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Invariant rings of the special orthogonal group have nonunimodal *h*-vectors

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To Sudhir Ghorpade, in celebration of his sixtieth birthday

For K an infinite field of characteristic other than two, consider the action of the special orthogonal group $\mathrm{SO}_t(K)$ on a polynomial ring via copies of the regular representation. When K has characteristic zero, Boutot's theorem implies that the invariant ring has rational singularities; when K has positive characteristic, the invariant ring is F-regular, as proven by Hashimoto using good filtrations. We give a new proof of this, viewing the invariant ring for $\mathrm{SO}_t(K)$ as a cyclic cover of the invariant ring for the corresponding orthogonal group; this point of view has a number of useful consequences, for example, it readily yields the a-invariant and information on the Hilbert series. Indeed, we use this to show that the h-vector of the invariant ring for $\mathrm{SO}_t(K)$ need not be unimodal.

Keywords: Invariant rings; Gorenstein rings; Hilbert series; h-vectors.

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1. Introduction

Let X be an $n \times n$ symmetric matrix of indeterminates over a field K, and let $I_{t+1}(X)$ denote the ideal of the polynomial ring $K[X]$ generated by the size $t+1$

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minors of X. For t a positive integer with $t+1 \leq n$, we refer to $K[X]/I_{t+1}(X)$ as a *symmetric determinantal ring*. The ring $K[X]/I_{t+1}(X)$ is a Cohen–Macaulay normal domain of dimension

$$\binom{n+1}{2} - \binom{n+1-t}{2},$$

as proven in [25]. These rings have been studied extensively, in part because they arise as invariant rings for the natural action of the orthogonal group

$$\mathrm{O}_t(K) := \{M \in \mathrm{GL}_t(K) \mid M^{\mathrm{tr}}M = \mathrm{id}\} \tag{1}$$

as follows: for Y a $t \times n$ matrix of indeterminates, $\mathrm{O}_t(K)$ acts K-linearly on $K[Y]$ via

$$M\colon Y \longmapsto MY \quad \text{for } M \in \mathrm{O}_t(K).$$

This is a right action of $\mathrm{O}_t(K)$ on the polynomial ring $K[Y]$, corresponding to a left action of $\mathrm{O}_t(K)$ on the affine space $\mathbb{A}_K^{t\times n}$. Note that $Y^{\mathrm{tr}}Y \longmapsto Y^{\mathrm{tr}}M^{\mathrm{tr}}MY = Y^{\mathrm{tr}}Y$ for $M \in \mathrm{O}_t(K)$, so the entries of $Y^{\mathrm{tr}}Y$ are invariant under the action; when the field K is infinite of characteristic other than two, the invariant ring is precisely the K-algebra generated by the entries of $Y^{\mathrm{tr}}Y$, see [14, Theorem 5.6], and is isomorphic to the symmetric determinantal ring $K[X]/I_{t+1}(X)$ via the entrywise map $X \longmapsto Y^{\mathrm{tr}}Y$. We use this to identify the rings $K[X]/I_{t+1}(X)$ and $K[Y^{\mathrm{tr}}Y]$.

By [17, 18], the ring $R := K[Y^{\mathrm{tr}}Y]$ has class group $\mathbb{Z}/2$, and is Gorenstein precisely when $n \equiv t+1 \mod 2$. Taking $\mathfrak{p}$ to be a prime ideal that serves as a generator for the class group, it follows that the symbolic power $\mathfrak{p}^{(2)}$ is isomorphic to R. We choose an explicit isomorphism $\mathfrak{p}^{(2)} \cong R$ so that the cyclic cover of R with respect to $\mathfrak{p}$ is precisely the invariant ring for the action of the special orthogonal group $\mathrm{SO}_t(K)$. This gives a straightforward approach towards studying the invariant ring $K[Y]^{\mathrm{SO}_t(K)}$, for example towards determining its a-invariant and information regarding the Hilbert series.

When K is an infinite field of characteristic two, the groups $\mathrm{O}_t(K)$ and $\mathrm{SO}_t(K)$ coincide when taking $\mathrm{O}_t(K)$ to be the group as defined in (1); the invariant ring in this case is

$$K\left[Y^{\mathrm{tr}}Y,\ \sum_{i=1}^{t} y_{ij} \;\middle|\; 1 \leq j \leq n\right],$$

see [31, Proposition 17], and a presentation is provided by [31, Proposition 23]. The reader is warned that there are varying definitions used for the orthogonal group in characteristic two, see for example [30, p. 10].

