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Flat morphisms with regular fibers do not preserve F-rationality

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Abstract. For each prime integer $p > 0$, we construct a standard graded F-rational ring R, over a field K of characteristic p, such that $R \otimes_K \overline{K}$ is not F-rational. By localizing, we obtain a flat local homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ such that R is F-rational, $S/\mathfrak{m}S$ is regular (in fact, a field), but S is not F-rational. In the process, we also obtain standard graded F-rational rings R for which $R \otimes_K R$ is not F-rational.

1. Introduction

Let $\mathcal{P}$ denote a local property of noetherian rings. The following types of *ascent* have been studied extensively; recall that for K a field, a noetherian K-algebra A is *geometrically regular* over K if $A \otimes_K L$ is regular for each finite extension field L of K.

($\mathrm{ASC_I}$) For a flat local homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ of excellent local rings, if R is $\mathcal{P}$ and the closed fiber $S/\mathfrak{m}S$ is regular, then S is $\mathcal{P}$.

($\mathrm{ASC_{II}}$) For a flat local homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ of excellent local rings, if R is $\mathcal{P}$ and the closed fiber $S/\mathfrak{m}S$ is geometrically regular over $R/\mathfrak{m}$, then S is $\mathcal{P}$.

Our main interest here is when $\mathcal{P}$ is F-rationality, a property rooted in Hochster and Huneke's theory of tight closure [14]: a local ring $(R, \mathfrak{m})$ of positive prime characteristic is *F-rational* if R is Cohen–Macaulay and each ideal generated by a system of parameters for R is tightly closed. Smith [22] proved that F-rational rings have rational singularities, while Hara [11] and Mehta–Srinivas [19] independently proved that rings with rational singularities have F-rational type. Rational singularities of characteristic zero satisfy ($\mathrm{ASC_I}$), as proven by Elkik, see Théorème 5 in [5].

In the situation of ($\mathrm{ASC_{II}}$), geometric regularity of the closed fiber $R/\mathfrak{m} \to S/\mathfrak{m}S$ implies that of each fiber

$$k(\mathfrak{p}) \to S \otimes_R k(\mathfrak{p}) \quad \text{for } \mathfrak{p} \in \operatorname{Spec} R,$$

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see [3], p. 297. The ascent ($\mathrm{ASC_{II}}$) holds for F-rationality; this, and its variations, are due to Vélez (Theorem 3.1 in [23]), Enescu (Theorem 2.27 in [6]), Hashimoto (Theorem 6.4 in [12]), and Aberbach–Enescu (Theorem 4.3 in [2]). A common thread amongst these is that each affirmative answer requires assumptions along the lines that the fibers are *geometrically* regular.

The situation is similar for F-injectivity in this regard; a local ring $(R, \mathfrak{m})$ of positive prime characteristic is *F-injective* if the Frobenius action on local cohomology modules

$$F : H^k_{\mathfrak{m}}(R) \to H^k_{\mathfrak{m}}(R)$$

is injective for each $k \geqslant 0$. Datta and Murayama, see Theorem A in [4], proved that if $(R, \mathfrak{m})$ is F-injective, and $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local map such that $S/\mathfrak{m}S$ is Cohen–Macaulay and *geometrically* F-injective over $R/\mathfrak{m}$, then S is F-injective; see also Theorem 4.3 in [7] and Corollary 5.7 in [12]. We present examples demonstrating that the geometric assumptions are indeed required, i.e., that F-rationality and F-injectivity do not satisfy ($\mathrm{ASC_I}$):

Theorem 1.1. *For each prime integer $p > 0$, there exists a flat local map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ of excellent local rings of characteristic p such that the ring R is F-rational, $S/\mathfrak{m}S$ is regular, but S is not F-rational or even F-injective.*

Enescu had earlier demonstrated that F-injectivity does not satisfy ($\mathrm{ASC_I}$), though the examples on p. 3075 of [7] are not normal; the question of whether normal F-injective rings satisfy ($\mathrm{ASC_I}$) has been raised earlier, see, e.g., Question 8.1 in [20], and is settled in the negative by Theorem 1.1. There is a more recent notion, *F-anti-nilpotence*, developed in the papers [8, 17, 18]; in view of the implications

$$F\text{-rational} \implies F\text{-anti-nilpotent} \implies F\text{-injective},$$

Theorem 1.1 also shows that F-anti-nilpotence does not satisfy ($\mathrm{ASC_I}$).

