# Continuous Time Quantum Walks on Graphs: Group State Transfer

Luke C. Brown[a,b], William J. Martin[a,*], Duncan Wright[a]

[a]Department of Mathematical Sciences, Worcester Polytechnic Institute, Massachusetts, United States
[b]Department of Mathematics, Drexel University, Pennsylvania, United States

**Abstract**

We introduce the concept of group state transfer on graphs, summarize its relationship to other concepts in the theory of quantum walks, set up a basic theory, and discuss examples.

Let $X$ be a graph with adjacency matrix $A$ and consider quantum walks on the vertex set $V(X)$ governed by the continuous time-dependent unitary transition operator $U(t) = \exp(itA)$. For $S, T \subseteq V(X)$, we say $X$ admits "group state transfer" from $S$ to $T$ at time $\tau$ if the submatrix of $U(\tau)$ obtained by restricting to columns in $S$ and rows not in $T$ is the all-zero matrix. As a generalization of perfect state transfer, fractional revival and periodicity, group state transfer satisfies natural monotonicity and transitivity properties. Yet non-trivial group state transfer is still rare; using a compactness argument, we prove that bijective group state transfer (the optimal case where $|S| = |T|$) is absent for almost all $\tau$. Focusing on this bijective case, we obtain a structure theorem, prove that bijective group state transfer is "monogamous", and study the relationship between the projections of $S$ and $T$ into each eigenspace of the graph.

Group state transfer is obviously preserved by graph automorphisms and this gives us information about the relationship between the setwise stabilizer of $S \subseteq V(X)$ and the stabilizers of certain vertex subsets $\mathsf{F}(S,t)$ and $\mathsf{I}(S,t)$. The operation $S \mapsto \mathsf{F}(S,t)$ is sufficiently well-behaved to give us a topology on $V(X)$; this is simply the topology of subsets for which bijective group state transfer occurs at time $t$. We illustrate non-trivial group state transfer in bipartite graphs with integer eigenvalues, in joins of graphs, and in symmetric double stars. The Cartesian product allows us to build new examples from old ones.

*Keywords:* Quantum walk, State transfer, Graph eigenvalues.
*2010 MSC:* 05C50, 81P16.

## 1. Introduction

Theoretical investigations in quantum computing and quantum information theory have given rise to a number of interesting questions in algebraic graph theory and nearby areas of combinatorics. Quantum walks on graphs, in particular, seem both fundamental to our understanding of how to control the evolution of finite-dimensional quantum mechanical systems and quite amenable to study using the standard tools of spectral graph theory. Since their introduction in 1998 by Farhi and Gutman [1] as a powerful alternative to classical Markov random processes, continuous time quantum walks on graphs and weighted graphs have received much attention as researchers attempt to understand the potential advantages of quantum computation over classical computation. While Farhi and Gutman allowed for a sparse real Hamiltonian expressible as a sum of Hamiltonians each acting on a limited number of underlying qubits, Childs proved in 2006 that we may restrict attention to Hamiltonians that are simply adjacency matrices of graphs having maximum degree three and still efficiently simulate any quantum circuit [2].

With the path on two vertices as a classical motivating example [3], Christandl, et al. [4] first demonstrated perfect quantum state transfer (PST) between vertices at arbitrary distance $d$ using the product of $d$

*Corresponding author.
E-mail address: martin@wpi.edu (W.J. Martin).

such paths to obtain the $d$-cube. Graph theorists specializing in spectral techniques soon developed a theory around such questions (see [5]), showing that perfect state transfer is quite rare. Attention then broadened to include closely related phenomena such as periodicity and fractional revival as well as approximations such as pretty good state transfer [6, 5], among other interesting behavior of quantum walks on graphs such as uniform mixing. With path-length distance between vertices as a reasonable surrogate for physical distance between components in an implementation of a quantum circuit, the hope of finding perfect state transfer between vertices far apart in a relatively small unweighted graph seems to have been dashed. Perfect state transfer is not only rare, but the number of vertices must grow at least in proportion to the cube of the distance between the endpoints (and possibly at a much larger rate) [7].

*Overview of the paper.* The present work is an outgrowth of the undergraduate senior thesis [8][1] of the first author (LCB), completed in April 2019 under the supervision of the second author (WJM). In this paper, motivated more by a desire to extend the theory than by any particular physical application, we introduce "group state transfer" by which any initial state supported on one set $S$ of vertices is carried to some state supported on another set $T$. In full generality, group state transfer occurs everywhere: every graph $X$ admits such state transfer from the empty set to any subset of vertices and from any set of vertices to the entire vertex set $V(X)$. We call these cases "trivial". In Lemma 3.1, we see how group state transfer behaves with respect to intersections, unions, complements, and time reversal. If $X$ admits group state transfer from $S$ to $T$ at time $\tau$ then, at time $\tau$, $X$ admits group state transfer from any subset of $S$ to any superset of $T$. This naturally leads (Section 3) to a partial order on such pairs with maximal pairs of particular interest. A compactness argument is used in Lemma 3.3 to show that for all but finitely many values of $\tau$ in any finite interval $[t_0, t_1]$, the only maximal elements are the trivial ones $(\emptyset, \emptyset)$ and $(V(X), V(X))$. In most strongly regular graphs, only trivial situations arise (Proposition 3.4).

The fundamental inequality $|S| \leq |T|$ in Lemma 3.2 can be viewed as an entropy bound and we focus on bijective group state transfer, where $|S| = |T|$, in Section 3.1. Using Lemma 3.3, we prove (Theorem 3.7) that bijective group state transfer is "monogamous" in the sense that, aside from $S$ itself, a set $S$ can be transferred to at most one other vertex subset of the same size. Whenever we have bijective group state transfer from $S$ to some other set at time $\tau$, we have group state transfer from $S$ to itself at time $2\tau$ — i.e., $S$ is "periodic at $2\tau$". Godsil showed that the complement of a periodic set is again periodic; we show that the collection of vertex subsets periodic at time $\tau$ is closed under intersection and union. A fundamental restriction on perfect state transfer is the idea of "parallel vertices" [9, Section 6.5]. Analogous to this, we show in Lemma 4.1 that, if $X$ admits bijective group state transfer from $S$ to $T$ and $E_r$ is any primitive idempotent of the adjacency algebra of $X$, then there is an $|S| \times |S|$ unitary matrix mapping the columns of $E_r$ indexed by $S$ to the columns of $E_r$ indexed by $T$.

Given a set $S$ of vertices and a time $t$, there are natural targets $R = \mathsf{I}(S, -t)$ and $T = \mathsf{F}(S, t)$ for group state transfer to and from $S$, respectively. In Theorem 5.1, we consider these maps $\mathsf{I}(\cdot, \cdot)$ and $\mathsf{F}(\cdot, \cdot)$ and a time-dependent topology on the vertices of $X$ whose clopen sets are those $S \subseteq V(X)$ for which bijective group state transfer occurs in $X$ at time $t$ from $S$ to $\mathsf{F}(S, t)$ (Corollary 5.2). This leads into some results in Section 6 revealing how group state transfer behaves with respect to the automorphism group of the graph $X$.

Turning toward examples, Section 7 explores the Cartesian product and join of two graphs. In Proposition 7.1, we show that if graph $X$ admits group state transfer from $S$ to $T$ at time $\tau$ and graph $Y$ admits group state transfer from $S'$ to $T'$ at time $\tau$, then the Cartesian product $X \square Y$ admits group state transfer from $S \times S'$ to $T \times T'$ at time $\tau$. In a simple reformulation of work of Coutinho and Godsil [9], we find non-trivial group state transfer from $V(X)$ to itself in any join $X + Y$ (Proposition 7.4). In Section 8, we list some further examples. For instance, in any bipartite graph $X$ whose eigenvalue ratios are all odd integers, we see group state transfer from one bipartite half to the other. Also in Theorem 8.1, we see periodicity on each bipartite half under weaker conditions. Periodicity is also shown in the symmetric double star in Proposition 8.2. We finish the paper with a few more examples and a list of open problems.

[1]The Major Qualifying Project (MQP) at Worcester Polytechnic Institute is a campus-wide capstone requirement of all undergraduates.

Our exposition of group state transfer is complemented well by the study of *real state transfer* in Godsil [10] and, as we indicate in this paper, the existence of $(S,S)$-GST may sometimes be yielded as a consequence of real state transfer, with bounds on the minimum period $\tau$ due to Godsil.

## 2. Preliminaries

Throughout, $X = (V(X), E(X))$ is a finite simple undirected graph on $n$ vertices with adjacency matrix $A$. For simplicity, we will sometimes write $V(X) = \{1,\dots,n\}$. When $a$ and $b$ are joined by an edge, we write $a \sim b$ or $(ab \in E(G))$ and we use $X(a) = \{b \in V(X) \mid a \sim b\}$ to denote the neighborhood of $a$ in $X$. The distance between $a$ and $b$ in $X$, denoted $\partial(a,b)$, is the length of a shortest path joining the two.

The unitary time-dependent transition operator $U(t) = U_X(t)$ is given by

$$U(t) = \exp(itA) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} A^k$$

where $t$ is any real number. As shown, for example, by Coutinho and Godsil in their text [9], the spectral decomposition of $A$ carries over to a useful expression for $U(t)$. Throughout, we suppose that graph $X$ has $d+1$ distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. We denote by $E_r$ the matrix representing orthogonal projection onto the eigenspace belonging to $\theta_r$, $V_r = \{\varphi \in \mathbb{C}^n \mid A\varphi = \theta_r\varphi\}$. Then we have $A = \sum_{r=0}^{d} \theta_r E_r$ where the various projections sum to the identity: $\sum_{r=0}^{d} E_r = I$. This gives [9, Section 1.5]

$$U(t) = \sum_{r=0}^{d} e^{it\theta_r} E_r \ . \tag{1}$$

## 3. Group state transfer and a partial order on subset pairs

We now give the central definition of this paper. We say graph $X$ admits group state transfer from $S \subseteq V(X)$ to $T \subseteq V(X)$ at time $\tau$ if the evolution operator $U(\tau)$ carries every initial state vector whose support is contained in $S$ to some vector whose support is contained in $T$.