Section 2 includes some generalities on cyclic covers; these are used in Sec. 3 where we compute the a-invariant of $K[Y]^{\mathrm{SO}_t(K)}$ and also record a proof that this ring is F-regular. Section 4 is devoted to the h-vector of $K[Y]^{\mathrm{SO}_t(K)}$, i.e. the coefficients of the numerator of its Hilbert series: the key result here is that this invariant ring is a semistandard graded Gorenstein normal domain, for which the h-vector need not be unimodal; the context for this is discussed as well in Sec. 4.

2. Cyclic Covers and F-Regularity

Let R be a normal domain. By a *divisorial ideal* of R, we mean a nonzero intersection of fractional principal ideals. Let $\mathfrak{a}$ be a divisorial ideal that has finite order m when viewed as an element of the divisor class group of R. Then $\mathfrak{a}^{(m)} = \alpha R$, for an element α in the fraction field of R. Set

$$T := 1/\alpha^{1/m}, \tag{2}$$

which is an element in an algebraic closure of the fraction field of R; the choice of α or the mth root is not unique. The *cyclic cover* of R with respect to $\mathfrak{a}$ is the ring

$$\widetilde{R} := R[\mathfrak{a}T,\ \mathfrak{a}^{(2)}T^2,\ \mathfrak{a}^{(3)}T^3,\ \ldots],$$

viewed as a subring of $R[T]$. Since

$$\mathfrak{a}^{(m+k)}T^{m+k} = \alpha\mathfrak{a}^{(k)}T^{m+k} = \mathfrak{a}^{(k)}T^k$$

for each $k \geq 0$, the ring $\widetilde{R}$ is a finitely generated reflexive R-module; specifically, one has an R-module isomorphism

$$\widetilde{R} \cong R \oplus \mathfrak{a} \oplus \mathfrak{a}^{(2)} \oplus \cdots \oplus \mathfrak{a}^{(m-1)}.$$

When the ring R is $\mathbb{N}$-graded and $\mathfrak{a}$ is a homogeneous divisorial ideal of finite order m, there exists a homogeneous element α with $\mathfrak{a}^{(m)} = \alpha R$, and the $\mathbb{N}$-grading on R extends to a $\mathbb{Q}$-grading on $\widetilde{R}$ obtained by setting

$$\deg T := -(\deg \alpha)/m.$$

It turns out that this is a $\mathbb{Q}_{\geq 0}$-grading on $\widetilde{R}$, and that $[\widetilde{R}]_0 = R_0$, see [32, Proposition 4.2].

Suppose that the characteristic of R is zero or relatively prime to m, and that $\mathfrak{p}$ is a height one prime ideal of R. Then the ideal $\mathfrak{a}R_{\mathfrak{p}}$ is principal; take r to be a generator. Since $r^m = \alpha u$, for u a unit in $R_{\mathfrak{p}}$, it follows that

$$\widetilde{R}_{\mathfrak{p}} = R_{\mathfrak{p}}[rT] \cong R_{\mathfrak{p}}[u^{1/m}],$$

so $R_{\mathfrak{p}} \longrightarrow \widetilde{R}_{\mathfrak{p}}$ is étale. In particular, under this assumption on the characteristic, the ring $\widetilde{R}_{\mathfrak{p}}$ is regular for each height one prime of R; since each $\mathfrak{a}^{(k)}$ is reflexive, the ring $\widetilde{R}$ also satisfies the Serre condition S_2, and is hence a normal domain. By [36, Theorem 2.7], F-regularity is preserved under finite extensions that are étale at height one primes, so one has:

Theorem 2.1 (Watanabe). *Let R be an $\mathbb{N}$-graded ring that is finitely generated over a field R_0 of characteristic $p > 0$, and let $\widetilde{R}$ be the cyclic cover of R with respect to a homogeneous ideal of finite order relatively prime to p. Then, if R is F-regular, so is $\widetilde{R}$.*

The restriction on the characteristic is removed in [9, Theorem C]. For the theory of F-regularity in the graded setting, we point the reader towards [22]. When R is an $\mathbb{N}$-graded ring finitely generated over a field R_0 of positive characteristic, the notions of weak F-regularity, F-regularity, and strong F-regularity all coincide as proven in [27], so we do not make a distinction between these in the present paper.