It is worth mentioning that the rings R in Theorem 1.1 are necessarily not Gorenstein, since F-rational Gorenstein rings are F-regular by Theorem 4.2 in [15], and F-regularity satisfies ($\mathrm{ASC_I}$) by Theorem 3.6 in [1]. Another subtlety is that such examples can only exist over imperfect fields, since ($\mathrm{ASC_I}$) and ($\mathrm{ASC_{II}}$) coincide when $R/\mathfrak{m}$ is a perfect field, and F-rationality satisfies ($\mathrm{ASC_{II}}$).

Some preliminary results are recorded in Section 2, including an extension of a criterion for F-rationality due to Fedder and Watanabe [9]. In Section 3, we construct two families of examples that each imply Theorem 1.1: the first has the advantage that the proofs are more transparent, though the transcendence degree of the imperfect field over $\mathbb{F}_p$ increases with the characteristic p; the second family accomplishes the desired with transcendence degree one, independent of the characteristic $p > 0$, though the calculations are more involved. The examples in Section 3 are constructed as standard graded rings, with the relevant properties preserved under passing to localizations. In the process, we also obtain standard graded F-rational rings R, with the degree zero component being a field K of positive characteristic, such that the enveloping algebra $R \otimes_K R$ is not F-rational.

2. Preliminaries

Following [13], p. 125, a local ring of positive prime characteristic is *F-rational* if it is a homomorphic image of a Cohen–Macaulay ring, and each ideal generated by a system of parameters is tightly closed. It follows from this definition that an F-rational local ring is Cohen–Macaulay, see Theorem 4.2 in [15], so the notion coincides with that in Section 1. Moreover, an F-rational local ring is a normal domain. A localization of an F-rational local ring at a prime ideal is again F-rational; with this in mind, a noetherian ring of positive prime characteristic – which is not necessarily local – is *F-rational* if its localization at each maximal ideal (equivalently, at each prime ideal) is F-rational.

For the case of interest in this paper, let R be an $\mathbb{N}$-graded Cohen–Macaulay normal domain, such that the degree zero component is a field K of characteristic $p > 0$, and R is a finitely generated K-algebra. Then R is F-rational if and only if the ideal generated by some (equivalently, any) homogeneous system of parameters for R is tightly closed; see Theorem 4.7 in [16] and the remark preceding it. An equivalent formulation in terms of local cohomology, following Proposition 3.3 in [21], is described next.

Fix a homogeneous system of parameters $x_1, \ldots, x_d$ for R, i.e., a sequence of $d := \dim R$ homogeneous elements that generate an ideal with radical the homogeneous maximal ideal $\mathfrak{m}$ of R. The local cohomology module $H^d_{\mathfrak{m}}(R)$ may then be computed using a Čech complex on $x_1, \ldots, x_d$ as

$$H^d_{\mathfrak{m}}(R) = \frac{R_{x_1 \cdots x_d}}{\sum_i R_{x_1 \cdots \hat{x}_i \cdots x_d}}.$$

This module admits a natural $\mathbb{Z}$-grading, where the cohomology class

$$\eta := \left[\frac{r}{x_1^k \cdots x_d^k}\right] \in H^d_{\mathfrak{m}}(R), \tag{2.1}$$

for $r \in R$ a homogeneous element, has

$$\deg \eta := \deg r - k \sum_{i=1}^{d} \deg x_i .$$

The Frobenius endomorphism $F \colon R \to R$ induces a map

$$F : H^d_{\mathfrak{m}}(R) \to H^d_{F(\mathfrak{m})}(R) = H^d_{\mathfrak{m}}(R)$$

that is the *Frobenius action* on $H^d_{\mathfrak{m}}(R)$; this is simply the map

$$\eta = \left[\frac{r}{x_1^k \cdots x_d^k}\right] \longmapsto F(\eta) = \left[\frac{r^p}{x_1^{kp} \cdots x_d^{kp}}\right]. \tag{2.2}$$

Since R is Cohen–Macaulay by assumption, R is F-injective precisely when the map (2.2) is injective.

