{-l+\min\{k-1,r+1\}}\big(|\bar g|_{H^{k+1}} + |\bar K|_{H^{r+1}}\big), \quad -1 \leqslant l \leqslant r. \tag{1.4}$$

Here, $\|\cdot\|_{H_h^l}$ denotes the elementwise H^l-norm. In [12], we considered an alternative integral representation that relies on the distributional covariant *incompatibility* operator, $\mathrm{inc}_g = \mathrm{curl}_g\,\mathrm{curl}_g$, which is related to the HHJ method by $\mathrm{inc}_g\,\sigma = -\mathrm{div}_g\mathrm{div}_g(\mathbb{S}_g\sigma)$. We showed a convergence rate increase (by one order) compared to (1.4) if g_h is the canonical Regge interpolant (defined by Li in [18], reproduced in (3.1) below) of the exact metric $\bar g$ and K_h is assumed to be in $\mathring{\mathcal{V}}_h^{k+1}$,

$$\|K_h - \bar K\|_{H_h^l} \leqslant C\, h^{-l+k}\big(|\bar g|_{W^{k+1,\infty}} + |\bar K|_{H^k}\big), \qquad -1 \leqslant l \leqslant k.$$

However, this convergence rate, when compared against the best approximation capabilities of the space, is not the theoretical optimum; for $K_h \in \mathring{\mathcal{V}}_h^{k+1}$ we obtain only L^2-convergence of order k instead of $k+2$.

In this paper we show that an increased, optimal convergence rate for the lifting of the distributional Gauss curvature K_h and its densitized version $K_h\omega_h$ is obtained when considering Lagrange elements $\mathring{\mathcal{V}}_h^k$ of one polynomial degree less assuming at least linear elements, $k \geqslant 1$, are used. Our analysis relies heavily on the properties of the canonical Regge interpolant [18] preserving specific moments at edges and elements. Therefore, the results only hold for the canonical Regge interpolant. For more general metric approximations in Reg_h^k, the estimate (1.4) cannot generally be improved. The technique of analysis in this work differs from our earlier work [12] in that we use an integral representation directly for the difference between the curvatures of the exact and the approximated metrics, instead of employing an interpolation from the Euclidean metric as in (1.3). This allows us to bypass a delicate "Christoffel orthogonality property," which was a key step in our prior work (see [12, Lemma 6.10]). As in [12], our current analysis also relies on the distributional covariant incompatibility operator $\widetilde{\mathrm{inc}_g}$, but now we rely specifically on its distributional L^2-like adjoint $\widetilde{\mathrm{rot}\,\mathrm{rot}_g}$. The latter simplifies the curvature error analysis compared to [12] (even if it does not provide estimates for $\widetilde{\mathrm{inc}_g}$-approximation, which we did in [12]).

This paper is structured as follows. In the next section we quickly review differential geometry notions we use, distributional covariant derivatives, and the distributional Gauss curvature. Section 3 is devoted to the error analysis of the lifted (densitized) Gauss curvature in the H^{-1}- and stronger Sobolev norms. In Section 4 we present numerical examples confirming the proved convergence rates.

2. Notation

Let $\Omega \subset \mathbb{R}^2$ be an open domain with a smooth metric tensor $\bar{g}$ providing a Riemannian manifold structure $(\Omega, \bar{g})$. Consider a triangulation $\mathscr{T}$ of Ω consisting of possibly curved triangles. Denote the set of all edges and vertices by $\mathscr{E}$ and $\mathscr{V}$, respectively. We split $\mathscr{E}$ into edges lying on the boundary $\partial\Omega$, given by $\mathscr{E}_\partial$, and inner ones $\mathring{\mathscr{E}} = \mathscr{E} \backslash \mathscr{E}_\partial$. Analogously we define $\mathscr{V}_\partial$ and $\mathring{\mathscr{V}}$. We assume to be given an approximation of $\bar{g}$, denoted g_h, defined on the triangulation $\mathscr{T}$. The subscript h indicates that g_h is defined with respect to the triangulation $\mathscr{T}$, where h can be related to the maximal element size. All quantities computed from the exact metric $\bar{g}$ will be marked by an overline "$\bar{\cdot}$" throughout the paper.

2.1. Regge metric.

Let $\mathfrak{X}(T)$, $\Lambda^k(T)$, and $\mathcal{T}_l^k(T)$ denote the set of smooth vector fields, k-form fields, and (k,l)-tensor fields on a submanifold T of Ω, respectively. Here, smoothness signifies infinite differentiability at interior points and continuous differentiability up to (including) the boundary. In such symbols, replacement of the manifold T by a collection of subdomains such as the triangulation $\mathscr{T}$, yields the piecewise smooth analogue with respect to the collection. For example, $\mathcal{T}_l^k(\mathscr{T})$ is the Cartesian product of $\mathcal{T}_l^k(T)$ over an enumeration of all $T \in \mathscr{T}$. Analogously, $\Lambda^1(\mathscr{T}) = \mathcal{T}_0^1(\mathscr{T})$ and $\mathfrak{X}(\mathscr{T}) = \mathcal{T}_1^0(\mathscr{T})$. Let $\mathcal{S}(\mathscr{T}) = \{\sigma \in \mathcal{T}_0^2(\mathscr{T}) : \sigma(X,Y) = \sigma(Y,X) \text{ for } X,Y \in \mathfrak{X}(\mathscr{T})\}$. Functions in $\mathcal{S}(\mathscr{T})$ are symmetric covariant 2-tensors on Ω with no continuity over element interfaces in general. We define $\mathcal{S}^+(\mathscr{T})$ as the subset of positive definite symmetric 2-tensors. For coordinate computations, we use coordinates x^1, x^2 and Einstein's summation convention of repeated indices. Let the accompanying coordinate frame and coframe be denoted by ∂_i and dx^i. We assume that these coordinates preserve orientation, i.e., the orientation of Ω is given by the ordering (∂_1, ∂_2). We use standard operations on 2-manifold spaces such as the exterior derivative $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ (see e.g. [17, 20, 28]).

Every $E \in \mathring{\mathscr{E}}$ is of the form $E = \partial T_+ \cap \partial T_-$ for two elements $T_\pm \in \mathscr{T}$. We say that a $\sigma \in \mathcal{S}(\mathscr{T})$ has "tangential-tangential continuity" or *"tt-continuity"* if $\sigma|_{T_+}(X,Y) = \sigma|_{T_-}(X,Y)$ for all tangential vector fields $X, Y \in \mathfrak{X}(E)$ for every E in $\mathring{\mathscr{E}}$ (i.e., $\sigma(X,Y)$ is single-valued on all $E \in \mathring{\mathscr{E}}$). This leads to the definition of the (infinite-dimensional) *Regge space*

$$\mathrm{Reg}(\mathscr{T}) = \{\sigma \in \mathcal{S}(\mathscr{T}) : \sigma \text{ is } tt\text{-continuous}\} \tag{2.1}$$

and its subset of *Regge metrics*

$$\mathrm{Reg}^+(\mathscr{T}) = \{\sigma \in \mathrm{Reg}(\mathscr{T}) : \sigma(X, X) > 0 \text{ for all } 0 \neq X \in \mathfrak{X}(\mathscr{T})\}.$$

The approximate metric g_h is assumed to be in $\mathrm{Reg}^+(\mathscr{T})$.

2.2. **Differential geometry.** Let the unique Levi–Civita connection generated by $\bar{g}$ be denoted by $\bar{\nabla}$. Note that it is standard to extend the Levi–Civita connection $\bar{\nabla}$ from vector fields to tensor fields (see e.g., [17, Lemma 4.6]) so that the Leibniz rule holds.

Following the sign convention of [17], recall that the *Riemann curvature tensor* $\bar{\mathcal{R}} \in \mathcal{T}^4_0(\Omega)$ of the manifold is defined by

$$\bar{\mathcal{R}}(X, Y, Z, W) = \bar{g}(\bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, W), \quad X, Y, Z, W \in \mathfrak{X}(\Omega).$$

Recall that the *Gauss curvature* of Ω is given by

$$\bar{K} := K(\bar{g}) = \frac{\bar{\mathcal{R}}(X, Y, Y, X)}{\bar{g}(X, X)\bar{g}(Y, Y) - \bar{g}(X, Y)^2},$$

where X and Y are some linearly independent vector fields and the value $\bar{K}$ is independent of their choice.

We will also require the *geodesic curvature* along a curve Γ in the manifold $(\Omega, \bar{g})$. Let $\hat{\tau}$ denote the $\bar{g}$-normalized tangent vector of Γ and $\hat{\nu}$ the $\bar{g}$-orthonormal vector such that $(\hat{\tau}, \hat{\nu})$ builds a right-handed coordinate system. Then

$$\bar{\kappa} := \kappa(\bar{g}) = \bar{g}(\bar{\nabla}_{\hat{\tau}} \hat{\tau}, \hat{\nu}) = -\bar{g}(\bar{\nabla}_{\hat{\tau}} \hat{\nu}, \hat{\tau})$$

gives the signed geodesic curvature of Γ. The element volume 2-form $\bar{\omega}$ and edge volume 1-form $\bar{\omega}_E$, $E \in \mathscr{E}$, read in coordinates

$$\bar{\omega} = \sqrt{\det \bar{g}}\, dx^1 \wedge dx^2, \qquad \bar{\omega}_E = \sqrt{\bar{g}(\tau, \tau)}\, d\tau, \tag{2.2}$$

where $\tau \in \mathfrak{X}(E)$ denotes the Euclidean normalized tangent vector at edge E and $d\tau$ is the associated 1-form. We use also the abbreviation of e.g. $\bar{g}_{\tau\tau} := \bar{g}(\tau, \tau)$.

