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Quantum isomorphism of graphs from association schemes

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Dedicated to the memory of Kai-Uwe Schmidt

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A B S T R A C T

We show that any two Hadamard graphs on the same number of vertices are quantum isomorphic. This follows from a more general recipe for showing quantum isomorphism of graphs arising from certain association schemes. The main result is built from three tools. A remarkable recent result [20] of Mančinska and Roberson shows that graphs G and H are quantum isomorphic if and only if, for any planar graph F, the number of graph homomorphisms from F to G is equal to the number of graph homomorphisms from F to H. A generalization of partition functions called "scaffolds" [23] affords some basic reduction rules such as series-parallel reduction and can be applied to counting homomorphisms. The final tool is the classical theorem of Epifanov showing that any plane graph can be reduced to a single vertex and no edges by extended series-parallel reductions and Delta-Wye transformations. This last sort of transformation is available to us in the case of exactly triply regular association schemes. The paper includes open problems and directions for future research.

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1. Introduction

The pioneering work of Bell [5] showing that no hidden variable theory can fully explain quantum mechanical correlations of spatially separated entangled particles has been experimentally verified; related experimental work of Aspect, Clauser and Zeilinger earned the Nobel Prize in Physics in 2022. Over recent decades, Bell's ideas have been refined and generalized in many directions, with one application being the study of quantum games. In one of the most basic quantum games, two physically separated players who share both a strategy and a quantum state, but are otherwise unable to communicate, are fed classical questions from a referee and respond with classical answers. Bell-type inequalities identify a gap between the winning probability of the optimal classical strategy and the corresponding probability in this quantum version. A particularly attractive class of quantum games are the graph homomorphism games [19]. These games are intuitive, cast in the familiar language of computer science protocols. They generate new lines of investigation in graph theory; for example, the quantum chromatic number of a graph is sandwiched between the classical chromatic number and the vector chromatic number. And, perhaps surprisingly, the natural definition of the quantum analogue of the automorphism group of a graph happens to be a compact quantum group.

This notion of quantum isomorphism of graphs was introduced by Atserias et al. [1] in their study of a non-local graph isomorphism game with two quantum players. In that paper, the first known construction is given for non-isomorphic graphs that are quantum isomorphic. Further examples were discovered by S. Schmidt [27] using Godsil-McKay switching; Schmidt constructs a family of pairs of strongly regular graphs on 120 vertices that are quantum isomorphic but pairwise non-isomorphic. (Schmidt also lists several references proposing alternative approaches, but these have led to no new examples so far.) In this paper, we give a different strategy of finding quantum isomorphic but non-isomorphic graphs using association schemes and scaffolds. We show how this approach implies that any two Hadamard graphs on the same number of vertices are quantum isomorphic. The feature we exploit is exact triple regularity, a property closely tied to the study of spin models [15].

Two graphs G and H, with adjacency matrices A_G and A_H respectively, are isomorphic if and only if there exists a permutation matrix P such that

$$PA_G = A_H P.$$

Lovász's classical result states that two graphs are isomorphic if and only if they have the same number of graph homomorphisms from any graph [17].

Let $\mathcal{A}$ be a C^*-algebra with unity $\mathbf{1}$ and let $\mathcal{U} = (u_{ij})$ be an $n \times n$ matrix with entries in $\mathcal{A}$. We call $\mathcal{U}$ a *quantum permutation matrix* if it satisfies $u_{ij}^2 = u_{ij}^* = u_{ij}$ and $\sum_{k=1}^{n} u_{ik}u_{jk} = \delta_{i,j}\mathbf{1} = \sum_{k=1}^{n} u_{ki}u_{kj}$ for all $1 \le i, j \le n$. See, for example, [20,30,31] for much more on quantum permutation matrices. Two graphs, G and H, are *quantum* *isomorphic* if and only if there exists a quantum permutation matrix $\mathcal{U}$ with entries in some C^*-algebra $\mathcal{A}$ such that

$$\mathcal{U}A_G = A_H\mathcal{U}$$

where operations are performed in $\mathcal{A}$. Note that when $\mathcal{A} = \mathbb{C}, \mathcal{U}$ is a permutation matrix and G is isomorphic to H. See [1, p323] for a pair of non-isomorphic graphs on 24 vertices that are quantum isomorphic. Only two families of examples are known of quantum isomorphic but non-isomorphic graphs; the graphs in the first family [1] are constructed based on a reduction from linear binary constraint system games to isomorphism games, and the graphs in the second family [27] are constructed via Godsil-McKay switching on two particular strongly regular graphs with parameters $(120, 63, 30, 36)$.

Given graphs F and G, we use $\mathsf{hom}(F, G)$ to denote the number of graph homomorphisms from F to G. Mančinska and Roberson give the following remarkable characterization of quantum isomorphic graphs.

Theorem 1.1 *([20]). Two graphs, G and H, are quantum isomorphic if and only if*

$$\mathsf{hom}(F, G) = \mathsf{hom}(F, H)$$

for any planar graph F.

Earlier, Atserias et al. [1] showed that it is undecidable to determine if two graphs are quantum isomorphic. Putting these together, given any graphs G and H, the problem of determining if there exists a planar F such that $\mathsf{hom}(F, G) \neq \mathsf{hom}(F, H)$ is undecidable.

For $\varphi : V(F) \to V(G)$, we have

$$\prod_{\{a,b\}\in E(F)} (A_G)_{\varphi(a),\varphi(b)} = \begin{cases} 1 & \text{if } \varphi \text{ is a graph homomorphism from } F \text{ to } G, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\mathsf{hom}(F, G) = \sum_{\varphi:V(F)\to V(G)} \prod_{\{a,b\}\in E(F)} (A_G)_{\varphi(a),\varphi(b)}, \tag{1.1}$$

which is the scaffold $\mathsf{S}(F, \emptyset; w)$ on the graph F with no root node and a weight function w that maps every edge of F to the matrix A_G. Please see Section 2 for some background on scaffolds and association schemes.

In Section 3, we consider the case where A_G belongs to the Bose-Mesner algebra of an exactly triply regular association scheme. We apply Epifanov's theorem to express the scaffold $\mathsf{S}(F, \emptyset; w)$ on any connected planar graph F in terms of the Delta-Wye parameters of the association scheme. This observation leads to our main result.

Theorem 3.3. *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be exactly triply regular symmetric association schemes that have the same Delta-Wye parameters (Definition 2.6). Let G' be a graph in $(Y, \mathcal{S})$ corresponding to*[1] *G in $(X, \mathcal{R})$. Then G and G' are quantum isomorphic.*

We focus on Hadamard graphs in Section 4. In the construction of spin models from Hadamard graphs [26], Nomura computes the Delta-Wye parameters of the association scheme of a Hadamard graph and shows that these parameters depend only on the number of vertices. Hence, the association schemes of two Hadamard graphs of the same order have the same Delta-Wye parameters. From [15] and [26], we see that the association scheme of a Hadamard matrix is exactly triply regular which leads to the following result, which has also been obtained independently by Gromada [13] via entirely different means.

Theorem 4.3. *Any two Hadamard graphs of the same order are quantum isomorphic.*

To our knowledge, the Hadamard graphs of the same order (≥ 64) are the first examples of three or more mutually quantum isomorphic but not isomorphic graphs. Merchant proved [25] that, provided at least one Hadamard matrix of order n exists, there are at least $2^{2n-16-6\log n}$ inequivalent Hadamard matrices of order $2n$. And McKay [24] proved that, given two Hadamard matrices H and H', if H' is inequivalent to both H and $H^\top$, then H and H' produce non-isomorphic Hadamard graphs. The longstanding Hadamard conjecture, claiming that there exists a Hadamard matrix of order $4m$ for each positive integer m would then imply that there are an exponential number (in m) non-isomorphic Hadamard graphs on $32m$ vertices.

