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Turnpike properties for stochastic linear-quadratic optimal control problems with periodic coefficients

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Abstract

This paper is concerned with the turnpike property for a class of stochastic linear-quadratic (LQ, for short) optimal control problems with periodic coefficients. The stability and stabilizability of the control system are studied, followed by the discussion of the existence and uniqueness of periodic solutions. A deterministic periodic LQ problem is introduced and solved, whose optimal pair, together with a pair of correction processes, serves as the turnpike limit of the stochastic problem. It is shown that the turnpike limit is periodic in the distribution sense. In the special case of constant coefficients, the turnpike limit turns out to have stationary distributions, with the expectation being the solution to a static optimization problem.

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1. Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathscr{F}_t\}_{t\geqslant 0}$ which satisfies the usual conditions. Suppose that on $(\Omega, \mathscr{F}, \mathbb{P})$ a standard one-dimensional Brownian motion $W = \{W(t), \mathscr{F}_t;\ t \geqslant 0\}$ is defined. Thus, $\mathbb{F}$ could be larger than the natural filtration generated by W. Consider the following controlled linear stochastic differential equation (SDE, for short)

$$\begin{cases} dX(t) = [A(t)X(t) + B(t)u(t) + b(t)]dt + [C(t)X(t) + D(t)u(t) + \sigma(t)]dW(t), \\ X(0) = x, \end{cases} \tag{1.1}$$

and the quadratic cost functional

$$\begin{aligned} J_T(x; u(\cdot)) \triangleq \mathbb{E}\int_0^T \Bigg[\Bigg\langle \begin{pmatrix} Q(t) & S(t)^\top \\ S(t) & R(t) \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \Bigg\rangle \\ + 2\Bigg\langle \begin{pmatrix} q(t) \\ r(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \Bigg\rangle \Bigg] dt. \end{aligned} \tag{1.2}$$

In the above, $A(\cdot),\ C(\cdot),\ Q(\cdot) \in L^\infty(0,\infty;\mathbb{R}^{n\times n})$, $B(\cdot),\ D(\cdot) \in L^\infty(0,\infty;\mathbb{R}^{n\times m})$, $S(\cdot) \in L^\infty(0,\infty;\mathbb{R}^{m\times n})$, $R(\cdot) \in L^\infty(0,\infty;\mathbb{R}^{m\times m})$, $b(\cdot),\ \sigma(\cdot),\ q(\cdot) \in L^\infty(0,\infty;\mathbb{R}^{n})$, and $r(\cdot) \in L^\infty(0,\infty;\mathbb{R}^{m})$, where $L^\infty(0,\infty;\mathbb{H})$ denotes the space of essentially bounded Lebesgue measurable functions from $[0,\infty)$ into $\mathbb{H}$, some Euclidian space. The functions $Q(\cdot)$ and $R(\cdot)$ are symmetric matrix-valued. In (1.2), the superscript $\top$ denotes the transpose of matrices, and $\langle\cdot\,,\cdot\rangle$ denotes the Frobenius inner product of two matrices (in possibly different spaces). The vector $x \in \mathbb{R}^n$ in (1.1) is called an *initial state*, and the process $u(\cdot)$, called a *control*, is selected from the following space:

$$\mathscr{U}[0,T] \triangleq \left\{ u : [0,T]\times\Omega \to \mathbb{R}^m \;\middle|\; u \in \mathbb{F} \text{ and } \mathbb{E}\int_0^T |u(t)|^2 dt < \infty \right\},$$

where the notation $u \in \mathbb{F}$ means that $u(\cdot)$ is progressively measurable with respect to the filtration $\mathbb{F}$, and $|\cdot|$ is the norm induced by the Frobenius inner product. For a fixed time horizon $T > 0$, the *stochastic linear-quadratic (LQ, for short) optimal control problem* on $[0,T]$ can be stated as follows.

Problem (SLQ)$_T$. For any given initial state $x \in \mathbb{R}^n$, find a control $\bar u_T(\cdot) \in \mathscr{U}[0,T]$ such that

$$J_T(x; \bar u_T(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}[0,T]} J_T(x; u(\cdot)) \equiv V_T(x). \tag{1.3}$$

The process $\bar u_T(\cdot)$ in (1.3) (if exists) is called an *open-loop optimal control* of Problem (SLQ)$_T$ for the initial state x, the corresponding state process $\bar X_T(\cdot)$ is called an *open-loop optimal state process*, the pair $(\bar X_T(\cdot), \bar u_T(\cdot))$ is called an *open-loop optimal pair*, and the function $V_T(\cdot)$ is called the *value function* of Problem (SLQ)$_T$.

The study of stochastic LQ optimal control problems began with the works of Kushner [14] and Wonham [26] in the 1960s. Since then, many researchers have carried out corresponding research and analysis on this topic, for both the definite (meaning that $Q(\cdot) \geqslant 0$ and $R(\cdot) > 0$) and indefinite (if the positive semi-definite/definite conditions are not assumed for $Q(\cdot)$ and $R(\cdot)$) situations; see, for example, [8,1,4,28,21] and the references cited therein. It is well-known by now that under proper conditions, Problem $(\mathrm{SLQ})_T$ is uniquely open-loop solvable, by which we mean that an open-loop optimal control uniquely exists, whose open-loop optimal control admits a closed-loop representation and can be constructed explicitly in terms of the solution to a differential Riccati equation. The open-loop optimal control usually depends on the initial state x, as well as the time horizon T.

The purpose of this paper is to analyze the limiting behavior of the optimal pair $(\bar{X}_T(\cdot), \bar{u}_T(\cdot))$ as the time horizon T tends to infinity, which, as we shall see later, exhibits the so called *exponential turnpike property*. Such a property was recently discovered by Sun–Wang–Yong [20] for stochastic LQ optimal control problems with constant coefficients, in which it was shown that the following is satisfied:

$$\big|\mathbb{E}\bar{X}_T(t) - x^*\big| + \big|\mathbb{E}\bar{u}_T(t) - u^*\big| \leqslant K\big[e^{-\lambda t} + e^{-\lambda(T-t)}\big], \quad \forall t \in [0, T] \tag{1.4}$$

for some constants $K, \lambda > 0$ that are independent of T, where (x^*, u^*), referred to as the *turnpike limit*, is the solution to a static optimization problem, which is independent of T and the (open-loop) optimal pair $(\bar{X}_T(\cdot), \bar{u}_T(\cdot))$. It is easy to see that (1.4) is not good enough and one hopes to have

$$\mathbb{E}\big|\bar{X}_T(t) - x^*\big| + \mathbb{E}\big|\bar{u}_T(t) - u^*\big| \leqslant K\big[e^{-\lambda t} + e^{-\lambda(T-t)}\big], \quad \forall t \in [0, T], \tag{1.5}$$

with the absolute value being inside of the expectation. Unfortunately, the estimate (1.5) is not true in general. For mean-field stochastic LQ optimal control problems (more general than Problem $(\mathrm{SLQ})_T$), Sun–Yong [22] introduced a pair of correction processes such that instead of (1.5), the following *strong exponential turnpike property* was proved:

$$\mathbb{E}\big[|\bar{X}_T(t) - \boldsymbol{X}^*(t)|^2 + |\bar{u}_T(t) - \boldsymbol{u}^*(t)|^2\big] \leqslant K\big[e^{-\lambda t} + e^{-\lambda(T-t)}\big], \quad \forall t \in [0, T], \tag{1.6}$$

where $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ is a pair of stochastic processes independent of T and $(\bar{X}_T(\cdot), \bar{u}_T(\cdot))$.

Having the above results, the main novelty and contribution of the current paper can be summarized as follows.

(i) The coefficients are all time-periodic, with a common period $\tau > 0$. The constant coefficient framework of [20] can be regarded as a special case of the current one. We have overcome the essential difficulties that the (time-varying) periodic feature of the coefficients brings in the stability and stabilizability of periodic stochastic linear systems, the existence and uniqueness of periodic solutions to SDEs, the periodic solvability of differential Riccati equations, etc.

(ii) Unlike in [20,22], we use a deterministic *periodic* LQ optimal control problem, denoted by Problem $(\mathrm{DLQ})_\tau$, to determine the turnpike limit for our Problem $(\mathrm{SLQ})_T$. This periodic optimal control problem is explicitly solved, whose optimal pair can be regarded as the analogue of the solution to the static optimization problem in [20] or [22].

(iii) It is shown that under proper conditions, the open-loop optimal pair $(\bar{X}_T(\cdot), \bar{u}_T(\cdot))$ of Problem $(\text{SLQ})_T$ satisfies (1.6), where $K, \lambda > 0$ are constants independent of T, and $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ is a pair of τ*-periodic* stochastic processes in the sense that $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ and $(\boldsymbol{X}^*(\tau+\cdot), \boldsymbol{u}^*(\tau+\cdot))$ have the same finite-dimensional distributions. The pair $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ can be explicitly determined as follows:

$$\boldsymbol{X}^*(\cdot) = X(\cdot) + x^*(\cdot), \quad \boldsymbol{u}^*(\cdot) = \Theta(\cdot)[X(\cdot) + x^*(\cdot)] + u^*(\cdot),$$

where $(x^*(\cdot), u^*(\cdot))$ is the τ-extension of the optimal pair of (deterministic) Problem $(\text{DLQ})_\tau$, $X(\cdot)$ is the τ-periodic solution of a linear SDE, and $\Theta(\cdot)$ is a completely determined matrix function; all of them are independent of T and the pair $(\bar{X}_T(\cdot), \bar{u}_T(\cdot))$.

(iv) When the coefficients are time invariant, our results reduce to the case studied in [22] without the mean-field, and further we show that the correction processes introduced in [22] can be chosen to have invariant distributions.

The turnpike property was initially realized by Ramsey [18] and by von Neumann [16], in the early of last century, for the optimal solution to a dynamic optimization problem in infinite time horizon in studying growth problems of economy systems. The name "turnpike" was firstly coined by Dorfman–Samuelson–Solow [9] in 1958, suggested by a similar feature of toll highway in the United States. In the recent years, numerous relevant results were developed for deterministic optimal control problems of both finite and infinite dimensional systems. Typical studies include, but not limited to, Porretta–Zuazua [17], Damm–Grune–Stieler–Worthmann [7], Trélat–Zuazua [24], Zaslavski [29,30], Trélat–Zhang [23], Grune–Guglielmi [12], Lou–Wang [15], Breiten–Pfeiffer [2], Grune–Guglielmi [13], Sakamoto–Zuazua [19], and Faulwasser–Grune [10]. For a more complete review, we refer the reader to Carlson–Haurie–Leizarowitz [3] and Zuazua [31]. As for stochastic optimal control problems, to the best of our knowledge, [20] is the first attempt to discover the corresponding turnpike properties, followed by a deeper and more general study [22].

