Ann. Inst. H. Poincaré
Anal. Non Linéaire (Online first)
DOI 10.4171/AIHPC/76

Forward-modulated damping estimates and nonlocalized stability of periodic Lugiato–Lefever waves

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Abstract. In an interesting recent analysis, Haragus–Johnson–Perkins–de Rijk have shown modulational stability under localized perturbations of steady periodic solutions of the Lugiato–Lefever equation (LLE), in the process pointing out a difficulty in obtaining standard "nonlinear damping estimates" on modulated perturbation variables to control regularity of solutions. Here, we point out that in place of standard "inverse-modulated" damping estimates, one can alternatively carry out a damping estimate on the "forward-modulated" perturbation, noting that norms of forward- and inverse-modulated variables are equivalent modulo absorbable errors, thus recovering the classical argument structure of Johnson–Noble–Rodrigues–Zumbrun for parabolic systems. This observation seems of general use in situations of delicate regularity. Applied in the context of (LLE), it gives the stronger result of stability and asymptotic behavior with respect to nonlocalized perturbations.

1. Introduction

In the interesting recent work [4], building on linear analysis in [3], Haragus, Johnson, Perkins, and de Rijk study nonlinear modulational stability of steady periodic solutions of the Lugiato–Lefever equations (LLE), a model for pattern formation in an optical medium in a cavity under excitement by laser pumping. A general framework for the passage from linear to nonlinear modulational stability has been set up in [8, 10] and related works, and the authors loosely follow this path. However, they find it necessary to modify the approach substantially in the treatment of regularity, substituting for the usual nonlinear modulational damping a combination of "tame" unmodulated estimates with exponentially decaying linear terms, and using in an important way semilinearity of the underlying equations (LLE). This strategy, introduced in [1] in the context of multi-D stability of planar periodic wave trains, is shown in [4] to be sufficient also to resolve the regularity issues arising in the treatment of one-dimensional stability of Lugiato–Lefever waves, thus adding a new and useful element in the tool kit for modulational stability, applicable to the semilinear case.

The authors in passing pose the question whether a modulated damping estimate is obtainable at all for (LLE). And, indeed, this is not just of academic interest. For, the

2020 Mathematics Subject Classification. 35B10.
Keywords. Lugiato–Lefever equations, stability of periodic waves.

unmodulated estimates obtained in [4], though sufficient to close their stability argument, are substantially weaker than those that would be given by a standard modulated damping estimate. Specifically, as we discuss in detail below, the argument of [4] gives control of the H^{s_0} norm of the modulated perturbation v by its L^2 norm times an arbitrarily slowly growing algebraic factor, modulo the L^2 norm of the modulating phase perturbation γ. By contrast, a standard damping estimate controls the H^{s_0} norm of v by its (exact) L^2 norm, modulo the L^2 norm of the *derivative of* γ.

A central point in the study of modulational stability is that v and derivatives of the phase perturbation γ exhibit comparable decay, whereas γ itself decays more slowly, by a factor of $(1+t)^{1/2}$. Indeed, this is the motivation for separating out phase perturbation in the first place, with the goal being to obtain a nonlinear iteration involving only faster-decaying v and $\partial_{x,t}\gamma$, hence more likely to close [9]. Thus, *the* $L^2 \to H^{s_0}$ *control afforded by damping modulo* $\|\partial_x \gamma\|_{L^2}$ *is in principle sharper by a factor* $(1+t)^{-1/2}$ *than that afforded modulo* $\|\gamma\|_{L^2}$ *by the strategy of* [4].

Heuristically, the phase perturbation is expected to satisfy an approximate Burgers equation [2,5], hence to decay approximately as solutions of the heat equation, a property that has been validated rigorously in various settings in [6,7,11]. In the case considered in [4] of *localized initial perturbations* $\tilde v$ of a background periodic wave $\bar u$, one has, writing $u = \bar u + \tilde v$ and defining the modulated perturbation variable

$$v(x,t) = u(x+\gamma(x,t),t) - \bar u(x),$$

that $v \sim \tilde v + \bar u_x \gamma$, with $\bar u_x \sim 1$, hence L^1 localization $\|v\|_{L^1}, \|\tilde v\|_{L^1} < \infty$ on $v, \tilde v$ imposes L^1 localization $\|\gamma\|_{L^1} < +\infty$ on γ as well, leading to decay rate $\|\gamma(\cdot,t)\|_{L^2} \lesssim (1+t)^{-1/4}$. Thus, one expects $\|\tilde v\|_{L^2} \sim \|\gamma\|_{L^2} \lesssim (1+t)^{-1/4}$ and $\|v\|_{L^2} \sim \|\partial_{x,t}\gamma\|_{L^2} \lesssim (1+t)^{-3/4}$. And, indeed, this is the result proved in [4] for initial perturbations $\tilde v_0$ sufficiently small in $L^1 \cap H^4$ of a smooth spatially periodic standing-wave solution of (LLE) satisfying the standard *diffusive spectral stability* condition of Schneider [6,7,9,12,13]. Though not stated in the theorem, the main derivative bounds obtained in the proof are $\|v\|_{H^2} \lesssim 1$ and $\|v\|_{H^4} \lesssim (1+t)^{1/4}$, with H^{s_0} corresponding to H^2 being the crucial bound needed to close the nonlinear iteration. Thus, the respective error terms $\|\gamma\|_{L^2}$ and $\|\partial_x\gamma\|_{L^2}$, being $\ll 1$ are both irrelevant, and so the difference in the context of this argument between tame and damping-type estimates is mainly in simplification/standardization of the argument and not in the finally obtained result.[1]