Paley's construction of Hadamard matrix of order 28 gives a vertex transitive Hadamard graph. Hence this graph is quantum vertex transitive and its quantum orbital coherent configuration is homogeneous [18, Corollary 3.8 and Theorem 3.10]. By Theorem 4.3 and Theorem 4.6 of [18], any Hadamard graph of order 112 has a homogeneous quantum orbital coherent configuration and is therefore quantum vertex transitive. In particular, the Hadamard graph constructed from Had.28.101 of http://neilsloane.com/hadamard/ has a cyclic automorphism group, $\mathbb{Z}_2$, which gives an affirmative answer to Problem 3.10 of [31]. Any Hadamard matrix has at least two automorphisms (I, I) and $(-I, -I)$, so the automorphism group of a Hadamard graph is non-trivial [24]. The problem of finding an asymmetric graph with a non-trivial quantum automorphism group remains open [31, Problem 3.9].

We discuss additional open problems and directions for future research in Section 5.

[1] See Definition 2.7 below.

2. Association schemes and scaffolds

A symmetric association scheme is a partition of the edges of a complete graph into regular graphs whose adjacency matrices span a vector space closed and commutative under multiplication (a "Bose-Mesner algebra"). The regularity imposed by the definition, and by extra assumptions such as triple regularity, facilitate counting of pairs, triples and m-tuples of vertices forming prescribed configurations. While the idea has been used informally in the community for decades, the concept of a "scaffold" was recently introduced in [23] to treat these counts of m-tuples algebraically. A simple example of such a formal sum of configuration counts is the degree sequence, $A_G\mathbf{1}$, of a graph G. (Here, $\mathbf{1}$ denotes the all ones vector of appropriate length.) Scaffolds allow us to take linear combinations of m-vertex counts and to apply local change-of-basis operations on such configurations. In this section, we introduce these concepts, and set up notation and terminology that we will use as we work with exactly triply regular association schemes later.

2.1. Definitions and our notation

A d-class *symmetric association scheme* [9,4,6,12,8,22] is an ordered pair $(X, \mathcal{R})$ where X is a nonempty finite set and $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a partition of $X \times X$ into non-empty relations satisfying

- $R_0 = \{(x, x) \mid x \in X\}$ is the identity relation;
- for each i, $0 \le i \le d$, we have $R_i^\top = R_i$ where $R_i^\top = \{(y, x) \mid (x, y) \in R_i\}$;
- there exist *intersection numbers* p_{ij}^k, $0 \le i, j, k \le d$ satisfying

$$|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| = p_{ij}^k$$

  whenever $(x, y) \in R_k$.

Note that, since all relations are symmetric, we have $p_{ij}^k = p_{ji}^k$ for all i, j, k; all symmetric association schemes are *commutative*.

For $x, y \in X$ and $0 \le i \le d$, we write $x \overset{i}{\sim} y$ to mean $(x, y) \in R_i$.

Denote by $\mathsf{Mat}_X(\mathbb{C})$ the algebra of all matrices with rows and columns indexed by the set X having complex entries. We define *adjacency matrices* (or *Schur idempotents*[2]) of the association scheme, $A_0, \ldots, A_d \in \mathsf{Mat}_X(\mathbb{C})$ by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } x \overset{i}{\sim} y, \\ 0 & \text{otherwise.} \end{cases}$$

[2] Note that the sum of any subset of these is also an idempotent under the Schur product; these $d + 1$ matrices are *minimal Schur idempotents*.

These satisfy

- $\sum_{i=0}^{d} A_i = J$, the all ones matrix
- $A_i \circ A_j = \delta_{i,j} A_i$ where $\circ$ is the entrywise or *Hadamard/Schur* product;
- $A_0 = I$;
- for each i, $0 \le i \le d$, $A_i^\top = A_i$;
- $\mathbb{A} = \mathrm{span}_{\mathbb{R}}(\{A_0, \ldots, A_d\})$ is closed and commutative under matrix multiplication: there exist p_{ij}^k, $0 \le i, j, k \le d$ satisfying $A_i A_j = A_j A_i = \sum_{k=0}^{d} p_{ij}^k A_k$.

We call $\mathbb{A}$ the *Bose-Mesner algebra* of the association scheme $(X, \mathcal{R})$.

Up to a choice of ordering of relations and ordering of vertices, the correspondence between Bose-Mesner algebras and association schemes is immediate and we often work with the adjacency matrix A_i in place of the graph (X, R_i).

The vector space $\mathbb{A} = \mathrm{span}_{\mathbb{R}}\{A_0, \ldots, A_d\}$ has a second basis of *primitive (matrix) idempotents* $\{E_0, \ldots, E_d\}$: $\sum_{j=0}^{d} E_j = I$, $E_i E_j = \delta_{ij} E_i$, $E_i = E_i^\top$. The eigenvalues P_{ji} and the dual eigenvalues Q_{ji} of the association scheme satisfy

$$A_i = \sum_{j=0}^{d} P_{ji} E_j \quad \text{and} \quad E_i = \frac{1}{|X|} \sum_{j=0}^{d} Q_{ji} A_j. \tag{2.1}$$

Since $\mathbb{A}$ is closed under entrywise multiplication, there exist *Krein parameters* q_{ij}^k, $0 \le i, j, k \le d$ satisfying $E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k$. (See [6, Section 2.2-2.3].)

2.2. Scaffolds

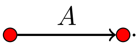
Let $(X, \mathcal{R})$ be an association scheme with Bose-Mesner algebra $\mathbb{A}$. These matrices act, in the obvious way, on the *standard module* $V = \mathbb{C}^X$ of all complex-valued functions on X with standard basis of column vectors $\{\hat{x} \mid x \in X\}$. This space is equipped with the corresponding positive definite Hermitian inner product $\langle v, w \rangle = v^\dagger w$ (where $\cdot^\dagger$ denotes conjugate transpose) satisfying $\langle \hat{x}, \hat{y} \rangle = \delta_{x,y}$ for $x, y \in X$. We identify V with its dual space $V^\dagger$ of linear functionals and view matrices in $\mathsf{Mat}_X(\mathbb{C})$ as second order tensors. More generally, we will presently define a scaffold with m roots (or m^{th} order scaffold) as a certain type of tensor belonging to

$$V^{\otimes m} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{m}$$

with standard basis consisting of simple tensors of the form $\hat{x}_1 \otimes \hat{x}_2 \otimes \cdots \otimes \hat{x}_m$ where $x_1, x_2, \ldots, x_m \in X$.

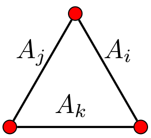
For a graph $F = (V(F), E(F))$, an ordered set $R = \{r_1, \ldots, r_m\}$ of nodes in F called *roots*, and a function $w : E(F) \to \mathsf{Mat}_X(\mathbb{C})$, we define

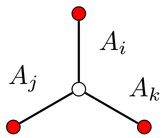
$$\mathsf{S}(F,R;w) = \sum_{\varphi:V(F)\to X} \left(\prod_{\substack{e\in E(F)\\ e=\{a,b\}}} w(e)_{\varphi(a),\varphi(b)} \right) \widehat{\varphi(r_1)} \otimes \widehat{\varphi(r_2)} \otimes \cdots \otimes \widehat{\varphi(r_m)}. \quad (2.2)$$

We call F the "diagram" of the scaffold $\mathsf{S}(F,R;w)$, use red solid nodes to depict the roots, and label each edge e with the matrix $w(e)$. We identify the scaffold $\mathsf{S}(F,R;w)$ with this pictorial representation of its data, being careful to consistently order the roots by spacial placement when two scaffolds appear in the same equation. For instance, the matrix $A = [A_{xy}] \in \mathsf{Mat}_X(\mathbb{C})$, viewed as the second-order tensor $\sum_{x,y\in X} A_{xy}\ \hat{x}\otimes\hat{y}$, is denoted by 

. In this paper, all examples are symmetric matrices. So we will omit the directions on edges of the diagrams.[3]

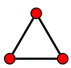
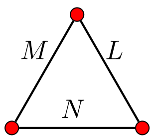
When studying an association scheme $(X,\mathcal{R})$ with adjacency matrices A_i, two families of third order scaffolds of fundamental importance [28,15] are

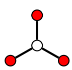
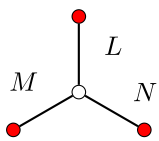


$$= \sum_{\substack{x,y,z\in X\\ x\overset{i}{\sim}z,\, x\overset{j}{\sim}y,\, y\overset{k}{\sim}z}} \hat{x}\otimes\hat{y}\otimes\hat{z}\ ,$$


$$= \sum_{\substack{x,y,z,u\in X\\ x\overset{i}{\sim}u,\, y\overset{j}{\sim}u,\, z\overset{k}{\sim}u}} \hat{x}\otimes\hat{y}\otimes\hat{z}\ . \quad (2.3)$$

We next consider the vector space spanned by all scaffolds with a given diagram and edge weights in $\mathbb{A}$ [23, Section 3.2]. In particular, define

- $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \operatorname{span}\Big\{$ 

$\Big|\ L,M,N\in\mathbb{A}\Big\}$,
- $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \operatorname{span}\Big\{$ 

$\Big|\ L,M,N\in\mathbb{A}\Big\}$.