2.3. **Finite element spaces.** Let $\hat{T} \subset \mathbb{R}^2$ denote the reference triangle and define $\mathcal{P}^k(\hat{T})$ as the set of polynomials of degree up to k on $\hat{T}$. For $T \in \mathscr{T}$ let $\Phi_T : \hat{T} \to T \in \mathcal{P}^k(\hat{T}, \mathbb{R}^2)$ denote the diffeomorphic mapping from the reference to the physical element.

We define the Regge finite element space as a subspace of $\mathrm{Reg}(\mathscr{T})$ (2.1) by

$$\mathrm{Reg}_h^k = \{\sigma \in \mathrm{Reg}(\mathscr{T}) : \text{for all } T \in \mathscr{T},\ \sigma|_T = \sigma_{ij} dx^i \otimes dx^j \text{ with } \sigma_{ij} \circ \Phi_T \in \mathcal{P}^k(\hat{T})\}. \tag{2.3}$$

Further, the Lagrange finite element space as a subspace of $\mathcal{V}(\mathscr{T})$ (1.2) is given by

$$\mathcal{V}_h^k = \{u \in \mathcal{V}(\mathscr{T}) : \text{ for all } T \in \mathscr{T},\ u|_T \circ \Phi_T \in \mathcal{P}^k(\hat{T})\},$$
$$\mathring{\mathcal{V}}_{h,\Gamma}^k = \{u \in \mathcal{V}_h^k : u|_\Gamma = 0\}, \quad \text{and} \quad \mathring{\mathcal{V}}_h^k = \mathring{\mathcal{V}}_{h,\partial\Omega}^k.$$

2.4. **Lifted distributional Gauss curvature.** For the reader's convenience we derive the (lifted) distributional Gauss curvature following [2, 12]. Since a $g_h \in \mathrm{Reg}^+(\mathscr{T})$ is smooth within each element $T \in \mathscr{T}$ we can compute elementwise its Gauss curvature $K(g_h)|_T$. It is only one contributor of the total distributional Gauss curvature as the jumps of g_h generate additional sources of curvature. Let for an edge $E \in \mathring{\mathscr{E}}$ the unique g_h-normal vector that points inward to elements $T_\pm \in \mathscr{T}$, such that $E = \partial T_+ \cap \partial T_-$, be denoted by $\hat{\nu}_E^{T_\pm}$. As g_h is only tt-continuous, $\hat{\nu}_E^{T_+} \neq -\hat{\nu}_E^{T_-}$ in general. Thus, the jump of the geodesic curvature

$$[\![\kappa]\!]_E = \kappa_{\hat{\nu}_E^{T_+}} + \kappa_{\hat{\nu}_E^{T_-}}$$

acts as a source of curvature at edges. If there is no chance of confusion we neglect the subscript and only write $[\![\cdot]\!]$ for the jump over edges.

Let $V \in \mathring{\mathscr{V}}$ be an interior vertex and $T \in \mathscr{T}$ a triangle containing V. Then there are two edges $E_\pm \in \mathring{\mathscr{E}} \cap \partial T$ such that $V = \partial E_+ \cap \partial E_-$. Denote $\hat{\tau}_V^{E_\pm}$ the g_h-normalized tangent vectors starting at V and pointing into $E_\pm$. We define the following angle function on V

$$\sphericalangle_V^T = \arccos(g_h|_T(\hat{\tau}_V^{E_+}, \hat{\tau}_V^{E_-}))$$

and the angle deficit at vertex $V \in \mathring{\mathscr{V}}$

$$\Theta_V = 2\pi - \sum_{T \in \mathscr{T}_V} \sphericalangle_V^T, \qquad \mathscr{T}_V = \{T \in \mathscr{T} \,:\, V \in T\}. \tag{2.4}$$

This function acts as a source of curvature on vertices. Note that for the smooth metric $\bar{g}$ there holds $\Theta_V = 0$.

Definition 2.1. Let $g_h \in \mathrm{Reg}^+(\mathscr{T})$ be a Regge metric. The *distributional densitized Gauss curvature* $\widetilde{K\omega}(g_h) : \mathring{\mathcal{V}}(\mathscr{T}) \to \mathbb{R}$ is defined for all $u \in \mathring{\mathcal{V}}(\mathscr{T})$

$$\widetilde{K\omega}(g_h)(u) = \sum_{T \in \mathscr{T}} \int_T K|_T \, u \, \omega_T + \sum_{E \in \mathring{\mathscr{E}}} \int_E [\![\kappa]\!] \, u \, \omega_E + \sum_{V \in \mathring{\mathscr{V}}} \Theta_V \, u(V), \tag{2.5}$$

where $K|_T$, κ, ω_T, ω_E, and Θ_V are evaluated with respect to g_h.

Remark 2.2. Note, that this generalizes the *densitized* Gauss curvature $K\omega$ (see e.g., [6]), not solely K. One can interpret (2.5) as a measure with support on triangles, edges, and vertices, cf. [26]. See Remark 3.3 for more on approximating just K.

We consider a discrete Riesz representative of the functional in (2.5), following Gawlik [9]. We also incorporate essential and natural Dirichlet Γ_D and Neumann Γ_N boundary conditions, as discussed in [12]. To this end, extend the definition of the angle deficit (2.4) and the jump of the geodesic curvature to boundary vertices and edges in the obvious manner: we define $[\![\kappa]\!]_E = \kappa$ for boundary edges $E \in \mathscr{E}_\partial$ and Θ_V for $V \in \mathscr{V}_\partial$ as in (2.4). Note that $[\![\kappa]\!]_E$ and Θ_V do not vanish for smooth metrics g at $E \in \mathscr{E}_\partial$ and $V \in \mathscr{V}_\partial$ in general. Instead, they are used to incorporate natural Neumann boundary conditions. We assume that on the Dirichlet boundary, the Gauss curvature $\bar{K}^D = \bar{K}|_{\Gamma_D}$ is prescribed. The Neumann boundary data is given by the functional $\widetilde{\kappa^N} : \mathcal{V}(\mathscr{T}) \to \mathbb{R}$

$$\widetilde{\kappa^N}(u) = \int_{\Gamma_N} \bar{\kappa} \, u \, \bar{\omega}_{\Gamma_N} + \sum_{V \in \mathscr{V} \cap \Gamma_N} \bar{\sphericalangle}_V^N \, u(V),$$

where $\bar{\sphericalangle}_V^N$ denotes the exterior angle, which is 2π minus the interior angle, measured with respect to $\bar{g}$ by the edges of Γ_N at V.

Definition 2.3. Let $g_h \in \mathrm{Reg}^+(\mathscr{T})$, $k \geqslant 1$ be an integer, and assume that the Dirichlet data $\bar{K}^D$ is the trace of a Lagrange finite element function in $\mathcal{V}_h^k$. The finite element curvature approximation $K_h := K_h(g_h)$ of degree k is the unique function in $\mathcal{V}_h^k$ determined by requiring that $K_h|_{\Gamma_D} = \bar{K}^D$ on Γ_D and for all $u_h \in \mathring{\mathcal{V}}_{h,\Gamma_D}^k$,

$$\int_\Omega K_h\, u_h\, \omega_h = \widetilde{K\omega}(g_h)(u_h) - \widetilde{\kappa}^N(u_h), \tag{2.6}$$

where we denote the volume form of g_h by $\omega_h := \omega(g_h)$.

Note that the difference between Definition 2.3 and [12, Definition 3.1] is the decrease of degree of approximation space of K_h from $\mathcal{V}_h^{k+1}$ to $\mathcal{V}_h^k$ and the additional requirement of $k \geqslant 1$.

2.5. **Distributional covariant differential operators.** In this section we review the definition of distributional covariant differential operators based on Regge metrics $g \in \mathrm{Reg}^+(\mathscr{T})$. We focus on the incompatibility operator and its adjoint with their coordinate expressions. For an introduction and discussion we refer to e.g. [12, Section 4]. First, we focus on pointwise covariant differential operators for a given smooth metric $g \in \mathcal{S}^+(\Omega)$.