The F-regularity of generic determinantal rings and of Plücker coordinate rings of Grassmannians is proven as [22, Theorem 7.14]; the proof therein is readily adapted to symmetric determinantal rings, as we show next. For a different approach, see [26, §4.1].

Theorem 2.2. *Let X be an $n \times n$ symmetric matrix of indeterminates over a field K of positive prime characteristic. Then the ring $K[X]/I_{t+1}(X)$ is F-regular.*

Proof. If $n \equiv t+1 \mod 2$, then $K[X]/I_{t+1}(X)$ is Gorenstein; otherwise, enlarge X to a symmetric matrix $\widetilde{X}$ of size $n+1$, in which case the ring $K[\widetilde{X}]/I_{t+1}(\widetilde{X})$ is Gorenstein, and contains $K[X]/I_{t+1}(X)$ as a pure subring. Since F-regularity is inherited by pure subrings, it suffices to prove the desired result when $R := K[X]/I_{t+1}(X)$ is Gorenstein.

The a-invariant of R is computed in [3] and [11], and recorded in the following section; in particular, $a(R) < 0$. We next claim that R is F-injective, equivalently F-pure, since the notions coincide in the Gorenstein case. This follows by [13, Theorem 2.1] in combination with the main result of [10] asserting that the "diagonal" initial ideal of $I_{t+1}(X)$ is square-free and defines a Cohen–Macaulay ring.

The F-regularity of R now follows from [22, Corollary 7.13], once we verify that the localization $R_{x_{ij}}$ is F-regular for each x_{ij}. Using the lemma below and induction on t, the localizations $R_{x_{11}}$ and R_Δ are F-regular; but then $R_\mathfrak{p}$ is F-regular if $\mathfrak{p}$ is a prime ideal such that $x_{11} \notin \mathfrak{p}$ or $\Delta \notin \mathfrak{p}$. It follows that $R_\mathfrak{p}$ is also F-regular if $x_{12} \notin \mathfrak{p}$. Since we have accounted for the diagonal variable x_{11} and the off-diagonal variable x_{12}, the symmetry implies that $R_{x_{ij}}$ is F-regular for each x_{ij}. □

Lemma 2.3. *Let $R := K[X]/I_{t+1}(X)$, where X is a symmetric $n \times n$ matrix of indeterminates. Then*:

(1) *The ring $R_{x_{11}}$ is isomorphic to a localization of a polynomial ring over $K[X']/I_t(X')$, where X' is a symmetric $(n-1) \times (n-1)$ matrix of indeterminates.*
(2) *For $\Delta := x_{11}x_{22} - x_{12}^2$, the ring R_Δ is isomorphic to a localization of a polynomial ring over $K[X']/I_{t-1}(X')$, for X' a symmetric $(n-2) \times (n-2)$ matrix of indeterminates.*

For a proof, see [24, Lemma 1.1]; the argument also appears implicitly in [28].

3. The a-Invariant

Let Y be a $t \times n$ matrix of indeterminates over a field K. In this section, we work with the grading on the subring $R := K[Y^{\mathrm{tr}}Y]$ that is induced by the standard grading on the polynomial ring $K[Y]$. Note that under the identification of $K[X]/I_{t+1}(X)$

with $K[Y^{\mathrm{tr}}Y]$, this corresponds to taking $\deg x_{ij} = 2$ for each i, j. With this grading, [3, Theorem 4.4] or [11, Theorem 2.4] imply that the a-invariant of R is

$$a(R) = \begin{cases} -t(n+1) & \text{if } n \equiv t \mod 2, \\ -tn & \text{if } n \not\equiv t \mod 2; \end{cases}$$

more generally, the graded canonical module of R is

$$\omega_R = \begin{cases} \mathfrak{p}(-tn+t) & \text{if } n \equiv t \mod 2, \\ R(-tn) & \text{if } n \not\equiv t \mod 2, \end{cases}$$

where $\mathfrak{p}$ is the ideal of $K[Y^{\mathrm{tr}}Y]$ generated by the maximal minors of the first t rows of $Y^{\mathrm{tr}}Y$, i.e. by the maximal minors of the product matrix

$$\begin{pmatrix} y_{11} & y_{21} & \cdots & y_{t1} \\ y_{12} & y_{22} & \cdots & y_{t2} \\ \vdots & \vdots & \vdots & \vdots \\ y_{1t} & y_{2t} & \cdots & y_{tt} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{t1} & y_{t2} & y_{t3} & \cdots & \cdots & y_{tn} \end{pmatrix}.$$