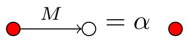
As explained in our introduction, scaffolds help us count homomorphisms.

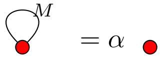
Lemma 2.1. *Let F be a graph, let G be a graph on vertex set X with adjacency matrix A. Define $w(e) = A$ for all edges e of F. Then $\mathsf{S}(F,\emptyset;w) = \mathsf{hom}(F,G)$, the number of graph homomorphisms from F to G.*

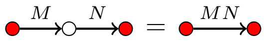
Proof. This lemma is an immediate consequence of (1.1): a function φ from $V(F)$ to $V(G)$ is a homomorphism if and only if $\prod_{\{a,b\}\in E(F)} A_{\varphi(a),\varphi(b)} = 1$. □

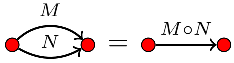
[3] In fact, our definition of a scaffold here is specialized to undirected graphs.

We will perform local operations on scaffolds that preserve their value, using Proposition 1.5 in [23]. These include loop removal, removal of a non-root vertex of degree one, and series and parallel reduction. One may use the definition to directly verify the following rules:

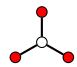
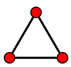
a: If M has constant row sum α, then 

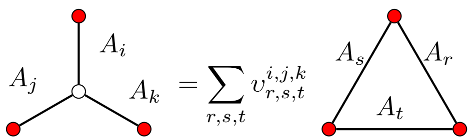
b: If M has constant diagonal α, then 

c: For $M, N \in \mathsf{Mat}_X(\mathbb{C})$, 

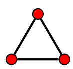
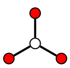
d: For $M, N \in \mathsf{Mat}_X(\mathbb{C})$, 

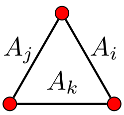
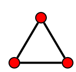
2.3. Extra regularity

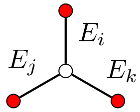
An association scheme $(X, \mathcal{R})$ with Bose-Mesner algebra $\mathbb{A}$ is *triply regular* if, for all $x, y, z \in X$ and all $0 \leq i, j, k \leq d$, $\upsilon(x,y,z) := \left| \left\{ u \in X \ : \ x \overset{i}{\sim} u, y \overset{j}{\sim} u, z \overset{k}{\sim} u \right\} \right|$ depends only on i, j, k and the three relations joining x, y, z and not on the choice of x, y, z themselves. Jaeger proved that $(X, \mathcal{R})$ is triply regular if and only if $\mathsf{W}\Big($  $; \ \mathbb{A}\Big) \subseteq \mathsf{W}\Big($  $; \ \mathbb{A}\Big)$ [15, Proposition 7(ii)]. If $(X, \mathcal{R})$ is triply regular and we use $\upsilon^{i,j,k}_{r,s,t}$ to denote $\upsilon(x,y,z)$ when $x \overset{r}{\sim} z$, $x \overset{s}{\sim} y$ and $y \overset{t}{\sim} z$, then the scaffold equation

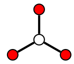


holds for all i, j, k.

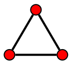
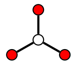
The association scheme $(X, \mathcal{R})$ is *dually triply regular* if $\mathsf{W}\Big($  $; \ \mathbb{A}\Big) \subseteq \mathsf{W}\Big($  $; \ \mathbb{A}\Big)$ [15, Proposition 8(ii)] and *exactly triply regular* if it is both triply regular and dually triply regular.

Theorem 2.2 *(Terwilliger [29] (see [23, Theorem 3.8])). Let $(X, \mathcal{R})$ be a symmetric association scheme with minimal Schur idempotents $A_0, \ldots, A_d$, primitive (matrix) idempotents $E_0, \ldots, E_d$, intersection numbers p^k_{ij} and Krein parameters q^k_{ij} $(0 \leq i, j, k \leq d)$. The set* $\left\{\right.$  $\left| \ p^k_{ij} > 0 \right\}$ *is an orthogonal basis for* $\mathsf{W}\Big($  $; \ \mathbb{A}\Big)$ *and*

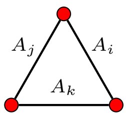
$\Bigg\{$ 

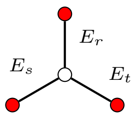
$\Big|\ q_{ij}^k > 0 \Bigg\}$ *is an orthogonal basis for* $\mathsf{W}\Big($  $;\ \mathbb{A}\Big)$. □

Let us denote by N_p the number of ordered triples (i, j, k) with $p_{ij}^k > 0$ and by N_q the number of ordered triples (i, j, k) with $q_{ij}^k > 0$. The following lemma follows from Jaeger's propositions.

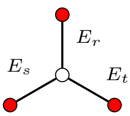
Lemma 2.3. *If* $(X, \mathcal{R})$ *is an exactly triply regular association scheme with Bose-Mesner algebra* $\mathbb{A}$*, then* $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ *. If* $N_p = N_q$ *and* $(X, \mathcal{R})$ *is either triply regular or dually triply regular, then* $(X, \mathcal{R})$ *is exactly triply regular.* □

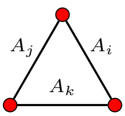
Definition 2.4. Let $(X, \mathcal{R})$ be an exactly triply regular d-class association scheme with Bose-Mesner algebra having an ordered basis $A_0, \ldots, A_d$ of adjacency matrices and an ordered basis $E_0, \ldots, E_d$ of primitive idempotents. The *Delta-Wye parameters* of $(X, \mathcal{R})$ are those $\left\{\sigma_{r,s,t}^{i,j,k} \middle| p_{ij}^k > 0,\ q_{rs}^t > 0\right\}$ and $\left\{\tau_{i,j,k}^{r,s,t} \middle| p_{ij}^k > 0,\ q_{rs}^t > 0\right\}$ satisfying the equations



$$= \sum_{q_{rs}^t > 0} \sigma_{r,s,t}^{i,j,k} \tag{2.4}$$


and



$$= \sum_{p_{ij}^k > 0} \tau_{i,j,k}^{r,s,t} \tag{2.5}$$


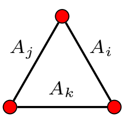
.

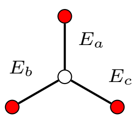
Remark 2.5. The two sets of coefficients are mutual inverses:

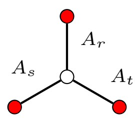
$$\sum_{q_{rs}^t > 0} \sigma_{r,s,t}^{i,j,k} \tau_{i',j',k'}^{r,s,t} = \delta_{i,i'} \delta_{j,j'} \delta_{k,k'}, \qquad \sum_{p_{ij}^k > 0} \tau_{i,j,k}^{r,s,t} \sigma_{r',s',t'}^{i,j,k} = \delta_{r,r'} \delta_{s,s'} \delta_{t,t'} .$$

So if one knows all $\sigma_{r,s,t}^{i,j,k}$, one may derive from these the parameters $\tau_{i,j,k}^{r,s,t}$ and conversely.