The rest of the paper is organized as follows. In Section 2, we introduce some notation, impose the assumptions, and collect a few lemmas that will be needed later. In Section 3, we study the stability and stabilizability of stochastic linear systems with periodic coefficients, and show that a unique periodic solution exists for stable systems in Section 4. In Section 5, we discuss the long-time behavior of the solution to the differential Riccati equation associated with Problem $(\text{SLQ})_T$. In Section 6, we introduce the deterministic periodic LQ optimal control problem and investigate its solvability. In Section 7, we establish the turnpike property for Problem $(\text{SLQ})_T$.

2. Preliminaries

In this paper, a vector always refers to a column vector if not specified otherwise. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we write the row vector $(f_{x_1}, \ldots, f_{x_n})$ as f_x. Let $\mathbb{R}^{n\times m}$ be the Euclidean space of $n \times m$ real matrices, equipped with the Frobenius inner product

$$\langle M, N\rangle \triangleq \text{tr}\,(M^\top N), \quad M, N \in \mathbb{R}^{n\times m},$$

where $M^\top$ is the transpose of M and $\text{tr}\,(M^\top N)$ is the trace of $M^\top N$. The norm induced by the Frobenius inner product is denoted by $|\cdot|$. Let $\mathbb{S}^n$ be the space of symmetric $n \times n$ real matrices and $\mathbb{S}^n_+$ the subset of $\mathbb{S}^n$ consisting of positive definite matrices. For two matrices $M, N \in \mathbb{S}^n$, we write $M \geqslant N$ if $M - N$ is positive semidefinite. The identity matrix of size n is denoted by I_n, which is often simply written as I when no confusion occurs. For a Euclidean space $\mathbb{H}$, we define

$$
\begin{aligned}
L^\infty(0,\infty;\mathbb{H}) &\triangleq \Big\{\varphi:[0,\infty)\to\mathbb{H} \;\Big|\; \varphi \text{ is Lebesgue essentially bounded}\Big\},\\
C([0,\infty);\mathbb{H}) &\triangleq \Big\{\varphi:[0,\infty)\to\mathbb{H} \;\Big|\; \varphi \text{ is continuous}\Big\},\\
C([0,T];\mathbb{H}) &\triangleq \Big\{\varphi:[0,T]\to\mathbb{H} \;\Big|\; \varphi \text{ is continuous}\Big\}.
\end{aligned}
$$

Also, recall that $(\Omega,\mathscr{F},\mathbb{P})$ is a complete probability space equipped with a filtration $\mathbb{F} = \{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions. For a random variable ξ, we write $\xi\in\mathscr{F}_t$ if ξ is $\mathscr{F}_t$-measurable; and for a stochastic process X, we write $X\in\mathbb{F}$ if it is progressively measurable with respect to the filtration $\mathbb{F}$.

Next, we introduce the assumptions that will be in force throughout the paper. For the simplicity of the presentation, we shall call a function $F(\cdot)\in L^\infty(0,\infty;\mathbb{S}^n)$ *uniformly positive definite* if for some constant $\delta>0$,

$$
F(t)\geqslant \delta I, \quad \text{a.e. } t\in[0,\infty).
$$

Now, we introduce the following basic assumptions on our Problem (SLQ)$_T$:

(A1) The time-varying coefficients in (1.1) and (1.2) are periodic functions with common period $\tau>0$ and satisfy the following boundedness condition:

$$
\begin{cases}
A(\cdot),C(\cdot)\in L^\infty(0,\infty;\mathbb{R}^{n\times n}), \quad B(\cdot),D(\cdot)\in L^\infty(0,\infty;\mathbb{R}^{n\times m}),\\
Q(\cdot)\in L^\infty(0,\infty;\mathbb{S}^n), \quad S(\cdot)\in L^\infty(0,\infty;\mathbb{R}^{m\times n}), \quad R(\cdot)\in L^\infty(0,\infty;\mathbb{S}^m),\\
b(\cdot),\sigma(\cdot),q(\cdot)\in L^\infty(0,\infty;\mathbb{R}^n), \quad r(\cdot)\in L^\infty(0,\infty;\mathbb{R}^m).
\end{cases}
$$

(A2) $R(\cdot)$ and $Q(\cdot)-S(\cdot)^\top R(\cdot)^{-1}S(\cdot)$ are uniformly positive definite.

It is standard that under (A1)–(A2), the state equation (1.1) is well-posed for all $x\in\mathbb{R}^n$ and $u(\cdot)\in\mathscr{U}[0,T]$, and the cost functional (1.2) is well-defined and uniformly convex in $\mathscr{U}[0,T]$. Therefore, Problem (SLQ)$_T$ is well-formulated and admits a unique open-loop optimal control (see, for example, [21]).

To conclude this section, we present some useful lemmas. Let

$$
\begin{gathered}
\mathcal{A}(\cdot),\mathcal{C}(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad \mathcal{B}(\cdot),\mathcal{D}(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m}),\\
\mathcal{G}\in\mathbb{S}^n, \quad \mathcal{Q}(\cdot)\in L^\infty(0,T;\mathbb{S}^n), \quad \mathcal{S}(\cdot)\in L^\infty(0,T;\mathbb{R}^{m\times n}), \quad \mathcal{R}(\cdot)\in L^\infty(0,T;\mathbb{S}^m),
\end{gathered}
$$

and let $\mathcal{X}(\cdot)$ be the solution to the matrix SDE

$$
\begin{cases}
d\mathcal{X}(t)=\mathcal{A}(t)\mathcal{X}(t)dt+\mathcal{C}(t)\mathcal{X}(t)dW(t), \quad t\in[0,T],\\
\mathcal{X}(0)=I.
\end{cases} \tag{2.1}
$$

Lemma 2.1. *The solution $\mathcal{P}(\cdot) \in C([0, T]; \mathbb{S}^n)$ to the ordinary differential equation (ODE, for short)*

$$\begin{cases} \dot{\mathcal{P}}(t) + \mathcal{P}(t)\mathcal{A}(t) + \mathcal{A}(t)^\top \mathcal{P}(t) + \mathcal{C}(t)^\top \mathcal{P}(t)\mathcal{C}(t) + \mathcal{Q}(t) = 0, \\ \mathcal{P}(T) = \mathcal{G} \end{cases}$$

admits the following representation:

$$\mathcal{P}(t) = \mathbb{E}\left\{[\mathcal{X}(T)\mathcal{X}(t)^{-1}]^\top \mathcal{G}[\mathcal{X}(T)\mathcal{X}(t)^{-1}] + \int_t^T [\mathcal{X}(s)\mathcal{X}(t)^{-1}]^\top \mathcal{Q}(s)[\mathcal{X}(s)\mathcal{X}(t)^{-1}]ds\right\}.$$

Consequently, if $\mathcal{G} \geqslant 0$ and $\mathcal{Q}(t) \geqslant 0$ for a.e. $t \in [0, T]$, then

$$\mathcal{P}(t) \geqslant 0, \quad \forall t \in [0, T].$$

Proof. See [28], pp. 320–321. □

Lemma 2.2. *Suppose that for $i = 1, 2$, $\mathcal{P}_i(\cdot) \in C([0, T]; \mathbb{S}^n)$ satisfies*

$$\mathcal{R}(t) + \mathcal{D}(t)^\top \mathcal{P}_i(t)\mathcal{D}(t) > 0, \quad a.e.\ t \in [0, T]$$

and the following Riccati differential equation:

$$\begin{aligned} &\dot{\mathcal{P}}_i + \mathcal{P}_i\mathcal{A} + \mathcal{A}^\top \mathcal{P}_i + \mathcal{C}^\top \mathcal{P}_i\mathcal{C} + \mathcal{Q} \\ &\quad - (\mathcal{P}_i\mathcal{B} + \mathcal{C}^\top \mathcal{P}_i\mathcal{D} + \mathcal{S}^\top)(\mathcal{R} + \mathcal{D}^\top \mathcal{P}_i\mathcal{D})^{-1}(\mathcal{B}^\top \mathcal{P}_i + \mathcal{D}^\top \mathcal{P}_i\mathcal{C} + \mathcal{S}) = 0. \end{aligned}$$

Then with $\bar{\mathcal{P}}(\cdot) \triangleq \mathcal{P}_2(\cdot) - \mathcal{P}_1(\cdot)$,

$$\begin{aligned} &\dot{\bar{\mathcal{P}}} + \bar{\mathcal{P}}(\mathcal{A} + \mathcal{B}\mathcal{K}_2) + (\mathcal{A} + \mathcal{B}\mathcal{K}_2)^\top \bar{\mathcal{P}} + (\mathcal{C} + \mathcal{D}\mathcal{K}_2)^\top \bar{\mathcal{P}}(\mathcal{C} + \mathcal{D}\mathcal{K}_2) \\ &\quad + (\mathcal{K}_2 - \mathcal{K}_1)^\top (\mathcal{R} + \mathcal{D}^\top \mathcal{P}_1\mathcal{D})(\mathcal{K}_2 - \mathcal{K}_1) = 0, \end{aligned}$$

where for $i = 1, 2$,

$$\mathcal{K}_i(t) \triangleq -\Big[\mathcal{R}(t) + \mathcal{D}(t)^\top \mathcal{P}_i(t)\mathcal{D}(t)\Big]^{-1}\Big[\mathcal{B}(t)^\top \mathcal{P}_i(t) + \mathcal{D}(t)^\top \mathcal{P}_i(t)\mathcal{C}(t) + \mathcal{S}(t)\Big].$$

Proof. The proof follows from a direct computation. We omit the details here. □

3. Stability and stabilizability

In this section we introduce the notions of (mean-square exponential) stability and stabilizability for stochastic linear systems with periodic coefficients. As we shall see in the subsequent sections, the stabilizability of the controlled state equation plays a key role in the turnpike property of Problem (SLQ)$_T$.

We start with the following definition.