For a 1-form $\alpha \in \Lambda^1(\Omega)$ and a (2,0)-tensor $\sigma \in \mathcal{T}_0^2(\Omega)$ the covariant curl operators $\mathrm{curl}_g : \Lambda^1(\Omega) \to \Lambda^0(\Omega)$ and $\mathrm{curl}_g : \mathcal{T}_0^2(\Omega) \to \Lambda^1(\Omega)$ read in coordinates [12]

$$\begin{aligned}
\mathrm{curl}_g(\alpha) &= \hat{\varepsilon}^{ij}\partial_i\alpha_j,\\
\mathrm{curl}_g(\sigma) &= \hat{\varepsilon}^{jk}(\partial_j\sigma_{ik} - \Gamma_{ji}^m\sigma_{mk})dx^i,
\end{aligned}$$

where $\hat{\varepsilon}^{ij} = \frac{1}{\sqrt{\det g}}\varepsilon^{ij}$ and ε^{ij} denotes the permuting symbol being 1, -1, or 0 if (i,j) is an even, odd, or no permutation of $(1,2)$, respectively. The covariant incompatibility operator $\mathrm{inc}_g = \mathrm{curl}_g \mathrm{curl}_g : \mathcal{T}_0^2(\Omega) \to \Lambda^0(\Omega)$ reads in coordinates

$$\mathrm{inc}_g(\sigma) = \hat{\varepsilon}^{qi}\hat{\varepsilon}^{jk}\left(\partial_j\partial_q\sigma_{ik} - \partial_q(\Gamma_{ji}^m\sigma_{mk}) - \Gamma_{lq}^l(\partial_j\sigma_{ik} - \Gamma_{ji}^m\sigma_{mk})\right).$$

Next, we consider for $f \in \Lambda^0(\Omega)$ and $X \in \mathfrak{X}(\Omega)$ the adjoint operators $\mathrm{rot}_g : \Lambda^0(\Omega) \to \mathfrak{X}(\Omega)$, $\mathrm{rot}_g : \mathfrak{X}(\Omega) \to \mathcal{T}_2^0(\Omega)$ and $\mathrm{rot}\,\mathrm{rot}_g = \mathrm{rot}_g\,\mathrm{rot}_g : \Lambda^0(\Omega) \to \mathcal{T}_2^0(\Omega)$. They read in coordinates [12]

$$\mathrm{rot}_g f = \hat{\varepsilon}^{iq}\partial_q f\,\partial_i = \frac{[\mathrm{rot}\, f]^i}{\sqrt{\det g}}\partial_i, \tag{2.7a}$$

$$\mathrm{rot}_g X = \hat{\varepsilon}^{jq}(\partial_q X^i + \Gamma_{qk}^i X^k)\partial_i\otimes\partial_j = \frac{[\mathrm{rot}[X]]^{ij} + \varepsilon^{jq}\Gamma_{qk}^i X^k}{\sqrt{\det g}}\partial_i\otimes\partial_j, \tag{2.7b}$$

$$\begin{aligned}
\mathrm{rot}\,\mathrm{rot}_g f &= \mathrm{rot}_g(\mathrm{rot}_g f)^{ij}\partial_i\otimes\partial_j = \frac{[\mathrm{rot}[\mathrm{rot}_g f]]^{ij} + \varepsilon^{jq}\Gamma_{qk}^i[\mathrm{rot}_g f]^k}{\sqrt{\det g}}\partial_i\otimes\partial_j\\
&= \frac{[\mathrm{rot}\,\mathrm{rot}\, f]^{ij} - [\mathrm{rot}\, f]^i\varepsilon^{jq}\Gamma_{lq}^l + \varepsilon^{jq}\Gamma_{qk}^i[\mathrm{rot}\, f]^k}{\det g}\partial_i\otimes\partial_j.
\end{aligned} \tag{2.7c}$$

In (2.7) we used so-called vector and matrix proxies $[\sigma] \in \mathbb{R}^{2\times 2}$ and $[X] \in \mathbb{R}^2$ for $\sigma \in \mathcal{T}_2^0(\Omega)$ and $X \in \mathfrak{X}(\Omega)$ [1]. These proxies consist of coefficients in the coordinate basis expansions. For example, $[\sigma]$ is the matrix, which (i,j)th entry is $\sigma^{ij} = \sigma(dx^i, dx^j)$. Then the standard two-dimensional Euclidean rotation operator applied to the vector $[X]$ is $\mathrm{rot}[X]^{ij} = \varepsilon^{jk}\partial_k X^i$.

There holds the integration by parts formulas for $f \in \Lambda^0(\Omega)$, $\alpha \in \Lambda^1(\Omega)$, $\sigma \in \mathcal{T}^2_0(\Omega)$, and $X \in \mathfrak{X}(\Omega)$

$$\begin{aligned}
\int_\Omega \langle \operatorname{curl}_g \sigma, X\rangle \omega &= \int_\Omega \langle \sigma, \operatorname{rot}_g X\rangle \omega + \int_{\partial\Omega} \sigma(X, \hat\tau)\, \omega_{\partial\Omega},\\
\int_\Omega \operatorname{curl}_g \alpha\, f\, \omega &= \int_\Omega \langle \alpha, \operatorname{rot}_g f\rangle \omega + \int_{\partial\Omega} \langle \alpha, \hat\tau\rangle f\, \omega_{\partial\Omega},\\
\int_\Omega \operatorname{inc}_g \sigma\, f\, \omega &= \int_\Omega \langle \operatorname{curl}_g \sigma, \operatorname{rot}_g f\rangle \omega + \int_{\partial\Omega} \langle \operatorname{curl}_g \sigma, \hat\tau\rangle f\, \omega_{\partial\Omega}\\
&= \int_\Omega \langle \sigma, \operatorname{rot}\operatorname{rot}_g f\rangle \omega + \int_{\partial\Omega} \big(\sigma(\operatorname{rot}_g f, \hat\tau) + \langle \operatorname{curl}_g \sigma, \hat\tau\rangle f\big)\, \omega_{\partial\Omega},
\end{aligned}$$

i.e., inc_g and $\operatorname{rot}\operatorname{rot}_g$ are L^2-adjoint with respect to the g-weighted L^2 inner product.

Above, we used the g inner product $\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$ extended from vector fields to arbitrary order tensors $\mathcal{T}^l_k(\Omega)$, e.g.

$$\langle \operatorname{curl}_g \sigma, X\rangle = (\operatorname{curl}_g \sigma)(X) \qquad \text{for all } \sigma \in \mathcal{T}^2_0(\Omega),\ X \in \mathfrak{X}(\Omega).$$

The following definition of the distributional covariant incompatibility operator has been derived in [12, Proposition 4.6]. A similar expression for the vertex contributions can be found in [5].

Definition 2.4. Let $g \in \operatorname{Reg}^+(\mathscr{T})$ and $u \in \mathcal{V}(\mathscr{T})$. The distributional incompatibility operator $\widetilde{\operatorname{inc}_g} : \operatorname{Reg}(\mathscr{T}) \to \mathcal{V}(\mathscr{T})'$ is defined by

$$(\widetilde{\operatorname{inc}_g}\, \sigma)(u) = \sum_{T\in\mathscr{T}} \bigg[\int_T \operatorname{inc}_g(\sigma)\, u\, \omega_T - \int_{\partial T} u \langle \operatorname{curl}_g \sigma + d(\sigma_{\hat\nu\hat\tau}), \hat\tau\rangle \omega_{\partial T} + \sum_{V\in\mathscr{V}_T} [\![\sigma_{\hat\nu\hat\tau}]\!]^T_V\, u(V) \bigg], \tag{2.8}$$

where $\mathscr{V}_T = \{V \in \mathscr{V} : V \in T\}$ and, cf. e.g. [13],

$$[\![\sigma_{\hat\nu\hat\tau}]\!]^T_V = \big(\sigma|_T(\hat\nu^T_{E_+}, \hat\tau^{E_+}_V) + \sigma|_T(\hat\nu^T_{E_-}, \hat\tau^{E_-}_V)\big)(V).$$

The distributional covariant rot rot operator $\widetilde{\operatorname{rot}\operatorname{rot}_g} : \mathcal{V}(\mathscr{T}) \to \operatorname{Reg}(\mathscr{T})'$ is defined by

$$(\widetilde{\operatorname{rot}\operatorname{rot}_g}\, u)(\sigma) = \sum_{T\in\mathscr{T}} \bigg[\int_T \langle \operatorname{rot}\operatorname{rot}_g u, \sigma\rangle \omega_T + \int_{\partial T} \sigma_{\hat\tau\hat\tau} \langle \nabla_g u, \hat\nu\rangle \omega_{\partial T} \bigg]. \tag{2.9}$$

Note that one of the boundary terms in (2.8) admits a representation using the geodesic curvature $\kappa_{\hat\nu}$, namely

$$d(\sigma_{\hat\nu\hat\tau})(\hat\tau) = \nabla_{\hat\tau}\big(\sigma(\hat\nu, \hat\tau)\big) = (\nabla_{\hat\tau}\sigma)(\hat\nu, \hat\tau) + (\sigma(\hat\nu, \hat\nu) - \sigma(\hat\tau, \hat\tau))\, \kappa_{\hat\nu}.$$

Next we show that the (distributional) adjoint of inc_g is $\operatorname{rot}\operatorname{rot}_g$ in the following sense.