The element η as in (2.1) belongs to $0^*_{H^d_{\mathfrak{m}}(R)}$, the *tight closure* of zero in $H^d_{\mathfrak{m}}(R)$, if there exists a nonzero element $c \in R$ such that for all $e \in \mathbb{N}$, one has

$$cF^e(\eta) = 0$$

in $H^d_{\mathfrak{m}}(R)$. This translates as

$$cr^{p^e} \in (x_1^{kp^e}, \dots, x_d^{kp^e})R$$

for all $e \in \mathbb{N}$. In particular, R is F-rational precisely when

$$0^*_{H^d_{\mathfrak{m}}(R)} = 0.$$

It follows that an F-rational ring must be F-injective.

We next review Veronese subrings. Let S be an $\mathbb{N}$-graded ring for which the degree zero component is a field K, and S is a finitely generated K-algebra. Fix a positive integer n. Then the n-th *Veronese subring* of S is the ring

$$S^{(n)} := \bigoplus_{k \in \mathbb{N}} S_{nk}.$$

Set $R := S^{(n)}$. The extension $R \subseteq S$ is split, so if S is normal ring, then so is R. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R, and note that $\mathfrak{m}S$ is primary to the homogeneous maximal ideal $\mathfrak{n}$ of S. For all $i \leqslant d := \dim S = \dim R$, it follows that $H^i_{\mathfrak{m}}(R)$ is a direct summand of $H^i_{\mathfrak{m}}(S) = H^i_{\mathfrak{n}}(S)$, and hence that the ring R is Cohen–Macaulay whenever S is. Moreover, by Theorem 3.1.1 in [10], one has

$$H^d_{\mathfrak{m}}(R) = \bigoplus_{k \in \mathbb{Z}} [H^d_{\mathfrak{n}}(S)]_{nk}.$$

Suppose $S := K[x_0, \dots, x_d]/(f)$, where f is a homogeneous polynomial that is monic of degree m with respect to the indeterminate x_0. Then S is free over the polynomial subring $K[x_1, \dots, x_d]$, with basis $\{1, x_0, \dots, x_0^{m-1}\}$. The local cohomology module $H^d_{\mathfrak{n}}(S)$, as computed using a Čech complex on $x_1, \dots, x_d$, thus has a K-basis consisting of elements

$$\left[\frac{x_0^{\alpha_0}}{x_1^{\alpha_1+1} \cdots x_d^{\alpha_d+1}}\right] \in H^d_{\mathfrak{n}}(S) \tag{2.3}$$

where each α_i is a nonnegative integer, and $\alpha_0 \leqslant m-1$. When S is graded, by restricting to elements of appropriate degree, one obtains a basis for a graded component of $H^d_{\mathfrak{n}}(S)$, or for the local cohomology $H^d_{\mathfrak{m}}(R)$ of the Veronese subring R. Similarly, for the enveloping algebra $S \otimes_K S$, one has a K-basis as follows: use $y_0, \dots, y_d$ for the second copy of S, and consider the maximal ideal $\mathfrak{N} := (x_0, \dots, x_d, y_0, \dots, y_d)$ of $S \otimes_K S$. Then the local cohomology module $H^{2d}_{\mathfrak{N}}(S \otimes_K S)$ has a K-basis

$$\left[\frac{x_0^{\alpha_0} y_0^{\beta_0}}{x_1^{\alpha_1+1} \cdots x_d^{\alpha_d+1} y_1^{\beta_1+1} \cdots y_d^{\beta_d+1}}\right], \tag{2.4}$$

where each α_i, β_j is a nonnegative integer, $\alpha_0 \leqslant m-1$, and $\beta_0 \leqslant m-1$.

The following is a variation of Theorem 2.8 in [9] and Theorem 7.12 in [16], and is used in the proof of Theorem 3.2.

Theorem 2.1. *Let S be an $\mathbb{N}$-graded Cohen–Macaulay normal domain, such that the degree zero component is a field K of positive characteristic, and S is a finitely generated K-algebra. Let $\mathfrak{n}$ denote the homogeneous maximal ideal of S, and set $d := \dim S$.*

Suppose each nonzero element of $\mathfrak{n}$ has a power that is a test element, and that there exists an integer $n > 0$ such that the Frobenius action on

$$[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$$

is injective. Then the tight closure of zero in $H^d_{\mathfrak{n}}(S)$ is contained in $[H^d_{\mathfrak{n}}(S)]_{>-n}$.