Definition 3.1. Let $\Phi(t)$ be the fundamental matrix for the linear SDE

$$dX(t) = A(t)X(t)dt + C(t)X(t)dW(t), \quad t \geqslant 0, \tag{3.1}$$

that is, $\Phi(\cdot)$ is the solution of the matrix linear SDE

$$\begin{cases} d\Phi(t) = A(t)\Phi(t)dt + C(t)\Phi(t)dW(t), & t \geqslant 0, \\ \Phi(0) = I. \end{cases} \tag{3.2}$$

We say that the system (3.1), or $[A(\cdot), C(\cdot)]$, is *mean-square exponentially stable* if there exist constants $K, \lambda > 0$ such that

$$\mathbb{E}|\Phi(t)|^2 \leqslant Ke^{-\lambda t}, \quad \forall t \geqslant 0. \tag{3.3}$$

As far as the above notion of mean-square exponential stability is concerned, we do not need the coefficients $A(\cdot)$ and $C(\cdot)$ to be periodic. However, when the coefficients $A(\cdot)$ and $C(\cdot)$ are periodic (with the same period $\tau > 0$), we have some further interesting properties. Let us now explore them.

First, it is well-known that $\Phi(t)^{-1}$ exists for any $t \geqslant 0$, and satisfies the following SDE:

$$\begin{cases} d[\Phi(t)^{-1}] = \Phi(t)^{-1}[C(t)^2 - A(t)]dt - \Phi(t)^{-1}C(t)dW(t), & t \geqslant 0, \\ \Phi(0)^{-1} = I. \end{cases}$$

Next, we present the following lemma (recalling that $\tau > 0$ is the period).

Lemma 3.2. *Let* (A1) *hold. Then for any integer $j \geqslant 1$ and any $t \geqslant 0$,*

(i) *$\Phi(t + j\tau)\Phi(j\tau)^{-1}$ and $\Phi(t)$ are identically distributed,*
(ii) *$\Phi(t + j\tau)\Phi(j\tau)^{-1}$ and $\Phi(j\tau)$ are independent.*

Proof. We prove the result for the case $j = 1$; the general case can be proved in a similar way. Let $W_\tau(t) \triangleq W(t + \tau) - W(\tau)$ for $t \geqslant 0$. We know that $W_\tau = \{W_\tau(t); t \geqslant 0\}$ is also a standard Brownian motion, independent of $\mathscr{F}_\tau^W \triangleq \sigma(W(s); 0 \leqslant s \leqslant \tau)$. Further,

$$\begin{aligned} \Phi(t+\tau) &= \Phi(\tau) + \int_\tau^{t+\tau} A(s)\Phi(s)ds + \int_\tau^{t+\tau} C(s)\Phi(s)dW(s) \\ &= \Phi(\tau) + \int_0^t A(s+\tau)\Phi(s+\tau)ds + \int_0^t C(s+\tau)\Phi(s+\tau)dW_\tau(s) \\ &= \Phi(\tau) + \int_0^t A(s)\Phi(s+\tau)ds + \int_0^t C(s)\Phi(s+\tau)dW_\tau(s). \end{aligned}$$

Let $\Gamma(t) \triangleq \Phi(t + \tau)\Phi(\tau)^{-1}$. From the above we have

$$\begin{cases} d\Gamma(t) = A(t)\Gamma(t)dt + C(t)\Gamma(t)dW_\tau(t), \quad t \geqslant 0, \\ \Gamma(0) = I. \end{cases}$$

Thus, by the pathwise uniqueness, $\Gamma(t)$ and $\Phi(t)$ have the same distribution for every $t \geqslant 0$. Moreover, by the strong uniqueness, $\Gamma(\cdot)$ is adapted to the augmented filtration generated by $W_\tau(\cdot)$. Since $W_\tau(\cdot)$ and $\mathscr{F}_\tau^W$ are independent, so are $\Gamma(t)$ and $\Phi(\tau)$. □

We now present a characterization of the mean-square exponential stability.

Proposition 3.3. *Let* (A1) *hold.*

(i) *If* $[A(\cdot), C(\cdot)]$ *is mean-square exponentially stable, then for each* τ*-periodic function* $\Lambda(\cdot) \in L^\infty(0,\infty;\mathbb{S}^n)$*, the Lyapunov differential equation*

$$\dot P(t) + P(t)A(t) + A(t)^\top P(t) + C(t)^\top P(t)C(t) + \Lambda(t) = 0 \tag{3.4}$$

admits a unique τ*-periodic solution* $P(\cdot) \in C([0,\infty);\mathbb{S}^n)$*. Moreover, if* $\Lambda(t) \geqslant 0$ *almost everywhere (respectively,* $\Lambda(\cdot)$ *is uniformly positive definite), then* $P(t) \geqslant 0$ *for all* $t \geqslant 0$ *(respectively,* $P(\cdot)$ *is uniformly positive definite).*

(ii) *Suppose that for some* τ*-periodic, uniformly positive definite function* $\Lambda(\cdot) \in L^\infty(0,\infty;\mathbb{S}^n)$*, equation* (3.4) *admits a* τ*-periodic, uniformly positive definite solution* $P(\cdot) \in C([0,\infty);\mathbb{S}^n)$*. Then* $[A(\cdot), C(\cdot)]$ *is mean-square exponentially stable.*

Proof. (i) Consider the following linear ODE:

$$\begin{cases} \dot P(t) + P(t)A(t) + A(t)^\top P(t) + C(t)^\top P(t)C(t) + \Lambda(t) = 0, \quad t \in [0,\tau], \\ P(\tau) = M, \end{cases}$$

where $M \in \mathbb{S}^n$ is a constant matrix to be determined. Clearly, for each $M \in \mathbb{S}^n$ the above equation has a unique solution $P_M(\cdot) \in C([0,T];\mathbb{S}^n)$, which, by Lemma 2.1, has the following representation:

$$\begin{aligned} P_M(t) = \mathbb{E}\Bigg\{ &[\Phi(\tau)\Phi(t)^{-1}]^\top M[\Phi(\tau)\Phi(t)^{-1}] \\ &+ \int_t^\tau [\Phi(s)\Phi(t)^{-1}]^\top \Lambda(s)[\Phi(s)\Phi(t)^{-1}]ds \Bigg\}, \end{aligned} \tag{3.5}$$

where $\Phi(\cdot)$ is the solution of (3.2). In particular, at $t = 0$, we have

$$P_M(0) = \mathbb{E}\Big[\Phi(\tau)^\top M\Phi(\tau)\Big] + \mathbb{E}\int_0^\tau \Phi(s)^\top \Lambda(s)\Phi(s)ds \equiv \mathbb{E}\Big[\Phi(\tau)^\top M\Phi(\tau)\Big] + L.$$

So we need only show that there exists a unique $M \in \mathbb{S}^n$ such that

$$M = \mathbb{E}\Big[\Phi(\tau)^\top M \Phi(\tau)\Big] + L. \tag{3.6}$$

To this end, let $M_0 = L$ and iteratively define for $k = 1, 2, \dots$,

$$M_k = \mathbb{E}\Big[\Phi(\tau)^\top M_{k-1} \Phi(\tau)\Big] + L.$$

By Lemma 3.2, $\Phi(2\tau)\Phi(\tau)^{-1}$ and $\Phi(\tau)$ are independent and identically distributed. Thus,

$$M_{k-1} = \mathbb{E}\left([\Phi(2\tau)\Phi(\tau)^{-1}]^\top M_{k-2}[\Phi(2\tau)\Phi(\tau)^{-1}]\right) + L,$$

and hence

$$\begin{aligned} M_k &= \mathbb{E}\Big[\Phi(\tau)^\top \mathbb{E}\left([\Phi(2\tau)\Phi(\tau)^{-1}]^\top M_{k-2}[\Phi(2\tau)\Phi(\tau)^{-1}]\right)\Phi(\tau)\Big] \\ &\quad + \mathbb{E}\Big[\Phi(\tau)^\top L \Phi(\tau)\Big] + L \\ &= \mathbb{E}\Big[\Phi(2\tau)^\top M_{k-2} \Phi(2\tau)\Big] + \mathbb{E}\Big[\Phi(\tau)^\top L \Phi(\tau)\Big] + L. \end{aligned}$$

By induction, we get

$$M_k = \sum_{j=0}^{k} \mathbb{E}\Big[\Phi(j\tau)^\top L \Phi(j\tau)\Big].$$

Since $[A(\cdot), C(\cdot)]$ is mean-square exponentially stable, there exist constants $K, \lambda > 0$ such that

$$\left|\mathbb{E}\Big[\Phi(j\tau)^\top L \Phi(j\tau)\Big]\right| \leqslant K e^{-\lambda j \tau}, \quad \forall j \geqslant 0.$$

It follows immediately that M_k converges to a solution of (3.6) as $k \to \infty$. Moreover, if $\Lambda(t) \geqslant 0$ almost everywhere, then $L \geqslant 0$ and hence $M \geqslant 0$. By the representation (3.5), we have $P(t) \geqslant 0$ for all $t \geqslant 0$. If $\Lambda(\cdot)$ is uniformly positive definite, i.e., the condition

$$\Lambda(t) \geqslant \delta I, \quad \text{a.e. } t \geqslant 0$$

holds for some constant $\delta > 0$, then

$$L \triangleq \mathbb{E}\int_0^\tau \Phi(s)^\top \Lambda(s)\Phi(s)ds \geqslant \delta \int_0^\tau \mathbb{E}[\Phi(s)^\top \Phi(s)]ds \geqslant \delta \int_0^\tau \mathbb{E}[\Phi(s)]^\top \mathbb{E}[\Phi(s)]ds.$$

Note that

$$\begin{cases} d\mathbb{E}[\Phi(t)] = A(t)\mathbb{E}[\Phi(t)]dt, & t \geqslant 0, \\ \mathbb{E}[\Phi(0)] = I. \end{cases}$$

Consequently, the continuous function $\mathbb{E}[\Phi(\cdot)]$ is invertible, implying that $\mathbb{E}[\Phi(\cdot)]^\top\mathbb{E}[\Phi(\cdot)]$ is uniformly positive definite over the interval $[0,\tau]$. Let

$$\alpha \triangleq \inf_{0\leqslant t\leqslant\tau}\Big\{\mu(t):\mu(t) \text{ is the smallest eigenvalue of } \mathbb{E}[\Phi(t)]^\top\mathbb{E}[\Phi(t)]\Big\}>0.$$