Using the identification of $K[X]/I_{t+1}(X)$ with $K[Y^{\mathrm{tr}}Y]$, the ideal $\mathfrak{p}$ is prime of height one by [25, Theorem 1], and generates the class group of R by [17]. The symbolic power $\mathfrak{p}^{(2)}$ is the principal ideal of R generated by the determinant of the first t columns of the product matrix displayed above, i.e. $\mathfrak{p}^{(2)}$ is generated by the square of

$$\Delta := \det \begin{pmatrix} y_{11} & y_{21} & \cdots & y_{t1} \\ y_{12} & y_{22} & \cdots & y_{t2} \\ \vdots & \vdots & \vdots & \vdots \\ y_{1t} & y_{2t} & \cdots & y_{tt} \end{pmatrix}.$$

Choosing a unit as in (2), set

$$T := 1/\Delta.$$

The generators of $\mathfrak{p}T$ are then identified with the maximal minors of the matrix Y, so that the cyclic cover $\widetilde{R}$ of R with respect to $\mathfrak{p}$ is the subring of the polynomial ring $K[Y]$ generated by the entries of the product matrix $Y^{\mathrm{tr}}Y$ along with the maximal minors of Y. It is clear that these generators are fixed under the action of the special orthogonal group

$$M\colon Y \longmapsto MY \quad \text{for } M \in \mathrm{SO}_t(K).$$

When the field K is infinite of characteristic other than two, the invariant ring is precisely the K-algebra generated by these elements [14, Theorem 5.6].

We determine the graded canonical module of $\widetilde{R}$; while the semisimplicity of $\mathrm{SO}_t(K)$ may be used to verify that $\widetilde{R}$ is Gorenstein, [23, p. 123], our goal is to additionally obtain the a-invariant of $\widetilde{R}$. Since $\deg T = -t$, one has

$$\widetilde{R} = R \oplus \mathfrak{p}(t).$$

Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R. For an $\mathbb{N}$-graded R-module M, we use $\underline{\mathrm{Hom}}(M, R/\mathfrak{m})$ to denote its graded dual as in [19, p. 184]. Setting $d := \dim R$, the graded canonical module of $\widetilde{R}$ may be computed as

$$\omega_{\widetilde{R}} = \underline{\mathrm{Hom}}(H^d_{\mathfrak{m}}(\widetilde{R}),\ R/\mathfrak{m}) = \underline{\mathrm{Hom}}(H^d_{\mathfrak{m}}(R),\ R/\mathfrak{m}) \oplus \underline{\mathrm{Hom}}(H^d_{\mathfrak{m}}(\mathfrak{p}(t)),\ R/\mathfrak{m}).$$

The first term in this direct sum is ω_R, while the second is

$$\begin{aligned}
\underline{\mathrm{Hom}}(H^d_{\mathfrak{m}}(\mathfrak{p}(t)),\ R/\mathfrak{m}) &= \underline{\mathrm{Hom}}(H^d_{\mathfrak{m}}(\omega_R) \otimes_R \omega_R^{(-1)} \otimes_R \mathfrak{p}(t),\ R/\mathfrak{m}) \\
&= \mathrm{Hom}_R(\omega_R^{(-1)} \otimes_R \mathfrak{p}(t),\ \underline{\mathrm{Hom}}(H^d_{\mathfrak{m}}(\omega_R),\ R/\mathfrak{m})) \\
&= \mathrm{Hom}_R(\omega_R^{(-1)} \otimes_R \mathfrak{p}(t),\ R) \\
&= (\omega_R \otimes_R \mathfrak{p}^{(-1)}(-t))^{**},
\end{aligned}$$

where $(-)^{**}$ is the reflexive hull. Since $\mathfrak{p}^{(2)} = R(-2t)$, one has $\mathfrak{p}^{(-1)} = \mathfrak{p}(2t)$, so

$$(\omega_R \otimes_R \mathfrak{p}^{(-1)}(-t))^{**} = \begin{cases} R(-tn) & \text{if } n \equiv t \mod 2, \\ \mathfrak{p}(-tn+t) & \text{if } n \not\equiv t \mod 2. \end{cases}$$