Lemma 2.5. *Let $\sigma \in \operatorname{Reg}(\mathscr{T})$ and $u \in \mathcal{V}(\mathscr{T})$. There holds*

$$(\widetilde{\operatorname{inc}_g}\, \sigma)(u) = (\widetilde{\operatorname{rot}\operatorname{rot}_g}\, u)(\sigma).$$

Proof. This follows by integration by parts on each $T \in \mathscr{T}$:

$$
\begin{aligned}
(\widetilde{\operatorname{inc}_g}\,\sigma)(u) &= \sum_{T\in\mathscr{T}} \Big[\int_T \operatorname{inc}_g(\sigma)\, u\, \omega_T - \int_{\partial T} u \,\langle \operatorname{curl}_g \sigma + d(\sigma_{\hat\nu\hat\tau}), \hat\tau\rangle\, \omega_{\partial T} + \sum_{V\in\mathscr{V}_T} [\![\sigma_{\hat\nu\hat\tau}]\!]_V^T\, u(V) \Big] \\
&= \sum_{T\in\mathscr{T}} \Big[\int_T \operatorname{inc}_g(\sigma)\, u\, \omega_T - \int_{\partial T} \big(u \,\langle \operatorname{curl}_g \sigma, \hat\tau\rangle - \sigma_{\hat\nu\hat\tau} \nabla_{\hat\tau} u\big)\, \omega_{\partial T} \Big] \\
&= \sum_{T\in\mathscr{T}} \Big[\int_T \langle \operatorname{curl}_g \sigma, \operatorname{rot}_g u\rangle\, \omega_T + \int_{\partial T} \sigma_{\hat\nu\hat\tau} \nabla_{\hat\tau} u\, \omega_{\partial T} \Big] \\
&= \sum_{T\in\mathscr{T}} \Big[\int_T \langle \sigma, \operatorname{rot}\operatorname{rot}_g u\rangle\, \omega_T + \int_{\partial T} \big(\sigma(\hat\tau, \operatorname{rot}_g u) + \sigma_{\hat\nu\hat\tau} \nabla_{\hat\tau} u\big)\, \omega_{\partial T} \Big] \\
&= \sum_{T\in\mathscr{T}} \Big[\int_T \langle \sigma, \operatorname{rot}\operatorname{rot}_g u\rangle\, \omega_T + \int_{\partial T} \sigma_{\hat\tau\hat\tau} \nabla_{\hat\nu} u\, \omega_{\partial T} \Big] = (\widetilde{\operatorname{rot}\operatorname{rot}_g}\, u)(\sigma).
\end{aligned}
$$

□

In [12] we proved an integral representation of the densitized Gauss curvature using a parametrization starting from the Euclidean metric δ

$$
\widetilde{K\omega}(g)(u) = -\frac{1}{2}\int_0^1 b(\delta + t(g-\delta); g-\delta, u)\, dt, \qquad \text{with } b(g;\sigma,u) = (\widetilde{\operatorname{inc}_g}\,\sigma)(u)
$$

and used its integrand to derive convergence results. In this work, we follow the approach of [10, 11, 13] and consider directly the integral representation of the error as follows. Let $g(t) = \bar g + t(g_h - \bar g)$ and $\sigma = g'(t) = g_h - \bar g$. Then there holds the integral representation of the error

$$
\big(\widetilde{K\omega}(g_h) - \bar K\bar\omega\big)(u) = -\frac{1}{2}\int_0^1 (\widetilde{\operatorname{inc}_{g(t)}}\,\sigma)(u)\, dt. \tag{2.10}
$$

To derive error estimates, one important part will be analyzing the integrand of (2.10), or, more precisely, its adjoint $(\widetilde{\operatorname{inc}_{g(t)}}\,\sigma)(u) = (\widetilde{\operatorname{rot}\operatorname{rot}_{g(t)}}\, u)(\sigma)$.

3. Error analysis

In this section we prove a priori error estimates for the lifted densitized Gauss curvature $K_h\omega_h$ and the Gauss curvature K_h. First, we consider the H^{-1}-norm as basis and then show estimates also for the stronger Sobolev norms L^2 and H^r, $r \geqslant 1$. Let $\Omega \subset \mathbb{R}^2$ be a domain with a given exact metric tensor $\bar g$ and corresponding exact Gauss curvature $\bar K = K(\bar g)$. For simplicity, we assume in this section that homogeneous Dirichlet data $\bar K^D = 0$ is described on the whole boundary, $\Gamma_D = \partial\Omega$.

3.1. **Statement of main theorem.** We consider a sequence of quasiuniform (hence shape-regular) affine-equivalent triangulations $\{\mathscr{T}_h\}_{h>0}$ with maximal mesh-size $h = \max_{T\in\mathscr{T}_h} h_T$, where $h_T = \operatorname{diam}(T)$. On the triangulations a sequence of Regge metrics $\{g_h\}_{h>0}$ with $g_h \in \operatorname{Reg}_h^k$, $k \geqslant 0$ (defined in (2.3)) is given. To be precise, we assume that g_h is the canonical interpolant of $\bar g$. This interpolant [18], denoted by $\mathcal{I}_h^{\operatorname{Reg},k} : W^{s,p}(\Omega, \mathbb{S}) \to \operatorname{Reg}_h^k$, $p \in [1,\infty]$,

$s \in (1/p, \infty]$, satisfies the following equations

$$\int_E (\mathcal{I}_h^{\mathrm{Reg},k}\sigma)_{\tau\tau}\, q \,\mathrm{dl} = \int_E \sigma_{\tau\tau}\, q \,\mathrm{dl} \qquad \text{for all } q \in \mathcal{P}^k(E) \text{ and edges } E \text{ of } \partial T, \tag{3.1a}$$

$$\int_T \mathcal{I}_h^{\mathrm{Reg},k}\sigma : \rho \,\mathrm{da} = \int_T \sigma : \rho \,\mathrm{da} \qquad \text{for all } \rho \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}),\ T \in \mathscr{T}. \tag{3.1b}$$

Equations (3.1) can be interpreted as orthogonality requirements, preserving specific moments at edges and elements. Note that when ρ is a skew-symmetric matrix, both sides of (3.1b) vanish, so (3.1b) is nontrivial only for symmetric ρ.

Throughout, we use standard Sobolev spaces $W^{s,p}(\Omega)$ and their norms and seminorms for any $s \geqslant 0$ and $p \in [1, \infty]$. When the domain is Ω, we omit it from the norm notation if there is no chance of confusion. We also use the elementwise norms $\|u\|_{W_h^{s,p}}^p = \sum_{T\in\mathscr{T}_h} \|u\|_{W^{s,p}(T)}^p$, with the usual adaption for $p = \infty$. When $p = 2$, we put $\|\cdot\|_{H_h^s} = \|\cdot\|_{W_h^{s,2}}$. Let $D \subset \Omega$ and define

$$|\!|\!|\sigma|\!|\!|_{2,D} = \|\sigma\|_{L^2(D)} + h\|\sigma\|_{H_h^1(D)}. \tag{3.2}$$

If D is the whole domain Ω, we neglect the subscript in (3.2).

We write $a \lesssim b$ if there exists a mesh-size independent constant $C > 0$ which may depend on—unless otherwise stated—the domain Ω, the polynomial degree k, the shape regularity constant $\sigma(\mathscr{T}_h)$ of $\mathscr{T}_h$, the $W^{2,\infty}$-norm of $\bar{g}$, L^∞-norm of $\bar{g}^{-1}$, and the H^1-norm of $\bar{K}$ i.e.

$$C = C(\Omega, k, \sigma(\mathscr{T}_h), \|\bar{g}\|_{W^{2,\infty}}, \|\bar{g}^{-1}\|_{L^\infty}, \|\bar{K}\|_{H^1}). \tag{3.3}$$

We abbreviate the L^2-inner product of two scalar functions and the g-weighted inner product by

$$(u, v)_{L^2} := \int_\Omega uv \,\mathrm{da}, \qquad (u, v)_{L^2,g} := \int_\Omega uv\sqrt{\det g}\,\mathrm{da}, \qquad u, v \in L^2(\Omega).$$

Our main theorem reads as follows:

Theorem 3.1. *Let $k \geqslant 1$ be an integer, $\{\mathscr{T}_h\}_{h>0}$ a sequence of quasiuniform triangulations, $\{g_h\}_{h>0}$ a sequence of metric approximations $g_h = \mathcal{I}_h^{\mathrm{Reg},k}\bar{g}$ with $\bar{g} \in W^{2,\infty}(\Omega, \mathcal{S}^+)$, so that $\bar{K} \in L^2(\Omega)$, $\omega_h = \omega(g_h)$, and $K_h \in \mathring{\mathcal{V}}_h^k$ the lifted distributional Gauss curvature from (2.6). Suppose also that $\bar{K} = 0$ on the boundary $\partial\Omega$. Then there exists an $h_0 > 0$ such that for all $h \leqslant h_0$*

$$\|K_h\omega_h - \bar{K}\bar{\omega}\|_{H^{-1}} \leqslant Ch\big(|\!|\!|g_h - \bar{g}|\!|\!|_2 + \inf_{v_h\in\mathring{\mathcal{V}}_h^k} \|v_h - \bar{K}\|_{L^2} + \|g_h - \bar{g}\|_{L^\infty}\big),$$

where the constant C depends on Ω, the shape regularity, polynomial degree k, $\|\bar{g}\|_{W^{2,\infty}}$, and $\|\bar{g}^{-1}\|_{L^\infty}$. If additionally for $0 \leqslant l \leqslant k+1$, $\bar{g} \in W^{l,\infty}(\Omega, \mathcal{S})$ and $\bar{K} \in H^l(\Omega)$, then

$$\|K_h\omega_h - \bar{K}\bar{\omega}\|_{H^{-1}} \leqslant Ch^{l+1}\big(\|\bar{g}\|_{W^{l,\infty}} + \|\bar{K}\|_{H^l}\big).$$

Further convergence results in stronger Sobolev norms follow.

Corollary 3.2. *Under the assumptions of Theorem 3.1, there holds for $0 \leqslant l \leqslant k+1$, $0 \leqslant r \leqslant l$*

$$\begin{aligned}\|K_h\omega_h - \bar{K}\bar{\omega}\|_{H_h^r} &\leqslant Ch^{-r}\big(\|g_h - \bar{g}\|_{L^\infty} + |\!|\!|g_h - \bar{g}|\!|\!|_2 + \inf_{v_h\in\mathring{\mathcal{V}}_h^k} \|v_h - \bar{K}\|_{L^2} \\ &\qquad + h^l\|\bar{K}\|_{H^l} + \inf_{v_h\in\mathring{\mathcal{V}}_h^k} \|v_h - \bar{K}\bar{\omega}\|_{L^2} + h^l\|\bar{K}\bar{\omega}\|_{H^l}\big) \\ &\leqslant Ch^{l-r}\big(\|\bar{g}\|_{W^{l,\infty}} + \|\bar{K}\bar{\omega}\|_{H^l} + \|\bar{K}\|_{H^l}\big),\end{aligned}$$

where the constant $C > 0$ depends additionally on $\|\bar{K}\|_{H^1}$.