Then by (3.6),

$$M\geqslant L\geqslant\delta\alpha\tau I.$$

Further, by (3.5),

$$\begin{aligned}P(t)&\geqslant\mathbb{E}\Big\{[\Phi(\tau)\Phi(t)^{-1}]^\top M[\Phi(\tau)\Phi(t)^{-1}]\Big\}\\&\geqslant\delta\alpha\tau\,\mathbb{E}[\Phi(\tau)\Phi(t)^{-1}]^\top\mathbb{E}[\Phi(\tau)\Phi(t)^{-1}],\quad\forall t\in[0,\tau].\end{aligned}\tag{3.7}$$

Set for $s\in[t,\tau]$, $\Upsilon(s)\triangleq\Phi(s)\Phi(t)^{-1}$. Then

$$\begin{cases}d\mathbb{E}[\Upsilon(s)]=A(s)\mathbb{E}[\Upsilon(s)]ds,\quad s\in[t,\tau],\\ \mathbb{E}[\Upsilon(t)]=I.\end{cases}$$

If we let $\Psi(\cdot)$ be the solution to the matrix ODE

$$\begin{cases}d\Psi(s)=A(s)\Psi(s)ds,\quad s\in[0,\tau],\\ \Psi(0)=I,\end{cases}$$

then $\mathbb{E}[\Upsilon(s)]=\Psi(s)\Psi(t)^{-1}$. In particular, $\mathbb{E}[\Phi(\tau)\Phi(t)^{-1}]=\Psi(\tau)\Psi(t)^{-1}$. Let

$$\beta\triangleq\inf_{0\leqslant t\leqslant\tau}\Big\{\mu(t):\mu(t)\text{ is the smallest eigenvalue of }[\Psi(\tau)\Psi(t)^{-1}]^\top[\Psi(\tau)\Psi(t)^{-1}]\Big\}>0.$$

Then from (3.7) we have (noting that $P(\cdot)$ is τ-periodic)

$$P(t)\geqslant\delta^*I\triangleq\delta\alpha\beta\tau I>0,\quad\forall t\geqslant 0.$$

For the uniqueness, we need only show that $M=0$ is the unique solution of

$$M=\mathbb{E}\Big[\Phi(\tau)^\top M\Phi(\tau)\Big].\tag{3.8}$$

Suppose that M solves (3.8), then proceeding similarly as before, we get

$$M=\mathbb{E}\Big[\Phi(k\tau)^\top M\Phi(k\tau)\Big],\quad\forall k\geqslant 1,$$

from which we obtain

$$|M|\leqslant Ke^{-\lambda k\tau},\quad\forall k\geqslant 1.$$

Letting $k \to \infty$ gives $M = 0$.

(ii) Let $\Phi(\cdot)$ be the solution of (3.2). By Itô's rule,

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}[\Phi(t)^\top P(t)\Phi(t)] \\
&= \mathbb{E}\Big\{\Phi(t)^\top[\dot{P}(t) + P(t)A(t) + A(t)^\top P(t) + C(t)^\top P(t)C(t)]\Phi(t)\Big\} \\
&= -\mathbb{E}[\Phi(t)^\top \Lambda(t)\Phi(t)]. \qquad (3.9)
\end{aligned}$$

Since both $P(\cdot)$ and $\Lambda(\cdot)$ are τ-periodic, uniformly positive definite, there are constants $\alpha, \beta > 0$ such that

$$\alpha I \leqslant \Lambda(t),\ P(t) \leqslant \beta I, \quad \text{a.e. } t \in [0,\infty).$$

Thus, (3.9) implies

$$\frac{d}{dt}\mathbb{E}[\Phi(t)^\top P(t)\Phi(t)] \leqslant -\frac{\alpha}{\beta}\mathbb{E}[\Phi(t)^\top P(t)\Phi(t)], \quad \forall t \geqslant 0.$$

By Gronwall's inequality, the mean-square exponential stability of $[A(\cdot), C(\cdot)]$ follows. □

Corollary 3.4. *Let* (A1) *hold. If* $[A(\cdot), C(\cdot)]$ *is mean-square exponentially stable, then there exist constants* $K, \lambda > 0$ *such that for any* $t \geqslant s \geqslant 0$*, the following holds (comparing with* (3.3)*):*

$$\mathbb{E}|\Phi(t)\Phi(s)^{-1}|^2 \leqslant Ke^{-\lambda(t-s)},$$

where $\Phi(\cdot)$ *is the solution of* (3.2)*.*

Proof. By Proposition 3.3, there exists a τ-periodic, uniformly positive definite function $P(\cdot) \in C([0,\infty); \mathbb{S}^n)$ such that

$$\dot{P}(t) + P(t)A(t) + A(t)^\top P(t) + C(t)^\top P(t)C(t) + I = 0, \quad t \geqslant 0.$$

Fix $s \geqslant 0$ and set

$$\Gamma(t) \triangleq \Phi(t+s)\Phi(s)^{-1}, \quad \widetilde{W}(t) \triangleq W(t+s) - W(s); \quad t \geqslant 0.$$

Then

$$\begin{cases} d\Gamma(t) = A(t+s)\Gamma(t)dt + C(t+s)\Gamma(t)d\widetilde{W}(t), & t \geqslant 0, \\ \Gamma(0) = I. \end{cases}$$

By applying Itô's formula to $t \mapsto \Gamma(t)^\top P(t+s)\Gamma(t)$, we can obtain

$$\frac{d}{dt}\mathbb{E}[\Gamma(t)^\top P(t+s)\Gamma(t)] = -\mathbb{E}[\Gamma(t)^\top \Gamma(t)].$$

Let $\alpha, \beta > 0$ be such that

$$\alpha^{-1} I \leqslant P(t) \leqslant \beta^{-1} I, \quad \forall t \in [0, \infty).$$

Then the above implies

$$\frac{d}{dt}\mathbb{E}[\Gamma(t)^\top P(t+s)\Gamma(t)] \leqslant -\beta\, \mathbb{E}[\Gamma(t)^\top P(t+s)\Gamma(t)], \quad \forall t \geqslant 0.$$

By Gronwall's inequality, we have

$$\mathbb{E}[\Gamma(t)^\top \Gamma(t)] \leqslant \alpha \mathbb{E}[\Gamma(t)^\top P(t+s)\Gamma(t)] \leqslant \alpha e^{-\beta t} P(s) \leqslant \frac{\alpha}{\beta} e^{-\beta t} I, \quad \forall t \geqslant 0,$$

from which the desired result follows. □

We note that for the case $C(\cdot) = 0$, namely, for the ODE case, when $A(\cdot)$ is τ-periodic, by the well-known Floquet theory ([11,27]), there are clear characterizations of the fundamental matrix, denoted by $\hat{\Phi}(\cdot)$, and in particular, $\hat{\Phi}(\cdot)$ is not necessarily τ-periodic in general. Here is a simple example.

Example 3.5. Let

$$A(t) = \begin{pmatrix} -1 & \sin t \\ 0 & -1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Then, it is 2π-periodic. The corresponding fundamental matrix $\hat{\Phi}(\cdot)$ is given by

$$\hat{\Phi}(t) = e^{-t} \begin{pmatrix} 1 & 1 - \cos t \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},$$

which is not 2π-periodic, and it is exponentially stable.

Now for the general case, i.e., both $A(\cdot)$ and $C(\cdot)$ might be non-zero and τ-periodic, we do not expect the solution $\Phi(\cdot)$ of (3.2) to be τ-periodic (in the sense that $\Phi(\cdot)$ and $\Phi(\tau + \cdot)$ have the same distributions), but it still could be exponentially stable. The following example is a modification of Example 3.5.

Example 3.6. Consider $A(\cdot)$ as in Example 3.5 and $C(t) = aI$ for some constant a. Then (3.2) reads:

$$d\Phi(t) = A(t)\Phi(t)dt + a\Phi(t)dW(t),$$

whose solution is given by

$$\Phi(t) = e^{-\frac{a^2}{2}t + aW(t)} \hat{\Phi}(t), \quad t \geqslant 0.$$

Then,

$$\begin{aligned}\mathbb{E}\Big[\operatorname{tr}\Big(\Phi(t)^\top\Phi(t)\Big)\Big] &= \Big[2+(1-\cos t)^2\Big]\mathbb{E}\Big[e^{-(a^2+2)t+2aW(t)}\Big]\\ &= \Big[2+(1-\cos t)^2\Big]e^{(a^2-2)t}\to 0, \quad \text{as } t\to\infty,\end{aligned}$$

provided $a^2<2$. This also shows that $\Phi(t+\tau)$ and $\Phi(t)$ cannot have the same distribution for all $t\geqslant 0$. Otherwise, one at least has

$$\mathbb{E}[\Phi(t+\tau)]=\mathbb{E}[\Phi(t)], \quad t\geqslant 0,$$

which contradicts the mean-square exponential stability of $\Phi(\cdot)$. Therefore, $\Phi(\cdot)$ is not τ-periodic (see Definition 4.1).

Next, we consider the following controlled linear SDE, denoted by $[A(\cdot),C(\cdot);\,B(\cdot),\,D(\cdot)]$ (with common period $\tau>0$ for the coefficients):

$$dX(t)=[A(t)X(t)+B(t)u(t)]dt+[C(t)X(t)+D(t)u(t)]dW(t), \quad t\geqslant 0.$$

For this system, we introduce the following notion.

Definition 3.7. System $[A(\cdot),C(\cdot);\,B(\cdot),\,D(\cdot)]$, with the common period τ for the coefficients, is said to be *mean-square exponentially stabilizable* if there exists a τ-periodic function $\Theta(\cdot)\in L^\infty(0,\infty;\mathbb{R}^{m\times n})$ such that $[A(\cdot)+B(\cdot)\Theta(\cdot),C(\cdot)+D(\cdot)\Theta(\cdot)]$ is mean-square exponentially stable. In this case, $\Theta(\cdot)$ is called a *stabilizer* of $[A(\cdot),C(\cdot);\,B(\cdot),\,D(\cdot)]$.

The following result provides a characterization of the mean-square exponential stabilizability.