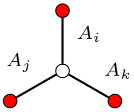
Applying (2.1) to Definition 2.4 gives the following equations

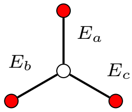


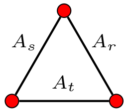
$$= \sum_{q_{ab}^c > 0} \sigma_{a,b,c}^{i,j,k}$$


$$= \frac{1}{|X|^3} \sum_{q_{ab}^c > 0} \sigma_{a,b,c}^{i,j,k} \sum_{r,s,t} Q_{ra} Q_{sb} Q_{tc} \tag{2.6}$$


and

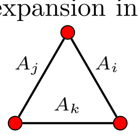


$$= \sum_{a,b,c} P_{ai} P_{bj} P_{ck}$$


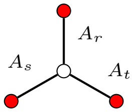
$$= \sum_{a,b,c} P_{ai} P_{bj} P_{ck} \sum_{p^t_{rs}>0} \tau^{a,b,c}_{r,s,t}$$


(2.7)

which we will use in the proof of Theorem 3.2. Note that, while the expansion in (2.7) is unique by Theorem 2.2, there are typically many ways to express



as a linear combination of the $(d+1)^3$ non-zero tensors



; it is important that we consistently use expression (2.6) in our proof.

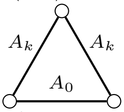
Definition 2.6. Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be exactly triply regular d-class association schemes. We say $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ *have the same Delta-Wye parameters* if there exist orderings $A_0, A_1, \ldots, A_d$ and $A'_0, A'_1, \ldots, A'_d$ of their respective adjacency matrices, and there exist orderings $E_0, E_1, \ldots, E_d$ and $E'_0, E'_1, \ldots, E'_d$ of their respective primitive idempotents such that every Delta-Wye parameter $\sigma^{i,j,k}_{r,s,t}$ for $(X, \mathcal{R})$ is equal to the corresponding Delta-Wye parameter for $(Y, \mathcal{S})$.

Definition 2.7. Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be exactly triply regular d-class association schemes with the same Delta-Wye parameters with respect to orderings $A_0, A_1, \ldots, A_d$ and $A'_0, A'_1, \ldots, A'_d$ of their respective adjacency matrices, and orderings $E_0, E_1, \ldots, E_d$ and $E'_0, E'_1, \ldots, E'_d$ of their respective primitive idempotents. The bijection $A_i \mapsto A'_i$ extends linearly to a vector space isomorphism $\kappa : \mathbb{A} \to \mathbb{A}'$ carrying a matrix $M = \sum_{j=0}^d c_j A_j$ to the matrix $\kappa(M) = \sum_{j=0}^d c_j A'_j$ which we denote M'. (Note that $\kappa(E_j) = E'_j$ by Proposition 2.8.) We call M' the matrix in the Bose-Mesner algebra of $(Y, \mathcal{S})$ *corresponding to* M. In the special case where M is the adjacency matrix of a graph G on vertex set X, the matrix M' is the adjacency matrix of some graph G' on vertex set Y; we call G' the graph in $(Y, \mathcal{S})$ *corresponding to* G.

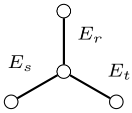
We do not know whether two exactly triply regular association schemes with the same intersection numbers must have the same Delta-Wye parameters. In Section 5, we ask if these parameters are functions of the p^k_{ij}'s.

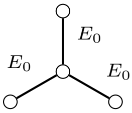
Proposition 2.8. *If $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ are exactly triply regular association schemes having the same Delta-Wye parameters, then they also have the same eigenvalues, dual eigenvalues, intersection numbers and Krein parameters under the appropriate consistent orderings of relations and primitive idempotents. The linear map κ in Definition 2.7 is a Bose-Mesner isomorphism: $\kappa(MN) = \kappa(M)\kappa(N)$ and $\kappa(M \circ N) = \kappa(M) \circ \kappa(N)$.*

Proof. We first show that all intersection numbers of an association scheme can be computed from its Delta-Wye parameters. From these we can obtain the eigenvalues, dual eigenvalues and Krein parameters (see [6, pp. 46,49]).

We compare the two sides of Equation (2.4) with all nodes made hollow (so that the tensor is just a scalar). By definition, 

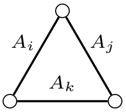
$= \sum_{x\in X}\sum_{y\in X}(A_k)_{xy} = |X| p^0_{kk}$ and since E_r has row sum zero for $r \neq 0$ and E_0 has row sum one, we use Rule a to find

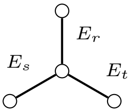
$$\sum_{q^t_{rs}>0} \sigma^{k,k,0}_{r,s,t}$$


$$= \sigma^{k,k,0}_{0,0,0}$$


$$= \sigma^{k,k,0}_{0,0,0} \frac{1}{|X|^3} \sum_{x,y,z,u\in X} 1 = |X| \sigma^{k,k,0}_{0,0,0}$$

showing $p^0_{kk} = \sigma^{k,k,0}_{0,0,0}$. By the same considerations, we also have

$$|X| p^0_{kk}\, p^k_{ij} =$$


$$= \sum_{q^t_{rs}>0} \sigma^{j,i,k}_{r,s,t}$$


$$= |X| \sigma^{j,i,k}_{0,0,0}$$

allowing us to compute p^k_{ij} from the Delta-Wye parameters.

Since $\kappa(A_i) = A'_i$ is a 01-matrix and both $A_i \circ A_j = \delta_{ij} A_i$ and $A'_i \circ A'_j = \delta_{ij} A'_i$ hold for this pair of bases, we have $\kappa(M \circ N) = \kappa(M) \circ \kappa(N)$ by linearity. Finally,

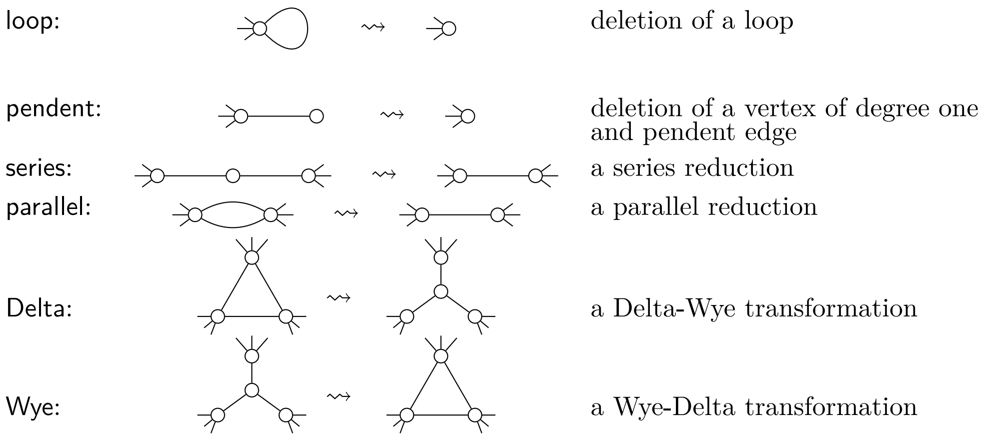
$$\kappa(A_i A_j) = \kappa\left(\sum_{k=0}^{d} p^k_{ij} A_k\right) = \sum_{k=0}^{d} p^k_{ij} A'_k = A'_i A'_j = \kappa(A_i)\kappa(A_j). \quad \square$$

3. Homomorphism counting

Given a planar graph F and a graph G, let w_G be the weight function mapping each edge of F to the adjacency matrix A_G. Using Lemma 2.1, we compute the scaffold $\mathsf{S}(F, \emptyset; w_G)$ to count the number of graph homomorphisms from F to G. Simplification of such counts is achieved using Epifanov's Theorem on plane graphs.

3.1. Epifanov's theorem

A *plane graph* [10, p83] is an embedding of a planar graph and we do not differentiate between embeddings equivalent under ambient isotopy. We allow the following local operations on plane graphs. Each modifies an embedded graph only within a closed disk with the understanding that this disk contains no part of the embedding other than what is shown.



Theorem 3.1 *(Epifanov (see [11] and [15, Proposition 5])). Let F be any connected plane graph. Then there exists a sequence of plane graphs $F_0, F_1, \ldots, F_\ell$ with the following properties.*

(i) *$F_0 = F$ and F_ℓ is a graph with one vertex and no edges*

(ii) *up to ambient isotopy, F_{h+1} is obtained from F_h by just one of the above local transformations (loop, pendent, series, parallel, Delta, or Wye), for $0 \leq h < \ell$.* □

3.2. A technical theorem and the main result

In this section, we assume $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ are exactly triply regular association schemes having the same Delta-Wye parameters with respect to orderings $A_0, \ldots, A_d$ and $A'_0, \ldots, A'_d$ of their respective adjacency matrices, and orderings $E_0, E_1, \ldots, E_d$ and $E'_0, E'_1, \ldots, E'_d$ of their respective primitive idempotents. We use $\mathbb{A}$ and $\mathbb{A}'$ to denote the Bose-Mesner algebras of $(X, \mathcal{R})$ and $(Y, \mathcal{S})$, respectively. To avoid confusion, we denote the composition of functions using the symbol $\bullet$.