Remark 3.3 (Convergence of pure Gauss curvature). In contrast to the densitized Gauss curvature, we refer to the $\bar{K}$ (without multiplication by the volume form) as the "pure Gauss curvature." For the error in the pure Gauss curvature, $\|K_h - \bar{K}\|_{H_h^r}$, $-1 \leqslant r \leqslant k$, the same convergence rates as proved in Theorem 3.1 and Corollary 3.2 are obtained: see Theorem 3.8 and Corollary 3.9 in Section 3.5.

Remark 3.4 (Optimal convergence). If we insert $l = k+1$ in Theorem 3.1, we obtain the convergence rate $\mathcal{O}(h^{k+2})$, which is of one order higher than $\mathcal{O}(h^{k+1})$ proved in [12, Theorem 6.5] and two orders higher compared to [9, Theorem 4.1]. Furthermore, using $l = k+1$ and $r = 0$ in Corollary 3.2 yields an L^2 convergence rate of $\mathcal{O}(h^{k+1})$, which is the "optimal" in the sense that it is the rate of convergence of the L^2 best approximation from $\mathring{\mathcal{V}}_h^k$. The requirement of at least linear elements, $k \geqslant 1$, cannot be relaxed to $k = 0$ as for $g_h \in \mathrm{Reg}_h^0$ there is no (non-trivial) Lagrange finite element function in $\mathring{\mathcal{V}}_h^0$. In [12] we observed that the pairing of the lowest order elements $g_h \in \mathrm{Reg}_h^0$ and $K_h \in \mathcal{V}_h^1$ does not lead to an improved L^2-convergence rate of $\mathcal{O}(h)$. In fact, our numerical examples in [12] showed that we may expect no convergence in the L^2-norm in general in this case.

3.2. **Basic estimates.** We need a number of preliminary estimates to proceed with our analysis. The approximation properties of the Regge elements are well understood. By the Bramble-Hilbert lemma, on any $T \in \mathscr{T}$, see [18, Theorem 2.5],

$$\|(\mathrm{id} - \mathcal{I}_h^{\mathrm{Reg},k})\rho\|_{W^{r,p}(T)} \leqslant C h^{l-r} |\rho|_{W^{l,p}(T)}, \tag{3.4}$$

for $p \in [1,\infty]$, $l \in (1/p, k+1]$, $r \in [0,l]$, $\rho \in W^{l,p}(T, \mathcal{S})$, and C depends on k, r, l, and the shape regularity $\sigma(T)$ of T. A similar estimate holds for the elementwise $L^2(T)$-projection into the space of polynomials of order k, which we denote by $\Pi_{L^2}^k$, see e.g. [3, Theorem 4.4.4],

$$\|(\mathrm{id} - \Pi_{L^2}^k) f\|_{L^p(T)} \leqslant h^l C |f|_{W^{l,p}(T)}$$

for $l \in (1/p, k+1]$, $f \in W^{l,p}(T)$, and C depends on k, l, and the shape regularity $\sigma(T)$ of T. The same holds if we replace T by an edge $E \in \mathscr{E}$ and the edge-wise L^2-projection denoted by $\Pi_{L^2}^{E,k}$.

Let $E \subset \partial T$ be an edge of T. We also need the following well-known estimates that follow from scaling arguments: for all $u \in H^1(T)$

$$\|u\|_{L^2(E)}^2 \lesssim h^{-1}\|u\|_{L^2(T)}^2 + h\|\nabla u\|_{L^2(T)}^2 \tag{3.5}$$

and for all $u \in \mathcal{P}^k(T)$,

$$\|u\|_{L^2(E)} \lesssim h^{-1/2}\|u\|_{L^2(T)}, \qquad |u|_{H^l(T)} \lesssim h^{-l}\|u\|_{L^2(T)}, \quad 1 \leqslant l \leqslant k. \tag{3.6}$$

The L^2-orthogonal projection with respect to $\bar{g}$ into Lagrange elements $P_{L^2}^k : L^2(\Omega) \to \mathring{\mathcal{V}}_h^k$ is defined via its orthogonality property

$$\int_\Omega \left(P_{L^2}^k u - u\right) v_h\, \bar{\omega} = 0, \qquad \text{for all } v_h \in \mathring{\mathcal{V}}_h^k. \tag{3.7}$$

It has the following well-known stability and approximation properties on quasiuniform meshes, see e.g. [8] or [9, Lemma 4.7],

$$\|P_{L^2}^k u\|_{L^2} \lesssim \|u\|_{L^2}, \qquad \text{for all } u \in L^2(\Omega), \tag{3.8a}$$

$$\|P_{L^2}^k u\|_{H^1} \lesssim \|u\|_{H^1}, \qquad \text{for all } u \in H_0^1(\Omega), \tag{3.8b}$$

$$\|P_{L^2}^k u - u\|_{L^2} \lesssim \inf_{u_h \in \mathring{\mathcal{V}}_h^k} \|u_h - u\|_{L^2} \lesssim h\,\|u\|_{H^1}, \qquad \text{for all } u \in H_0^1(\Omega). \tag{3.8c}$$

Since $g_h = \mathcal{I}_h^{\mathrm{Reg},k}\bar{g}$ approaches $\bar{g}$ as $h \to 0$, we tacitly assume throughout that h has become sufficiently small ($h \leqslant h_0$) to guarantee that the approximated metric g_h is positive definite throughout. Further, thanks to (3.4) (with $p = r = l = 2$ and $k \geqslant 1$) and (3.3) we have that $\sup_{T\in\mathscr{T}_h} \|g_h\|_{W^{2,\infty}(T)} \leqslant C$. The following estimates are a consequence of [9, 10, 11]: for $p \in [1,\infty]$, $t \in [0,1]$, $g(t) = \bar{g} + t(g_h - \bar{g})$, $l \in \{0,1,2\}$,

$$\|g(t) - \bar{g}\|_{W_h^{l,p}} + \|g^{-1}(t) - \bar{g}^{-1}\|_{W_h^{l,p}} + \|\sqrt{\det g(t)} - \sqrt{\det \bar{g}}\|_{W_h^{l,p}} \lesssim \|g_h - \bar{g}\|_{W_h^{l,p}}, \tag{3.9a}$$

$$\|g(t)\|_{W_h^{2,\infty}} + \|g(t)^{-1}\|_{L^\infty} + \|\sqrt{\det g(t)}\|_{L^\infty} + \|\sqrt{\det g(t)^{-1}}\|_{L^\infty} \lesssim 1. \tag{3.9b}$$

Further, for all x in the interior of any element $T \in \mathscr{T}$ and for all $u \in \mathbb{R}^2$, as well as for the L^2 inner product there holds the following equivalences

$$u'u \lesssim u'g(t)(x)u \lesssim u'u, \qquad\qquad (\cdot,\cdot)_{L^2} \lesssim (\cdot,\cdot)_{L^2,g(t)} \lesssim (\cdot,\cdot)_{L^2}.$$

3.3. **Analysis of distributional rotrot operator.** In this section, we derive improved convergence rates of the distributional covariant $\operatorname{rot}\operatorname{rot}_g$ operator (2.9). The proof strategy follows [12, Theorem 6.1, (6.3)], however, adapted from the distributional covariant curl to the rot rot operator.

Proposition 3.5. *Let $k \geqslant 1$ be an integer, $g \in \mathrm{Reg}^+(\mathscr{T})$, $\rho \in W^{s,p}(\Omega,\mathcal{S})$, $p \in [1,\infty]$, $s \in (1/p,\infty]$, $\rho_h = \mathcal{I}_h^{\mathrm{Reg},k}\rho$, and $u_h \in \mathring{\mathcal{V}}_h^k$. Then there holds*

$$|(\widetilde{\operatorname{rot}\operatorname{rot}_g} u_h)(\rho - \rho_h)| \leqslant Ch \|\!|\rho - \rho_h|\!\|_2 \|u_h\|_{H^1},$$

where the constant $C > 0$ depends on Ω, the mesh regularity, k, $\|g\|_{W_h^{2,\infty}}$, and $\|g^{-1}\|_{L^\infty}$.