Proposition 3.8. *Let* (A1) *hold, and let*

$$M(\cdot)\in L^\infty(0,\infty;\mathbb{S}^n), \quad N(\cdot)\in L^\infty(0,\infty;\mathbb{S}^m)$$

be two τ-periodic, uniformly positive definite functions. The system $[A(\cdot),C(\cdot);B(\cdot),D(\cdot)]$ is mean-square exponentially stabilizable if and only if the differential Riccati equation

$$\begin{aligned}&\dot P+PA+A^\top P+C^\top PC+M\\ &\quad-(PB+C^\top PD)(N+D^\top PD)^{-1}(B^\top P+D^\top PC)=0\end{aligned} \tag{3.10}$$

admits a τ-periodic, uniformly positive definite solution $P(\cdot)\in C([0,\infty);\mathbb{S}^n)$. In this case, the τ-periodic, uniformly positive definite solution to (3.10) *is unique, and the function $\Theta(\cdot)$ defined by*

$$\Theta(t)\triangleq-[N(t)+D(t)^\top P(t)D(t)]^{-1}[B(t)^\top P(t)+D(t)^\top P(t)C(t)] \tag{3.11}$$

is a stabilizer of $[A(\cdot),C(\cdot);\,B(\cdot),\,D(\cdot)]$.

Proof. *Sufficiency.* Suppose that (3.10) admits a τ-periodic, uniformly positive definite solution $P(\cdot) \in C([0,\infty);\mathbb{S}^n)$. Let $\Theta(\cdot)$ be defined by (3.11). Clearly, $\Theta(\cdot) \in L^\infty(0,\infty;\mathbb{R}^{m\times n})$ is τ-periodic, and the equation (3.10) can be rewritten as

$$\dot{P} + P(A+B\Theta) + (A+B\Theta)^\top P + (C+D\Theta)^\top P(C+D\Theta) + \Theta^\top N\Theta + M = 0.$$

By Proposition 3.3 (ii), $[A(\cdot)+B(\cdot)\Theta(\cdot), C(\cdot)+D(\cdot)\Theta(\cdot)]$ is mean-square exponentially stable.

Necessity. Let $\Theta_0(\cdot)$ be a stabilizer of $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$. Then by Proposition 3.3, the ODE

$$\begin{aligned}&\dot{P}_1 + P_1(A+B\Theta_0) + (A+B\Theta_0)^\top P_1 + (C+D\Theta_0)^\top P_1(C+D\Theta_0)\\&\quad + \Theta_0^\top N\Theta_0 + M = 0\end{aligned} \tag{3.12}$$

admits a unique τ-periodic, uniformly positive definite solution $P_1(\cdot) \in C([0,\infty);\mathbb{S}^n)$. Let

$$\Theta_1 \triangleq -(N + D^\top P_1 D)^{-1}(B^\top P_1 + D^\top P_1 C).$$

Then we can rewrite (3.12) as follows:

$$\begin{aligned}&\dot{P}_1 + P_1(A+B\Theta_1) + (A+B\Theta_1)^\top P_1 + (C+D\Theta_1)^\top P_1(C+D\Theta_1)\\&\quad + M + \Theta_1^\top N\Theta_1 + (\Theta_0-\Theta_1)^\top(N+D^\top P_1 D)(\Theta_0-\Theta_1) = 0.\end{aligned}$$

We see from Proposition 3.3 (ii) that $\Theta_1(\cdot)$ is a stabilizer of $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$. Now, inductively, set for $i = 1, 2, \cdots,$

$$\Theta_i \triangleq -(N + D^\top P_i D)^{-1}(B^\top P_i + D^\top P_i C), \quad A_i \triangleq A + B\Theta_i, \quad C_i \triangleq C + D\Theta_i,$$

and consider

$$\dot{P}_{i+1} + P_{i+1}A_i + A_i^\top P_{i+1} + C_i^\top P_{i+1}C_i + \Theta_i^\top N\Theta_i + M = 0. \tag{3.13}$$

By induction, we can see that for each $i \geqslant 1$, $\Theta_i(\cdot)$ is a stabilizer of $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$, and the ODE (3.13) has a unique τ-periodic, uniformly positive definite solution $P_{i+1}(\cdot)$. We claim that $\{P_i(\cdot)\}_{i=1}^\infty$ converges pointwise to a limit $P(\cdot)$ that is a τ-periodic, uniformly positive definite solution to the Riccati equation (3.10). To prove this, we set

$$\Delta_i \triangleq P_i - P_{i+1}, \quad \Lambda_i \triangleq \Theta_{i-1} - \Theta_i; \quad i \geqslant 1.$$

For $i \geqslant 1$, we have

$$\begin{aligned}-\dot{\Delta}_i &= \dot{P}_{i+1} - \dot{P}_i\\&= P_iA_{i-1} + A_{i-1}^\top P_i + C_{i-1}^\top P_iC_{i-1} + \Theta_{i-1}^\top N\Theta_{i-1}\\&\quad - P_{i+1}A_i - A_i^\top P_{i+1} - C_i^\top P_{i+1}C_i - \Theta_i^\top N\Theta_i\\&= \Delta_iA_i + A_i^\top\Delta_i + C_i^\top\Delta_iC_i + P_i(A_{i-1}-A_i) + (A_{i-1}-A_i)^\top P_i\end{aligned}$$

$$+ C_{i-1}^\top P_i C_{i-1} - C_i^\top P_i C_i + \Theta_{i-1}^\top N \Theta_{i-1} - \Theta_i^\top N \Theta_i. \tag{3.14}$$

It is easy to check that

$$\begin{cases} A_{i-1} - A_i = B\Lambda_i, \\ C_{i-1}^\top P_i C_{i-1} - C_i^\top P_i C_i = \Lambda_i^\top D^\top P_i D \Lambda_i + C_i^\top P_i D \Lambda_i + \Lambda_i^\top D^\top P_i C_i, \\ \Theta_{i-1}^\top N \Theta_{i-1} - \Theta_i^\top N \Theta_i = \Lambda_i^\top N \Lambda_i + \Lambda_i^\top N \Theta_i + \Theta_i^\top N \Lambda_i. \end{cases} \tag{3.15}$$

Note also that

$$B^\top P_i + D^\top P_i C_i + N\Theta_i = B^\top P_i + D^\top P_i C + (N + D^\top P_i D)\Theta_i = 0.$$

Then plugging (3.15) into (3.14) yields

$$\dot{\Delta}_i + \Delta_i A_i + A_i^\top \Delta_i + C_i^\top \Delta_i C_i + \Lambda_i^\top (N + D^\top P_i D)\Lambda_i = 0. \tag{3.16}$$

Since $[A_i(\cdot), C_i(\cdot)]$ is mean-square exponentially stable, it follows from Proposition 3.3 (i) that $\Delta_i(t) \geqslant 0$ for all $t \geqslant 0$. Thus, for each $i \geqslant 1$,

$$P_i(t) \geqslant P_{i+1}(t) \geqslant 0, \quad \forall t \geqslant 0.$$

By the monotone convergence theorem, the limit $P(t) \equiv \lim_{i\to\infty} P_i(t)$ exists for all $t \geqslant 0$. To show that the τ-periodic function $P(\cdot)$ is a uniformly positive definite solution to (3.10), we observe first that

$$P_{i+1}(t) = P_{i+1}(0) - \int_0^t \Big(P_{i+1}A_i + A_i^\top P_{i+1} + C_i^\top P_{i+1} C_i + \Theta_i^\top N \Theta_i + M\Big) ds. \tag{3.17}$$

Since as $i \to \infty$, we have

$$\Theta_i(t) \to -[N(t) + D(t)^\top P(t) D(t)]^{-1}[B(t)^\top P(t) + D(t)^\top P(t) C(t)] \equiv \Theta(t),$$
$$A_i(t) \to A(t) + B(t)\Theta(t), \qquad C_i(t) \to C(t) + D(t)\Theta(t),$$

the dominated convergence theorem implies that

$$P(t) = P(0) - \int_0^t \Big[P(A+B\Theta) + (A+B\Theta)^\top P + (C+D\Theta)^\top P (C+D\Theta) + \Theta^\top N \Theta + M\Big] ds.$$

By differentiating both sides of the above and substituting for Θ, we see that $P(\cdot)$ satisfies the Riccati equation (3.10).

Finally, let us show that the τ-periodic, uniformly positive definite solution to (3.10) is unique. Suppose that $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$ are two τ-periodic, uniformly positive definite solutions of (3.10). Set $\Sigma \triangleq \Pi_2 - \Pi_1$ and

$$\Upsilon_i \triangleq -(N + D^\top \Pi_i D)^{-1}(B^\top \Pi_i + D^\top \Pi_i C), \quad i = 1, 2.$$

Then by Lemma 2.2,

$$\begin{aligned}&\dot{\Sigma} + \Sigma(A + B\Upsilon_2) + (A + B\Upsilon_2)^\top \Sigma + (C + D\Upsilon_2)^\top \Sigma(C + D\Upsilon_2)\\&\quad + (\Upsilon_2 - \Upsilon_1)^\top (N + D^\top \Pi_1 D)(\Upsilon_2 - \Upsilon_1) = 0.\end{aligned}$$

Since $\Upsilon_2(\cdot)$ is a stabilizer of $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$ and $N(t) + D(t)^\top \Pi_1(t)D(t) \geqslant 0$ for almost all t, we conclude from Proposition 3.3 (i) that

$$\Pi_2(t) - \Pi_1(t) = \Sigma(t) \geqslant 0, \quad \forall t \geqslant 0.$$

Reversing the roles of $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$, we obtain the opposite inequality. □

We now further introduce the following assumption.

(A3) System $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$ is mean-square exponentially stabilizable.

4. Periodic solutions of stochastic linear systems

In this section, we consider the following linear SDE:

$$\begin{cases} dX(t) = [\mathcal{A}(t)X(t) + b(t)]dt + [\mathcal{C}(t)X(t) + \sigma(t)]dW(t), \quad t \geqslant 0,\\ X(0) = \xi, \end{cases} \tag{4.1}$$

where

$$\mathcal{A}(\cdot), \mathcal{C}(\cdot) \in L^\infty(0, \infty; \mathbb{R}^{n\times n}), \quad b(\cdot), \sigma(\cdot) \in L^\infty(0, \infty; \mathbb{R}^n)$$

are τ-periodic functions and $\xi \in \mathscr{F}_0$ is an $\mathbb{R}^n$-valued, square-integrable random vector. Note that $\mathbb{F}$ is not assumed to be the natural filtration generated by W. Thus, ξ is a random vector in general as $\mathscr{F}_0$ is non-trivial.