For *nonlocalized initial perturbations* on the other hand, as considered in [6–8,11], the phase perturbation γ is taken merely bounded in L^∞, with L^1 localization $\|\partial_x\gamma\|_{L^2} < \infty$ imposed, rather, on its derivative. This yields decay rates $\|v\|_{L^2}, \|\partial_x\gamma\|_{L^2} \lesssim (1+t)^{-1/4}$, but with $\|\gamma\|_{L^2} = +\infty$. Thus, in this context, $\|\gamma\|_{L^2}$ is clearly *not* a negligible error, and so the argument structure of [4] based on tame estimates of the unmodulated perturbation

[1]The simplification afforded by damping, however, is rather great, eliminating the need for integration by parts and mean value inequalities and most of the technical complications of the proof (cf. [4,9]).

$\tilde{v}$ does not suffice to close a nonlinear iteration. The sharper bound $\|v\|_{H^{s_0}} \sim \|\partial_x \gamma\|_{L^2} \lesssim (1+t)^{-1/4}$ afforded by damping estimates, however, does suffice, yielding both stability and asymptotic behavior.

> *This gives substantial motivation, of practical interest, to answering the question of* [4] *whether or not it is possible to obtain a modulated damping estimate for (LLE).*

We are not able to answer this question in the sense of the original damping estimates formulated in [8] and elsewhere. However, in the present brief note, motivated by the discussion of [4], we point out that (i) a modulated damping estimate *can* be obtained for (LLE) if one substitutes for the usual "inverse-modulated" perturbation variable [8] a "forward-modulated" version (see below for definitions), and (ii) inverse- and forward-modulated perturbation variables are equivalent in all relevant norms, modulo an absorbable high-derivative H^r norm of γ_x.

This gives an alternative approach to the regularity problem for (LLE) in the original spirit of [8] (see final section), which (i) removes much of the technical complexity of the argument and (ii) is not inherently limited to the semilinear case. More important, our approach yields significantly sharper bounds, allowing the treatment as in [8] of *nonlocalized perturbations* allowing different asymptotic limits of the phase shift γ, whereas the argument of [4] requires $\|\gamma\|_{L^2} < \infty$. Thus, we obtain at once a substantial simplification of the argument and a substantial generalization of the result.

We emphasize that our approach is not tied to (LLE) but applies for arbitrary systems. Indeed, we would propose as a useful option, substituting for the original framework of [8] the modified one of obtaining linear bounds in inverse-modulated variables, but damping-type energy estimates in forward-modulated ones where they may be easier to obtain. We believe this observation to be of general use in situations of delicate regularity, belonging in the multipurpose tool kit of [8].

2. Comparisons of techniques

We begin by comparing the various techniques in a general context.

2.1. Damping vs. tame estimates

A standard issue in the approach of [8] is closing a nonlinear iteration despite apparent loss of derivatives in the nonlinear (modulational) perturbation equations (displayed below for (LLE)). This has previously been addressed by the use of *nonlinear damping estimates* [8, §1.3]

$$\partial_t \mathcal{E}(v) \le -\eta \mathcal{E}(v) + C(\|v\|_{L^2_\alpha}^2 + \|\partial_{x,t}\gamma\|_{H^r_\alpha}), \tag{2.1}$$

$s < r$, where $\mathcal{E}(v)$ is an energy controlling a (possibly weighted, with weight denoted by α) Sobolev norm $\|v\|_{H^s_\alpha}^2$ for the modulated perturbation

$$v(x,t) = u(x+\gamma(x,t),t) - \bar{u}(x),$$

where u and $\bar{u}$ are perturbed and background solutions, and γ (see below) is a modulation parameter introduced in the analysis with arbitrarily high derivative control of the same order as $\|v\|_{L^2_\alpha}$. This yields by a Gronwall-type estimate control of $\|v\|_{H^s_\alpha}^2$ by $e^{-\eta t}\|v\|_{H^s_\alpha}^2(0)$ plus the integral of an exponentially decaying memory kernel against $C(\|v\|_{L^2_\alpha}^2 + \|\partial_{x,t}\gamma\|_{H^r_\alpha})$, thus effectively controlling H^s by L^2 decay.

In [4], the authors use an alternative approach introduced in [1], playing modulated and unmodulated perturbations against each other to obtain a result. The ingredients needed are exponential decay of high-frequency linear estimates for the unmodulated semigroup, plus the aforementioned semilinear structure, allowing the unmodulated perturbation to be estimated via Duhamel's principle thanks to the fact that there is no loss of derivatives in the unmodulated equation. Specifically, one attains on the unmodulated variable

$$\tilde{v}(x,t) = u(x,t) - \bar{u}(x)$$

the tame estimate $\|\tilde{v}\|_{H^s}(t) \le C(1+t)^{1/4}$, for arbitrary s so long as (i) $\|\tilde{v}\|_{H^{s_0}}$ remains small for some fixed s_0 ($s_0 = 2$ in the argument of [4]) and (ii) the undifferentiated unmodulated variable decays at the rate $\|\tilde{v}\|_{L^2}(t) \le C(1+t)^{-1/4}$ predicted by linear theory. By Sobolev interpolation, taking s high enough, one may estimate

$$\|\tilde{v}\|_{H^{s_0}}(t) \le C(1+t)^{-1/4+\varepsilon} \tag{2.2}$$

for $\varepsilon > 0$ as small as desired, nearly recovering the decay rate of $\|v\|_{L^2}(t)$.