Proof. The hypotheses ensure that S has a homogeneous system of parameters $x_1, \dots, x_d$, where each x_i is a test element; we compute local cohomology using a Čech complex on such a homogeneous system of parameters. Suppose the assertion of the theorem is false; then there exists a nonzero homogeneous element η in $0^*_{H^d_{\mathfrak{n}}(S)}$ with $\deg \eta \leqslant -n$. After possibly replacing the x_i by powers, we may assume that

$$\eta = \left[\frac{s}{x_1 \cdots x_d}\right],$$

for s a homogeneous element of S. Since each x_i is a test element, one has

$$x_i s^q \in (x_1^q, \dots, x_d^q)$$

for each $q = p^e$, and hence

$$s^q \in (x_1^q, \dots, x_d^q) :_R (x_1, \dots, x_d) = (x_1^q, \dots, x_d^q) + (x_1 \cdots x_d)^{q-1},$$

where the equality is because $x_1, \dots, x_d$ is a regular sequence. Since $F^e(\eta)$ is nonzero in view of the injectivity of the Frobenius action on $[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$, one has

$$s^q \notin (x_1^q, \dots, x_d^q).$$

This implies that $\deg s^q \geqslant \deg(x_1 \cdots x_d)^{q-1}$ for each $q = p^e$, which translates as

$$\deg s \geqslant \frac{q-1}{q} \deg(x_1 \cdots x_d).$$

Taking the limit $e \to \infty$ gives

$$\deg s \geqslant \deg(x_1 \cdots x_d),$$

so $\deg \eta \geqslant 0$. This contradicts $\deg \eta \leqslant -n < 0$. ∎

A ring S is *standard graded* if it is $\mathbb{N}$-graded, with the degree zero component being a field K, such that S is generated as a K-algebra by finitely many elements of S_1.

While Theorem 2.1 requires the injectivity of the Frobenius action on $[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$, additional hypotheses enable one to verify the injectivity of Frobenius on *one* graded component; the following corollary will be used in the proof of Theorem 3.2. Following [10], the *a-invariant* of a Cohen–Macaulay graded ring S, as in Theorem 2.1, is

$$a(S) := \max\{i \in \mathbb{Z} \mid [H^d_{\mathfrak{n}}(S)]_i \neq 0\}.$$

Corollary 2.2. *Let S be a standard graded Gorenstein normal domain, of characteristic $p > 0$, such that the homogeneous maximal ideal $\mathfrak{n}$ is an isolated singular point. Set $d := \dim S$. Suppose $a(S) < 0$, and that there exists an integer n with $-n \leqslant a(S)$ such that the Frobenius action*

$$F\colon [H^d_{\mathfrak{n}}(S)]_{-n} \to [H^d_{\mathfrak{n}}(S)]_{-np}$$

is injective. Then the Veronese subring $S^{(n)}$ is F-rational.

Proof. Because $\mathfrak{n}$ is an isolated singular point, each nonzero element of $\mathfrak{n}$ has a power that is a test element, and Theorem 2.1 is applicable. Since S is Gorenstein, each nonzero homogeneous element η of $[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$ has a nonzero multiple $s\eta$ in the socle of $H^d_{\mathfrak{n}}(S)$, which is the graded component $[H^d_{\mathfrak{n}}(S)]_{a(S)}$. As S is standard graded, such a multiplier $s \in S$ can be chosen to be a product of elements of degree one, therefore η has a nonzero multiple $s'\eta$ in $[H^d_{\mathfrak{n}}(S)]_{-n}$. Since $F(s'\eta)$ is nonzero, so is $F(\eta)$. It follows that the Frobenius action on $[H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$ is injective, so Theorem 2.1 implies that the tight closure of zero in $H^d_{\mathfrak{n}}(S)$ is contained in $[H^d_{\mathfrak{n}}(S)]_{>-n}$.