Proof. First, consider the element terms of (2.9). Comparing with coordinate expressions (2.7c) and (2.2) we can find smooth functions

$$F(g) = \frac{1}{\sqrt{\det g}}, \qquad [G(g)]^{ij}_k = \frac{1}{\sqrt{\det g}}\varepsilon^{jq}\left(\Gamma^i_{qk}(g) - \Gamma^l_{lq}(g)\,\delta^i_k\right),$$

such that

$$\begin{aligned}
\int_T \langle \operatorname{rot}\operatorname{rot}_g u_h, \rho - \rho_h\rangle \omega_T &= \int_T [\rho - \rho_h]_{ij}\left(F(g)[\operatorname{rot}\operatorname{rot} u_h]^{ij} + [G(g)]^{ij}_k[\operatorname{rot} u_h]^k\right) \mathrm{da}\\
&= \int_T [\rho-\rho_h]_{ij}[\operatorname{rot}\operatorname{rot} u_h]^{ij}\left((\Pi^1_{L^2} + \Pi^{1,\perp}_{L^2})F(g)\right)\\
&\qquad + [\rho-\rho_h]_{ij}[\operatorname{rot} u_h]^k\left((\Pi^0_{L^2} + \Pi^{0,\perp}_{L^2})[G(g)]^{ij}_k\right) \mathrm{da}\\
&= \int_T [\rho-\rho_h]_{ij}[\operatorname{rot}\operatorname{rot} u_h]^{ij}\Pi^{1,\perp}_{L^2}\left(F(g)\right)\\
&\qquad + [\rho-\rho_h]_{ij}[\operatorname{rot} u_h]^k\Pi^{0,\perp}_{L^2}\left([G(g)]^{ij}_k\right) \mathrm{da}\\
&\leqslant Ch_T^2\|\rho-\rho_h\|_{L^2(T)}|u_h|_{H^2(T)} + h_T\|\rho-\rho_h\|_{L^2(T)}\|u_h\|_{H^1(T)}\\
&\leqslant Ch_T\|\rho-\rho_h\|_{L^2(T)}\|u_h\|_{H^1(T)}.
\end{aligned}$$

Above, we split the nonlinear terms using L^2-projections $\Pi^k_{L^2}$ and co-projections $\Pi^{k,\perp}_{L^2}$ onto elementwise polynomials, and used their approximation property. Further, we exploited that the first $k-1$ moments of $\rho - \rho_h$ are zero (3.1b) and used inverse inequality (3.6).

Next, we focus on the element-boundary terms of (2.9). With the coordinate expressions for the g-normalized tangent and normal vector, see e.g. [12] with the Euclidean vectors (τ, ν) and the notation $g^{\nu\nu} = g^{ij}\nu_i\nu_j$

$$\hat{\tau}^i = \frac{1}{\sqrt{g_{\tau\tau}}}\tau^i, \qquad \hat{\nu}^i = \frac{g^{ij}\nu_j}{\sqrt{g^{\nu\nu}}}$$

we collect all terms depending on g in the nonlinear function

$$[H(g)]^i = \frac{1}{\sqrt{g_{\tau\tau}\, g^{\nu\nu}}} g^{ij}\nu_j.$$

We split H with the edgewise L^2-interpolant $\Pi_{L^2}^{E,k}$. Then, we use Hölder inequality, as well as the trace inequalities (3.5) and (3.6) to obtain

$$\begin{aligned}\int_{\partial T}(\rho-\rho_h)(\hat{\tau},\hat{\tau})\langle\nabla u_h,\hat{\nu}\rangle\,\omega_{\partial T} &= \int_{\partial T}(\rho-\rho_h)(\tau,\tau)\partial_i u_h\,\left((\Pi_{L^2}^{E,1}+\Pi_{L^2}^{E,1,\perp})[H(g)]^i\right)\,\mathrm{dl}\\ &\leqslant Ch_T^2\,\|\rho-\rho_h\|_{L^2(\partial T)}\|\nabla u_h\|_{L^2(\partial T)}\|g\|_{W^{2,\infty}(\partial T)}\\ &\leqslant Ch_T|\!|\!|\rho-\rho_h|\!|\!|_{2,T}\|u_h\|_{H^1(T)}.\end{aligned}$$

Summing over all elements $T \in \mathscr{T}$ finishes the proof. ☐

Due to Lemma 2.5 we obtain as a byproduct the convergence of the distributional covariant incompatibility operator.

Corollary 3.6. *Under the assumptions of Proposition 3.5, there holds*

$$\left|\left(\widetilde{\operatorname{inc}}_g\,(\sigma-\sigma_h)\right)(u_h)\right| \leqslant Ch|\!|\!|\sigma-\sigma_h|\!|\!|_2\|u_h\|_{H^1},$$

where the constant $C>0$ depends on Ω, the mesh regularity, k, $\|g\|_{W_h^{2,\infty}}$, and $\|g^{-1}\|_{L^\infty}$.

3.4. **Proof of Theorem 3.1.** We are now in position to prove our main theorem. The proof strategy is inspired by the proofs of [12, Theorem 6.5] and [9, Theorem 4.1].

Proof of Theorem 3.1. We start with the definition of the H^{-1}-norm noting that $K_h\omega_h$ and $\bar{K}\bar{\omega}$ are square integrable

$$\|K_h\omega_h-\bar{K}\bar{\omega}\|_{H^{-1}} = \sup_{u\in H_0^1(\Omega)}\frac{(K_h\omega_h-\bar{K}\bar{\omega},u)_{L^2}}{\|u\|_{H^1}}.$$

Next, we add and subtract the L^2-orthogonal interpolant (3.7) $u_h := P_{L^2}^k u$, $P_{L^2}^k : L^2(\Omega)\to\mathring{\mathcal{V}}_h^k$, to split the error into three parts

$$\begin{aligned}(K_h\omega_h-\bar{K}\bar{\omega},u)_{L^2} &= (K_h\omega_h-\bar{K}\bar{\omega},u_h)_{L^2}+(K_h\omega_h-\bar{K}\bar{\omega},u-u_h)_{L^2}\\ &= (K_h\omega_h-\bar{K}\bar{\omega},u_h)_{L^2}+(K_h-\bar{K},u-u_h)_{L^2,\bar{g}}+(K_h(\omega_h-\bar{\omega}),u-u_h)_{L^2}\\ &=: s_1+s_2+s_3.\end{aligned}$$

We use (2.6), the integral representation of the error (2.10) with $g(t) = \bar{g}+t(g_h-\bar{g})$ and $\sigma = g'(t) = g_h-\bar{g}$, and the adjoint of the distributional incompatibility operator Lemma 2.5

$$s_1 = \left(\widetilde{K\omega}(g_h)-\bar{K}\bar{\omega}\right)(u_h) = -\frac{1}{2}\int_0^1\left(\widetilde{\operatorname{inc}_{g(t)}\sigma}\right)(u_h)\,dt = -\frac{1}{2}\int_0^1\left(\widetilde{\operatorname{rot}\operatorname{rot}_{g(t)}}\,u_h\right)(\sigma)\,dt.$$

From Proposition 3.5 (setting $g=g(t)$ and $\rho=-\bar{g}$) we obtain together with the H^1-stability (3.8b)

$$|s_1| \lesssim h|\!|\!|\sigma|\!|\!|_2\|u_h\|_{H^1} \lesssim h|\!|\!|g_h-\bar{g}|\!|\!|_2\|u\|_{H^1}. \tag{3.10}$$

For s_2 we use the definition of the L^2-orthogonal interpolant (3.7) $u_h = P^k_{L^2} u$, Cauchy-Schwarz inequality, and the approximation property (3.8c) of u_h. For arbitrary $v_h \in \mathring{\mathcal{V}}^k_h$ there holds

$$(K_h - \bar{K}, u - u_h)_{L^2,\bar{g}} = (v_h - \bar{K}, u - u_h)_{L^2,\bar{g}} \lesssim \|v_h - \bar{K}\|_{L^2} \|u - u_h\|_{L^2} \lesssim h\|v_h - \bar{K}\|_{L^2}\|u\|_{H^1}$$

and thus,

$$|s_2| \lesssim h \inf_{v_h \in \mathring{\mathcal{V}}^k_h} \|v_h - \bar{K}\|_{L^2}\|u\|_{H^1}.$$

Before we turn to the third term s_3 we show that the lifted Gauss curvature K_h is bounded in the L^2-norm by using $K_h \in \mathring{\mathcal{V}}^k_h$ instead of u_h in estimate (3.10)

$$\begin{aligned}\|K_h\|^2_{L^2} &\lesssim (K_h\omega_h, K_h)_{L^2} = (K_h\omega_h - \bar{K}\bar{\omega}, K_h)_{L^2} + (\bar{K}, K_h)_{L^2,\bar{g}}\\ &\lesssim h\|g_h - \bar{g}\|_{L^2}\|K_h\|_{H^1} + \|\bar{K}\|_{L^2}\|K_h\|_{L^2}\\ &\lesssim \big(\|g_h - \bar{g}\|_{L^2} + \|\bar{K}\|_{L^2}\big)\|K_h\|_{L^2}.\end{aligned}$$

For the last inequality we used the inverse estimate (3.6). Dividing by $\|K_h\|_{L^2}$ yields the boundedness of K_h.

Using Hölder inequality, the approximation property (3.8c) of u_h, inequality (3.9a), and that $\|K_h\|_{L^2} \lesssim 1$ yields the following estimate for s_3

$$|s_3| \leqslant \|K_h\|_{L^2}\|\omega_h - \bar{\omega}\|_{L^\infty}\|u - u_h\|_{L^2} \lesssim h\|g_h - \bar{g}\|_{L^\infty}\|u\|_{H^1}.$$

Combining all results yields

$$\begin{aligned}\|K_h\omega_h - \bar{K}\bar{\omega}\|_{H^{-1}} &= \sup_{u \in H^1_0(\Omega)} \frac{(K_h\omega_h - \bar{K}\bar{\omega}, u)_{L^2}}{\|u\|_{H^1}}\\ &\lesssim h(|\!|\!|g_h - \bar{g}|\!|\!|_2 + \inf_{v_h \in \mathring{\mathcal{V}}^k_h} \|v_h - \bar{K}\|_{L^2} + \|g_h - \bar{g}\|_{L^\infty}).\end{aligned}$$

Using standard interpolation techniques we obtain the desired convergence rate for $0 \leqslant l \leqslant k+1$

$$\|K_h\omega_h - K\omega\|_{H^{-1}} \lesssim h^{l+1}\big(\|\bar{g}\|_{W^{l,\infty}} + \|\bar{K}\|_{H^l}\big).$$

□

3.5. **Analysis of the lifting of pure Gauss curvature.** To relate the error of the Gauss curvature with the densitized Gauss curvature we need the following result.