Putting it all together, one gets

$$\omega_{\widetilde{R}} = \begin{cases} \mathfrak{p}(-tn+t) \oplus R(-tn) & \text{if } n \equiv t \mod 2, \\ R(-tn) \oplus \mathfrak{p}(-tn+t) & \text{if } n \not\equiv t \mod 2, \end{cases}$$

so that

$$\omega_{\widetilde{R}} = \widetilde{R}(-tn),$$

i.e. $\widetilde{R}$ is Gorenstein with $a(\widetilde{R}) = -tn$. To summarize what we have at this stage:

Theorem 3.1. *Let Y be a $t \times n$ matrix of indeterminates over a field K of characteristic other than two. Let $\widetilde{R}$ denote the K-subalgebra of $K[Y]$ generated by the entries of the product matrix $Y^{\mathrm{tr}}Y$ along with the maximal minors of Y. Then $\widetilde{R}$ is a Gorenstein normal domain. When K has characteristic zero, the ring $\widetilde{R}$ has rational singularities; when K has positive characteristic, $\widetilde{R}$ is F-regular.*

With the $\mathbb{N}$-grading on $\widetilde{R}$ inherited from the standard grading on $K[Y]$, one has

$$a(\widetilde{R}) = -tn.$$

The fact that $\widetilde{R}$ has rational singularities in characteristic zero follows from Boutot's theorem [4]; the F-regularity in characteristic $p \geq 3$ follows by combining Theorems 2.1 and 2.2. For a different approach using good filtrations, see [20, Corollary 2].

Remark 3.2. The ring $\widetilde{R}$ in Theorem 3.1 has K-algebra generators in degree 2 and degree t; it admits a standard grading in the following two cases:

(i) When $t = 1$, index the entries of Y as $y_1, \dots, y_n$. The ring $R := K[Y^{\text{tr}}Y]$ is then the second Veronese subring of the polynomial ring $K[Y]$, i.e. the subring generated by the monomials $y_i y_j$. One has

$$\mathfrak{p} = (y_1^2, y_1y_2, \dots, y_1y_n)R \quad \text{and} \quad \mathfrak{p}^{(2)} = (y_1^2)R.$$

Taking $T := 1/y_1$, the cyclic cover $\widetilde{R}$ coincides with $K[Y]$ under the standard grading.

(ii) When $t = 2$, the K-algebra generators of $\widetilde{R}$ are the entries of $Y^{\text{tr}}Y$, and the size two minors of Y; these generators all have degree two, so the grading on $\widetilde{R}$ may be rescaled to a standard grading.

Remark 3.3. When t is even, the ring $\widetilde{R}$ in Theorem 3.1 has generators of even degree; rescaling by a factor of two, one obtains generators in degree one (the entries of $Y^{\text{tr}}Y$) and generators in degree $t/2$ (the maximal minors of Y); this is the grading considered in the following section. This is a *semistandard* grading on $\widetilde{R}$, i.e. an $\mathbb{N}$-grading under which the ring is integral over the K-subalgebra generated by its elements of degree one.

4. Nonunimodal *h*-Vectors

A description for the Hilbert function of a generic determinantal ring may be found in [1], while an expression for its Hilbert series is presented in [12]. In particular, for the numerator of the Hilbert series, known as the *h-polynomial*, one has both a combinatorial description (in terms on non-intersection paths with given number of turns) and an explicit compact (and determinantal!) formula. For pfaffian rings, the corresponding results are in [15, 16]. For symmetric determinantal rings one finds in [11] a combinatorial description of the h-polynomial, but no compact determinantal expression for it is known in general. However, for X a symmetric $n \times n$ matrix of indeterminates and $t+1 = n-1$, the expression of the h-polynomial of $K[X]/I_{t+1}(X)$ is easily obtained to be

$$\binom{2}{2} + \binom{3}{2}z + \cdots + \binom{n}{2}z^{n-2}, \tag{3}$$

see, for example, [11, Example 2.3(c)].