Set $R := S^{(n)}$. The hypotheses $-n \leqslant a(S) < 0$ give

$$H^d_{\mathfrak{m}}(R) \subseteq [H^d_{\mathfrak{n}}(S)]_{\leqslant -n}$$

where $\mathfrak{m}$ is the homogeneous maximal ideal of R. As the tight closure of zero in $H^d_{\mathfrak{m}}(R)$ is contained in the tight closure of zero in $H^d_{\mathfrak{n}}(S)$, the assertion follows. ∎

3. The examples

Theorem 3.1. *Fix a prime integer $p > 0$. Let $t_1, \dots, t_p$ be indeterminates over the field $\mathbb{F}_p$ and set $K := \mathbb{F}_p(t_1, \dots, t_p)$. Consider the hypersurface*

$$S := K[x_0, \dots, x_p] / (x_0^p - t_1 x_1^p - \cdots - t_p\, x_p^p)$$

with the standard $\mathbb{N}$-grading, and its p-th Veronese subring $R := S^{(p)}$. Then:

(1) *The ring R is F-rational.*

(2) *The rings $R \otimes_K K^{1/p}$ and $R \otimes_K \overline{K}$ are not F-injective, hence not F-rational.*

(3) *The enveloping algebra $R \otimes_K R$ is not F-injective, hence not F-rational.*

Proof. First consider the hypersurface

$$A := \mathbb{F}_p[t_1, \dots, t_p,\, x_0, \dots, x_p] / (x_0^p - t_1 x_1^p - \cdots - t_p\, x_p^p).$$

The Jacobian criterion shows A_{x_i} is regular for each i, so A is normal by Serre's criterion. By inverting an appropriate multiplicative set in A, one obtains the ring S, which therefore is also normal. Since R is a pure subring of the finite extension ring S, it follows that R is normal and Cohen–Macaulay.

Note that S is not F-injective: set $\mathfrak{n}$ to be the homogeneous maximal ideal of S; computing local cohomology $H^p_{\mathfrak{n}}(S)$ using a Čech complex on the system of parameters $x_1, \ldots, x_p$ for S, the cohomology class

$$\left[\frac{x_0}{x_1 \cdots x_p}\right] \in H^p_{\mathfrak{n}}(S)$$

maps to zero under the Frobenius action on $H^p_{\mathfrak{n}}(S)$. We shall see that the Frobenius action on $H^p_{\mathfrak{m}}(R)$, with $\mathfrak{m}$ the homogeneous maximal ideal of R, is however injective.

First note that $[H^p_{\mathfrak{m}}(R)]_{-p}$ is the socle of $H^p_{\mathfrak{m}}(R)$: it is the highest degree component, and any nonzero homogeneous element $\eta \in H^p_{\mathfrak{m}}(R)$ has a nonzero multiple $s\eta$ in the socle of $H^p_{\mathfrak{n}}(S)$, which is $[H^p_{\mathfrak{n}}(S)]_{-1}$; but then it has a nonzero multiple $s'\eta$ in

$$[H^p_{\mathfrak{n}}(S)]_{-p} = [H^p_{\mathfrak{m}}(R)]_{-p},$$

for s, s' homogeneous in S, in which case degree considerations imply that $s' \in R$.

To verify that the Frobenius action F on $H^p_{\mathfrak{m}}(R)$ is injective, it suffices to prove the injectivity of F on the socle $[H^p_{\mathfrak{m}}(R)]_{-p}$ which, following (2.3), is the K-vector space spanned by the cohomology classes

$$\eta_{\boldsymbol{\alpha}} := \left[\frac{x_0^{\alpha_1+\cdots+\alpha_p}}{x_1^{\alpha_1+1} \cdots x_p^{\alpha_p+1}}\right] \in [H^p_{\mathfrak{m}}(R)]_{-p},$$

where each α_i is a nonnegative integer, $\sum \alpha_i \leqslant p-1$, and $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_p)$. Since

$$x_0^p = t_1 x_1^p + \cdots + t_p x_p^p$$

in the ring S, one has

$$F(\eta_{\boldsymbol{\alpha}}) = \left[\frac{(t_1 x_1^p + \cdots + t_p x_p^p)^{\sum \alpha_i}}{x_1^{p\alpha_1+p} \cdots x_p^{p\alpha_p+p}}\right] = \frac{(\sum \alpha_i)!}{\alpha_1! \cdots \alpha_p!}\left[\frac{t_1^{\alpha_1} \cdots t_p^{\alpha_p}}{x_1^p \cdots x_p^p}\right], \tag{3.1}$$

where the latter equality uses the pigeonhole principle. The elements $t_1^{\alpha_1} \cdots t_p^{\alpha_p}$ of K, as $\boldsymbol{\alpha}$ varies subject to the conditions above, are linearly independent over the subfield K^p. It follows that for any nonzero K-linear combination η of the elements $\eta_{\boldsymbol{\alpha}}$, one has $F(\eta) \neq 0$. This proves that the ring R is F-injective.