Theorem 3.2. *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be exactly triply regular association schemes having the same Delta-Wye parameters and let $\kappa : \mathbb{A} \to \mathbb{A}'$ be as given in Proposition 2.8. Let F be a connected planar graph (possibly with loops and multiple edges) and consider any edge weights $w : E(F) \to \mathbb{A}$. Then $\mathsf{S}(F, \emptyset; w) = \mathsf{S}(F, \emptyset; \kappa \bullet w)$.*

Proof. Write $w' = \kappa \bullet w$ so that $w' : E(F) \to \mathbb{A}'$. First choose an embedding and view F as a plane graph. Let $F_0, F_1, \ldots, F_\ell$ be a sequence of plane graphs satisfying the conditions in Theorem 3.1.

Since $\mathbb{A} = \text{span}\{A_0, A_1, \ldots, A_d\}$ and $\mathbb{A}' = \text{span}\{A'_0, A'_1, \ldots, A'_d\}$, it is sufficient to show inductively that, for $h = 0, 1, \ldots, \ell$, there exist m_h, $\alpha_{h,m}$ and weight functions

$$w_{h,m} : E(F_h) \to \{A_0, A_1, \ldots, A_d\}$$

such that

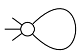
$$\mathsf{S}(F,\emptyset;w)=\sum_{m=1}^{m_h}\alpha_{h,m}\,\mathsf{S}(F_h,\emptyset;w_{h,m})\quad\text{and}\quad\mathsf{S}(F,\emptyset;w')=\sum_{m=1}^{m_h}\alpha_{h,m}\,\mathsf{S}(F_h,\emptyset;\kappa\bullet w_{h,m})\ . \tag{3.1}$$

Note that, since F_ℓ has no edges, $\mathsf{S}(F_\ell,\emptyset;\omega)=|X|$ for $\omega:\emptyset\to\mathsf{Mat}_X(\mathbb{C})$ and the result follows:

$$\mathsf{S}(F,\emptyset;w)=|X|\sum_{m=1}^{m_\ell}\alpha_{\ell,m}=|Y|\sum_{m=1}^{m_\ell}\alpha_{\ell,m}=\mathsf{S}(F,\emptyset;w').$$

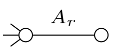
When $h=0$, Equation 3.1 follows because $w(e)\in\mathbb{A}$, for each $e\in E(F)$, and $w'=\kappa\bullet w$. For $h>0$, we show Equation 3.1 holds for appropriate $\alpha_{h,m}$ and $w_{h,m}$ by considering each type of local transformation occurring in Theorem 3.1 and its effect on the associated scaffolds. We include the edge weights in the following scaffolds to highlight the change of the weight functions due to the local transformations.

loop: F_{h+1} is obtained from F_h by deleting a loop e. Set $m_{h+1}=m_h$, consider $1\le m\le m_h$, and suppose $w_{h,m}(e)=A_r$. Define $w_{h+1,m}:E(F_{h+1})\to\mathbb{A}$ to be the restriction of $w_{h,m}$ to $E(F_h)\backslash\{e\}$ so that

$\mathsf{S}($  $A_r,\emptyset;w_{h,m})=\delta_{0,r}\,\mathsf{S}($  $,\emptyset;w_{h+1,m}),$ (3.2)

by Rule b in Section 2.2 above and the same equation holds after replacing $w_{h,m}$ and $w_{h+1,m}$ with $\kappa\bullet w_{h,m}$ and $\kappa\bullet w_{h+1,m}$ respectively. Summing over m with coefficients $\alpha_{h+1,m}=\delta_{0,r}\alpha_{h,m}$ yields Equations (3.1) with h replaced by $h+1$.

pendent: F_{h+1} is obtained from F_h by deletion of a degree one vertex and the sole incident edge e. Let $m_{h+1}=m_h$ and, for each $1\le m\le m_h$, setting $A_r=w_{h,m}(e)$, define $w_{h+1,m}:E(F_{h+1})\to\mathbb{A}$ to be the restriction of $w_{h,m}$ to $E(F_h)\backslash\{e\}$. Then we have

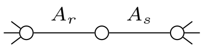
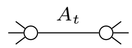
$\mathsf{S}($  $,\emptyset;w_{h,m})=p^0_{r,r}\,\mathsf{S}($  $,\emptyset;w_{h+1,m})$ (3.3)

by Rule a in Section 2.2 above. Since $A_r'=\kappa(A_r)$ has the same row sum as A_r, the same equation holds when $w_{h,m}$ and $w_{h+1,m}$ are replaced by $\kappa\bullet w_{h,m}$ and $\kappa\bullet w_{h+1,m}$, respectively. Choosing coefficients $\alpha_{h+1,m}=p^0_{r,r}\alpha_{h,m}$ for $1\le m\le m_h$ and summing gives the induction step in this case.

series: F_{h+1} is obtained from F_h by contraction of an edge e_1 in series with edge e_2 (their common endpoint being incident to no other edges). We have $E(F_{h+1})=E(F_h)\backslash\{e_1\}$. Let $m_{h+1}=(d+1)m_h$. Re-indexing to keep things simple define, for $1\le m\le m_h$ and $0\le t\le d$,

$$w^t_{h+1,m}(e) = \begin{cases} A_t & \text{if } e = e_2, \\ w_{h,m}(e) & \text{otherwise} \end{cases}$$

and $\alpha^t_{h+1,m} = p^t_{rs}\alpha_{h,m}$. Then, applying Rule c, we have the equation

$\mathsf{S}($  $, \emptyset; w_{h,j}) = \sum_{t=0}^{d} p^t_{r,s}\ \mathsf{S}($  $, \emptyset; w^t_{h+1,j}).$ (3.4)

Summing over $1 \le m \le m_h$ gives us

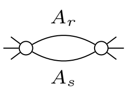
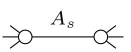
$$\sum_{m=1}^{m_h} \alpha_{h,m}\ \mathsf{S}(F_h, \emptyset; w_{h,m}) = \sum_{m=1}^{m_h}\sum_{t=0}^{d} \alpha^t_{h+1,m}\ \mathsf{S}(F_{h+1}, \emptyset; w^t_{h+1,m})$$

and similarly for edge weights in $\mathbb{A}'$:

$$\sum_{m=1}^{m_h} \alpha_{h,m}\ \mathsf{S}(F_h, \emptyset; \kappa \bullet w_{h,m}) = \sum_{m=1}^{m_h}\sum_{t=0}^{d} \alpha^t_{h+1,m}\ \mathsf{S}(F_{h+1}, \emptyset; \kappa \bullet w^t_{h+1,m})$$

using Proposition 2.8.

parallel: F_{h+1} is obtained from F_h by deletion of an edge e_1 which is in parallel to edge e_2. Let $m_{h+1} = m_h$ and, for each $1 \le m \le m_h$, define $w_{h+1,m} : E(F_{h+1}) \to \mathbb{A}$ to be the restriction of $w_{h,m}$ to $E(F_h)\backslash\{e_1\}$ and, assuming $w_{h,m}(e_1) = A_r$ and $w_{h,m}(e_2) = A_s$, set $\alpha_{h+1,m} = \delta_{r,s}\alpha_{h,m}$. Then we have, using Rule d,

$\mathsf{S}($  $, \emptyset; w_{h,j}) = \delta_{r,s}\ \mathsf{S}($  $, \emptyset; w_{h+1,j})$ (3.5)

so that

$$\sum_{m=1}^{m_h} \alpha_{h,m}\ \mathsf{S}(F_h, \emptyset; w_{h,m}) = \sum_{m=1}^{m_h} \alpha_{h+1,m}\ \mathsf{S}(F_{h+1}, \emptyset; w_{h+1,m})$$

and the same holds with edge weights replaced by $\kappa \bullet w_{h,m}$ and $\kappa \bullet w_{h+1,m}$ by Proposition 2.8.