**Definition 3.1.** *Let $X$ be a graph and let $S,T \subseteq V(X)$. We say that $X$ has $(S,T)$-*group state transfer*, or $(S,T)$-GST, at time $\tau \in \mathbb{R}$ if, for all $\psi \in \mathbb{C}^n$ such that* $\operatorname{Supp}\psi \subseteq S$*, the vector $\phi = U_X(\tau)\psi$ satisfies* $\operatorname{Supp}\phi \subseteq T$.

For $S \subseteq V(X)$, denote by $\langle S\rangle$ the subspace of $\mathbb{C}^n$ of vectors whose support is contained in $S$: $\langle S\rangle = \operatorname{span}\{e_a | a \in S\}$. For $S,T \subseteq V(X)$, we have $(S,T)$-GST at time $\tau$ if $U(\tau)\langle S\rangle \subseteq \langle T\rangle$.

*Familiar examples.* Trivial examples include $S = \emptyset$ and $T = V(X)$: for any $R \subseteq V(X)$ and for any $\tau \in \mathbb{R}$, we have both $(\emptyset, R)$-GST and $(R, V(X))$-GST at time $\tau$. Our definition of group state transfer, while having no direct physical motivation, generalizes some important phenomena that have received much attention in the quantum information theory community recently. The graph $X$ is said to be *periodic* at $a$ at time $\tau$ if $X$ has $(\{a\},\{a\})$-GST at time $\tau$ and, for $b \neq a$, we say that we have *perfect state transfer* ($ab$-PST) from $a$ to $b$ in $X$ at time $\tau$ if $X$ has $(\{a\},\{b\})$-GST at time $\tau$. The graph $X$ has *fractional revival* on $S = \{a,b\}$ at time $\tau$ if $X$ has $(\{a\},S)$-GST at time $\tau$. We use the term *proper fractional revival* when this holds with $U(\tau)_{a,b} \neq 0$. (I.e., $(\{a\},\{a,b\})$-GST occurs at time $\tau$ but $(\{a\},\{a\})$-GST does not.) It is already known that, if $X$ has $(\{a\},\{a,b\})$-GST at time $\tau$ then either $a$ is periodic or $X$ has $(\{b\},\{a,b\})$-GST at time $\tau$; see, e.g., Lemma 9.9.1 in [9]. So $(\{a\},\{a,b\})$-GST at time $\tau$ implies either that $X$ is periodic at $a$, PST occurs from $a$ to $b$, or we have proper fractional revival on $\{a,b\}$ in $X$ (all at time $\tau$). In [11], Chan, et al. say graph $X$ has *generalized fractional revival* from $a \in V(X)$ to $B \subseteq V(X)$ if $X$ admits $(\{a\},B)$-GST but does not admit $(\{a\},B')$-GST for any proper subset $B' \subset B$. For $S \subseteq V(X)$, the set $S$ is a *periodic subset* [9, Section 9.6] if $X$ has $(S,S)$-GST at at some time $\tau$ (in which case we say $S$ is *periodic at time* $\tau$)[2]. In [12], Chan et al. use the term $K$-*fractional revival* for a periodic subset and build a theory of $K$-fractional revival which includes a ratio condition on eigenvalues that we do not have and a number of examples having some overlap with the examples in [8]. (See also [11].)

*Basic results.* We begin with a number of elementary observations that already impose a good deal of structure on the group state transfer phenomenon.

**Lemma 3.1.** *Let $X$ be a simple undirected graph. Then*

*(a) $X$ admits $(S,V(X))$-GST at time $\tau$ for all $S \subseteq V(X)$ and all times $\tau$;*

*(b) $X$ admits $(\emptyset,T)$-GST at time $\tau$ for all $T \subseteq V(X)$ and all times $\tau$.*

*(c) $X$ has $(S,T)$-GST at time $\tau$ if and only if $X$ has $(\{a\},T)$-GST at time $\tau$ for every $a \in S$;*

*(d) if $S' \subseteq S$ and $T \subseteq T'$ and $(S,T)$-GST occurs at time $\tau$, then $(S',T')$-GST also occurs at time $\tau$;*

*(e) if, at time $\tau$, graph $X$ has $(S_1,T_1)$-GST and $(S_2,T_2)$-GST, then $X$ has both $(S_1 \cap S_2, T_1 \cap T_2)$-GST and $(S_1 \cup S_2, T_1 \cup T_2)$-GST at time $\tau$;*

*(f) if $X$ has $(R,S)$-GST at time $\sigma$ and $X$ has $(S,T)$-GST at time $\tau$, then $X$ has $(R,T)$-GST at time $\sigma+\tau$;*

*(g) $X$ has $(S,T)$-GST at time $\tau$ if and only if $X$ has $(V(X)\setminus T, V(X)\setminus S)$-GST at time $\tau$;*

*(h) $X$ has $(S,T)$-GST at time $\tau$ if and only if $X$ has $(S,T)$-GST at time $-\tau$.*

*Proof.* Parts *(a)* and *(b)* are vacuous. For part *(d)*, we simply observe that, if $U(\tau)\langle S\rangle \subseteq \langle T\rangle$, then $U(\tau)\langle S'\rangle \subseteq \langle T'\rangle$ since $S' \subseteq S$, $T \subseteq T'$ give $\langle S'\rangle \subseteq \langle S\rangle$ and $\langle T\rangle \subseteq \langle T'\rangle$, respectively. Part *(e)*: suppose $\varphi \in \langle S_1 \cup S_2\rangle = \langle S_1\rangle + \langle S_2\rangle$. Then $U(\tau)\varphi \in \langle T_1\rangle + \langle T_2\rangle = \langle T_1 \cup T_2\rangle$. (The preservation of intersections is proved in a similar manner.) Now *(c)* follows from *(d)* and *(e)*. Part *(f)* is also straightforward. Part *(g)* follows from the fact that $U(t)$ is a symmetric matrix. Part *(h)*: since $U(-\tau) = U(\tau)^{-1} = \overline{U(\tau)}$, we see that $U(\tau)$ and $U(-\tau)$ have precisely the same set of all-zero submatrices. □

**Example 3.1.** *Suppose graph $X$ admits $a_ib_i$-PST at time $\tau$ for $i = 1,\ldots,\ell$. Then, with $S = \{a_1,\ldots,a_\ell\}$ and $T = \{b_1,\ldots,b_\ell\}$, $X$ admits $(S,T)$-GST at $\tau$. For instance, the $d$-cube has PST at time $\pi/2$ from any vertex to its antipode. Let $S \subseteq V(X)$ and choose $T$ to consist of the antipodes of the elements of $S$; this provides us examples with $|S| = |T|$ taking any value up to $|V(X)| = n$ when $X$ is the $d$-cube.*

**Lemma 3.2.** *If graph $X$ has $(S,T)$-GST at $\tau$, then $|S| \le |T|$.*

*Proof.* Since $\varphi \mapsto U(\tau)\varphi$ is injective and $U(\tau)\langle S\rangle \subseteq \langle T\rangle$, we have $\dim\langle S\rangle \le \dim\langle T\rangle$. □

**Example 3.2.** *Let $(P,B)$ be a symmetric $(40,13,4)$ design with bipartite incidence graph[3] $X$ having eigenvalues $\pm 13$, $\pm 3$. For $a \in P$, we have $(\{a\},B)$-GST at time $\tau = \pi/2$ (Theorem 8.1(b)) but $(B,\{a\})$-GST can never occur by Lemma 3.2.*

---

[2]Note that, in [9], a graph $X$ is said to be "periodic" at time $\tau$ if $U(\tau)$ is a diagonal matrix; that is, every subset of $V(X)$ is periodic at time $\tau$.

[3]Here, $V(X) = P \cup B$ and $a \sim b$ if one of these, say $b \in B$, is a block containing point $a \in P$.

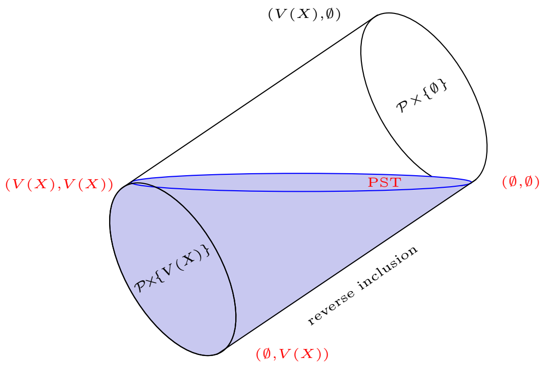


Figure 1: The partially ordered set on $(V(X) \times V(X), \preceq)$ with $(S', T') \preceq (S, T)$ when $S' \subseteq S$ and $T' \supseteq T$. Here, $\mathcal{P}$ denotes the power set of $V(X)$ ordered by containment. The blue region indicates pairs $(S,T)$ with $|S| \leq |T|$ and necessarily contains $\mathcal{ST}(X,\tau)$. Ideal GST occurs at the upper boundary of the blue region, with PST as a special case.

*The state transfer poset.* We now introduce the state transfer poset of a graph $X$. Writing $\mathcal{P}$ for the power set of $V(X)$,

$$\mathcal{P} = \mathcal{P}(V(X)) = \{S | S \subseteq V(X)\},$$

we begin with the poset $(\mathcal{P} \times \mathcal{P}, \preceq)$ with partial order relation $(S,T) \preceq (S',T')$ if $S \subseteq S'$ and $T' \subseteq T$. For each time $t$, the *state transfer poset* of $X$ at time $t$ is the subposet of this partially ordered set, depicted in Figure 1, consisting only of those pairs $(S,T)$ for which $X$ has $(S,T)$-GST at time $t$; this smaller partially ordered set is denoted $\mathcal{ST}(X,t)$. Note that, at any time $t$, $\mathcal{ST}(X,t)$ contains the *trivial* pairs $(\emptyset, T)$ for all $T \subseteq V(X)$ and $(S, V(X))$ for all $S \subseteq V(X)$ but may otherwise depend on $t$. One may alternatively view this collection of pairs $(S,T)$ for which $X$ has GST at time $t$ as a down-set (or "downward closed set") in the original poset $(\mathcal{P} \times \mathcal{P}, \preceq)$. This is nothing more than the poset formed by the all-zero submatrices of $U(t)$; we have $(S,T)$-GST at time $t$ precisely when the submatrix of $U(t)$ obtained by restricting to rows indexed by elements of $V(X) \setminus T$ and columns indexed by elements of $S$ has all entries zero.