One may now use Corollary 2.2 to conclude that R is F-rational; alternatively, one can also argue as follows: equation (3.1) shows that the image of $[H^p_{\mathfrak{m}}(R)]_{-p}$ under F lies in the K-span of the cohomology class

$$\mu := \left[\frac{1}{x_1^p \cdots x_p^p}\right],$$

so it suffices to verify that μ does not belong to the tight closure of zero in $H^p_{\mathfrak{m}}(R)$. This holds since no nonzero homogeneous form in R annihilates

$$F^e(\mu) = \left[\frac{1}{x_1^{p^{e+1}} \cdots x_p^{p^{e+1}}}\right]$$

for each $e \geqslant 0$.

For (2), let $\overline{R}$ denote either of $R \otimes_K K^{1/p}$ or $R \otimes_K \overline{K}$. Note that

$$t_2^{1/p}\left[\frac{x_0}{x_1^2 x_2 \cdots x_p}\right] - t_1^{1/p}\left[\frac{x_0}{x_1 x_2^2 x_3 \cdots x_p}\right]$$

is a nonzero element of $H^p_{\mathfrak{m}}(\overline{R})$, since it is a nontrivial linear combination of basis elements as in (2.3). However, its image under the Frobenius action is

$$\begin{aligned} t_2\left[\frac{t_1 x_1^p + \cdots + t_p x_p^p}{x_1^{2p} x_2^p \cdots x_p^p}\right] - t_1\left[\frac{t_1 x_1^p + \cdots + t_p x_p^p}{x_1^p x_2^{2p} x_3^p \cdots x_p^p}\right] \\ = t_2\left[\frac{t_1}{x_1^p x_2^p \cdots x_p^p}\right] - t_1\left[\frac{t_2}{x_1^p x_2^p \cdots x_p^p}\right] \end{aligned}$$

which, of course, is zero.

Lastly, for (3), write the enveloping algebra $S \otimes_K S$ of S as

$$K[x_0, \ldots, x_p, y_0, \ldots, y_p] / (x_0^p - t_1 x_1^p - \cdots - t_p x_p^p,\ y_0^p - t_1 y_1^p - \cdots - t_p y_p^p),$$

with the $\mathbb{N}^2$-grading under which $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$ for each i. Then

$$R \otimes_K R = \bigoplus_{k,l \in \mathbb{N}} [S \otimes_K S]_{(pk,pl)}.$$

Note that $R \otimes_K R$ admits a standard grading; let $\mathfrak{M}$ denote its homogeneous maximal ideal. Then $\mathfrak{M}(S \otimes_K S)$ is primary to $\mathfrak{N} := (x_0, \ldots, x_p, y_0, \ldots, y_p)$, the homogeneous maximal ideal of $S \otimes_K S$, and

$$H^{2p}_{\mathfrak{M}}(R \otimes_K R) = \bigoplus_{k,l \in \mathbb{N}} [H^{2p}_{\mathfrak{N}}(S \otimes_K S)]_{(pk,pl)}.$$

The cohomology class

$$\left[\frac{x_0 y_1 - x_1 y_0}{x_1^2 x_2 \cdots x_p\, y_1^2 y_2 \cdots y_p}\right] \in H^{2p}_{\mathfrak{M}}(R \otimes_K R)$$

is nonzero since it is a nontrivial linear combination of basis elements as in (2.4); however, it is readily seen to be in the kernel of the Frobenius action. ■

Note that $R \otimes_K K^{1/p}$ and $R \otimes_K \overline{K}$ in the previous theorem are not reduced: for example,

$$(x_0 - t_1^{1/p} x_1 - \cdots - t_p^{1/p} x_p)\, x_1 \cdots x_{p-1}$$

is a nonzero nilpotent element. This gives an alternative proof of (2), since F-injective rings are reduced by Remark 2.6 in [20].

In the examples provided by Theorem 3.1, the transcendence degree of K over $\mathbb{F}_p$ increases with p; for the interested reader, the following theorem gets around this, though the proof is perhaps more technical.