This argument is closed by "mean value inequalities" [4, Lem. 4.9] controlling L^2 norms of $v - \tilde{v} = u(x+\gamma,t) - u(x,t) \sim u_x\gamma$ and $\partial_x^{s_0}(v-\tilde{v}) =\sim \partial_x^{s_0+1}u\gamma$ by constant multiples of $\|\gamma\|_{L^2}$, together with integration by parts formulae [4, Lem. 4.8] effectively shifting derivatives in the Duhamel formulation for v from v-factors to harmless γ-factors in order to minimize the required bounds on v_x.

2.2. Forward-modulated damping

Here, we observe that the forward-modulated perturbation equation ((5.2) below) like the unmodulated one, involves no loss in derivatives, hence admits a damping estimate modulo higher-derivative terms in the modulation parameter γ. The forward- and inverse-modulated perturbation variables can be seen to decay at the same rates (Section 5.3 below), for a general choice of system. Thus, the substitution of forward-modulated damping estimates for the unmodulated estimate of [4] would appear both to streamline the argument a bit, and to apply in principle to a wider class of systems. (Indeed, as noted below, it can be applied in all cases of delicate regularity [4, 10, 16] treated so far.)

In the remainder of the paper, we give technical details filling in the outline above for (LLE).

3. Nonlinear perturbation system

The Lugiato–Lefever equation (LLE) is

$$\partial_t \psi = -i\beta\partial_x^2\psi - (1+i\alpha)\psi + i|\psi|^2\psi + F \tag{3.1}$$

with $\psi \in \mathbb{C}$ and parameters $\alpha, \beta, F \in \mathbb{R}$ and $F > 0$. Perturbing about a steady periodic solution ϕ, with $\psi = \phi + \tilde{v}$, and expanding $\tilde{v} = \tilde{v}_r + i\tilde{v}_i$ in real and imaginary parts gives, following [4],

$$\partial_t \begin{pmatrix} \tilde{v}_r \\ \tilde{v}_i \end{pmatrix} = \mathcal{A}[\phi]\partial_t \begin{pmatrix} \tilde{v}_r \\ \tilde{v}_i \end{pmatrix} + \mathcal{N}(\tilde{v}), \tag{3.2}$$

where $\mathcal{N}$ is quadratic order in $\tilde{v}$ and

$$\mathcal{A}[\phi] = -\mathrm{Id} + \mathcal{J}\mathcal{L}[\phi], \tag{3.3}$$

with

$$\begin{aligned} \mathcal{J} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \mathcal{L}[\phi] &= \begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta\partial_x^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix}. \end{aligned} \tag{3.4}$$

4. Unmodulated damping estimate

Following [4], define the (unmodulated) energy

$$\begin{aligned} \tilde{E}_j(t) &= \|\partial_x^j \tilde{v}\|_{L^2}^2 - \frac{1}{2\beta}\langle \mathcal{J}M[\phi]\partial_x^{j-1}\tilde{v}, \partial_x^{j-1}\tilde{v}\rangle, \\ M[\phi] &:= 2\begin{pmatrix} -2\phi_r\phi_i & \phi_r^2 - \phi_i^2 \\ \phi_r^2 - \phi_i^2 & 2\phi_r\phi_i \end{pmatrix} \end{aligned} \tag{4.1}$$

yielding, after some computation [4, App. A]

$$\partial_t \tilde{E}_j(t) = -2\tilde{E}_j(t) + R_1(t) + R_2(t),$$

where $R_1(t)$, comprising lower-order derivative bilinear terms, satisfies

$$|R_1(t)| \le C_1(\|\partial_x^{j-1}\tilde{v}\|_{L^2} + \|\tilde{v}\|_{L^2})(t),$$

and R_2 is a nonlinear residual, satisfying

$$|R_2(t)| \le C_2\|\partial_x^j \tilde{v}\|_{L^2}^2(t)(\|\partial_x^j\tilde{v}\|_{L^2} + \|\tilde{v}\|_{L^2})(t).$$

Combining, and using Sobolev interpolation, one obtains for $\|\partial_x^j \tilde{v}\|_{L^2}$ sufficiently small the unmodulated nonlinear damping estimate

$$\partial_t \tilde{E}(t) \le -\theta \tilde{E}(t) + C\|\tilde{v}\|_{L^2}^2(t), \quad \theta > 0, \tag{4.2}$$

yielding after integration,

$$\widetilde{E}(t) \le e^{-\theta t}\widetilde{E}(0) + C\int_0^t e^{-\theta(t-s)}\|\tilde{v}\|_{L^2}^2(s)\,ds, \tag{4.3}$$

so long as $\|\partial_x^j \tilde{v}\|_{L^2}(s)$ is sufficiently small on $0 \le s \le t$, hence, applying Sobolev interpolation once more, controlling $\|\partial_x^j \tilde{v}(t)\|_{L^2}$ by $\frac{1}{2}\widetilde{E}(t) + C\|\tilde{v}\|_{L^2}(t)$, and thus ultimately by $\|\tilde{v}\|_{L^2}(t)$.

Remark 4.1. As noted in [4, Rmk. 1.4], this unmodulated damping estimate can substitute for the tame estimates of [4] with little change in the argument, the advantage being the possibility (to be checked in individual cases) of extension to the quasilinear case.