Delta: F_{h+1} is obtained from F_h by replacing a Delta (the edges of a triangle) with a Wye (a new vertex adjacent only to the three vertices of that triangle). Let us treat $E(F_{h+1})$ as equal to $E(F_h)$ with the understanding that the three edges e_1, e_2, e_3 of the triangle are now the three edges of the Wye, with the convention that e_u in the second graph is incident to neither end of e_u in the first.

Fix an m, $1 \le m \le m_h$ and set $A_i = w_{h,m}(e_1)$, $A_j = w_{h,m}(e_2)$, and $A_k = w_{h,m}(e_3)$. Equation (2.6) can be written

$$\sum_{x,y,z\in X} (A_i)_{x,z}(A_j)_{x,y}(A_k)_{y,z}\ \hat{x}\otimes\hat{y}\otimes\hat{z}$$
$$= \sum_{r,s,t=0}^{d} \rho^{i,j,k}_{r,s,t} \sum_{w,x,y,z\in X} (A_r)_{w,x}(A_s)_{w,y}(A_t)_{w,z}\ \hat{x}\otimes\hat{y}\otimes\hat{z} \tag{3.6}$$

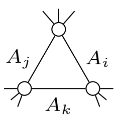
where $\rho^{i,j,k}_{r,s,t} = \frac{1}{|X|^3}\sum_{q^c_{ab}>0} \sigma^{i,j,k}_{a,b,c} Q_{ra}Q_{sb}Q_{tc}$ for all $0 \le i,j,k,r,s,t \le d$. Hence, for all $x,y,z \in X$,

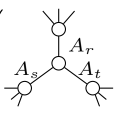
$$(A_i)_{x,z}(A_j)_{x,y}(A_k)_{y,z} = \sum_{r,s,t=0}^{d} \rho^{i,j,k}_{r,s,t} \sum_{w\in X} (A_r)_{w,x}(A_s)_{w,y}(A_t)_{w,z}. \tag{3.7}$$

As above, it will be simpler to allow several indices of summation on the right hand side of Equations (3.1). For each $0 \le r,s,t \le d$, define $w^{r,s,t}_{h+1,m} : E(F_{h+1}) \to \mathbb{A}$ via

$$w^{r,s,t}_{h+1,m}(e) = \begin{cases} A_s & \text{if } e = e_1 \\ A_t & \text{if } e = e_2 \\ A_r & \text{if } e = e_3 \\ w_{h,m}(e) & \text{otherwise.} \end{cases}$$

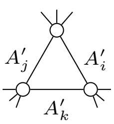
It follows from (3.7) (cf. [23, Proposition 1.5]) that

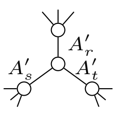
$\mathsf{S}\Bigg($ 

$,\ \emptyset\ ;\ w_{h,m}\Bigg) = \sum_{r,s,t=0}^{d} \rho^{i,j,k}_{r,s,t}\ \mathsf{S}\Bigg($ 

$,\ \emptyset\ ;\ w^{r,s,t}_{h+1,m}\Bigg).$ (3.8)

Using Proposition 2.8, we likewise obtain

$\mathsf{S}\Bigg($ 

$,\ \emptyset\ ;\ \kappa \bullet w_{h,m}\Bigg) = \sum_{r,s,t=0}^{d} \rho^{i,j,k}_{r,s,t}\ \mathsf{S}\Bigg($ 

$,\ \emptyset\ ;\ \kappa \bullet w^{r,s,t}_{h+1,m}\Bigg).$ (3.9)

Summing over m with coefficients $\alpha^{r,s,t}_{h+1,m} = \rho^{i,j,k}_{r,s,t}\alpha_{h,m}$, we obtain our induction step for the Delta-Wye transformation.

Wye: F_{h+1} is obtained from F_h by replacing a Wye with a Delta. We employ the same conventions as in the previous case.

Fix m and locate those i, j, k for which $A_i = w_{h,m}(e_1)$, $A_j = w_{h,m}(e_2)$, and $A_k = w_{h,m}(e_3)$.

With $\varrho^{i,j,k}_{r,s,t} = \sum_{a,b,c=0}^{d} P_{ai}P_{bj}P_{ck}\tau^{a,b,c}_{r,s,t}$, Equation (2.7) can be written

$$\sum_{w,x,y,z\in X} (A_i)_{w,x}(A_j)_{w,y}(A_k)_{w,z}\ \hat{x}\otimes\hat{y}\otimes\hat{z} = \sum_{p^t_{rs}>0} \varrho^{i,j,k}_{r,s,t}(A_r)_{x,z}(A_s)_{x,y}(A_t)_{y,z}\ \hat{x}\otimes\hat{y}\otimes\hat{z}. \tag{3.10}$$

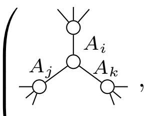
(In fact, $\varrho^{i,j,k}_{r,s,t} = \upsilon^{i,j,k}_{r,s,t}$ defined in Section 2.3.) Hence, for all $x, y, z \in X$,

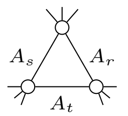
$$\sum_{w\in X} (A_i)_{w,x}(A_j)_{w,y}(A_k)_{w,z} = \sum_{p^t_{rs}>0} \varrho^{i,j,k}_{r,s,t}\ (A_r)_{x,z}(A_s)_{x,y}(A_t)_{y,z}. \tag{3.11}$$

For each $0 \le r, s, t \le d$, define $w^{r,s,t}_{h+1,m} : E(F_{h+1}) \to \mathbb{A}$ via

$$w^{r,s,t}_{h+1,m}(e) = \begin{cases} A_t & \text{if } e = e_1 \\ A_r & \text{if } e = e_2 \\ A_s & \text{if } e = e_3 \\ w_{h,m}(e) & \text{otherwise} \end{cases}$$

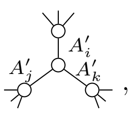
and everything goes through as in the previous case, giving

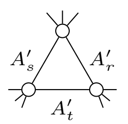
$$\mathsf{S}\left(\right.$$


$$\left.,\ \emptyset\ ;\ w_{h,m}\right) = \sum_{r,s,t=0}^{d} \varrho^{i,j,k}_{r,s,t}\ \mathsf{S}\left(\right.$$


$$\left.,\ \emptyset\ ;\ w^{r,s,t}_{h+1,m}\right) \tag{3.12}$$

and

$$\mathsf{S}\left(\right.$$


$$\left.,\ \emptyset\ ;\ \kappa\bullet w_{h,m}\right) = \sum_{r,s,t=0}^{d} \varrho^{i,j,k}_{r,s,t}\ \mathsf{S}\left(\right.$$


$$\left.,\ \emptyset\ ;\ \kappa\bullet w^{r,s,t}_{h+1,m}\right). \tag{3.13}$$

Summing over m and using $\alpha^{r,s,t}_{h+1,m} = \varrho^{i,j,k}_{r,s,t}\alpha_{h,m}$, we establish our induction step. □

We are ready to present our main theorem. By "a graph in" an association scheme $(X, \mathcal{R})$ we mean a graph with vertex set X whose adjacency relation is a union of non-identity basis relations from $\mathcal{R}$.

Theorem 3.3. *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be exactly triply regular symmetric association schemes that have the same Delta-Wye parameters. Let G' be a graph in $(Y, \mathcal{S})$ corresponding to G in $(X, \mathcal{R})$. Then G and G' are quantum isomorphic.*

Proof. By Proposition 2.8, the map κ given in Definition 2.7 is a Bose-Mesner isomorphism from $\mathbb{A}$ to $\mathbb{A}'$.

Given any planar graph F, we define functions w such that $w(e) = A_G$ for all $e \in E(F)$ and $\kappa \bullet w(e) = A_{G'}$ for all $e \in E(F)$. By Lemma 2.1 and Theorem 3.2, we have

$$\mathsf{hom}(F, G) = \mathsf{S}(F, \emptyset; w) = \mathsf{S}(F, \emptyset; \kappa \bullet w) = \mathsf{hom}(F, G').$$

By Theorem 1.1, we conclude that G and G' are quantum isomorphic. □

A weaker result holds for association schemes with the same parameters if we allow only the loop, pendent, series and parallel transformations.