Theorem 3.2. *Fix a prime integer $p > 0$. Let t be an indeterminate over the field $\mathbb{F}_p$ and set $K := \mathbb{F}_p(t)$. Consider the hypersurface*

$$S := K[w, x, y, z_1, \dots, z_{p-1}] / \big(w^{p+1} - tx^{p+1} - xy^p - \textstyle\sum_{i=1}^{p-1} z_i^{p+1}\big)$$

with the standard $\mathbb{N}$-grading, and set $R := S^{(p)}$. Then:

(1) *The ring R is F-rational.*

(2) *The rings $R \otimes_K K^{1/p}$ and $R \otimes_K \overline{K}$ are not F-injective, hence not F-rational.*

(3) *The enveloping algebra $R \otimes_K R$ is not F-injective, hence not F-rational.*

Proof. We begin with the hypersurface

$$A := \mathbb{F}_p[t, w, x, y, z_1, \dots, z_{p-1}] / \big(w^{p+1} - tx^{p+1} - xy^p - \textstyle\sum_i z_i^{p+1}\big).$$

The Jacobian criterion shows that, up to radical, the defining ideal of the singular locus of A contains $(w, x, y, z_1, \dots, z_{p-1})$. The ring S is obtained from A by inverting an appropriate multiplicative set; it follows that S has an isolated singular point at its homogeneous maximal ideal $\mathfrak{n}$. In particular, S is normal by Serre's criterion.

To prove that R is F-rational, it suffices by Corollary 2.2 to verify that

$$F : [H^{p+1}_{\mathfrak{n}}(S)]_{-p} \to [H^{p+1}_{\mathfrak{n}}(S)]_{-p^2} \tag{3.2}$$

is injective. Using the Čech complex on $x, y, z_1 \dots, z_{p-1}$, the vector space $[H^{p+1}_{\mathfrak{n}}(S)]_{-p}$ has a K-basis, as in (2.3), consisting of cohomology classes

$$\eta_{\alpha,\beta,\boldsymbol{\gamma}} := \left[\frac{w^{1+\alpha+\beta+\sum \gamma_i}}{x^{\alpha+1}\, y^{\beta+1} \prod_i z_i^{\gamma_i+1}}\right],$$

where $\alpha, \beta, \gamma_1, \dots, \gamma_{p-1}$ are nonnegative integers with $\alpha + \beta + \sum \gamma_i \leqslant p - 1$. The ring S admits a $(\mathbb{Z}/(p+1))^{p+1}$-grading with

$$\deg z_i = e_i, \quad \deg w = e_p \quad \text{and} \quad \deg x = e_{p+1} = \deg y,$$

where $e_1, \dots, e_{p+1}$ denote standard basis vectors modulo $p + 1$. Since $\gcd(p,\ p + 1) = 1$, the action (3.2) maps distinct multigraded components to distinct multigraded components, so it suffices to verify the injectivity componentwise. Note that

$$\deg \eta_{\alpha,\beta,\boldsymbol{\gamma}} = \Big(- \gamma_1 - 1, \dots, -\gamma_{p-1} - 1, 1 + \alpha + \beta + \sum_i \gamma_i, -\alpha - \beta - 2\Big)$$

with respect to the multigrading. Thus, for fixed nonnegative integers k and γ_i with

$$0 \leqslant k + \sum_i \gamma_i \leqslant p - 1,$$

a homogeneous element of $[H^{p+1}_{\mathfrak{n}}(S)]_{-p}$ with multidegree

$$\Big(- \gamma_1 - 1,\ \dots,\ -\gamma_{p-1} - 1,\ 1 + k + \sum_i \gamma_i,\ -k - 2\Big)$$

has the form

$$\sum_{\alpha+\beta=k} c_\alpha\, \eta_{\alpha,\beta,\boldsymbol{\gamma}},$$

where α and β are nonnegative integers with $\alpha+\beta=k$, and $c_\alpha \in K$.

Set $m := k + \sum \gamma_i$, and suppose that the above element

$$\sum_{\alpha+\beta=k} c_\alpha\, \eta_{\alpha,\beta,\boldsymbol{\gamma}} = \sum_{\alpha+\beta=k} c_\alpha x^\beta y^\alpha \left[\frac{w^{m+1}}{x^{k+1}\, y^{k+1} \prod_i z_i^{\gamma_i+1}} \right] \tag{3.3}$$

belongs to the kernel of the Frobenius action. Then

$$\Big(\sum_{\alpha+\beta=k} c_\alpha^p x^{\beta p} y^{\alpha p} \Big) w^{(m+1)p}$$

belongs to the ideal

$$\big(x^{(k+1)p}, y^{(k+1)p}, z_1^{(\gamma_1+1)p}, \dots, z_{p-1}^{(\gamma_{p-1}+1)p} \big) S.$$