5. Modulated damping estimates

5.1. Inverse modulation

We first recall the standard "centered" or "inverse-modulated" perturbation

$$v(x,t) := \psi(x+\gamma(x,t),t) - \phi(x) \approx \tilde{v} + (\phi_x + \tilde{v}_x)\gamma \tag{5.1}$$

serving as the primary perturbation variable in [4], where the phase-modulation γ is chosen in nonlocal fashion so as to remove the principal time-asymptotic part of $\tilde{v}$, thus minimizing v.

As the choice of γ concerns long-time behavior, there is a great deal of flexibility in its short-time behavior – in particular, *γ may be chosen so that it and all derivatives are bounded in short time and decaying at an optimal linear rate in long time.* See, e.g., [8, §2.2], for a general description of this strategy. Writing

$$\psi(x,t) = (\phi + v)(x-\gamma,t),$$

computing derivatives

$$\begin{aligned}
\partial_t\psi(x,t) &= (\phi_t + v_t)(x-\gamma,t) - (\phi_x + v_x)(x-\gamma,t)\partial_t\gamma,\\
\partial_x\psi(x,t) &= (\phi_x + v_x)(x-\gamma,t) - (\phi_x + v_x)(x-\gamma,t)\partial_x\gamma,
\end{aligned}$$

etc., and substituting into (3.1), yields, after a computation as in (3.2), a v-equation consisting of the one for $\tilde{v}$ together with new terms $(\phi_x + v_x)\partial_x\gamma$ and $(\phi_x + v_x)\partial_t\gamma$ and their derivatives in x.

Terms involving only ϕ and γ are harmless, as derivatives of ϕ are bounded and derivatives of γ are of the same order in L^2 as v itself [4, 10]. However, a persistent issue in problems without parabolic smoothing is the appearance of terms involving products of the highest derivative of ϕ and v terms. These can sometimes be treated by judicious rearrangement/construction of the energy functional [10,16]; however, even when it succeeds, this can cost a great deal of additional effort.

In the present case (LLE), as pointed out in [4], inverse modulation changes the perturbation equations from semilinear to quasilinear form, seemingly preventing such a nonlinear damping estimate altogether; see the discussion in [4, App. A]. For this reason, the authors restrict to tame estimates on the slower-decaying but favorable regularity unmodulated variable $\tilde{v}$, coupling this via an auxiliary argument as in [1] to their linearized bounds on the inverse-modulated perturbation v to obtain ultimately, optimal nonlinear bounds on $\|v\|_{L^p}$. As they point out, they could alternatively substitute a damping estimate on the unmodulated variable $\tilde{v}$.

5.2. Forward modulation

A more natural modulated variable in many ways is the forward-modulated variable

$$\begin{aligned} \bar{v} &:= \psi(x,t) - \phi(x-\gamma,t) \\ &\approx \tilde{v} + \phi_x(x-\gamma,t)\gamma \\ &\approx v - \phi_{xx}\gamma^2. \end{aligned} \tag{5.2}$$

Indeed, the decay of $\bar{v}$, corresponding to a description of the behavior of ψ as a modulation of ϕ, is the usual end goal for stability/behavior of periodic waves [8]. However, the perturbation equations for $\bar{v}$ contain the shifted linear operator $\overline{\mathcal{L}} = \mathcal{A}[\phi(\cdot-\gamma,\cdot)]$ in place of $\mathcal{L} = \mathcal{A}[\phi]$, giving, after "centering" to recover the fixed linear operator $\mathcal{L}$, error terms of order γ times derivatives of ϕ, which are not sufficiently rapidly decaying to close a nonlinear perturbation argument.

For this reason, it is the inverse-modulated variable v that is typically used in the stability analysis [2, 8, 11], with bounds on $\bar{v}$ recovered after, by comparison with v, using the fact that v and $\bar{v}$ are related by the change of coordinates

$$x \to x - \gamma(x,t), \tag{5.3}$$

with γ and all derivatives small in L^p, $1 \le p \le \infty$. This comparison is formalized in [8, Lem. 2.7], stating that v controls $\bar{v}$ in all L^p norms, provided that ϕ_x, γ, and γ_x are bounded in L^∞ and γ_x in L^p, with $\|\gamma_x\|_{L^\infty} < 1$ (the latter guaranteeing invertibility of (5.3)).

5.3. Forward vs. inverse modulation bounds

We now make two small but useful observations. First, we note that the argument for [8, Lem. 2.7] gives not only L^p control of $\bar{v}$ by v but *equivalence of L^p norms* modulo derivatives of γ, hence, by differentiation/induction, *equivalence of H^s norms* modulo suitable derivatives of γ as well. Recall [8, §2.3], that derivatives of γ are harmless in the derivation of nonlinear damping estimates (see Section 5.4 below). We formalize this observation in the following pair of results.