Definition 4.1. The solution $X(\cdot)$ of (4.1) is said to be τ*-periodic* if $X(\cdot)$ and $X(\tau + \cdot)$ have the same finite-dimensional distributions, i.e., for any integer $m \geqslant 1$, real numbers $0 \leqslant t_1 < t_2 < \cdots < t_m < \infty$, and $A_1, \dots, A_m \in \mathscr{B}(\mathbb{R}^n)$, we have

$$\mathbb{P}[X(\tau + t_1) \in A_1, \dots, X(\tau + t_m) \in A_m] = \mathbb{P}[X(t_1) \in A_1, \dots, X(t_m) \in A_m].$$

We shall show that if $[\mathcal{A}(\cdot), \mathcal{C}(\cdot)]$ is mean-square exponentially stable, there exists a unique initial distribution with finite second moment such that the solution of (4.1) is τ-periodic. For this, let us denote by $\mathcal{P}(\mathbb{R}^n)$ the set of probability measures on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, equipped with the L^2-Wasserstein distance:

$$d(\mu_1, \mu_2) \triangleq \inf\left\{\sqrt{\mathbb{E}|\xi_1 - \xi_2|^2}\ \Big|\ \xi_i \text{ is a random variable in } \mathbb{R}^n \text{ with } \mu_{\xi_i} = \mu_i;\ i = 1, 2\right\},$$

where μ_{ξ_i} denotes the distribution of ξ_i. It is well known that $(\mathcal{P}(\mathbb{R}^n), d)$ is a complete, separable metric space (see, for example, [25]).

Proposition 4.2. *Suppose that* $[\mathcal{A}(\cdot), \mathcal{C}(\cdot)]$ *is mean-square exponentially stable. Then there exists a unique initial distribution* $\bar{\nu}$ *with finite second moment such that the solution of* (4.1) *is* τ*-periodic.*

Proof. Let us first use the fixed-point theorem to show that there exists a unique initial distribution $\bar{\nu}$ such that the corresponding solution of (4.1) satisfies $\mu_{X(\tau)} = \bar{\nu}$. To this end, let ν_1 and ν_2 be two distributions with finite second moments, and let ξ_1 and ξ_2 be $\mathscr{F}_0$-measurable random variables whose distributions are ν_1 and ν_2, respectively. Denote by $X_i(\cdot)$ the solution of (4.1) corresponding to ξ_i. Then $\widehat{X}(\cdot) \triangleq X_1(\cdot) - X_2(\cdot)$ satisfies

$$\begin{cases} d\widehat{X}(t) = \mathcal{A}(t)\widehat{X}(t)dt + \mathcal{C}(t)\widehat{X}(t)dW(t), \quad t \geqslant 0, \\ \widehat{X}(0) = \hat{\xi} \triangleq \xi_1 - \xi_2. \end{cases}$$

Since $[\mathcal{A}(\cdot), \mathcal{C}(\cdot)]$ is mean-square exponentially stable, by Proposition 3.3, the ODE

$$\dot{P}(t) + P(t)\mathcal{A}(t) + \mathcal{A}(t)^\top P(t) + \mathcal{C}(t)^\top P(t)\mathcal{C}(t) + I = 0$$

admits a unique τ-periodic, uniformly positive definite solution $P(\cdot) \in C([0, \infty); \mathbb{S}^n)$. By applying Itô's rule to $t \mapsto \langle P(t)\widehat{X}(t), \widehat{X}(t)\rangle$, we obtain

$$\frac{d}{dt}\mathbb{E}\langle P(t)\widehat{X}(t), \widehat{X}(t)\rangle = -\mathbb{E}|\widehat{X}(t)|^2.$$

Let $\lambda(t) > 0$ be the largest eigenvalue of $P(t)$. Then

$$\frac{d}{dt}\mathbb{E}\langle P(t)\widehat{X}(t), \widehat{X}(t)\rangle \leqslant -\frac{1}{\lambda(t)}\mathbb{E}\langle P(t)\widehat{X}(t), \widehat{X}(t)\rangle.$$

By Gronwall's inequality,

$$\mathbb{E}\langle P(t)\widehat{X}(t), \widehat{X}(t)\rangle \leqslant \mathbb{E}\langle P(0)\widehat{X}(0), \widehat{X}(0)\rangle \exp\left[-\int_0^t \frac{1}{\lambda(s)}ds\right], \quad \forall t \in [0, \tau].$$

In particular, noting that $P(\tau) = P(0)$, we have

$$\mathbb{E}\langle P(0)\widehat{X}(\tau), \widehat{X}(\tau)\rangle \leqslant \mathbb{E}\langle P(0)\widehat{X}(0), \widehat{X}(0)\rangle \exp\left[-\int_0^\tau \frac{1}{\lambda(s)}ds\right]. \tag{4.2}$$

Now we define a new distance $d_P(\cdot, \cdot)$ on $\mathcal{P}(\mathbb{R}^n)$ by

$$d_P(\mu_1,\mu_2) \triangleq \inf\Big\{\sqrt{\mathbb{E}\langle P(0)(\xi_1-\xi_2),\xi_1-\xi_2\rangle}\ \Big|\ \xi_i \text{ is a random variable}$$
$$\text{in } \mathbb{R}^n \text{ with } \mu_{\xi_i}=\mu_i;\ i=1,2\Big\}.$$

Since $P(0)>0$, the metrics $d(\cdot\,,\cdot)$ and $d_P(\cdot\,,\cdot)$ are equivalent. Further, the inequality (4.2) shows that the mapping $\Gamma:\mathcal{P}(\mathbb{R}^n)\to\mathcal{P}(\mathbb{R}^n)$ defined by

$$\Gamma(\nu)\triangleq\mu_{X(\tau;\nu)},$$

where $X(\cdot\,;\nu)$ is the solution of (4.1) with initial distribution ν, is a contraction mapping with respect to the distance $d_P(\cdot\,,\cdot)$. Thus, by the fixed-point theorem, there exists a unique initial distribution $\bar\nu$ such that the solution of (4.1) satisfies $\mu_{X(\tau;\bar\nu)}=\bar\nu$.

To see that the solution of (4.1) with the initial distribution $\bar\nu$ is τ-periodic, choose a $\xi\in\mathscr{F}_0$ with $\mu_\xi=\bar\nu$, and let $X(\cdot)$ be the corresponding solution. Then

$$\begin{aligned}
X(\tau+t)&=X(\tau)+\int_\tau^{\tau+t}[\mathcal{A}(s)X(s)+b(s)]ds+\int_\tau^{\tau+t}[\mathcal{C}(s)X(s)+\sigma(s)]dW(s)\\
&=X(\tau)+\int_0^t[\mathcal{A}(\tau+s)X(\tau+s)+b(\tau+s)]ds\\
&\quad+\int_0^t[\mathcal{C}(\tau+s)X(\tau+s)+\sigma(\tau+s)]d[W(\tau+s)-W(\tau)]\\
&=X(\tau)+\int_0^t[\mathcal{A}(s)X(\tau+s)+b(s)]ds+\int_0^t[\mathcal{C}(s)X(\tau+s)+\sigma(s)]dW_\tau(s).
\end{aligned}$$

Thus, $X_\tau(\cdot)\triangleq X(\tau+\cdot)$ is the solution to the following SDE:

$$\begin{cases} dX_\tau(t)=[\mathcal{A}(t)X_\tau(t)+b(t)]dt+[\mathcal{C}(t)X_\tau(t)+\sigma(t)]dW_\tau(t), \quad t\geqslant 0,\\ X_\tau(0)=X(\tau).\end{cases}$$

Since $\mu_{X(\tau)}=\mu_\xi=\bar\nu$ and $W_\tau=\{W_\tau(t);t\geqslant 0\}$ is also a Brownian motion, it follows from the uniqueness in the sense of probability law that $X_\tau(\cdot)$ and $X(\cdot)$ have the same finite-dimensional distributions. The desired result is therefore proved. □

5. Exponential stability of the Riccati equation

The convergence property of the solution to the Riccati equation associated with Problem $(\text{SLQ})_T$ plays a key role in establishing the turnpike property. We address this convergence issue in this section.

First, let us recall the following result, which is concerned with the solvability of Problem $(\text{SLQ})_T$ (for fixed T) and whose proof can be found in [21, Chapter 2].

Lemma 5.1. *Let* (A1)–(A2) *hold. Then the differential Riccati equation*

$$\begin{cases} \dot{P}_T + P_T A + A^\top P_T + C^\top P_T C + Q \\ \qquad - (P_T B + C^\top P_T D + S^\top)(R + D^\top P_T D)^{-1}(B^\top P_T + D^\top P_T C + S) = 0, \\ P_T(T) = 0 \end{cases} \tag{5.1}$$

admits a unique solution $P_T(\cdot) \in C([0,T];\mathbb{S}^n)$ *satisfying* $P_T(t) \geqslant 0$ *for all* $t \in [0,T]$. *Furthermore, for each initial state* $x \in \mathbb{R}^n$, *Problem (SLQ)*$_T$ *admits a unique optimal control given by*

$$\bar{u}_T(t) = \Theta_T(t)\bar{X}_T(t) + \phi_T(t), \tag{5.2}$$

where $\Theta_T(\cdot)$ *and* $\phi_T(\cdot)$ *are defined by*

$$\Theta_T(t) \triangleq -\Big[R(t) + D(t)^\top P_T(t)D(t)\Big]^{-1}\Big[B(t)^\top P_T(t) + D(t)^\top P_T(t)C(t) + S(t)\Big] \tag{5.3}$$

and

$$\phi_T(t) \triangleq -\Big[R(t) + D(t)^\top P_T(t)D(t)\Big]^{-1}\Big[B(t)^\top \varphi_T(t) + D(t)^\top P_T(t)\sigma(t) + r(t)\Big], \tag{5.4}$$

respectively, with $\varphi_T(\cdot)$ *being the solution of the following ODE:*

$$\begin{cases} \dot{\varphi}_T(t) + [A(t) + B(t)\Theta_T(t)]^\top \varphi_T(t) + [C(t) + D(t)\Theta_T(t)]^\top P_T(t)\sigma(t) \\ \qquad + \Theta_T(t)^\top r(t) + P_T(t)b(t) + q(t) = 0, \quad t \in [0,T], \\ \varphi_T(T) = 0. \end{cases} \tag{5.5}$$

The following result gives some properties of the family $\{P_T(\cdot)\}_{T\geqslant 0}$.