Except at times $\tau = 2\pi k/n$ with $k \in \mathbb{Z}$, the complete graph $K_n$ admits no non-trivial GST for $n \geq 3$. In Figure 2, we give the state transfer poset for the path on two vertices $X = K_2$ at time $\tau = \pi/2$.

*The extremal case.* Let us say that $X$ has *maximal group state transfer* from $S$ to $T$ at time $\tau$ if $X$ has $(S,T)$-GST at $\tau$ and, whenever $X$ has $(S',T')$-GST at $\tau$ for $S \subseteq S'$ and $T' \subseteq T$, $S' = S$ and $T' = T$. Focusing on a more rare situation, we say $X$ has *bijective group state transfer* from $S$ to $T$ at time $\tau$ if $X$ has $(S,T)$-GST at $\tau$ and $|S| = |T|$. Lemma 3.2 tells us that bijective implies maximal. Given $S \subseteq V(X)$, the maximal element of $\mathcal{ST}(X,t)$ of the form $(S,T)$ is $(S, \mathsf{F}(S,t))$ where

$$\mathsf{F}(S,t) = \left\{ a \in V(X) \;\middle|\; (\exists \varphi \in \langle S \rangle)(e_a^\top U(t)\varphi \neq 0) \right\};$$

that is, $X$ has $(S,T)$-GST at time $t$ if and only if $T \supseteq \mathsf{F}(S,t)$.

*Smallest non-trivial elements of the poset.* The most common (and least interesting) case of non-trivial GST (i.e., where $S \neq \emptyset$ and $T \neq V(X)$) occurs where $U(\tau)$ has some entry equal to zero: $X$ exhibits $(\{a\}, V(X) \setminus \{b\})$-GST at time $\tau$ if and only if $U(\tau)_{b,a} = 0$. Even this fails almost everywhere.

**Lemma 3.3.** *Assume $X$ is a connected graph. In any interval $[t_0, t_1]$ of finite length, there are only finitely many $t$ for which $\mathcal{ST}(X,t)$ contains non-trivial pairs.*

*Proof.* We need only show that, for $a, b \in V(X)$, $U(t)_{b,a} = 0$ for at most finitely many values of $t \in [t_0, t_1]$. Assume not. By compactness, there exists a convergent sequence $\{t_k\}_{k=1}^{\infty}$ of values all satisfying $U(t_k)_{b,a} = 0$. Define

$$f(t) = \sum_{r=0}^{d} e^{i\theta_r t} (E_r)_{b,a} \ .$$

Then $f(t)$ is analytic and $f(t_k) = 0$ for all $k$. So, defining $t^* = \lim_{k\to\infty} t_k$, we obtain $f(t^*) = 0$ by continuity. Similarly, every derivative of $f$ is zero at $t^*$. Since $f$ is analytic, it must be the zero function. But we are assuming that $X$ is connected, so some element of the adjacency algebra $\langle A\rangle$ has a nonzero value in its $(b, a)$-position. Since the $\theta_r$ are algebraic integers, they remain distinct when reduced modulo $2\pi$; thus, there is some $\epsilon > 0$ for which $U(\epsilon)$ has $d + 1$ distinct eigenvalues. Therefore the set $\{U(t) \mid t \in \mathbb{R}\}$, closed under multiplication, generates $\langle A\rangle$ and there must be some time $t$ at which $U(t)_{b,a} \neq 0$, giving us the desired contradiction. □

*Strongly regular graphs.* For some graphs $X$, there is no value of $t$ in $(0, 2\pi)$ for which $\mathcal{ST}(X, t)$ is non-trivial, as we now illustrate.

A graph $X$ is *strongly regular* with parameters $(\nu, \kappa, \lambda, \mu)$ if $|V(X)| = \nu$ and $|X(a) \cap X(b)| = \kappa, \lambda, \mu$, accordingly, as $a = b$, $a \sim b$ and $b \notin \{a\} \cup X(a)$, respectively. We say $X$ is an $\mathrm{srg}(\nu, \kappa, \lambda, \mu)$. Write $A_0 = I$, $A_1 = A$ and $A_2 = J - I - A$; these form a vector space basis for the adjacency algebra of $X$. Standard tools (e.g., [13, Chapter 10]) give us the eigenvalues:

$$\theta_0 = \kappa, \qquad \theta_1 = \frac{1}{2}\left(\lambda - \mu + \sqrt{\Delta}\right), \qquad \theta_2 = \frac{1}{2}\left(\lambda - \mu - \sqrt{\Delta}\right)$$

where $\Delta = (\mu - \lambda)^2 + 4(\kappa - \mu)$. The respective eigenvalue multiplicities for $\theta_1$ and $\theta_2$ are

$$f = \frac{1}{2}\left(\nu - 1 + \frac{(\nu-1)(\mu-\lambda) - 2\kappa}{\sqrt{\Delta}}\right), \qquad g = \frac{1}{2}\left(\nu - 1 - \frac{(\nu-1)(\mu-\lambda) - 2\kappa}{\sqrt{\Delta}}\right).$$

Except when $f = g$, $\theta_1$ and $\theta_2$ must be integers. It is well-known that $E_0 = \frac{1}{\nu} J$,

$$\begin{aligned} E_1 &= \frac{1}{\nu}\left(f A_0 + \frac{f\theta_1}{\kappa} A_1 + \frac{f(1+\theta_1)}{\kappa + 1 - \nu} A_2\right), \\ E_2 &= \frac{1}{\nu}\left(g A_0 + \frac{g\theta_2}{\kappa} A_1 + \frac{g(1+\theta_2)}{\kappa + 1 - \nu} A_2\right). \end{aligned}$$

Choose a base vertex $b \in V(X)$ and define the $\nu \times 3$ matrix $H$ whose columns are $e_b$, $Ae_b$ and $A_2 e_b$. Since the partition according to distance from $b$ is equitable, we have $AH = HB$ for

$$B = \begin{bmatrix} 0 & \kappa & 0 \\ 1 & \lambda & \kappa - 1 - \lambda \\ 0 & \mu & \kappa - \mu \end{bmatrix}.$$

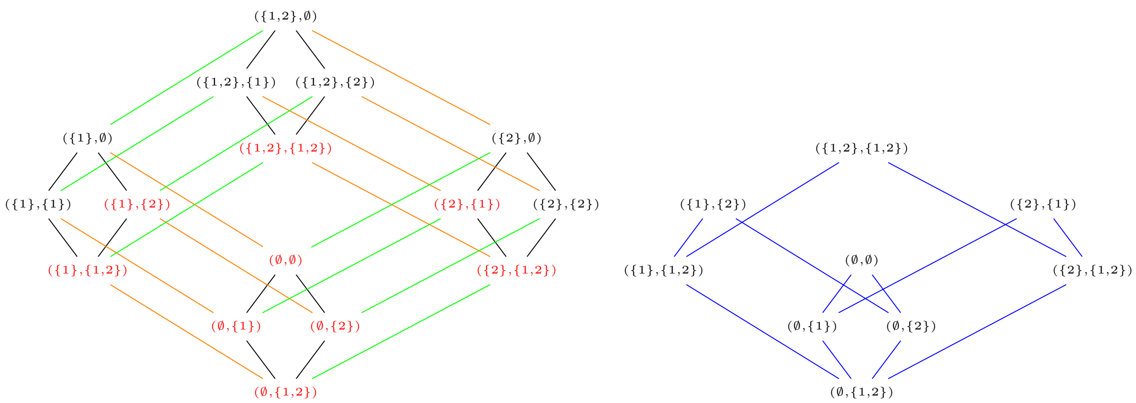


Figure 2: The poset $(\mathcal{P}(V(K_2)) \times \mathcal{P}(V(K_2)), \preceq)$ on the left (reverse inclusion highlighted in black) with the subposet identified in red giving us the state transfer poset $\mathcal{ST}(K_2, \frac{\pi}{2})$ for $K_2$ at time $\tau = \pi/2$.

Omitting details, we find that $U(t) = \exp(itA) = e^{it\kappa}E_0 + e^{it\theta_1}E_1 + e^{it\theta_2}E_2$ satisfies $U(t)H = HU'(t)$ where $U'(t) = e^{it\kappa}F_0 + e^{it\theta_1}F_1 + e^{it\theta_2}F_2$ with

$$F_0 = \frac{1}{\nu}\begin{bmatrix} 1 & \kappa & \nu-1-\kappa \\ 1 & \kappa & \nu-1-\kappa \\ 1 & \kappa & \nu-1-\kappa \end{bmatrix}, \qquad F_1 = \frac{f}{\nu}\begin{bmatrix} 1 & \theta_1 & -1-\theta_1 \\ \frac{\theta_1}{\kappa} & \frac{\kappa+\lambda\theta_1+\mu(-1-\theta_1)}{\kappa} & \frac{-\theta_1-\kappa-\lambda\theta_1+\mu(1+\theta_1)}{\kappa} \\ \frac{-1-\theta_1}{\nu-1-\kappa} & \frac{-\theta_1-\kappa-\lambda\theta_1+\mu(1+\theta_1)}{\nu-1-\kappa} & \frac{2\theta_1+1+\kappa+\lambda\theta_1+\mu(-1-\theta_1)}{\nu-1-\kappa} \end{bmatrix}$$

and

$$F_2 = \frac{g}{\nu}\begin{bmatrix} 1 & \theta_2 & -1-\theta_2 \\ \frac{\theta_2}{\kappa} & \frac{\kappa+\lambda\theta_2+\mu(-1-\theta_2)}{\kappa} & \frac{-\theta_2-\kappa-\lambda\theta_2+\mu(1+\theta_2)}{\kappa} \\ \frac{-1-\theta_2}{\nu-1-\kappa} & \frac{-\theta_2-\kappa-\lambda\theta_2+\mu(1+\theta_2)}{\nu-1-\kappa} & \frac{2\theta_2+1+\kappa+\lambda\theta_2+\mu(-1-\theta_2)}{\nu-1-\kappa} \end{bmatrix}.$$

If the system is in initial state $e_b$ at time zero, then at time $t$, the state of the system is given by $U(t)e_b = HU'(t)\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$. So $U(t)e_b$ is constant on the neighbors of $b$ and and on the non-neighbors of $b$. Define

$$\begin{aligned} h_0(t) &= e^{i\kappa t} + fe^{i\theta_1 t} + ge^{i\theta_2 t} \\ h_1(t) &= e^{i\kappa t} + (f\theta_1/\kappa)e^{i\theta_1 t} + (g\theta_2/\kappa)e^{i\theta_2 t} \\ h_2(t) &= e^{i\kappa t} + f(1+\theta_1)/(\kappa+1-\nu)e^{i\theta_1 t} + g(1+\theta_2)/(\kappa+1-\nu)e^{i\theta_2 t}\ . \end{aligned}$$

Then $e_a^\top U(t)e_b = \frac{1}{\nu}h_\delta(t)$ where $\delta \in \{0,1,2\}$ is the distance from $a$ to $b$ in $X$. This tells us that GST almost never occurs on strongly regular graphs.