As in Remark 3.3, an $\mathbb{N}$-grading on a ring A is *semistandard* if A is a finitely generated algebra over a field $K := A_0$, and A is integral over the K-subalgebra generated by its elements of degree one. This condition ensures that the Hilbert series of A may be written as a rational function

$$\frac{h_0 + h_1 z + h_2 z^2 + \cdots + h_k z^k}{(1-z)^{\dim A}}, \quad \text{where } h_i \in \mathbb{Z} \quad \text{and} \quad h_k \neq 0.$$

The coefficients of the numerator, i.e. of the h-polynomial, form the *h-vector* $(h_0, \dots, h_k)$ of the ring A. When A is Cohen–Macaulay, it is readily seen that each h_i is nonnegative; when A is Gorenstein, the h-vector is a palindrome, i.e. $h_i = h_{k-i}$ for each $0 \leq i \leq k$. In this case, the h-vector is said to be *unimodal* if

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor k/2 \rfloor}.$$

Unimodality results reflect interesting geometric and combinatorial properties; they figure prominently in Ehrhart theory. Following his proof of the Anand–Dumir–Gupta conjectures regarding the enumeration of magic squares [33, 35], Stanley asked if the h-vector of the corresponding affine semigroup ring is unimodal. This was indeed proven to be the case by Athanasiadis [2], see also [8]. While Mustaţă and Payne [29] have constructed examples of Gorenstein normal affine semigroup rings for which the h-vector is not unimodal, these are not standard graded, and the following remains unresolved:

Conjecture 4.1. *The h-vector of a standard graded Gorenstein domain is unimodal.*

This is due to Stanley [34, Conjecture 4(a)], see also [5, Conjecture 1], [6, Conjecture 5.1], [7, p. 36], and [21, Conjecture 1.5]. We show that invariant rings for the action of $\mathrm{SO}_t(K)$ yield examples of "naturally occurring" semistandard graded Gorenstein normal domains, for which the h-vector is not unimodal:

Theorem 4.2. *Consider a $2m \times (2m+2)$ matrix of indeterminates Y over a field K of characteristic other than two. Let $\widetilde{R}$ denote the K-subalgebra of $K[Y]$ generated by the entries of the product matrix $Y^{\mathrm{tr}}Y$ and the maximal minors of Y, where the generators are assigned degree 1 and degree m respectively. If $m \geq 2$, the h-vector of $\widetilde{R}$ is not unimodal.*

Proof. Viewing the subring $R := K[Y^{\mathrm{tr}}Y]$ as a symmetric determinantal ring and using the expression (3), one see that R has Hilbert series

$$\frac{\binom{2}{2} + \binom{3}{2}z + \cdots + \binom{2m+2}{2}z^{2m}}{(1-z)^{2m^2+5m}}.$$

The ring R is not Gorenstein; the Hilbert series of R yields that of ω_R, from which it follows that the cyclic cover $\widetilde{R}$ has Hilbert series

$$\frac{\left[\binom{2}{2} + \binom{3}{2}z + \cdots + \binom{2m+2}{2}z^{2m}\right] + \left[\binom{2m+2}{2}z^{m} + \binom{2m+1}{2}z^{m+1} + \cdots + \binom{2}{2}z^{3m}\right]}{(1-z)^{2m^2+5m}}.$$

Hence

$$h_m - h_{m+1} \;=\; \left[\binom{m+2}{2} + \binom{2m+2}{2}\right] - \left[\binom{m+3}{2} + \binom{2m+1}{2}\right] \;=\; m-1,$$

so the h-vector of $\widetilde{R}$ is not unimodal; for a specific example, the case $m = 2$ yields the nonunimodal h-vector

$$(1,\ 3,\ 6,\ 10,\ 15,\ 0,\ 0) + (0,\ 0,\ 15,\ 10,\ 6,\ 3,\ 1) \;=\; (1,\ 3,\ 21,\ 20,\ 21,\ 3,\ 1).$$

$\square$

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