Since $w^{(m+1)p} = w^{p-m}\, w^{(p+1)m}$ and $1 \leqslant p-m \leqslant p$, it follows that

$$\Big(\sum_{\alpha+\beta=k} c_\alpha^p x^{\beta p} y^{\alpha p} \Big) \Big(t x^{p+1} + x y^p + \sum_{i=1}^{p-1} z_i^{p+1} \Big)^m \tag{3.4}$$

belongs to the monomial ideal

$$\big(x^{(k+1)p}, y^{(k+1)p}, z_1^{(\gamma_1+1)p}, \dots, z_{p-1}^{(\gamma_{p-1}+1)\, p} \big) \tag{3.5}$$

in the polynomial ring $K[x, y, z_1, \dots, z_{p-1}]$. Bearing in mind that $m = k + \sum \gamma_i$, the terms in the multinomial expansion of (3.4) that include the monomial

$$\prod_i z_i^{(p+1)\,\gamma_i}$$

constitute the polynomial

$$\binom{m}{k, \gamma_1, \dots, \gamma_{p-1}} \Big(\sum_{\alpha+\beta=k} c_\alpha^p x^{\beta p} y^{\alpha p} \Big) (t x^{p+1} + x y^p)^k \prod_i z_i^{(p+1)\gamma_i}$$

which, therefore, also belongs to the monomial ideal (3.5). But then

$$\Big(\sum_{\alpha+\beta=k} c_\alpha^p x^{\beta p} y^{\alpha p} \Big) (t x^{p+1} + x y^p)^k \in \big(x^{(k+1)p}, y^{(k+1)p} \big)$$

in the polynomial ring $K[x, y]$. This implies that the coefficient of $x^{kp+k} y^{kp}$ in the polynomial above must be zero, i.e., that

$$\sum_{\alpha+\beta=k} \binom{k}{\alpha} c_\alpha^p t^\alpha = 0.$$

Since $c_\alpha^p \in K^p$ for each α, and $k < [K^p(t) : K^p] = p$, this forces each c_α to be zero. But then the element (3.3) is zero, so the map (3.2) is indeed injective as claimed. This completes the proof of (1).

For (2), let $\mathfrak{m}$ denote the homogeneous maximal ideal of R, and let $\overline{R}$ denote either of $R \otimes_K K^{1/p}$ or $R \otimes_K \overline{K}$. Then

$$\left[\frac{w^2}{x^2 y \prod_i z_i}\right] - t^{1/p}\left[\frac{w^2}{xy^2 \prod_i z_i}\right] \in H^{p+1}_{\mathfrak{m}}(\overline{R})$$

is a nontrivial linear combination of basis elements as in (2.3). The ring $\overline{R}$ is not F-injective since under the Frobenius action on $H^{p+1}_{\mathfrak{m}}(\overline{R})$, this element maps to

$$\left[\frac{w^{p-1}tx}{x^p y^p \prod_i z_i^p}\right] - t\left[\frac{w^{p-1}x}{x^p y^p \prod_i z_i^p}\right] = 0.$$

For (3), use w', x', y', z_i' for the second copy of S, and proceed as in the proof of Theorem 3.1. Using $\mathfrak{M}$ for the homogeneous maximal ideal of $R \otimes_K R$, the cohomology class

$$\left[\frac{(ww')^2\,(x'y - xy')}{(xx'yy')^2 \prod_i z_i \prod_i z_i'}\right] \in H^{2p+2}_{\mathfrak{M}}(R \otimes_K R)$$

is a nontrivial linear combination of basis elements as in (2.4), and is in the kernel of the Frobenius action on $H^{2p+2}_{\mathfrak{M}}(R \otimes_K R)$. It follows then that the ring $R \otimes_K R$ is not F-injective. ∎

Theorem 1.1 follows readily from the results of this section.

Proof of Theorem 1.1. Let K and R be as in Theorem 3.1 or in Theorem 3.2, and let $S := R \otimes_K K^{1/p}$ or $R \otimes_K \overline{K}$. An example is then obtained after localizing at the homogeneous maximal ideals; note that the closed fiber is the field $K^{1/p}$ or $\overline{K}$ in the respective cases. ∎

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