**Proposition 3.4.** *Let $X$ be a connected strongly regular graph with non-trivial $(S,T)$-GST at time $\tau \in (0, 2\pi)$. Then one of the following occurs:*

*(a) $\kappa, \theta_1, \theta_2$ are all integers divisible by some $D \geq 2$ and $\tau = 2\ell\pi/D$ where $\ell$ is an integer, $0 < \ell < D$;*

*(b) $(\nu, \kappa, \lambda, \mu) = (n, n-m, n-2m, n-m)$, $X$ is complete multipartite, the complement of a disjoint union of $|\theta_2| = \frac{n}{m} > 2$ complete graphs $K_m$, $\tau = 2\pi\ell/m$;*

*(c) $(\nu, \kappa, \lambda, \mu) = (2m, m, 0, m)$, $X$ is complete bipartite and $\tau = \pi/D$ where $D$ is any positive divisor of $\kappa$;*

*(d) $(\nu, \kappa, \lambda, \mu) = (4m+1, 2m, m-1, m)$ and $\tau = 2\pi B/\nu$ for some integer $B$ satisfying $\cos\left(\pi B\nu^{-1/2}\right) = -1/4m$.*

*Proof.* We have done most of the work already. Part *(c)* is handled in Theorem 8.1 below.

First note that $1 + f + g = \nu$; if $|f - g| > 1$, then $h_0(t)$ is never zero, by the triangle inequality. Likewise, since

$$1 + f\frac{\theta_1}{\kappa} + g\frac{\theta_2}{\kappa} = 1 + f\frac{\theta_1+1}{\kappa+1-\nu} + g\frac{\theta_2+1}{\kappa+1-\nu} = 0,$$

we can only have $h_2(t) = 0$ when $e^{i\kappa t} = e^{i\theta_1 t} = e^{i\theta_2 t}$, in which case $U(t)e_b = e_b$. These are the only times at which $h_1(t) = 0$ with the exception of complete multipartite graphs $X = \overline{|\theta_2| K_m}$ (where $\theta_1 = 0$) in which case we obtain $(\{b\}, V(X) \setminus X(b))$-GST at times $t = 2\pi\ell/m$, maximal for $\ell$ odd.

In the case $f = g$, it is well-known that the parameters $(\nu, \kappa, \lambda, \mu)$ are as given in case *(d)* with $f = g = 2\mu$ and $\theta_1, \theta_2 = \frac{1}{2}(-1 \pm \sqrt{\nu})$. To obtain $h_0(\tau) = 0$, we must have $\tau(\theta_1 - \kappa) + \tau(\theta_2 - \kappa)$ an integer multiple of $2\pi$. Writing $\tau = -2\pi B/\nu$, we need

$$e^{i\tau(\theta_1-\kappa)} + e^{i\tau(\theta_1-\kappa)} = -1/\kappa$$

which gives us the condition $\cos\left(\pi B/\sqrt{4\mu+1}\right) = -1/4\mu$ and no such examples are known.

The only case that remains to consider is $(\{b\}, V(X)\setminus\{b\})$-GST in the case where $|f-g| = 1$. Aleksandar Jurišić [pers. communication] showed that the strongly regular graph parameters with $|f-g| = 1$ are precisely those in the family

$$(\nu, \kappa, \lambda, \mu) = (4m^2 + 4m + 2,\ 2m^2 + m,\ m^2 - 1,\ m^2)$$

where $m$ is a positive integer. And now a simple parity argument shows $U(t)_{b,b} \neq 0$ for all real $t$. □

**Example 3.3.** *The line graph of the complete graph $X = L(K_n)$ has eigenvalues $\kappa = 2n - 4$, $\theta_1 = n - 4$ and $\theta_2 = -2$. For $n$ even, case (a) holds and we have $U(\pi) = I$.*

A regular graph $X$ is *distance-regular* if the partition according to distance from any vertex is an equitable partition (and, hence, all these partitions admit the same quotient matrix $B$ [14]). Connected strongly regular graphs are precisely the distance-regular graphs of diameter $d = 2$. The analysis above for strongly regular graphs extends to distance-regular graphs in the following way: if $X$ is a distance-regular graph of diameter $d$ and $X$ admits $(S,T)$-GST at time $\tau$, then there exist $i_1, \ldots, i_k \in \{0, 1, \ldots, d\}$, $k > 0$, for which

$$T \supseteq \{v \in V(X) | (\exists u \in S,\ 1 \le j \le k)\, (\partial(u,v) = i_j)\}$$

where $\partial(u,v)$ denotes path-length distance between $u$ and $v$ in $X$.

*3.1. Bijective group state transfer*

*Block matrices.* Let us consider the block structure of $U_X(\tau)$ when $X$ admits $(S,T)$-GST at time $\tau$. For convenience, assume the vertex set $V(X) = \{1, \ldots, n\}$ is ordered so that

$$S \setminus T = \{1, \ldots, n_1\}, \quad I = S \cap T = \{n_1 + 1, \ldots, n_2\}, \quad T \setminus S = \{n_2 + 1, \ldots n_3\}$$

where $1 \le n_1 \le n_2 \le n_3 \le n$. Partition the rows and columns accordingly and write

$$U(\tau) = U_X(\tau) = \begin{matrix} \\ T\left\{\begin{matrix}\\ \\ \\ \end{matrix}\right. \\ \\ \end{matrix} \begin{bmatrix} \overbrace{U_{11} \quad U_{12}}^{S} & U_{13} & U_{14} \\ U_{21} \quad U_{22} & U_{23} & U_{24} \\ U_{31} \quad U_{32} & U_{33} & U_{34} \\ U_{41} \quad U_{42} & U_{43} & U_{44} \end{bmatrix} = \left[\begin{array}{c|c|c|c} 0 & 0 & U_{13} & U_{14} \\ \hline U_{21} & U_{22} & U_{23} & U_{24} \\ \hline U_{31} & U_{32} & U_{33} & U_{34} \\ \hline 0 & 0 & U_{43} & U_{44} \end{array}\right]$$

using the hypothesis of $(S,T)$-GST. Since $U(\tau)$ is a symmetric matrix, we have

$$U(\tau) = \left[\begin{array}{c|c|c|c} 0 & 0 & U_{13} & 0 \\ \hline 0 & U_{22} & U_{23} & 0 \\ \hline U_{31} & U_{32} & U_{33} & U_{34} \\ \hline 0 & 0 & U_{43} & U_{44} \end{array}\right]$$

with $U_{31} = U_{13}^\top$, $U_{32} = U_{23}^\top$, $U_{43} = U_{34}^\top$, and $U_{jj}$ symmetric for $j = 2, 3, 4$.

A by-product of this calculation is a second proof of Lemma 3.1*(g)*: if $X$ has $(S,T)$-GST at time $\tau$, then $X$ has $(V(X) \setminus T, V(X) \setminus S)$-GST at time $\tau$.

The Frobenius norm of $U_{31}$ is $n_1$, so the sum of the squared moduli of the entries of $U_{13}$ is also $n_1$, giving another proof that $|S| \le |T|$. If $|S| = |T|$, then $U_{j3} = 0$ is forced for $j = 2, 3, 4$. So, for $|S| = |T|$, we have

$$U(\tau) = \left[\begin{array}{c|c|c|c} 0 & 0 & U_{13} & 0 \\ \hline 0 & U_{22} & 0 & 0 \\ \hline U_{31} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & U_{44} \end{array}\right]$$

with $U_{22}$ and $U_{44}$ symmetric unitary matrices. This gives us the following result[4]

**Theorem 3.5.** *Assume that graph $X$ has $(S,T)$-GST at time $\tau$ and $|S| = |T|$. Write $I = S \cap T$. Then*

[4]Godsil [personal communication] studied the case of a periodic subset (where $S = T$), showing not only that $V(X) \setminus S$ is also periodic but proving that $Q_S U(\tau) Q_S$ belongs to the center of the algebra $Q_S \mathcal{A} Q_S$ where $Q_S = \sum_{a \in S} e_a e_a^\top$ is the diagonal matrix projecting $\mathbb{C}^n$ orthogonally onto $\langle S \rangle$.

(a) *X has* $(T,S)$*-GST at* $\tau$*;*

(b) *X has* $(S\setminus I, T\setminus I)$*-GST at* $\tau$*;*

(c) *X has* $(T\setminus I, S\setminus I)$*-GST at* $\tau$*;*

(d) *I is periodic at* $\tau$*;*

(e) *both S and T are periodic at time* $2\tau$*;*

(f) *the set* $R = V(X)\setminus S\cup T$ *is periodic at* $\tau$*.* □

**Corollary 3.6.** *If X is a graph with* $(S,T)$*-GST at time* $\tau$ *such that* $|S| = |T|$ *but* $S \neq T$*, then there exist non-empty disjoint* $S', T' \subseteq V(X)$ *for which* $|S'| = |T'|$ *and X has* $(S',T')$*-GST at time* $\tau$*.* □

In 2011, Kay [15] showed that perfect state transfer is *monogamous*: if $a,b,c \in V(X)$ and $X$ has both $ab$-PST and $ac$-PST, then $c = b$. In [10, Corollary 5.3], Godsil generalized this to mixed states with real density matrices. We now generalize this in a different direction.