Lemma 3.7. *Under the assumptions of Theorem 3.1, there holds for* $0 \leqslant l \leqslant k+1$

$$\begin{aligned}\|K_h(\omega_h - \bar{\omega})\|_{H^{-1}} &\lesssim h\,\|g_h - \bar{g}\|_{L^\infty}\|K_h\|_{H^1_h}\\ &\lesssim h^{l+1}\|\bar{g}\|_{W^{l,\infty}}\|K_h\|_{H^1_h}.\end{aligned}$$

Proof. By noting that $\omega_h - \bar{\omega} = [g_h - \bar{g}]_{ij}[F(g_h, \bar{g})]^{ij}$ with the smooth function

$$[F(g_h, \bar{g})] = \frac{1}{\sqrt{\det g_h} + \sqrt{\det \bar{g}}}\begin{bmatrix} (g_h)_{22} & \frac{1}{2}(\bar{g} + g_h)_{12}\\ \frac{1}{2}(\bar{g} + g_h)_{12} & \bar{g}_{11}\end{bmatrix},$$

and using that for $k \geqslant 1$ the constant moment of the difference $g_h - \bar{g}$ is zero due to (3.1b), we obtain

$$\begin{aligned}(K_h(\omega_h - \bar{\omega}), u)_{L^2} &= \int_\Omega [g_h - \bar{g}]_{ij}[F(g_h, \bar{g})]^{ij} K_h u \,\mathrm{da}\\ &= \int_\Omega [g_h - \bar{g}]_{ij} \Big((\Pi^0_{L^2} + \Pi^{0,\perp}_{L^2}) \big([F(g_h, \bar{g})]^{ij} K_h u\big)\Big) \,\mathrm{da}\\ &= \int_\Omega [g_h - \bar{g}]_{ij} \Pi^{0,\perp}_{L^2} \big([F(g_h, \bar{g})]^{ij} K_h u\big) \,\mathrm{da}\\ &\leqslant \|g_h - \bar{g}\|_{L^\infty} \|\Pi^{0,\perp}_{L^2} \left(F(g_h, \bar{g}) K_h u\right)\|_{L^1}\\ &\lesssim h \|g_h - \bar{g}\|_{L^\infty} \|K_h\|_{H^1_h} \|u\|_{H^1},\end{aligned}$$

finishing the proof. □

By Lemma 3.7, inverse estimate (3.6), and boundedness of K_h in L^2, $\|K_h\|_{H^1_h} \lesssim h^{-1}\|K_h\|_{L^2} \lesssim h^{-1}$, we can deduce a suboptimal convergence rate of the error of the pure lifted Gauss curvature

$$\begin{aligned}\|K_h - \bar{K}\|_{H^{-1}} &\lesssim \|K_h \bar{\omega} - \bar{K}\bar{\omega}\|_{H^{-1}}\\ &\leqslant \|K_h \omega_h - \bar{K}\bar{\omega}\|_{H^{-1}} + \|K_h(\bar{\omega} - \omega_h)\|_{H^{-1}}\\ &\lesssim h\big(\|\!|g_h - \bar{g}|\!\|_2 + \inf_{v_h \in \mathring{\mathcal{V}}^k_h} \|v_h - \bar{K}\|_{L^2} + \|g_h - \bar{g}\|_{L^\infty}\big) + h\,\|g_h - \bar{g}\|_{L^\infty}\|K_h\|_{H^1_h}\\ &\lesssim h^{l+1}\big(\|\bar{g}\|_{W^{l+1,\infty}} + \|\bar{K}\|_{H^l}\big), \qquad\qquad 0 \leqslant l \leqslant k \text{ (not } k+1).\end{aligned} \tag{3.11}$$

To correct the convergence of the lifted Gauss curvature and to prove optimal rates for the (densitized) lifted Gauss curvature in stronger Sobolev norms we consider a bootstrapping-like technique. First, we can easily adapt the proof of [9, p. 1818] and [12, Corollary 6.6] to deduce for $0 \leqslant l \leqslant k$ and $0 \leqslant r \leqslant l$

$$\begin{aligned}\|K_h - \bar{K}\|_{H^r_h} &\lesssim h^{-r}\big(\|\!|g_h - \bar{g}|\!\|_2 + h^{-1}\|g_h - \bar{g}\|_{L^\infty} + \inf_{v_h \in \mathring{\mathcal{V}}^k_h} \|v_h - K\|_{L^2} + h^l |\bar{K}|_{H^l}\big)\\ &\lesssim h^{l-r}\big(\|\bar{g}\|_{W^{l+1,\infty}} + \|\bar{K}\|_{H^l}\big)\end{aligned}$$

and therefore the boundedness in the elementwise H^1-norm for $l = r = 1$ and $k \geqslant 1$,

$$\|K_h\|_{H^1_h} \leqslant \|K_h - \bar{K}\|_{H^1_h} + \|\bar{K}\|_{H^1} \lesssim \|\bar{g}\|_{W^{2,\infty}} + \|\bar{g}\|_{H^1} + \|\bar{K}\|_{H^1} \lesssim 1.$$

Thus, instead of using the inverse inequality in (3.11) we directly obtain the improved convergence rate:

Theorem 3.8. *Under the assumptions of Theorem 3.1, there holds for* $0 \leqslant l \leqslant k+1$

$$\begin{aligned}\|K_h - \bar{K}\|_{H^{-1}} &\lesssim h\big(\|\!|g_h - \bar{g}|\!\|_2 + \inf_{v_h \in \mathring{\mathcal{V}}^k_h} \|v_h - \bar{K}\|_{L^2} + \|g_h - \bar{g}\|_{L^\infty}\big)\\ &\lesssim h^{l+1}\big(\|\bar{g}\|_{W^{l,\infty}} + \|\bar{K}\|_{H^l}\big).\end{aligned}$$

This yields also improved rates in stronger norms for the lifted Gauss curvature:

Corollary 3.9. *Under the assumptions of Theorem 3.1, there holds for all* $0 \leqslant l \leqslant k+1$ *and* $0 \leqslant r \leqslant l$

$$\begin{aligned}\|K_h - \bar{K}\|_{H^r_h} &\lesssim h^{-r}\big(\|g_h - \bar{g}\|_{L^\infty} + \|\!|g_h - \bar{g}|\!\|_2 + \inf_{v_h \in \mathring{\mathcal{V}}^k_h} \|v_h - \bar{K}\|_{L^2} + h^l |\bar{K}|_{H^l}\big)\\ &\lesssim h^{l-r}\big(\|\bar{g}\|_{W^{l,\infty}} + \|\bar{K}\|_{H^l}\big).\end{aligned}$$

Proof. Follows analogously to the proof of [12, Corrolary 6.6] and [9, p. 1818] as the error in stronger Sobolev norms is traced back to the H^{-1}-norm. See also the proof of Corollary 3.2 below. □

3.6. **Proof of Corollary 3.2.** To prove the desired rates for the densitized Gauss curvature we first note that for the L^2-norm there directly holds with Lemma 3.7, $\|K_h\|_{H^1_h} \lesssim 1$, and Corollary 3.9 for $0 \leqslant l \leqslant k+1$

$$\begin{aligned}\|K_h\omega_h - \bar{K}\bar{\omega}\|_{L^2} &\lesssim \|K_h(\omega_h - \bar{\omega})\|_{L^2} + \|K_h - \bar{K}\|_{L^2}\\ &\lesssim h\|g_h - \bar{g}\|_{L^\infty} + \|g_h - \bar{g}\|_{L^\infty} + \|\!|g_h - \bar{g}|\!\|_2 + \inf_{v_h \in \mathring{\mathcal{V}}_h^k} \|v_h - \bar{K}\|_{L^2} + h^l|\bar{K}|_{H^l}\\ &\lesssim h^l\big(\|\bar{g}\|_{W^{l,\infty}} + \|\bar{K}\|_{H^l}\big).\end{aligned}$$

With the improved L^2 error estimate at hand we can prove optimal convergence rates in stronger Sobolev spaces.

Proof of Corollary 3.2. Let $u_h \in \mathring{\mathcal{V}}_h^k$ be the Scott–Zhang interpolant [24] of $\bar{K}\bar{\omega}$. Then there holds, analogously to the proof of [9, p. 1818],

$$\begin{aligned}|K_h\omega_h - \bar{K}\bar{\omega}|_{H^r_h} &\leqslant |K_h\omega_h - u_h|_{H^r_h} + |u_h - \bar{K}\bar{\omega}|_{H^r_h}\\ &\lesssim h^{-r}\|K_h\omega_h - u_h\|_{L^2} + h^{l-r}\|\bar{K}\bar{\omega}\|_{H^l}\\ &\leqslant h^{-r}\big(\|K_h\omega_h - \bar{K}\bar{\omega}\|_{L^2} + \|\bar{K}\bar{\omega} - u_h\|_{L^2} + h^l\|\bar{K}\bar{\omega}\|_{H^l}\big)\\ &\lesssim h^{-r}\Big(\|g_h - \bar{g}\|_{L^\infty} + \|\!|g_h - \bar{g}|\!\|_2 + \inf_{v_h \in \mathring{\mathcal{V}}_h^k} \|v_h - \bar{K}\|_{L^2}\\ &\qquad + h^l\|\bar{K}\|_{H^l} + \inf_{v_h \in \mathring{\mathcal{V}}_h^k} \|v_h - \bar{K}\bar{\omega}\|_{L^2} + h^l\|\bar{K}\bar{\omega}\|_{H^l}\Big)\\ &\lesssim h^{l-r}\big(\|\bar{g}\|_{W^{l,\infty}} + \|\bar{K}\|_{H^l} + \|\bar{K}\bar{\omega}\|_{H^l}\big).\end{aligned}$$

□

4. Numerical examples

In this section we confirm, by numerical examples, that the theoretical convergence rates from Theorem 3.1, Corollary 3.2, Theorem 3.8, and Corollary 3.9 are sharp. All experiments were performed in the open source finite element software NGSolve[1] [22, 23], where the Regge elements are available.