Proposition 5.2. *Let* (A1)–(A2) *hold. The solution* $P_T(\cdot)$ *has the following properties:*

(i) $P_T(\cdot)$ *is nondecreasing in* T, *that is,*

$$P_{T_1}(t) \leqslant P_{T_2}(t), \quad \forall 0 \leqslant t \leqslant T_1 \leqslant T_2 < \infty. \tag{5.6}$$

(ii) *For any* $0 \leqslant t \leqslant T < \infty$,

$$P_{T+\tau}(t+\tau) = P_T(t). \tag{5.7}$$

Proof. (i) Let $T_2 > T_1 > 0$ and denote $P_{T_i}(\cdot)$ simply by $P_i(\cdot)$ $(i = 1,2)$. Define

$$\bar{P}(t) \triangleq P_2(t) - P_1(t), \quad t \in [0, T_1],$$

and set for $i = 1,2$,

$$\Theta_i(t) \triangleq -\Big[R(t) + D(t)^\top P_i(t)D(t)\Big]^{-1}\Big[B(t)^\top P_i(t) + D(t)^\top P_i(t)C(t) + S(t)\Big].$$

Then on $[0, T_1]$, we have by Lemma 2.2,

$$\begin{cases} \dot{\bar{P}} + \bar{P}(A + B\Theta_2) + (A + B\Theta_2)^\top \bar{P} + (C + D\Theta_2)^\top \bar{P}(C + D\Theta_2) \\ \quad + (\Theta_2 - \Theta_1)^\top (R + D^\top P_1 D)(\Theta_2 - \Theta_1) = 0, \\ \bar{P}(T_1) \geqslant 0. \end{cases}$$

Applying Lemma 2.1 yields (5.6).

(ii) Set for $t \in [0, T]$,

$$\Pi(t) \triangleq P_{T+\tau}(t + \tau).$$

Then $\Pi(T) = 0$, and

$$\begin{aligned} &\dot{\Pi}(t) + \Pi(t)A(t) + A(t)^\top \Pi(t) + C(t)^\top \Pi(t)C(t) + Q(t) \\ &\quad - \Big[\Pi(t)B(t) + C(t)^\top \Pi(t)D(t) + S(t)^\top\Big] \\ &\quad \times \Big[R(t) + D(t)^\top \Pi(t)D(t)\Big]^{-1} \Big[B(t)^\top \Pi(t) + D(t)^\top \Pi(t)C(t) + S(t)\Big] = 0, \end{aligned}$$

since the coefficients are all τ-periodic by (A1). By the uniqueness of solutions to (5.1), we have $\Pi(t) = P_T(t)$ and hence (5.7). □

By Proposition 5.2 (i), the limit

$$P_\infty(t) \triangleq \lim_{T\to\infty} P_T(t) \geqslant 0 \tag{5.8}$$

exists for all $t \geqslant 0$. We are interested in determining the equation satisfied by $P_\infty(\cdot)$ and how fast $P_T(t)$ converges to $P_\infty(t)$. To address this question, let us consider the differential Riccati equation

$$\begin{aligned} &\dot{P} + PA + A^\top P + C^\top PC + Q \\ &\quad - (PB + C^\top PD + S^\top)(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S) = 0 \end{aligned} \tag{5.9}$$

over the infinite time horizon $[0, \infty)$.

Proposition 5.3. *Let* (A1)–(A3) *hold. Then the Riccati equation* (5.9) *admits a unique τ-periodic, uniformly positive definite solution $P(\cdot) \in C([0, \infty); \mathbb{S}^n)$. Moreover, the matrix-valued function $\Theta(\cdot)$ defined by*

$$\Theta(t) \triangleq -\Big[R(t) + D(t)^\top P(t)D(t)\Big]^{-1} \Big[B(t)^\top P(t) + D(t)^\top P(t)C(t) + S(t)\Big] \tag{5.10}$$

is a stabilizer of $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$.

Proof. Consider the following ODE:

$$\begin{aligned} &\dot{P} + P\mathcal{A} + \mathcal{A}^\top P + \mathcal{C}^\top P\mathcal{C} + \mathcal{Q} \\ &\quad - (PB + \mathcal{C}^\top PD)(R + D^\top PD)^{-1}(B^\top P + D^\top P\mathcal{C}) = 0, \end{aligned} \tag{5.11}$$

where

$$\mathcal{A}(\cdot) \triangleq (A - BR^{-1}S)(\cdot), \quad \mathcal{C}(\cdot) \triangleq (C - DR^{-1}S)(\cdot), \quad \mathcal{Q}(\cdot) \triangleq (Q - S^\top R^{-1}S)(\cdot).$$

By a direct computation, one can verify that (5.9) and (5.11) are equivalent. Since $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$ is stabilizable, so is $[\mathcal{A}(\cdot), \mathcal{C}(\cdot); B(\cdot), D(\cdot)]$. Thus, by Proposition 3.8, the ODE (5.11), and hence (5.9), has a unique τ-periodic, uniformly positive definite solution $P(\cdot)$. Again, by Proposition 3.8,

$$\begin{aligned} &-\Big[R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)\Big]^{-1}\Big[B(\cdot)^\top P(\cdot) + D(\cdot)^\top P(\cdot)C(\cdot) + S(\cdot)\Big] + R(\cdot)^{-1}S(\cdot) \\ &\quad = -\Big[R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)\Big]^{-1}\Big[B(\cdot)^\top P(\cdot) + D(\cdot)^\top P(\cdot)\mathcal{C}(\cdot)\Big] \end{aligned}$$

is a stabilizer of $[\mathcal{A}(\cdot), \mathcal{C}(\cdot); B(\cdot), D(\cdot)]$, which implies that the function $\Theta(\cdot)$ defined by (5.10) is a stabilizer of $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$. □

Let $P_T(\cdot)$ be the unique solution of (5.1), and let $P(\cdot)$ be the unique τ-periodic, uniformly positive definite solution to (5.9). The following result shows that $P_\infty(t) = P(t)$.

Proposition 5.4. *Let* (A1)–(A3) *hold. Then*

$$\lim_{T\to\infty} P_T(t) = P(t), \quad \forall t \geqslant 0.$$

Proof. Set

$$\Sigma_T(t) = P(t) - P_T(t), \quad t \in [0, T].$$

Then $\Sigma_T(T) \geqslant 0$. Proceeding similarly to the proof of Proposition 5.2 (i), we obtain

$$P_T(t) \leqslant P(t), \quad \forall 0 \leqslant t \leqslant T < \infty.$$

This shows that $P_\infty(t) \leqslant P(t)$. On the other hand, by Proposition 5.2 (ii),

$$P_\infty(\tau + t) = \lim_{T\to\infty} P_T(\tau + t) = \lim_{T\to\infty} P_{T+\tau}(\tau + t) = P_\infty(t).$$

Thus, $P_\infty(\cdot)$ is τ-periodic. To see $P_\infty(\cdot) = P(\cdot)$, it suffices to show that $P_\infty(\cdot)$ satisfies the same equation as $P(\cdot)$. For this, we observe that for any $0 \leqslant s \leqslant t \leqslant T$,

$$P_T(t) - P_T(s) = -\int_s^t \Big[P_T A + A^\top P_T + C^\top P_T C + Q - (P_T B + C^\top P_T D + S^\top)$$

$$\times (R + D^\top P_T D)^{-1}(B^\top P_T + D^\top P_T C + S)\Big]dr.$$

Letting $T \to \infty$, we obtain from the dominated convergence theorem that

$$P_\infty(t) - P_\infty(s) = -\int_s^t \Big[P_\infty A + A^\top P_\infty + C^\top P_\infty C + Q - (P_\infty B + C^\top P_\infty D + S^\top)$$

$$\times (R + D^\top P_\infty D)^{-1}(B^\top P_\infty + D^\top P_\infty C + S)\Big]dr.$$

Differentiating with respect to t, we get the desired result. □

To see how quickly $P_T(t)$ converges to $P(t)$, let us present the following lemma first.

Lemma 5.5. *Let $a > 0$ and $g : [0, a] \to [0, \infty)$ be a continuous function satisfying*

$$g(t) \leqslant K + \int_0^t e^{-\lambda s} g(s)^2 ds, \quad \forall t \in [0, a],$$

where $K, \lambda > 0$ are two constants. If $K < \lambda$, then

$$g(t) \leqslant \frac{\lambda K}{\lambda - K}, \quad \forall t \in [0, a].$$

Proof. Set for $t \in [0, a]$,

$$G(t) \triangleq \int_0^t e^{-\lambda s} g(s)^2 ds.$$

Then $G(0) = 0$ and

$$e^{\lambda t} G'(t) = g(t)^2 \leqslant [K + G(t)]^2,$$

or equivalently,

$$d\left[\frac{-1}{K + G(t)}\right] \leqslant e^{-\lambda t}, \quad \forall t \in [0, a].$$

Integration gives

$$\frac{1}{K}-\frac{1}{K+G(t)}\leqslant\int_0^t e^{-\lambda s}ds\leqslant\frac{1}{\lambda},\qquad\forall t\in[0,a].$$

The rest of the proof is clear. □

Now we state and prove the main result of this section, which shows that $P_T(t)$ converges exponentially to $P(t)$.