**Theorem 3.7.** *If X is a connected graph and X admits* $(S,R)$*-GST at time* $\sigma$ *and* $(S,T)$*-GST at time* $\tau$ *with* $|R| = |S| = |T|$*, then* $R \in \{S,T\}$*.*

*Proof.* We first prove that $\sigma$ and $\tau$ must be commensurable real numbers. If not, then the set of remainders $\{\rho_k = k\tau \pmod{\sigma} \mid k \in \mathbb{Z}\ \}$ (where $\rho_k$ satisfies $0 \le \rho_k < \sigma$ and $(k\tau - \rho_k)/\sigma \in \mathbb{Z}$) must be infinite. Re-index to a subsequence of $\mathbb{Z}^+$ if necessary so that, with $\ell(k) = (k\tau - \rho_k)/\sigma$, we have $\tau_k = k\tau - \ell(k)\sigma$ converging to some point $\tau^* \in [0,\sigma)$. Applying Lemma 3.1*(f)* and Theorem 3.5*(a,e)*, we find infinitely many distinct times at which $(K,L)$-GST occurs for some $K, L \in \{R,S,T\}$, contradicting Lemma 3.3. So there must be some distinct $k$ and $k'$ for which $\rho_k = \rho_{k'}$ and we have $\sigma = (k-k')\tau/(\ell(k)-\ell(k'))$.

Since $\sigma$ and $\tau$ are commensurable, there exist nonzero integers $k,\ell$ such that $\ell\sigma = k\tau$ and, without loss of generality, $\ell$ is odd. At time $\ell\sigma$, $X$ admits $(S,R)$-GST and either $(S,S)$-GST or $(S,T)$-GST. Thus $R = S$ or $R = T$. □

## 4. Eigenspace geometry

Let $X$ be a graph on $n$ vertices with adjacency matrix $A$ and spectral decomposition $A = \sum_{r=0}^{d} \theta_r E_r$ with $\theta_0,\dots,\theta_d$ distinct. The *adjacency algebra* $\mathcal{A} = \operatorname{span}_{\mathbb{C}}\{I_n, A, A^2, \dots\} = \left\{\sum_{r=0}^{d} \alpha_r A^r \,\middle|\, \alpha_0,\dots,\alpha_d \in \mathbb{C}\right\}$ of $X$ contains $E_0,\dots,E_d$ as well as $U_X(t)$ for each $t \in \mathbb{R}$. This is properly contained in the *centralizer algebra* $\mathcal{C}(A) = \{M \in \mathbb{C}^{n\times n} | MA = AM\}$ of $A$. The permutation matrices in $\mathcal{C}(A)$ are simply those representing elements of the automorphism group, $\{P_\sigma \mid \sigma \in \operatorname{Aut}(X)\}$.

*The action of* $U(\tau)$ *on an eigenspace.* Suppose $X$ admits $(S,T)$-GST at time $\tau$ with $|S| = |T|$. As in the previous section, write $U(\tau)$ in block form and partition $E_r$ into blocks in the same way:

$$U = U(\tau) = \left[\begin{array}{c|c|c|c} 0 & 0 & U_{13} & 0 \\ \hline 0 & U_{22} & 0 & 0 \\ \hline U_{31} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & U_{44} \end{array}\right], \qquad E = E_r = \left[\begin{array}{c|c|c|c} E_{11} & E_{12} & E_{13} & E_{14} \\ \hline E_{21} & E_{22} & E_{23} & E_{24} \\ \hline E_{31} & E_{32} & E_{33} & E_{34} \\ \hline E_{41} & E_{42} & E_{43} & E_{44} \end{array}\right].$$

Abbreviating $e^{i\tau\theta_r} = \lambda_r$, the equations $EU = UE = \lambda_r E$ give us a system of equations relating the various blocks

$$\begin{aligned}
E_{13}U_{31} &= U_{13}E_{31} &= \lambda_r E_{11} \qquad & E_{11}U_{13} &= U_{13}E_{33} &= \lambda_r E_{13}\\
E_{23}U_{31} &= U_{22}E_{21} &= \lambda_r E_{21} \qquad & E_{21}U_{13} &= U_{22}E_{23} &= \lambda_r E_{23}\\
E_{33}U_{31} &= U_{31}E_{11} &= \lambda_r E_{31} \qquad & E_{31}U_{13} &= U_{31}E_{13} &= \lambda_r E_{33}\\
E_{43}U_{31} &= U_{44}E_{41} &= \lambda_r E_{41} \qquad & E_{41}U_{13} &= U_{44}E_{43} &= \lambda_r E_{43}
\end{aligned}$$

$$\begin{aligned}
E_{12}U_{22} &= U_{13}E_{32} &= \lambda_r E_{12} \qquad & E_{14}U_{44} &= U_{13}E_{34} &= \lambda_r E_{14}\\
E_{22}U_{22} &= U_{22}E_{22} &= \lambda_r E_{22} \qquad & E_{24}U_{44} &= U_{22}E_{24} &= \lambda_r E_{24}\\
E_{32}U_{22} &= U_{31}E_{12} &= \lambda_r E_{32} \qquad & E_{34}U_{44} &= U_{31}E_{14} &= \lambda_r E_{34}\\
E_{42}U_{22} &= U_{44}E_{42} &= \lambda_r E_{42} \qquad & E_{44}U_{44} &= U_{44}E_{44} &= \lambda_r E_{44}
\end{aligned}$$

where we know that both $E$ and $U$ are symmetric and $U$ is unitary. So $U_{13}, U_{22}, U_{31}, U_{44}$ are all unitary. This shows that $S \setminus I$ and $T \setminus I$ are "parallel" subsets in the following sense.

**Lemma 4.1.** *Let $X$ be a graph with adjacency matrix $A$ having spectral decomposition $A = \sum_{r=0}^{d} \theta_r E_r$ with $\theta_0, \ldots, \theta_d$ distinct. Let $S, T \subseteq V(X)$ with $|S| = |T|$ having orthogonal projections $Q_S = \sum_{a \in S} e_a e_a^\top$ and $Q_T = \sum_{a \in T} e_a e_a^\top$ onto $\langle S \rangle$ and $\langle T \rangle$, respectively. If $X$ admits $(S,T)$-GST, then, for each $r = 0, \ldots, d$, there exists a unitary matrix $N_r$ such that $E_r Q_S N_r = E_r Q_T$. In particular* $\operatorname{span}\{E_r e_a \mid a \in S\} = \operatorname{span}\{E_r e_a \mid a \in T\}$.

*Proof.* Write $M = \lambda_r^{-1} U_{13}$, so that

$$\begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \\ E_{41} \end{bmatrix} M = \begin{bmatrix} E_{13} \\ E_{23} \\ E_{33} \\ E_{43} \end{bmatrix}$$

from above. Choose

$$N' = \left[\begin{array}{c|c|c|c} M & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 \\ \hline 0 & 0 & M^{-1} & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right], \quad P = \left[\begin{array}{c|c|c|c} 0 & 0 & I & 0 \\ \hline 0 & I & 0 & 0 \\ \hline I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right], \quad \text{so that} \quad E_r N' = \left[\begin{array}{c|c|c|c} E_{13} & E_{12} & E_{11} & E_{14} \\ \hline E_{23} & E_{22} & E_{21} & E_{24} \\ \hline E_{33} & E_{32} & E_{31} & E_{34} \\ \hline E_{43} & E_{42} & E_{41} & E_{44} \end{array}\right]$$

and $N_r = N'P$ satisfies $E_r Q_S N_r = E_r Q_T$ as desired. □

## 5. Discrete topology

The elementary structure seen in Lemma 3.1 motivates us to fix a time $t$ and view those sets $S$ for which there exists bijective $(S,T)$-GST at time $t$ for some $T$ as "closed sets". As there is a rich history of topological methods in combinatorics (see Björner, [16, Chapter 34] for an early survey), we hope this viewpoint will help us understand the connection between GST phenomena in related graphs.

*Three maps on subsets of vertices.* In Section 3, we introduced a time-dependent function $\mathsf{F} : \mathcal{P} \to \mathcal{P}$ given by

$$\mathsf{F}(S,t) = \{a \in V(X) \mid e_a \not\perp U(t)\langle S \rangle\}.$$

Mirroring this, consider

$$\mathsf{I}(S,t) = \{a \in V(X) \mid e_a \in U(t)\langle S \rangle\}.$$

Immediately, we see that the following are equivalent for $S, T \subseteq V(X)$:

- $X$ has $(S,T)$-GST at time $\tau$;
- $\mathsf{F}(S,\tau) \subseteq T$;
- $S \subseteq \mathsf{I}(T, -\tau)$.

Now define the $t$-*closure* of $S \subseteq V(X)$ as

$$\mathcal{C}\!\ell_t(S) = \mathsf{F}\left(\mathsf{F}(S,t), -t\right).$$

**Example 5.1.** *As an example, consider the path $X = P_3$. For convenience, write $V(X) = \{1,2,3\}$ with $1 \sim 2 \sim 3$ so that*

$$A = A(X) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

*Clearly $A^3 = 2A$, so $P_3$ has minimal polynomial $p(z) = z^3 - 2z$ and eigenvalues $0, \pm\sqrt{2}$. We compute*

$$e^{iAt} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{1}{2}\cos(t\sqrt{2})\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{i}{\sqrt{2}}\sin(t\sqrt{2})\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

*Observe that, for $a = 1,2,3$, $P_3$ exhibits $(\{a\},\{4-a\})$-GST at $t = (2m+1)\pi/\sqrt{2}$ ($m \in \mathbb{Z}$), exhibits $(\{a\},\{a\})$-GST at $t = m\pi\sqrt{2}$ ($m \in \mathbb{Z}$), and that $P_3$ exhibits only trivial GST at any other time. Consequently, for every subset $S \subseteq V(X)$ and every $t = n\pi/\sqrt{2}$ ($n \in \mathbb{Z}$) we have $\mathcal{C}\!\ell_t(S) = S$. But when $\frac{\sqrt{2}}{\pi}t \notin \mathbb{Z}$, $\mathcal{C}\!\ell_t(\{a\}) = V(X)$ for each $a \in V(X)$. (Note that this example also illustrates Theorem 8.1 below; $X$ is bipartite and, for $\alpha = 1/\sqrt{2}$, $\alpha A$ has integer eigenvalues.)*