Lemma 5.1 ([8]). *Let γ be bounded with $\|\gamma_x\|_{L^\infty(\mathbb{R})}<1$. Then the change of coordinates* $\mathrm{Id}-\gamma$ *is invertible, with inverse* $(\mathrm{Id}-\gamma)^{-1}=\mathrm{Id}+\tilde\gamma$*, satisfying for all* $1\le p\le\infty$*,*

$$\begin{aligned}\|\psi-\phi\circ(\mathrm{Id}-\gamma)^{-1}\|_{L^p(\mathbb{R})}&\le(1+\|\gamma_x\|_{L^\infty(\mathbb{R})})^{\frac1p}\|\psi\circ(\mathrm{Id}-\gamma)-\phi\|_{L^p(\mathbb{R})},\\ \|\psi-\phi\circ(\mathrm{Id}+\gamma)\|_{L^p(\mathbb{R})}&\le(1+\|\gamma_x\|_{L^\infty(\mathbb{R})})^{\frac1p}\|\psi\circ(\mathrm{Id}-\gamma)-\phi\|_{L^p(\mathbb{R})}\\ &\quad+\|\phi_x\|_{L^\infty(\mathbb{R})}(1+\|\gamma_x\|_{L^\infty(\mathbb{R})})^{\frac1p}\|\gamma\|_{L^\infty(\mathbb{R})}\|\gamma_x\|_{L^p(\mathbb{R})}\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}\|\psi-\phi\circ(\mathrm{Id}-\gamma)^{-1}\|_{L^p(\mathbb{R})}&\ge(1-\|\gamma_x\|_{L^\infty(\mathbb{R})})^{-\frac1p}\|\psi\circ(\mathrm{Id}-\gamma)-\phi\|_{L^p(\mathbb{R})},\\ \|\psi-\phi\circ(\mathrm{Id}+\gamma)\|_{L^p(\mathbb{R})}&\ge(1-\|\gamma_x\|_{L^\infty(\mathbb{R})})^{-\frac1p}\|\psi\circ(\mathrm{Id}-\gamma)-\phi\|_{L^p(\mathbb{R})}\\ &\quad-\|\phi_x\|_{L^\infty(\mathbb{R})}(1+\|\gamma_x\|_{L^\infty(\mathbb{R})})^{\frac1p}\|\gamma\|_{L^\infty(\mathbb{R})}\|\gamma_x\|_{L^p(\mathbb{R})}.\end{aligned}\tag{5.5}$$

Proof. We follow the argument of [8, Lem. 2.7]. By the implicit function theorem and boundedness of γ, the map $\mathrm{Id}-\gamma$ is invertible. Let us write its inverse $\mathrm{Id}+\tilde\gamma$. Since the Jacobian of $\mathrm{Id}+\tilde\gamma$ is bounded below by $(1+\|\gamma_x\|_{L^\infty(\mathbb{R})})^{-1}$, and above by $(1-\|\gamma_x\|_{L^\infty(\mathbb{R})})^{-1}$, we have

$$\|[\psi\circ(\mathrm{Id}-\gamma)-\phi]\circ(\mathrm{Id}+\tilde\gamma)\|_{L^p(\mathbb{R})}\ \le\ (1+\|\gamma_x\|_{L^\infty(\mathbb{R})})^{\frac1p}\|\psi\circ(\mathrm{Id}-\gamma)-\phi\|_{L^p(\mathbb{R})},$$

giving the first part of (5.4). The first part of (5.5) follows similarly. Splitting $\psi-\phi\circ(\mathrm{Id}+\gamma)$ as

$$\psi-\phi\circ(\mathrm{Id}+\gamma)\ =\ [\psi\circ(\mathrm{Id}-\gamma)-\phi]\circ(\mathrm{Id}+\tilde\gamma)\ +\ \phi\circ(\mathrm{Id}+\tilde\gamma)-\phi\circ(\mathrm{Id}+\gamma),$$

and applying the intermediate value theorem then yields

$$\|\phi\circ(\mathrm{Id}+\tilde\gamma)-\phi\circ(\mathrm{Id}+\gamma)\|_{L^p(\mathbb{R})}\ \le\ \|\phi_x\|_{L^\infty(\mathbb{R})}\|\tilde\gamma-\gamma\|_{L^p(\mathbb{R})}.$$

But, from the identity $\tilde\gamma=\gamma_x\circ(\mathrm{Id}+\tilde\gamma)$ we have $\tilde\gamma(x)-\gamma(x)=\tilde\gamma(x)\int_0^1\gamma(x+t\tilde\gamma(x))\,dt$, from which Hölder's inequality gives

$$\|\tilde\gamma-\gamma\|^p_{L^p(\mathbb{R})}\le\|\tilde\gamma\|^p_{L^\infty(\mathbb{R})}\int_0^1\|\gamma_x\circ(\mathrm{Id}+t\tilde\gamma)\|^p_{L^p(\mathbb{R})}\,dt.$$

This gives the second part of (5.4), since $\|\tilde\gamma\|_{L^\infty(\mathbb{R})}\le\|\gamma\|_{L^\infty(\mathbb{R})}$ and, for $t\in[0,1]$, $\mathrm{Id}+t\tilde\gamma$ is invertible with a Jacobian bounded below by $(1+\|\gamma_x\|_{L^\infty(\mathbb{R})})^{-1}$. The second part of (5.5) follows similarly. ∎

Remark 5.2. Note the asymmetry in bounds (5.4)–(5.5), in that they involve derivatives of ϕ and γ only, and not of ψ (a potential problem, not a priori controlled) or $\tilde\gamma$ (harmless, controlled by γ).