Theorem 3.4. *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be symmetric d-class association schemes with the same parameters, say $p_{ij}^k = {p'}_{ij}^k$ $(0 \leq i, j, k \leq d)$ with respect to fixed orderings of the relations. Let G' be a graph in $(Y, \mathcal{S})$ corresponding to G in $(X, \mathcal{R})$. For any series-parallel graph F, $\mathsf{hom}(F, G) = \mathsf{hom}(F, G')$.* □

If one writes $G \mathrel{\mathring{=}} H$ when $\mathsf{hom}(F, G) = \mathsf{hom}(F, H)$ for all series-parallel graphs F, it is not clear if there is an independent characterization of this equivalence relation, along the lines of [20, Section 1.1]. (See also [21].) From known results, we can see that $G \mathrel{\mathring{=}} H$ implies that G and H are fractionally isomorphic (since all trees are series-parallel graphs) but does not imply that G and H are quantum isomorphic (as the Shrikhande graph and 4×4 grid illustrate).

4. Hadamard graphs

Hadamard graphs are very closely related to Hadamard matrices, which have received much attention, and are themselves a well-studied family of distance-regular graphs, see [6, Section 1.8]. Our goal in this section is to show that Theorem 3.3 applies to any pair of Hadamard graphs of the same order. In [13], Gromada proves the same result using diagrammatic calculus. Please see [2,3] for quantum symmetries of complex Hadamard matrices.

4.1. Quantum isomorphism

A Hadamard matrix is an $n \times n$ ± 1 matrix H satisfying

$$HH^\top = nI.$$

Given an $n \times n$ Hadamard matrix, we construct a graph G on vertex set

$$X = \{r_1^+, r_1^-, \ldots, r_n^+, r_n^-,\ c_1^+, c_1^-, \ldots, c_n^+, c_n^-\},$$

where r_i^+ is adjacent to c_j^+ and r_i^- is adjacent to c_j^- if $H_{ij} = 1$, r_i^+ is adjacent to c_j^- and r_i^- is adjacent to c_j^+ if $H_{ij} = -1$. This graph, called *a Hadamard graph of order* $4n$, is a distance-regular graph of diameter four with intersection array

$$\{n, n-1, \frac{n}{2}, 1;\ 1, \frac{n}{2}, n-1, n\}.$$

For $j = 0, \ldots, 4$, we use A_j to denote the j^{th} distance matrix of G. Then $A_0, \ldots, A_4$ are the adjacency matrices of a 4-class symmetric association scheme. The matrix of eigenvalues of this association scheme is

$$P = \begin{bmatrix} 1 & n & 2n-2 & n & 1 \\ 1 & \sqrt{n} & 0 & -\sqrt{n} & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -\sqrt{n} & 0 & \sqrt{n} & -1 \\ 1 & -n & 2n-2 & -n & 1 \end{bmatrix}.$$

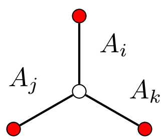
Since $P^2 = 4nI$, this association scheme is formally self-dual which means $P_{ij} = Q_{ij}$ and $p_{ij}^k = q_{ij}^k$, for $i, j, k = 0, \ldots, 4$. See [15, p138]. The intersection numbers are given in $L_i = [p_{ij}^k]_{k,j}$ with $L_0 = I$ and $L_1, \ldots, L_4$ listed in order as

$$\begin{bmatrix} 0 & n & 0 & 0 & 0 \\ 1 & 0 & n-1 & 0 & 0 \\ 0 & n/2 & 0 & n/2 & 0 \\ 0 & 0 & n-1 & 0 & 1 \\ 0 & 0 & 0 & n & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2n-2 & 0 & 0 \\ 0 & n-1 & 0 & n-1 & 0 \\ 1 & 0 & 2n-4 & 0 & 1 \\ 0 & n-1 & 0 & n-1 & 0 \\ 0 & 0 & 2n-2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & n & 0 \\ 0 & 0 & n-1 & 0 & 1 \\ 0 & n/2 & 0 & n/2 & 0 \\ 1 & 0 & n-1 & 0 & 0 \\ 0 & n & 0 & 0 & 0 \end{bmatrix},$$

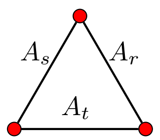
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that $N_p = N_q = 35$.

In Nomura's construction of spin models from Hadamard graphs [26], he proves that the association scheme of Hadamard graphs are triply regular by computing all parameters $\upsilon_{r,s,t}^{i,j,k}$ satisfying



$$= \sum_{r,s,t} \upsilon_{r,s,t}^{i,j,k}$$



.

Applying Lemma 2.3 gives the following result due to Jaeger [15, Section 7.3].

Lemma 4.1. *The association scheme of a Hadamard graph is exactly triply regular.* □

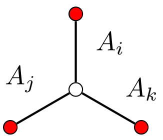
Further, all of the parameters $\upsilon_{r,s,t}^{i,j,k}$ depend only on n. For instance, when $i = j = k = 1$,

$$\upsilon_{0,0,0}^{1,1,1} = n, \quad \upsilon_{0,2,2}^{1,1,1} = \upsilon_{2,0,2}^{1,1,1} = \upsilon_{2,2,0}^{1,1,1} = \frac{n}{2}, \quad \upsilon_{2,2,2}^{1,1,1} = \frac{n}{4}, \tag{4.1}$$

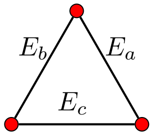
and $\upsilon_{r,s,t}^{1,1,1} = 0$ for all other r, s and t.

Lemma 4.2. *The association schemes of Hadamard graphs of order* $4n$ *have the same Delta-Wye parameters.*

Proof. From [26], the coefficients on the right-hand side of



$$= \sum_{r,s,t} \upsilon_{r,s,t}^{i,j,k} \sum_{a,b,c} P_{ar} P_{bs} P_{ct}$$



depend only on n. Hence the association schemes of any two Hadamard graphs of order $4n$ have the same Delta-Wye parameters. □

Our next result follows immediately from Theorem 3.3.

Theorem 4.3. *Any two Hadamard graphs of order* $4n$ *are quantum isomorphic.* □

Remark 4.4. Given a Hadamard graph of order $4n$, let $s, t_0, t_1, t_2, t_3, t_4$ be complex numbers satisfying

$$s^2 + 2(2n-1)s + 1 = 0, \quad t_0^2 = \frac{2\sqrt{n}}{(4n-1)s+1}, \quad t_1^4 = 1, \quad t_2 = st_0, \quad t_3 = -t_1$$
$$\text{and} \quad t_4 = t_0.$$

Then the matrices $W_+ = \sum_{j=0}^{4} t_j A_j$ and $W_- = \sum_{j=0}^{4} t_j^{-1} A_j$ form a spin model [26]. In [16], Jones constructed from each link diagram a plane graph F with signed edges and showed that the scaffold $\mathsf{S}(F, \emptyset; w)$, where

$$w(e) = \begin{cases} W_+ & \text{if the sign of } e \text{ is } + \\ W_- & \text{if the sign of } e \text{ is } - \end{cases},$$

is a link invariant with some simple normalization.

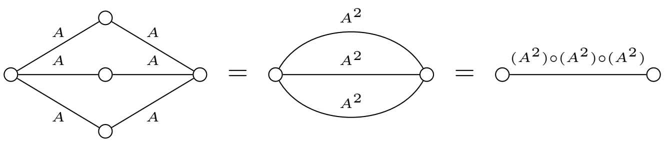
Using Theorem 3.2, we reproduce Jaeger's proof that the spin models from any Hadamard graphs of the same order give the same link invariant [15, Proposition 22].

4.2. Examples of homomorphism counts

We have seen that scaffolds of order zero include homomorphism counts and we have given a general recipe for reducing these scaffolds to sums of single-vertex scaffolds.