**Theorem 5.1.** *Let $X$ be a graph and let $S, T \subseteq V(X)$. Then, for any $t \in \mathbb{R}$,*

*(a)* $S \subseteq T$ *implies* $\mathsf{F}(S,t) \subseteq \mathsf{F}(T,t)$;

*(b)* $\mathsf{F}(S \cap T, t) \subseteq \mathsf{F}(S,t) \cap \mathsf{F}(T,t)$;

*(c)* $\mathsf{F}(S \cup T, t) = \mathsf{F}(S,t) \cup \mathsf{F}(T,t)$;

*(d)* $S \subseteq T$ *implies* $\mathsf{I}(S,t) \subseteq \mathsf{I}(T,t)$;

*(e)* $\mathsf{I}(S \cap T, t) = \mathsf{I}(S,t) \cap \mathsf{I}(T,t)$;

*(f)* $\mathsf{I}(S \cup T, t) \supseteq \mathsf{I}(S,t) \cup \mathsf{I}(T,t)$;

*(g)* $S \subseteq T$ *implies* $\mathcal{C}\!\ell_t(S) \subseteq \mathcal{C}\!\ell_t(T)$;

*(h)* $S \subseteq \mathcal{C}\!\ell_t(S)$;

*(i)* $\mathcal{C}\!\ell_t(S \cap T) \subseteq \mathcal{C}\!\ell_t(S) \cap \mathcal{C}\!\ell_t(T)$;

*(j)* $\mathcal{C}\!\ell_t(S \cup T) = \mathcal{C}\!\ell_t(S) \cup \mathcal{C}\!\ell_t(T)$.

*Proof.* The proofs are all elementary. We include proofs of *(b)*, *(f)*, and *(h)*-*(j)* and note that *(a)*, *(d)*, *(g)* follow from *(b)*, *(e)*, *(i)*, respectively. First, we prove part *(b)*: if $u \in \mathsf{F}(S \cap T, t)$ then there is some $v \in S \cap T$ with $e_u^\top U(t) e_v \neq 0$. Since $v \in S$, $u \in \mathsf{F}(S,t)$ and since $v \in T$, $u \in \mathsf{F}(T,t)$. For *(f)*, take $u \in \mathsf{I}(S,t)$ so that $e_u \in U(t)\langle S\rangle$, giving $e_u \in U(t)\langle S\rangle + U(t)\langle T\rangle = U(t)\langle S \cup T\rangle$ and repeat this with $S$ and $T$ swapped. To prove *(h)*, take $u \in S$ and set $\varphi = U(t)e_u \in U(t)\langle S\rangle$. Then $\mathrm{Supp}(\varphi) \subseteq \mathsf{F}(S,t)$ giving $\varphi \in \langle \mathsf{F}(S,t)\rangle$ which, in turn, implies $e_u \in \mathsf{F}(\mathsf{F}(S,t), -t) = \mathcal{C}\!\ell_t(S)$. Part *(i)* (resp., *(j)*) follows by applying *(b)* (resp., *(c)*) twice. □

*Discrete topology.* Let us say that $S \subseteq V(X)$ is *closed at time $t$* (or simply $t$-*closed*) if $S = \mathcal{C}\!\ell_t(S)$ and *open at time $t$* if $V(X) \setminus S$ is closed at time $t$. Note immediately that $S$ is $t$-closed if and only if $X$ admits bijective group state transfer from $S$ to $\mathsf{F}(S,t)$ at time $t$ by Lemma 3.2. Combining Lemma 3.1*(g)* and Theorem 3.5*(a)*, we see that $S$ is $t$-closed if and only if $S$ is $t$-open. From Lemma 3.3, we know that, for most $t$, we obtain only the indiscrete topology $\{\emptyset, V(X)\}$ and Example 3.1 illustrates a case where the discrete topology arises: at time $t = \pi/2$, every vertex subset of the $d$-cube is both $t$-open and $t$-closed.

**Corollary 5.2.** *Let $X$ be a graph. At each time $t$, the $t$-open sets form a topology on $V(X)$.* □

*Proof.* Both $\emptyset$ and $V(X)$ are $t$-closed for all $t$. By parts *(h)* and *(i)* of Theorem 5.1, the intersection of any two $t$-closed sets is $t$-closed and, by part *(j)*, the union of any two $t$-closed sets is $t$-closed. □

**Definition 5.1.** *For graphs $X$ and $Y$, a function $\alpha : V(X) \to V(Y)$ is* continuous at times $t_0$ and $t_1$ *(or* continuous at time $t_0 = t_1$*) if $\alpha^{-1}(S)$ is $t_0$-open in $V(X)$ for every $t_1$-open $S \subseteq V(Y)$.*

It is worthwhile to consider some examples of functions that are continuous in this sense. Proposition 6.1 yields that any automorphism of $X$ is continuous at each time $t$. The identity function $V(X) \to V(X)$ is continuous at times $kt$ and $t$ for each integer $k$ and each $t \in \mathbb{R}$ (Lemma 3.1*(f)* and Theorem 3.5*(a,e)*). Since, at time $t_0 = k\frac{\pi}{2}$ ($k \in \mathbb{Z}$), the topology of $t$-open sets on the $n$-cube is the discrete topology, any graph homomorphism from the $n$-cube to a graph $Y$ is continuous at times $t_0$ and any $t_1$. We will see below that the projection map from a Cartesian product of graphs to any individual factor is continuous relative to the two topologies at time $t$. Returning to our previous example of $P_3$, the topology of $t$-open sets on $P_3$ is trivial for almost all $t \geq 0$, but at each $t = n\pi/\sqrt{2}$, the topology is $\{\emptyset, S, T, V\}$. As a result, for instance, the functions $f : V(P_3) \to \mathbb{R}$ that are continuous with respects to the GST-induced topology at $t = n\pi/\sqrt{2}$, where $\mathbb{R}$ is endowed the the natural topology, are exactly those functions for which $f(1) = f(3)$.

## 6. GST and the automorphism group

We continue with a graph $X$ on vertex set $V(X) = \{1, \ldots, n\}$ and adjacency matrix $A$. Using $\mathcal{S}_n$ to denote the symmetric group, we denote by $\mathrm{Aut}(X)$ the automorphism group of $X$: if $P_\sigma$ is the permutation matrix representing the bijection $\sigma : V(X) \to V(X)$ sending $a \in X$ to $a^\sigma$, then $\mathrm{Aut}(X) = \{\sigma \in \mathcal{S}_n \mid P_\sigma A = AP_\sigma\}$. For $a \in V(X)$ and $H \leq \mathrm{Aut}(X)$, the orbit of $a$ under $H$ will be denoted $\mathcal{O}_H(a) = \{a^\eta \mid \eta \in H\}$ and, writing $S^\eta = \{a^\eta \mid a \in S\}$, the orbit of $S \subseteq V(X)$ under $H$ will be denote $\mathcal{O}_H(S) = \{S^\eta \mid \eta \in H\}$. The setwise stabilizer of $S$ is $\mathrm{Stab}(S) = \{\sigma \in \mathrm{Aut}(X) \mid S^\sigma = S\}$.

**Proposition 6.1.** *Let $X$ be a graph, $S, T \subseteq V(X)$. Assume $X$ admits $(S,T)$-GST at time $\tau$. Then*

*(a) for any $\sigma \in \mathrm{Aut}(X)$, $X$ admits $(S^\sigma, T^\sigma)$-GST at time $\tau$;*

*(b) setting $H = \mathrm{Stab}(S)$ and $T' = \bigcap_{\eta \in H} T^\eta$, $X$ admits $(S, T')$-GST at time $\tau$;*

*(c) if $|S| = |T|$, then $\mathrm{Stab}(S) = \mathrm{Stab}(T)$.*

*Proof.* For part *(a)*, $u \in S^\sigma$ gives $U(\tau)e_u = P_\sigma U(\tau) P_\sigma^{-1} e_u = P_\sigma \psi$ for some $\psi \in \langle T \rangle$. Now part *(b)* follows by applying *(a)* to each $\sigma \in H$ and using Lemma 3.1*(e)*. Part *(c)* follows from *(b)* using Theorems 3.2 and 3.5 *(a)*. □

Using this, together with Lemma 3.1*(c)*, we have

**Corollary 6.2.** *If $X$ has $(u,v)$-PST at $\tau$, then $X$ has both $(\mathcal{O}(u), \mathcal{O}(v))$-GST at $\tau$ and $(\mathcal{O}(v), \mathcal{O}(u))$-GST at $\tau$, where $\mathcal{O}(u)$ and $\mathcal{O}(v)$ denote the orbit under any subgroup $H$ of $\mathrm{Aut}(X)$.* □

**Proposition 6.3.** *Let $X$ be a graph, $S \subseteq V(X)$; write $R = \mathsf{I}(S,t)$ and $T = \mathsf{F}(S,t)$. Then*

*(a) $\mathrm{Stab}(S) \leq \mathrm{Stab}(R)$ and $\mathrm{Stab}(S) \leq \mathrm{Stab}(T)$;*

*(b) $|\mathcal{O}(S)| \geq |\mathcal{O}(R)|$ and $|\mathcal{O}(S)| \geq |\mathcal{O}(T)|$.*

*Proof.* Suppose $\sigma \in \mathrm{Stab}(S)$. For $v \in T$, locate $\psi \in \langle S \rangle$ with $e_v^\top U(t)\psi \neq 0$. Then $e_{v^\sigma} = P_\sigma e_v$ and $e_{v^\sigma}^\top U(t)\varphi = e_v^\top U(t)\psi \neq 0$ for $\varphi = P_\sigma \psi \in \langle S \rangle$ since $\sigma \in \mathrm{Stab}(S)$. This shows $v^\sigma \in T$. On the other hand, if $v \in R$, then $\varphi = U(t)e_v \in \langle S \rangle$ so $U(t)e_{v^\sigma} = U(t)P_\sigma e_v = P_\sigma \varphi \in \langle S \rangle$ since $\sigma$ stabilizes $S$. This shows that $\sigma$ stabilizes $R$. Part *(b)* now follows by the Orbit-Stabilizer Theorem. □

Lemma 3.3 tells us that we almost always have $R = \emptyset$ and $T = V(X)$; in such cases, the above result is vacuous.