Corollary 5.3. *Let γ be bounded with $\|\gamma_x\|_{L^\infty(\mathbb{R})} < 1$, and $\|\gamma_x\|_{H^{k+1}} \leq C_1$. Then, for some $C > 0$,*

$$\begin{aligned} C^{-1}\|\psi \circ (\mathrm{Id}-\gamma) - \phi\|_{H^k(\mathbb{R})} - \|\gamma_x\|_{H^{k+1}} \\ \leq \|\psi - \phi \circ (\mathrm{Id}+\gamma)\|_{H^k(\mathbb{R})} \\ \leq C\big(\|\psi \circ (\mathrm{Id}-\gamma) - \phi\|_{H^k(\mathbb{R})} + \|\gamma_x\|_{H^{k+1}}\big). \end{aligned} \tag{5.6}$$

Proof. This follows readily by induction on the order of derivatives k, using the chain rule, Lemma 5.1, and Sobolev embedding to control L^∞ norms of derivatives of γ. ■

- Thus, *we are free to use v or $\bar{v}$ alternatively, as is most convenient,* in $L^p \cap H^j$ estimates of any type of the (either inverse or forward) modulation remainder, *in particular for nonlinear damping*.

5.4. Forward damping estimate

Second, we note that (high-frequency) damping estimates, particularly in situations [4, 10, 16], do not proceed as in low-frequency estimates by separating out a centered, linearized part from a nonlinear residual, but rather by energy estimates based on the symmetric/antisymmetric structure of the equations. Thus, there appears to be no inherent disadvantage, and in the present case considerable advantage as we shall see, in working with (uncentered) *forward-modulated equations* to obtain a nonlinear damping estimate.

In particular, in the case of the Lugiato–Lefever equations, the forward-modulated perturbation equations for $\bar{v}$ become, writing $\psi(x,t) = \phi(x-\gamma,t) - v(x,t)$, computing derivatives

$$\begin{aligned} \partial_t\psi(x,t) &= v_t(x,t) - (\phi_t - \phi_x\partial_t\gamma)(x-\gamma,t), \\ \partial_x\psi(x,t) &= v_t(x,t) - (\phi_x(1+\partial_x\gamma))(x-\gamma,t), \end{aligned}$$

etc., and substituting into (3.1), after a computation like that of (3.2),

$$\partial_t \begin{pmatrix} \bar{v}_r \\ \bar{v}_i \end{pmatrix} = \mathcal{A}[\phi(\cdot-\gamma,\cdot)]\partial_t \begin{pmatrix} \bar{v}_r \\ \bar{v}_i \end{pmatrix} + \bar{\mathcal{N}}(\bar{v}) + \mathcal{R}(\phi(\cdot-\gamma,\cdot),\gamma), \tag{5.7}$$

where $\bar{\mathcal{N}}$ is of quadratic order in $\bar{v}$, $\mathcal{A}$ is as in (3.3)–(3.4), and $\mathcal{R}$ involves products of derivatives of order ≤ 2 of ϕ against derivatives of order between 1 and 2 of γ.

Performing an energy estimate essentially identical to that in Section 4, we thus find for the modulated energy,

$$\bar{E}_j(t) = \|\partial_x^j \bar{v}\|_{L^2}^2 - \frac{1}{2\beta}\langle \mathcal{J} M[\phi(\cdot-\gamma,\cdot)]\partial_x^{j-1}\bar{v}, \partial_x^{j-1}\bar{v}\rangle,$$

M as in (4.1), the estimate

$$\partial_t \bar{E}_j(t) = -2\bar{E}_j(t) + R_1(t) + R_2(t) + R_3(t),$$

where R_1, R_2 as before satisfy

$$|R_1(t)| \le C_1(\|\partial_x^{j-1}\bar v\|_{L^2} + \|\bar v\|_{L^2})(t), \quad |R_2(t)| \le C_2\|\partial_x^j \bar v\|_{L^2}^2(t)(\|\partial_x^j \bar v\|_{L^2} + \|\bar v\|_{L^2})(t),$$

and the new term R_3 satisfies

$$\begin{aligned} |R_3(t)| &\le C_3\|\partial_{x,t}\gamma\|_{H^{j+2}}(t)\|\phi(\cdot - \gamma(\cdot,t),t)\|_{W^{j+3,\infty}}\|\partial_x^{j-1}\bar v\|_{L^2}(t) \\ &\le \tilde C_3(\|\partial_{x,t}\gamma\|_{H^{j+2}}^2 + \|\partial_x^{j-1}\bar v\|_{L^2}^2)(t), \end{aligned}$$

using integration by parts in the first line, and Young's inequality in the second.

Combining, and using Sobolev interpolation, one obtains for $\|\partial_x^j \bar v\|_{L^2}$, $\|\partial_{x,t}\gamma\|_{H^{j+2}}$, sufficiently small the forward-modulated nonlinear damping estimate

$$\partial_t \bar E(t) \le -\theta \bar E + C(\|\bar v\|_{L^2}^2 + \|\partial_{x,t}\gamma\|_{H^{j+2}}^2)(t), \quad \theta > 0, \tag{5.8}$$

yielding after integration

$$\bar E(t) \le e^{-\theta t}\bar E(0) + C\int_0^t e^{-\theta(t-s)}(\|\bar v\|_{L^2}^2 + \|\partial_{x,t}\gamma\|_{H^{j+2}}^2)(s)\, ds, \tag{5.9}$$

so long as $\|\partial_x^j \bar v\|_{L^2}$ and $\|\partial_{x,t}\gamma\|_{H^{j+2}}$ are sufficiently small on $0 \le s \le t$, hence controlling $\|\partial_x^j \bar v\|(t)$ by exponential slaving to $\|\bar v\|_{L^2}(t)$ and $\|\partial_{x,t}\gamma\|_{H^{j+2}}$: an exact analog of the standard inverse-modulation estimate derived in [8, Prop. 2.5, §2.3] in the parabolic case. Summing over $0 \le j \le k$ gives the following analog of [8, Prop. 2.5, §2.3].