We consider the numerical example proposed in [9], where on the square $\Omega = (-1,1)\times(-1,1)$ the smooth Riemannian metric tensor

$$\bar{g}(x,y) = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_x f \partial_y f & 1 + (\partial_y f)^2 \end{pmatrix}$$

with $f(x,y) = \frac{1}{2}(x^2+y^2) - \frac{1}{12}(x^4+y^4)$ is defined. This metric corresponds to the surface induced by the embedding $(x,y) \mapsto (x,y,f(x,y))$ and its exact Gauss curvature is given by

$$\bar{K}(x,y) = \frac{81(1-x^2)(1-y^2)}{(9 + x^2(x^2-3)^2 + y^2(y^2-3)^2)^2}.$$

To test also the case of non-homogeneous Dirichlet and Neumann boundary conditions we follow [12] and consider only one quarter $\Omega = (0,1)\times(0,1)$ and define the right and bottom boundaries as Dirichlet and the remaining parts as Neumann boundary. We start with a

[1] www.ngsolve.org

structured mesh consisting of 2^{2l+1} triangles with maximal mesh-size $h = \max_T h_T = \sqrt{2}\,2^{-l}$ (and minimal edge length 2^{-l}) for $l = 0, 1, \dots$. To avoid possible super-convergence properties due to a structured grid, we perturb all internal points of the triangular mesh by a uniform distribution in the range $[-\frac{h}{2^{2.5}}, \frac{h}{2^{2.5}}]$. The geodesic curvature on the left boundary is exactly zero, whereas on the top boundary we have

$$\bar{\kappa}|_{\Gamma_{\text{left}}} = 0, \qquad \bar{\kappa}|_{\Gamma_{\text{top}}} = \frac{-27(x^2-1)y(y^2-3)}{(x^2(x^2-3)^2+9)^{3/2}\sqrt{x^2(x^2-3)^2+y^2(y^2-3)^2+9}}.$$

The vertex expressions $\bar{\sphericalangle}_V^N$ at the vertices of the Neumann boundary can directly be computed by measuring the angle $\arccos(\bar{g}(\hat{\tau}_V^1, \hat{\tau}_V^2))$.

To illustrate our theorems, we must use $g_h = \mathcal{I}_h^{\mathrm{Reg},k}\bar{g}$. In implementing the Regge interpolant, the moments on the edges have to coincide exactly: see (3.1). To this end, we use a high enough integration rule for interpolating $\bar{g}$ for minimizing the numerical integration errors.

We compute and report the curvature error in the L^2-norm, namely $\|\bar{K} - K_h\|_{L^2}$ and $\|\bar{K}\bar{\omega} - K_h\omega_h\|_{L^2}$. We also report the H^{-1}-norm of the errors. They can be computed by solving e.g. for $w \in H_0^1(\Omega)$ such that $-\Delta w = \bar{K} - K_h$ and observing that

$$\|\bar{K} - K_h\|_{H^{-1}} = \|w\|_{H^1}.$$

Of course the right-hand side can generally be computed only approximately. To avoid extraneous errors, we approximate w using Lagrange finite elements of two degrees more, i.e., $w_h \in \mathcal{V}_h^{k+2}$ when $K_h \in \mathcal{V}_h^k$.

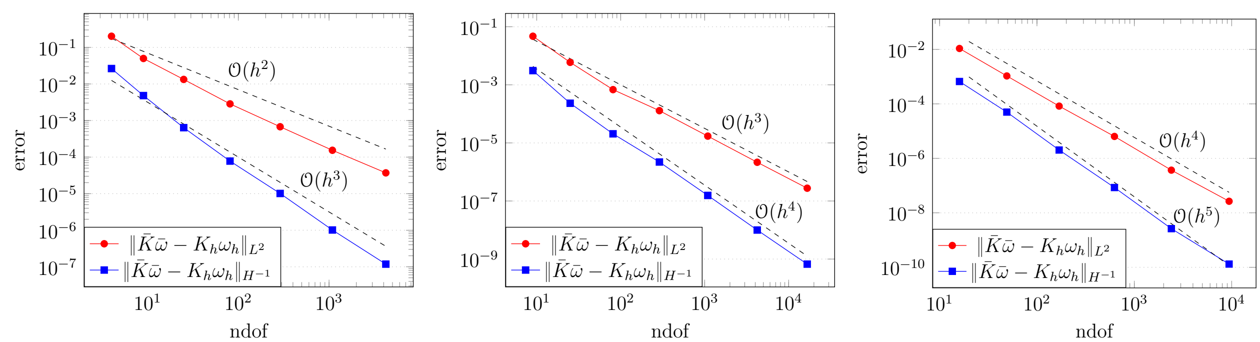
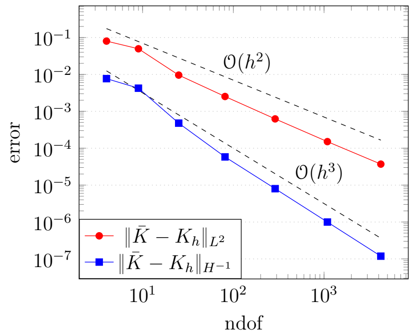
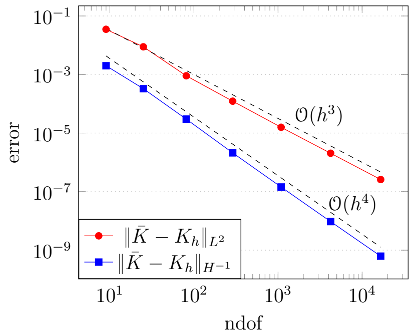


FIGURE 1. Convergence of lifted densitized Gauss curvature with respect to number of degrees of freedom (ndof) in different norms for Regge elements $g_h \in \mathrm{Reg}_h^k$ of order $k = 1, 2, 3$.

We start by approximating $\bar{g}$ with linear Regge elements $g_h \in \mathrm{Reg}_h^1$. As shown in Figure 1 (left), we obtain the stated quadratic convergence in the L^2-norm and cubic rate in the weaker H^{-1}-norm, in agreement with Theorem 3.1. When increasing the approximation order of Regge elements to quadratic and cubic polynomials we observe the appropriate increase of the convergence rates: see Figure 1 (middle and right), confirming that the results stated in Theorem 3.1 and Corollary 3.2 are sharp. For the error of the pure Gauss curvature we practically obtain the same behavior as stated by Theorem 3.8 and Corollary 3.9, cf. Figure 2. Only in the pre-asymptotic regime the error is smaller compared to the densitized Gauss curvature.

We conclude with a few additional remarks on the lifting degree. Attempting to increase the degree for the curvature approximation, say by placing K_h in $\mathcal{V}_h^{k+1}$ or $\mathcal{V}_h^{k+2}$, while the metric g_h remains in Reg_h^k, need not generally produce additional orders of convergence. This is because





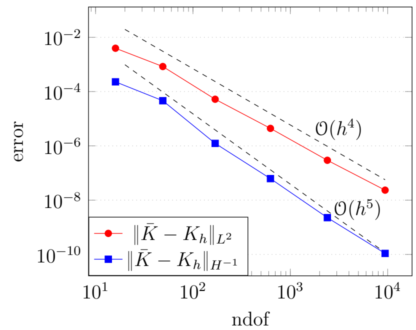


FIGURE 2. Convergence of pure lifted Gauss curvature with respect to number of degrees of freedom (ndof) in different norms for Regge elements $g_h \in \mathrm{Reg}_h^k$ of order $k = 1, 2, 3$.

the orthogonality properties of the canonical Regge interpolant, namely (3.1a)–(3.1b), may not be fulfilled in such cases. Indeed, we numerically observed loss of two orders of convergence when K_h is placed in $\mathcal{V}_h^{k+2}$ instead of $\mathcal{V}_h^k$. In [12], where we used $K_h \in \mathcal{V}_h^{k+1}$, one order less is obtained, again due to the orthogonality properties of the canonical Regge interpolant.

Finally, when reducing the polynomial degree of the curvature approximation from $\mathcal{V}_h^k$ to $\mathcal{V}_h^{k-1}$, $k \geqslant 2$, while keeping the metric in Reg_h^k, we observed that the convergence rates reduce by one order. Note that the orthogonality properties of the canonical Regge interpolant are still fulfilled. Nevertheless, the overall approximation ability of the space is reduced so that the convergence rate in the H^{-1}-norm decreases from $\mathcal{O}(h^{k+2})$ to $\mathcal{O}(h^{k+1})$.

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