## 7. Products and joins

**Proposition 7.1.** *Let $X_1$ and $X_2$ be connected graphs. Assume that $X_1$ has $(S_1, T_1)$-GST at time $\tau$ and $X_2$ has $(S_2, T_2)$-GST at time $\tau$. Then $X_1 \square X_2$ has $(S_1 \times S_2, T_1 \times T_2)$-GST at $\tau$, where $\square$ denotes the Cartesian graph product.*

*Proof.* Let $U_1(t) = U_{X_1}(t)$ and $U_2(t) = U_{X_2}(t)$. We know from [9, Lemma 1.3.1] that

$$U_{X_1 \square X_2}(t) = U_1(t) \otimes U_2(t).$$

Suppose that graph $X_1$ has $(S_1, T_1)$-GST at $\tau$, and graph $X_2$ has $(S_2, T_2)$-GST at $\tau$. If $(a_1, a_2) \in S_1 \times S_2$, then we may write $e_{(a_1,a_2)} = e_{a_1} \otimes e_{a_2}$ and we compute

$$U_{X_1 \square X_2}(t) e_{(a_1,a_2)} = (U_1(t) \otimes U_2(t)) (e_{a_1} \otimes e_{a_2}) = (U_1(t) e_{a_1}) \otimes (U_2(t) e_{a_2}).$$

Since $U_1(t)e_{a_1} \in \langle T_1 \rangle$ and $U_2(t)e_{a_2} \in \langle T_2 \rangle$, we have $U_{X_1 \square X_2}(t) e_{(a_1,a_2)} \in \langle T_1 \times T_2 \rangle$. □

As a special case, we have the following, using Lemma 3.1*(a)*.

**Proposition 7.2.** *Let $X$ and $Y$ be connected graphs, so that $X$ has $(S, T)$-GST at $\tau$. Then $X \square Y$ has $(S \times V(Y), T \times V(Y))$-GST at $\tau$.* □

One curious consequence comes in the form of the following corollary.

**Corollary 7.3.** *If $\pi_i : (a_1, a_2) \mapsto a_i$ is the projection from $X \square Y$ onto $X$ or $Y$ in the cases $i = 1$ and $i = 2$, respectively, then $\pi_i$ is continuous with respects to the GST-induced topology at each time $t$.*

*Proof.* Without loss of generality, take $i = 1$ and suppose $S \subseteq V(X)$ is $t$-closed in $X$. Then $X$ admits bijective $(S, \mathsf{F}(S, t))$-GST at time $t$ and therefore $X \square Y$ admits bijective $(S \times V(Y), \mathsf{F}(S, t) \times V(Y))$-GST at time $t$. Therefore

$$\pi_1^{-1}(S) = S \times V(Y)$$

is $t$-closed, so for any $t$-closed $S$ in $X$, $\pi_1^{-1}(S)$ is $t$-closed in $X \square Y$. □

*The join.* Let $X_1$ and $X_2$ be connected graphs on disjoint vertex sets and define $X = X_1 + X_2$ to be the graph on vertex set $V(X) = V(X_1) \cup V(X_2)$ with edge set $E(X) = E(X_1) \cup E(X_2) \cup \{ab \mid a \in V(X_1),\ b \in V(X_2)\}$. The graph $X$ is the *join* of $X_1$ and $X_2$. Denoting the adjacency matrices of the three graphs by $A(X_1)$, $A(X_2)$ and $A(X)$, we have

$$A(X) = \left[ \begin{array}{c|c} A(X_1) & J \\ \hline J^\top & A(X_2) \end{array} \right]$$

where $J$ is the all ones matrix with $n_1 = |V(X_1)|$ rows and $n_2 = |V(X_2)|$ columns. In the case that $X_1$ and $X_2$ are regular graphs, a basis of eigenvectors for $A(X)$ can be derived from eigenbases for $A(X_1)$ and $A(X_2)$ as shown, for example, in [9, Section 12.1–2]; from this, the following result is immediate.

**Proposition 7.4.** *Assume $X$ is the join of the $k_1$-regular graph $X_1$ on $n_1$ vertices and the $k_2$-regular graph $X_2$ on $n_2$ vertices. Let $\Delta = (k_1 - k_2)^2 + 4n_1n_2$. Then $X$ admits $(V(X_1), V(X_1))$-GST and $(V(X_2), V(X_2))$-GST at time $\tau = 2\ell\pi/\sqrt{\Delta}$ for each integer $\ell$.*

*Proof.* The equitable partition $\{V(X), V(Y)\}$ of the vertex set of $X + Y$ has quotient matrix $\begin{bmatrix} k_1 & n_2 \\ n_1 & k_2 \end{bmatrix}$ with eigenvalues $\theta_0, \theta_1 = \frac{1}{2}\left(k_1 + k_2 \pm \sqrt{\Delta}\right)$. Set

$$j_1 = \left[ \begin{array}{c} \mathbf{1} \\ \hline \mathbf{0} \end{array} \right], \qquad j_2 = \left[ \begin{array}{c} \mathbf{0} \\ \hline \mathbf{1} \end{array} \right], \qquad u_0 = n_2 j_1 + (\theta_0 - k_1) j_2, \qquad u_1 = n_2 j_1 + (\theta_1 - k_1) j_2 \ .$$

Then $A(X)u_h = \theta_h u_h$ $(h = 0, 1)$. It follows that every eigenvector $v$ of $X_i$ orthogonal to the all-ones vector gives us an eigenvector of $X$ by simply extending by an all-zero vector. So we may expand

$$A(X) = \sum_{r=0}^{d} \theta_r E_r \qquad \text{where} \quad E_0 = \frac{1}{\|u_0\|^2} u_0 u_0^\top, \quad E_1 = \frac{1}{\|u_1\|^2} u_1 u_1^\top$$

and each $E_r$ is block diagonal for $r \geq 2$. Since $\sum_{r=0}^{d} E_r = I$, $E_0 + E_1$ is block diagonal as well. So, for any $\ell \in \mathbb{Z}$, we may take $\tau = 2\ell\pi/\sqrt{\Delta}$ so that $e^{i\theta_0\tau} = e^{i\theta_1\tau}$ and $U(\tau) = \sum_{r=0}^{d} e^{i\theta_r\tau} E_r$ is block diagonal. This guarantees that $U(\tau)\langle V(X_1)\rangle = \langle V(X_1)\rangle$ as desired. □ □

## 8. Examples

In previous sections we have seen mostly trivial examples of group state transfer, but also those cases that arise from perfect state transfer. We now discuss non-trivial examples of this phenomenon.

**Example 8.1.** *The six-cycle $C_6$ can be expressed as the tensor product $C_3 \times P_2$ of $C_3$ and $P_2$. If we number the vertices $1, 2, 3, 4, 5, 6$ in cyclic order and set $S = \{1, 3, 5\}$, then $C_6$ admits $(S, S)$-GST at time $\pi$.*

This is a special case of the following phenomenon.

**Theorem 8.1.** *Let $X$ be a connected bipartite graph with bipartition $V(X) = V_0 \cup V_1$.*

*(a) If, for some $\alpha > 0$, all eigenvalues of $\alpha A$ are integers, then $X$ admits $(V_0, V_0)$-GST and $(V_1, V_1)$-GST at time $\tau = \pi\alpha$;*

*(b) If, for some $\alpha > 0$, all eigenvalues of $\alpha A$ are odd integers, then $X$ admits $(V_0, V_1)$-GST and $(V_1, V_0)$-GST at time $\tau = \pi\alpha/2$.*

*Proof.* Suppose $\theta_r$ is an eigenvalue of $X$ whose projector has block form $E_r = \left[\begin{array}{c|c} F_{00} & F_{01} \\ \hline F_{10} & F_{11} \end{array}\right]$. Since $X$ is bipartite, there is an index $r'$ such that $\theta_{r'} = -\theta_r$ and $E_{r'} = \left[\begin{array}{c|c} F_{00} & -F_{01} \\ \hline -F_{10} & F_{11} \end{array}\right]$. So

$$e^{i\theta_r\tau} \left[\begin{array}{c|c} F_{00} & F_{01} \\ \hline F_{10} & F_{11} \end{array}\right] + e^{-i\theta_r\tau} \left[\begin{array}{c|c} F_{00} & -F_{01} \\ \hline -F_{10} & F_{11} \end{array}\right] = \left[\begin{array}{c|c} (e^{i\theta_r\tau} + e^{-i\theta_r\tau})F_{00} & (e^{i\theta_r\tau} - e^{-i\theta_r\tau})F_{01} \\ \hline (e^{i\theta_r\tau} - e^{-i\theta_r\tau})F_{10} & (e^{i\theta_r\tau} + e^{-i\theta_r\tau})F_{11} \end{array}\right].$$

Let us assume first that $A$ is invertible so that $A = \sum_{\theta_r > 0} (\theta_r E_r + \theta_{r'} E_{r'})$. Let us first consider case *(b)*: at time $\tau = \pi\alpha/2$, $e^{i\theta_r\tau} = \pm i$, $e^{-i\theta_r\tau} = \mp i$ and the diagonal blocks of $e^{i\theta_r\tau}E_r + e^{i\theta_{r'}\tau}E_{r'}$ vanish. Similarly, in case *(a)*, the off-diagonal blocks of $e^{i\theta_r\tau}E_r + e^{i\theta_{r'}\tau}E_{r'}$ vanish at time $\tau = \pi\alpha/2$. Summing over the positive eigenvalues $\theta_r$ gives our result, except in case *(a)* where $A$ is singular. To finish the argument we note that the zero eigenspace of a bipartite graph admits a basis of eigenvectors each supported on just one of $V_0$, $V_1$. So the orthogonal projection $E_0$ is a block diagonal matrix and this does not affect the block diagonal structure of $U(\tau)$. □

**Remark 8.1.** *The existence of a time $\tau$ at which GST exists in part (a) of Theorem 8.1 (but not its value) is also a corollary of Godsil's Theorem 2.2 and Lemma 2.3 in [10]. The results in [10] also imply periodicity of $V_0$ and $V_1$ at time $2\tau$ in part (b) above.*

*The symmetric double star.* In [17], Fan and Godsil study pretty good state transfer on graphs composed of gluing together two stars. Let $X$ be the graph (denoted $S_{k,k}$ in [17]) on vertex set $V(X) = \{1, \ldots, n\}$ where $n = 2k + 2$ with $E(X) = \{1a \mid 2 \leq a \leq k + 2\} \cup \{2a \mid k + 3 \leq a \leq 2k + 2\}$ and adjacency matrix

$$A = \left[\begin{array}{c|c|c|c} 0 & 1 & \mathbf{1}^\top & 0 \\ \hline 1 & 0 & 0 & \mathbf{1}^\top \\ \hline \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & \mathbf{1} & 0 & 0 \end{array}\right]$$

where $\mathbf{1}$ is the $k \times 1$ matrix of all ones. The eigenvalues are given, for example, in [17];

$$\theta_0 = \frac{1}{2}\left(1+\sqrt{4k+1}\right),\ \theta_1 = \frac{1}{2}\left(-1+\sqrt{4k+1}\right),\ \theta_2 = 0,\ \theta_3 = \frac{1}{2}\left(1-\sqrt{4k+1}\right),\ \theta_4 = \frac{1}{2}\left(-1-\sqrt{4k+1}\right)$$

with each nonzero eigenvalue having multiplicity one.