Lemma 5.4. *For the forward-modulated perturbation variable $\bar v$ about periodic wave ϕ of (LLE),*

$$\begin{aligned} \|\bar v\|_{H^k}^2(t) &\le Ce^{-\theta t}\|\bar v\|_{H^k}^2(0) \\ &\quad + C\int_0^t e^{-\theta(t-s)}(\|\bar v\|_{L^2}^2 + \|\partial_{x,t}\gamma\|_{H^{k+2}}^2)(s)\, ds, \quad \theta > 0, \end{aligned} \tag{5.10}$$

provided $\|\bar v\|_{H^k}(s)$ and $\|\partial_{x,t}\gamma\|_{H^{k+2}}^2(s)$ are sufficiently small on $0 \le s \le t$.

Remark 5.5. One may check that a similar approach likewise yields a forward-modulated damping estimate in the delicate cases treated previously in [10, 16] by inverse-modulated damping.

6. Conclusion and applications

Combining the observations of Sections 5.3 and 5.4, we see that a useful general strategy is to use inverse-modulated variables to obtain linearized estimates, and forward-modulated variables for nonlinear damping estimates, the first convenient for decay and the second for regularity. In particular, as described in the introduction, this allows the treatment of stability of periodic Lugiato–Lefever waves by the original techniques described in [8, 9], playing off linearized estimates with nonlinear damping, without the need for the additional techniques/estimates introduced in [4]. Specifically, combining Corollary 5.3 and Lemma 5.4, we have the following nonlinear damping estimate on the inverse-modulated perturbation variable v for (LLE).

Theorem 6.1. *For the inverse-modulated perturbation variable v about a periodic wave ϕ of (LLE),*

$$\begin{aligned}\|v\|^2_{H^k}(t) \le Ce^{-\theta t}\|v\|^2_{H^k}(0) + C\int_0^t e^{-\theta(t-s)}(\|v\|^2_{L^2} + \|\partial_{x,t}\gamma\|^2_{H^{k+2}})(s)\,ds\\ + C\|\partial_x\gamma\|^2_{H^{k+1}}(t)\end{aligned} \tag{6.1}$$

for $\theta > 0$ provided $\|v\|_{H^k}(s)$ and $\|\partial_{x,t}\gamma\|^2_{H^{k+2}}(s)$ are sufficiently small on $0 \le s \le t$.

Proof. From Corollary 5.3 and Lemma 5.4, we have immediately (6.1) for $\|\bar{v}\|_{H^k}(s)$ and $\|\partial_{x,t}\gamma\|^2_{H^{k+2}}(s)$ sufficiently small on $0 \le s \le t$. Observing by a second application of Corollary 5.3 that $\|\bar{v}\|_{H^k}(s)$ is controlled by $\|v\|_{H^k}(s)$ and $\|\partial_{x,t}\gamma\|_{H^{k+1}}(s)$, we are done. ∎

The damping estimate (6.1) differs from the standard one (5.10) of [8] only by the addition of the final term $C\|\partial_x\gamma\|^2_{H^{k+1}}(t)$, which is of the same order in the usual nonlinear decay argument as the integral term $C\int_0^t e^{-\theta(t-s)}\|\partial_{x,t}\gamma\|^2_{H^{k+2}}(s)\,ds$, hence harmless. It follows that, indeed, the nonlinear iteration argument of [4] can be closed, alternatively, using the more standard nonlinear damping estimate (6.1) in place of the coupled tame estimates/integration by parts and mean value inequalities used there.

6.1. Stability and asymptotic behavior

The incorporation of Theorem 6.1 in the stability analysis of [4] for localized perturbations simplifies the arguments but does not much affect the results. More important, having recovered the missing ingredient of nonlinear damping, we can apply the full machinery developed in [6–8] to also obtain new results on stability and asymptotic behavior of periodic (LLE) waves with respect to nonlocalized perturbations.

Theorem 6.2 (Stability). *Let ϕ be a smooth spatially periodic standing-wave solution of (LLE) that is diffusively spectrally stable in the sense of Schneider [12, 13], and let ψ be a perturbation such that*

$$E_0 := \|\psi(\cdot - h_0(\cdot), 0) - \phi(\cdot)\|_{L^1(\mathbb{R})\cap H^3(\mathbb{R})} + \|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^3(\mathbb{R})}$$

is sufficiently small, for some choice of phase modulation h_0 such that $h_0(-\infty) = -h_0(\infty)$.[2] Then ψ exists for all $t > 0$, and, for some phase function $\gamma(x,t)$ and $2 \le p \le \infty$,

$$\begin{aligned}\|\psi(\cdot - \gamma(\cdot,t),t) - \phi(\cdot)\|_{L^p(\mathbb{R})}, \quad \|\nabla_{x,t}\gamma(\cdot,t)\|_{L^p(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)},\\ \|\psi(\cdot - \gamma(\cdot,t),t) - \phi(\cdot)\|_{H^3(\mathbb{R})} \lesssim E_0(1+t)^{-\frac{1}{4}},\end{aligned} \tag{6.2}$$

[2] Achievable without loss of generality by a shift in $\bar{u}$ [8].

and

$$\|\psi(\cdot,t)-\phi(\cdot)\|_{L^\infty(\mathbb{R})}, \quad \|\gamma(\cdot,t)\|_{L^\infty(\mathbb{R})} \lesssim E_0. \tag{6.3}$$