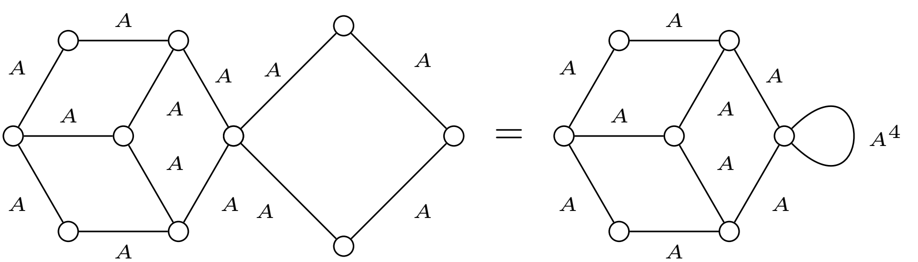
Here, we give two concrete examples where we compute $\mathsf{hom}(F, G)$ where F is a small planar graph and G is a Hadamard graph on $4n$ vertices. In the first example, F is a series-parallel graph and the Delta-Wye equations are not needed.

Example 4.5. The number of homomorphisms from the complete bipartite graph $K_{2,3}$ into a Hadamard graph G on $4n$ vertices with adjacency matrix $A = A_1$ is $\mathsf{hom}(K_{2,3}, G) = n^4(n+3)$, the sum of entries of the matrix $(A_1^2) \circ (A_1^2) \circ (A_1^2)$ using the expansion $A_1^2 = nA_0 + \frac{n}{2}A_2$:

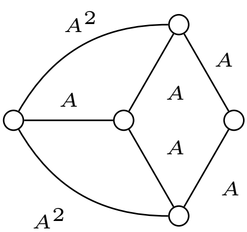


Now let's work with a more complicated example where the extended series-parallel reduction rules are applied in conjunction with the scaffold expansion afforded by triple regularity.

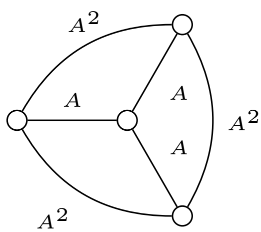
Example 4.6. Consider the graph F, below, obtained by taking a 1-clique sum of a 4-cycle and the graph obtained by deleting one vertex of the 3-cube. Let A be the adjacency matrix of a Hadamard graph G on $4n$ vertices with Bose-Mesner algebra having basis of Schur idempotents $\{A_0 = I, A_1 = A, A_2, A_3, A_4\}$ ordered according to distance in G. We compute, using scaffold rules c (series reduction) and b (loop removal),

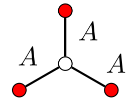


$= \mathsf{tr}(A^4)$



$= 2n^3(n+1)$



As a consequence of exact triple regularity, we have from (4.1) 

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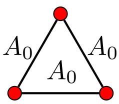
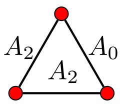
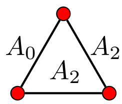
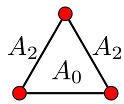
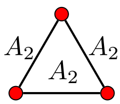
n 

$+\frac{n}{2}$ 

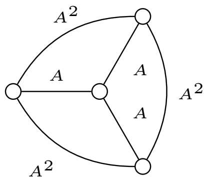
$+\frac{n}{2}$ 

$+\frac{n}{2}$ 

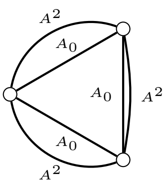
$+\frac{n}{4}$ 

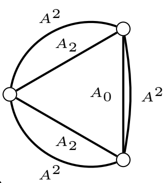
.

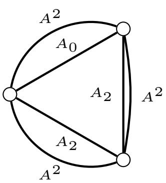
We make this substitution to find

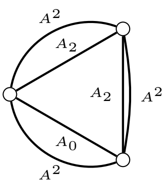
$\mathsf{hom}(F,G) = \quad 2n^3(n+1)$ 

$= 2n^3(n+1)\,\cdot$

$\Big[\, n$ 

$+\frac{n}{2}\Big($ 

$+$ 

$+$ 

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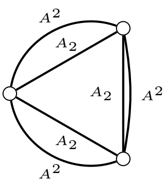
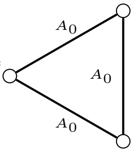
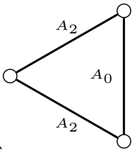
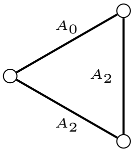
$+\frac{n}{4}$ 

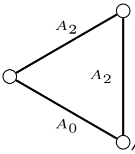
$\Big]$

and, with $A^2 \circ A_0 = nA_0$ and $A^2 \circ A_2 = \frac{n}{2}A_2$, we apply Rule d (parallel reduction) to arrive at

$= 2n^3(n+1)\Big[\, n^4$ 

$+\frac{n^4}{8}\Big($ 

$+$ 

$+$ 

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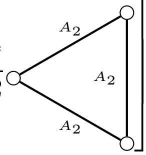
$+\frac{n^4}{32}$ 

$\Big]$

$$= 2n^3(n+1)\left[n^4\mathsf{tr}(I) + \frac{n^4}{8}\cdot 3\cdot \mathsf{tr}(A_2^2) + \frac{n^4}{32}\mathsf{tr}(A_2^3)\right]$$
$$= n^{11} + 4n^{10} + 7n^9 + 4n^8.$$

5. Discussion and open problems

In this paper, we have used association schemes as a place to search for graphs where subconfiguration counts are somewhat under control. We have relied on a theorem of Mančinska and Roberson that shows it is sufficient to count homomorphisms into G from any planar graph F and also on a theorem of Epifanov that gives a reduction procedure for any planar graph F involving moves on just 1, 2, or 3 edges. The algebraic effect of single-edge and 2-edge reductions can be computed in any association scheme, but the moves involving three edges — the Delta-Wye transformations — seem only manageable in the case of exactly triply regular association schemes. Fortunately, the association scheme of any Hadamard graph has this property.

In Section 4.1, we use homomorphism counting to show two Hadamard graphs of the same order, G and H, are quantum isomorphic. So we know that there exists a quantum permutation matrix $\mathcal{U}$ that satisfies $\mathcal{U}A_G = A_H\mathcal{U}$. While there is no guarantee that there exists such a $\mathcal{U}$ whose entries belong to a finite-dimensional C^*-algebra, we ask if there exists a quantum permutation matrix $\mathcal{P}$ with entries in $\mathsf{Mat}_d(\mathbb{C})$ for some d satisfying $\mathcal{P}(A_G \otimes I_d) = (A_H \otimes I_d)\mathcal{P}$; such a solution would give a perfect quantum strategy for the non-local isomorphism game in the quantum tensor framework.

The example from [1] and Hadamard graphs are graphs in association schemes. While the schemes arising from Hadamard graphs are exactly triply regular, the conjugacy class scheme of the symmetric group S_4 is not: here $N_p = 42$ while $N_q = 43$ so it cannot be triply regular. The two graphs in [1] both belong to this association scheme. We are interested in the common properties shared by distinct association schemes that contain quantum isomorphic graphs. In particular, do they have to be both exactly triply regular with the same Delta-Wye parameters? We ask for more pairs of exactly triply regular association schemes with the same Delta-Wye parameters, which will give more examples of quantum isomorphic graphs. A related question is whether the Delta-Wye parameters of an exactly triply regular association scheme are determined by its intersection numbers.

In Section 6 of [1], the authors mention their first example of quantum isomorphic graphs can be constructed using both their method as well as a version of the Cai, Fürer and Immerman construction. The Cai, Fürer and Immerman construction is designed to produce non-isomorphic graphs that are indistinguishable by the Weisfeiler-Lehman algorithm [7]. A natural question is whether two non-isomorphic Hadamard graphs of the same order are distinguishable by the d-dimensional Weisfeiler-Lehman algorithm, for some d.

The association schemes supporting spin models that give the Kauffman polynomial or the Hadamard spin models are formally self-dual and exactly triply regular, [14] and [26]. Does the Bose-Mesner algebra of a formally self-dual exactly triply regular association scheme always contain a spin model? Conversely, is the Nomura algebra of a spin model exactly triply regular? In [15], Jaeger asked for examples of exactly triply regular association schemes that are not formally self-dual or a proof that such an association scheme cannot exist.

Data availability

No data was used for the research described in the article.

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