**Proposition 8.2.** *At time $\tau = 2\pi/\sqrt{4k+1}$, the symmetric double star $X$ admits $(S,S)$-GST for $S = \{1,2\}$.*

*Proof.* Let $\sigma_0 = \sigma_3 = +1$ and $\sigma_1 = \sigma_4 = -1$ and note that $\theta_r^2 - k = \sigma_r\theta_r$ for $r \neq 2$. The orthogonal projection onto the eigenspace of $A$ belonging to $\theta_r$ ($r \neq 2$) is

$$E_r = \frac{1}{4k+2\sigma_r\theta_r}\left[\begin{array}{r|r|r|r} \theta_r^2 & \sigma_r\theta_r^2 & \theta_r\mathbf{1}^\top & \sigma_r\theta_r\mathbf{1}^\top \\ \hline \sigma_r\theta_r^2 & \theta_r^2 & \sigma_r\theta_r\mathbf{1}^\top & \theta_r\mathbf{1}^\top \\ \hline \theta_r\mathbf{1} & \sigma_r\theta_r\mathbf{1} & J & \sigma_r J \\ \hline \sigma_r\theta_r\mathbf{1} & \theta_r\mathbf{1} & \sigma_r J & J \end{array}\right]$$

where $J$ is the $k \times k$ matrix of all ones. The null space of $A$ is orthogonal to $\langle S\rangle$ so $E_2$ plays no role here. Since

$$\theta_3\tau = \theta_0\tau - 2\pi, \quad \theta_4\tau = \theta_1\tau - 2\pi, \qquad \theta_4 = -\theta_0, \quad \theta_3 = -\theta_1,$$

we have

$$e^{i\theta_3\tau} = e^{i\theta_0\tau}, \quad e^{i\theta_4\tau} = e^{i\theta_1\tau} = \overline{e^{i\theta_0\tau}}\ .$$

This gives us

$$U(\tau) = e^{i\theta_0\tau}(E_0+E_3) + e^{i\theta_1\tau}(E_1+E_4) + E_2$$

having its first two rows equal to

$$\left[\begin{array}{c|c|c|c} U(\tau)_{11} & U(\tau)_{12} & \alpha\mathbf{1}^\top & \beta\mathbf{1}^\top \\ \hline U(\tau)_{21} & U(\tau)_{22} & \beta\mathbf{1}^\top & \alpha\mathbf{1}^\top \end{array}\right]$$

where

$$\alpha = \sum_{r=0}^{4} \frac{e^{i\theta_r\tau}\theta_r}{4k+2\sigma_r\theta_r}, \qquad \beta = \sum_{r=0}^{4} \frac{e^{i\theta_r\tau}\sigma_r\theta_r}{4k+2\sigma_r\theta_r}\ .$$

Writing $K = 4k+1$, we have

$$\frac{\theta_0}{4k+2\theta_0} + \frac{\theta_3}{4k+2\theta_3} = \frac{1}{2}\left[\frac{1+\sqrt{K}}{K+\sqrt{K}} + \frac{1-\sqrt{K}}{K-\sqrt{K}}\right] = 0$$

and

$$\frac{\theta_1}{4k-2\theta_1} + \frac{\theta_4}{4k-2\theta_4} = \frac{1}{2}\left[\frac{-1+\sqrt{K}}{K-\sqrt{K}} + \frac{-1-\sqrt{K}}{K+\sqrt{K}}\right] = 0$$

giving us $\alpha = \beta = 0$ as desired. □

Denote by $T_i$ the leaf vertices of the symmetric double star $X$ adjacent to vertex $i$ ($i = 1,2$). At time $\tau = \pi$, $X$ admits $(S,S)$-GST for $S = \{1\} \cup T_2$ and $S = \{2\} \cup T_1$ and, at time $\tau = 2\pi$, $(S,S)$-GST for each of $S = \{1\}$, $\{2\}$, $T_1$ and $T_2$.

## 9. Some problems

We now list some questions that we consider worthy of study.

1. Which graph products respect group state transfer?
2. If $X$ is a path on five or more vertices, can non-trivial bijective $(S, T)$-GST ever occur?
3. Which graph homomorphism are continuous with respect to the topologies of $t$-open sets?
4. Does case *(d)* of Proposition 3.4 ever occur?
5. Suppose $X$ is a double star with $S$ and $T$ the natural partition of the vertices of degree one (elements of $S$ (resp., $T$) are pairwise at distance two in $X$ and each $a \in S$ is at distance three from each $b \in T$. In what cases does $X$ admit $(S, T)$-GST?
6. Assume $X$ admits $(S, T)$-GST at time $\tau$ with $S \cap T = \emptyset$. When is there a weighted quotient graph $\bar{X}$ admitting PST from the sole image of $S$ to the sole image of $T$?
7. Is it true that, for almost all graphs $X$, the poset $\mathcal{ST}(X, t)$ is trivial for all $t \neq 0$?
8. Suppose $X$ admits $(S, T)$-GST at time $\tau$ and let $\delta$ denote the minimum distance from $a$ to $b$ over all $a \in S$, $b \in T$. Must $|V(X)|$ grow exponentially with $\delta$?

### Acknowledgements

We thank Ada Chan, Aleksandar Jurišić, and Chris Larsen for helpful discussions and thank the referee for useful comments that led to an improved manuscript. The work of WJM was supported through a grant from the National Science Foundation (DMS Award #1808376) which is gratefully acknowledged.

## References

[1] E. Farhi, S. Gutmann, Quantum computation and decision trees, Phys. Rev. A 58 (1998) 915–928. doi:10.1103/PhysRevA.58.915.
URL https://link.aps.org/doi/10.1103/PhysRevA.58.915

[2] A. M. Childs, Universal computation by quantum walk, Phys. Rev. Lett. 102 (2009) 180501. doi:10.1103/PhysRevLett.102.180501.
URL https://link.aps.org/doi/10.1103/PhysRevLett.102.180501

[3] S. Bose, Quantum communication through an unmodulated spin chain, Phys. Rev. Lett. 91 (2003) 207901. doi:10.1103/PhysRevLett.91.207901.
URL https://link.aps.org/doi/10.1103/PhysRevLett.91.207901

[4] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, A. J. Landahl, Perfect transfer of arbitrary states in quantum spin networks, Phys. Rev. A 71 (2005) 032312. doi:10.1103/PhysRevA.71.032312.
URL https://link.aps.org/doi/10.1103/PhysRevA.71.032312

[5] C. Godsil, State transfer on graphs, Discrete Mathematics 312 (1) (2012) 129–147. doi:https://doi.org/10.1016/j.disc.2011.06.032.
URL https://www.sciencedirect.com/science/article/pii/S0012365X11002974

[6] L. Vinet, A. Zhedanov, Almost perfect state transfer in quantum spin chains, Phys. Rev. A 86 (2012) 052319. doi:10.1103/PhysRevA.86.052319.
URL https://link.aps.org/doi/10.1103/PhysRevA.86.052319

[7] G. Coutinho, Quantum walks and the size of the graph, Discrete Mathematics 342 (10) (2019) 2765–2769. doi:10.1016/j.disc.2018.04.011.
URL https://doi.org/10.1016%2Fj.disc.2018.04.011

[8] L. Brown, Continuous-time quantum walks: Group state transfer on graphs, https://digital.wpi.edu/show/00000250m, major Qualifying Project (2019).

[9] G. Coutinho, C. Godsil, Graph spectra and continuous quantum walks, University of Waterloo, lecture Notes (2021).

[10] C. Godsil, Real state transfer (2017). doi:10.48550/ARXIV.1710.04042.
URL https://arxiv.org/abs/1710.04042

[11] A. Chan, G. Coutinho, C. Tamon, L. Vinet, H. Zhan, Quantum fractional revival on graphs, Discrete Appl. Math. 269 (2019) 86–98. doi:10.1016/j.dam.2018.12.017.
URL https://doi.org/10.1016/j.dam.2018.12.017

[12] A. Chan, G. Coutinho, W. Drazen, O. Eisenberg, C. Godsil, G. Lippner, M. Kempton, C. Tamon, H. Zhan, Fundamentals of fractional revival in graphs (2020). doi:10.48550/ARXIV.2004.01129.
URL https://arxiv.org/abs/2004.01129

[13] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, NY, 2001. doi:https://doi.org/10.1007/978-1-4613-0163-9.

[14] C. Godsil, J. Shawe-Taylor, Distance-regularised graphs are distance-regular or distance-biregular, Journal of Combinatorial Theory, Series B 43 (1) (1987) 14–24.
URL https://eprints.soton.ac.uk/259848/

[15] A. Kay, Basics of perfect communication through quantum networks, Phys. Rev. A 84 (2011) 022337. doi:10.1103/PhysRevA.84.022337.
URL https://link.aps.org/doi/10.1103/PhysRevA.84.022337

[16] A. Björner, Topological Methods, Vol. 1,2, Elsevier, Amsterdam, 1995, Ch. 34, pp. 1819–1872.

[17] X. Fan, C. Godsil, Pretty good state transfer on double stars (2012). doi:10.48550/ARXIV.1206.0082.
URL https://arxiv.org/abs/1206.0082