Proof. Combining the linearized estimates of [8, §3] with estimate (6.1) of Theorem 6.1, and applying word for word the arguments of [6, 7] (a special case of the more general [8]), we obtain the result; see specifically the proof of [8, Thm. 1.10]. ■

Remark 6.3. We note that the more detailed linearized estimates of [8, §3] follow from the same Bloch decomposition/spectral preparation as do those of [3, 4]; compare [8, §2] and [4, §2–3]. A subtle difference in the two analyses is that the stronger nonlinear bounds of the nonlinear damping approach, effectively controlling $\|v\|_{H^3}$ by $\|v\|_{L^2}$, allow the use of weaker linear bounds. In particular, one may obtain by Prüss' theorem exponential $H^1 \to H^1$ bounds instead of $L^2 \to L^2$ bounds on the high-frequency part of the solution operator, yielding by Sobolev embedding/interpolation exponential $H^1 \to L^p$ bounds for all $p \geq 2$. Since H^1 is controlled by L^2 in the nonlinear iteration, this serves the same purpose as would an $L^2 \to L^\infty$ bound, allowing one to obtain L^p bounds on $\bar v$ for higher norms $p \geq 2$. To obtain such bounds in the tame estimate framework of [4] would require obtaining $L^2 \to L^\infty$ bounds, which may or may not be true.

In Theorem 6.2, $\psi(\cdot-\gamma(\cdot,t),t)-\phi(\cdot)$ corresponds to the modulated variable v in the previous sections, and $\psi(\cdot,t)-\phi(\cdot)$ to $\tilde v$, both decaying more slowly by a factor $(1+t)^{1/2}$ than their counterparts in the case of localized perturbations treated in [4]. In particular, the phase γ is bounded only in L^∞, having infinite L^p norm for any $p<\infty$. These estimates are in fact sharp, as we now show.

Periodic standing waves occur in a one-parameter family. Taking the base wave ϕ without loss of generality to be period 1, parametrize this family as $\phi^k(kx)$, where $k = 1/X$ is the wave number, with X equal to the period. Recall now the formal, Whitham approximation

$$u(x,t) \approx \phi^{\kappa(x,t)}(\Psi(x,t)) \tag{6.4}$$

[2, 5, 14, 15], where the wave number $\kappa := \Psi_x$ satisfies the Whitham equation

$$\kappa_t - (\omega_0(\kappa))_x = (d(\kappa)\kappa_x)_x, \tag{6.5}$$

with $\omega(\kappa) \equiv 0$ the time frequency associated with the family of periodic traveling waves – here, identically zero – and d is a diffusion term determined by formal asymptotic expansion.

Following [2, 7, 11], define the quadratic-order approximate equation

$$k_t = k_* d(k_*) k_{xx}, \tag{6.6}$$

approximately governing a small perturbation $k = k_*\gamma_x$ of the type we seek, and define

$$h(x) := \int_{-\infty}^x k(x). \tag{6.7}$$

Then we have the following description of L^p-asymptotic behavior.

Theorem 6.4 (Asymptotic behavior). *Let $\eta > 0$. Under the assumptions of Proposition 6.2, let k satisfy the quadratic approximant (6.6) of the second-order Whitham modulation equations (6.5) with initial data $k|_{t=0} = k_* \partial_x h_0$, let h be as in (6.7), and let γ be the phase prescribed in the proof of Theorem 6.2 (see [6]). Then, for $t > 0$ and $2 \le p \le \infty$,*

$$\begin{aligned} \|\psi(\cdot - \gamma(\cdot,t),t) - \phi^{k_*(1+\gamma_x(\cdot,t))}(\cdot)\|_{L^p(\mathbb{R})} &\lesssim E_0 \ln(2+t)(1+t)^{-\frac{3}{4}}, \\ \|k_* \partial_x \gamma(t) - k(t)\|_{L^p(\mathbb{R})} &\lesssim E_0 (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}, \\ \|\gamma(t) - h(t)\|_{L^p(\mathbb{R})} &\lesssim E_0 (1+t)^{-\frac{1}{2}(1-1/p)+\eta}. \end{aligned} \tag{6.8}$$

Proof. Again, this follows by combining the linearized estimates of [8, §3] with estimate (6.1) of Theorem 6.1, and applying word for word the arguments of [6, 7] (a special case of the more general [8]). See specifically the proof of [8, Thm. 1.12]. ∎

Note that k and h both satisfy a heat equation, with localized, and nonlocalized behavior. When $k|_{t=0}$ has a first moment in L^1, its solution thus decays in all L^p to a heat kernel, while h converges to an error function. In particular, both γ and $\tilde{v} = \psi - \phi$ have infinite L^p norm for all $p < \infty$, in agreement with the estimates stated in Theorem 6.2. Thus, the tame estimate argument used in [4], based on finiteness of $\|\gamma\|_{L^2}$ among other things, does not suffice to treat this case. Indeed, as described in Section 2.1, the tame estimate approach of [4] gives up an arbitrarily small amount of time-algebraic decay, so that even if carried out in L^∞, where γ remains bounded, the estimates derived by this technique would still be unbounded and the argument not closed.

Acknowledgments. We thank the referees for their careful reading and helpful suggestions.

Funding. Research of K.Z. was partially supported under NSF grant no. DMS-0300487.

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Received 1 March 2022; revised 8 August 2022; accepted 10 August 2022.

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