Theorem 5.6. *Let* (A1)–(A3) *hold. Then there exist constants $K,\lambda>0$, independent of T, such that*

$$|P_T(t)-P(t)|\leqslant Ke^{-\lambda(T-t)},\qquad\forall t\in[0,T].$$

Proof. Set $\Sigma_T(\cdot)\triangleq P(\cdot)-P_T(\cdot)$. Then by Lemma 2.2,

$$\begin{aligned}&\dot\Sigma_T+\Sigma_T(A+B\Theta)+(A+B\Theta)^\top\Sigma_T+(C+D\Theta)^\top\Sigma_T(C+D\Theta)\\&\quad+(\Theta-\Theta_T)^\top(R+D^\top P_TD)(\Theta-\Theta_T)=0,\end{aligned}$$

where $\Theta_T(\cdot)$ and $\Theta(\cdot)$ are defined by (5.3) and (5.10), respectively. Let

$$\Lambda_T\triangleq(\Theta-\Theta_T)^\top(R+D^\top P_TD)(\Theta-\Theta_T).$$

Since $R+D^\top P_TD\geqslant R$ and

$$\begin{aligned}\Theta-\Theta_T&=(R+D^\top P_TD)^{-1}(B^\top P_T+D^\top P_TC+S)\\&\quad-(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)\\&=-(R+D^\top P_TD)^{-1}(B^\top\Sigma_T+D^\top\Sigma_TC)\\&\quad+\Big[(R+D^\top P_TD)^{-1}-(R+D^\top PD)^{-1}\Big](B^\top P+D^\top PC+S)\\&=-(R+D^\top P_TD)^{-1}(B^\top\Sigma_T+D^\top\Sigma_TC)\\&\quad+(R+D^\top P_TD)^{-1}D^\top\Sigma_TD(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)\\&=-(R+D^\top P_TD)^{-1}[B^\top\Sigma_T+D^\top\Sigma_T(C+D\Theta)],\end{aligned}$$

we conclude that

$$|\Lambda_T(t)|\leqslant K_1|\Sigma_T(t)|^2,\qquad\forall 0\leqslant t\leqslant T<\infty\tag{5.12}$$

for some constant $K_1>0$ that is independent of T. Let $\Phi_\Theta(\cdot)$ be the solution to the following SDE:

$$\begin{cases}d\Phi_\Theta(t)=[A(t)+B(t)\Theta(t)]\Phi(t)dt+[C(t)+D(t)\Theta(t)]\Phi(t)dW(t),\quad t\geqslant0,\\ \Phi_\Theta(0)=I.\end{cases}$$

By Proposition 5.3, $[A(\cdot)+B(\cdot)\Theta(\cdot), C(\cdot)+D(\cdot)\Theta(\cdot)]$ is mean-square exponentially stable. Thus, by Corollary 3.4, there exist constants $K_2, \lambda > 0$ such that

$$\mathbb{E}|\Phi_\Theta(s)\Phi_\Theta(t)^{-1}|^2 \leqslant K_2 e^{-\lambda(s-t)}, \quad \forall s \geqslant t \geqslant 0.$$

According to Proposition 5.4, we can choose an integer $N > 0$ such that

$$\rho \triangleq K_2|\Sigma_N(0)| \leqslant \frac{\lambda}{2K_1K_2}.$$

Now, to prove the result, we need only show that for any $T \geqslant N+\tau$, the inequality

$$|\Sigma_T(t)| \leqslant K e^{-\lambda(T-t)}, \quad \forall t \in [0,T] \tag{5.13}$$

holds for some constant $K > 0$ independent of T. For this, let $T \geqslant N+\tau$ be fixed but arbitrary, and let $k \geqslant 1$ be the largest integer such that $N + k\tau \leqslant T$. By Proposition 5.2 and the definition of $\Sigma_T(\cdot)$, we have

$$0 \leqslant \Sigma_T(k\tau) \leqslant \Sigma_{N+k\tau}(k\tau) = \Sigma_N(0),$$

and thereby $K_2|\Sigma_T(k\tau)| \leqslant K_2|\Sigma_N(0)| = \rho$. On the other hand, from Lemma 2.1 we see that for $t \leqslant k\tau$,

$$\begin{aligned}\Sigma_T(t) = \mathbb{E}\Big\{&[\Phi_\Theta(k\tau)\Phi_\Theta(t)^{-1}]^\top \Sigma_T(k\tau)[\Phi_\Theta(k\tau)\Phi_\Theta(t)^{-1}] \\ &+ \int_t^{k\tau} [\Phi_\Theta(s)\Phi_\Theta(t)^{-1}]^\top \Lambda_T(s)[\Phi_\Theta(s)\Phi_\Theta(t)^{-1}]ds\Big\},\end{aligned} \tag{5.14}$$

which, together with (5.12), implies

$$\begin{aligned}|\Sigma_T(t)| &\leqslant |\Sigma_T(k\tau)|\cdot\mathbb{E}|\Phi_\Theta(k\tau)\Phi_\Theta(t)^{-1}|^2 + K_1\int_t^{k\tau}\mathbb{E}|\Phi_\Theta(s)\Phi_\Theta(t)^{-1}|^2\cdot|\Sigma_T(s)|^2 ds \\ &\leqslant \rho e^{-\lambda(k\tau-t)} + K_1K_2\int_t^{k\tau} e^{-\lambda(s-t)}|\Sigma_T(s)|^2 ds, \quad \forall 0 \leqslant t \leqslant k\tau.\end{aligned}$$

Set $K_3 \triangleq K_1K_2$ and

$$g(t) \triangleq K_3 e^{\lambda t}|\Sigma_T(k\tau - t)|, \quad 0 \leqslant t \leqslant k\tau.$$

Then for any $0 \leqslant t \leqslant k\tau$,

$$g(t) \leqslant K_3\rho + \int_0^t e^{-\lambda s} g(s)^2 ds \leqslant \frac{\lambda}{2} + \int_0^t e^{-\lambda s} g(s)^2 ds.$$

According to Lemma 5.5, we have (noting that $T \leqslant N + (k+1)\tau$)

$$
\begin{aligned}
|\Sigma_T(t)| &= \frac{1}{K_3} e^{-\lambda(k\tau - t)} g(k\tau - t) \leqslant \frac{\lambda}{K_3} e^{-\lambda(k\tau - t)} = \frac{\lambda}{K_3} e^{\lambda(T - k\tau)} e^{-\lambda(T-t)} \\
&\leqslant \frac{\lambda}{K_3} e^{\lambda(N+\tau)} e^{-\lambda(T-t)}, \quad \forall t \in [0, k\tau].
\end{aligned} \tag{5.15}
$$

For $t \in [k\tau, T]$, since

$$
N + k\tau \leqslant T \leqslant N + (k+1)\tau,
$$

we have $0 \leqslant t - k\tau \leqslant N + \tau$, and

$$
0 \leqslant \Sigma_T(t) \leqslant \Sigma_{N+(k+1)\tau}(t) = \Sigma_{N+\tau}(t - k\tau).
$$

Let $K_4 \triangleq \max_{s\in[0,N+\tau]} |\Sigma_{N+\tau}(s)|$. Then

$$
|\Sigma_T(t)| \leqslant |\Sigma_{N+\tau}(t - k\tau)| \leqslant K_4 \leqslant K_4 e^{\lambda(N+\tau)} e^{-\lambda(T-t)}, \quad \forall t \in [k\tau, T]. \tag{5.16}
$$

Combining (5.15) and (5.16) yields the desired (5.13). □

The following is a direct consequence of Theorem 5.6.

Corollary 5.7. *Let $\Theta_T(\cdot)$ and $\Theta(\cdot)$ be defined by (5.3) and (5.10), respectively. Then there exist constants $K, \lambda > 0$, independent of T, such that*

$$
|\Theta_T(t) - \Theta(t)| \leqslant K e^{-\lambda(T-t)}, \quad \forall t \in [0, T].
$$

6. A deterministic periodic LQ optimal control problem

In this section we introduce a deterministic periodic LQ optimal control problem and establish its solvability. As we shall see in the next section, the optimal pair of this periodic optimal control problem serves as the turnpike limit of Problem (SLQ)$_T$ in the expectation sense.

Let $P(\cdot) \in C([0, \infty); \mathbb{S}^n)$ be the unique τ-periodic, uniformly positive definite solution to the Riccati equation (5.9) and $\Theta(\cdot)$ the matrix defined by (5.10). Let

$$
\widehat{A}(t) \triangleq A(t) + B(t)\Theta(t), \quad \widehat{C}(t) \triangleq C(t) + D(t)\Theta(t), \tag{6.1}
$$

$$
\begin{aligned}
f(t,x,u) \triangleq{}& \langle Q(t)x, x\rangle + 2\langle S(t)x, u\rangle + \langle R(t)u, u\rangle + 2\langle q(t), x\rangle + 2\langle r(t), u\rangle \\
&+ \langle P(t)[C(t)x + D(t)u + \sigma(t)], C(t)x + D(t)u + \sigma(t)\rangle,
\end{aligned} \tag{6.2}
$$

$$
g(t,x,u) \triangleq f(t, x, \Theta(t)x + u), \tag{6.3}
$$

and denote by $L^2(0, \tau; \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued functions that are square-integrable over $[0, \tau]$ (recall that τ is the period in (A1)). Consider the ODE

$$\begin{cases} \dot{x}(t) = \widehat{A}(t)x(t) + B(t)u(t) + b(t), \quad t \in [0, \tau], \\ x(0) = x(\tau), \end{cases} \tag{6.4}$$

and the cost functional

$$J_\tau(u(\cdot)) \triangleq \int_0^\tau g(t, x(t), u(t))dt. \tag{6.5}$$

By Proposition 5.3, we know that the solution of the matrix ODE

$$\begin{cases} d\widehat{\Phi}(t) = \widehat{A}(t)\widehat{\Phi}(t)dt, \quad t \geqslant 0, \\ \widehat{\Phi}(0) = I \end{cases} \tag{6.6}$$

is exponentially stable, i.e., there exist constants $K, \lambda > 0$ such that

$$|\widehat{\Phi}(t)| \leqslant Ke^{-\lambda t}, \quad \forall t \geqslant 0. \tag{6.7}$$

Before introducing the deterministic periodic LQ optimal control problem, let us first present the following result concerning the well-posedness of the ODE (6.4).

Proposition 6.1. *Let* (A1)–(A3) *hold. Then for any* $u(\cdot) \in L^2(0, \tau; \mathbb{R}^m)$*, the ODE* (6.4) *has a unique solution.*

Proof. By the variation of constants formula, $x(\cdot)$ is a solution of (6.4) if and only if

$$\begin{cases} x(t) = \widehat{\Phi}(t)x(0) + \widehat{\Phi}(t)\int_0^t \widehat{\Phi}(s)^{-1}\Big[B(s)u(s) + b(s)\Big]ds, \quad t \in [0, \tau], \\ x(\tau) = x(0), \end{cases}$$

or equivalently, if and only if

$$\begin{cases} x(t) = \widehat{\Phi}(t)x(0) + \widehat{\Phi}(t)\int_0^t \widehat{\Phi}(s)^{-1}\Big[B(s)u(s) + b(s)\Big]ds, \quad t \in [0, \tau], \\ \Big[I - \widehat{\Phi}(\tau)\Big]x(0) = \widehat{\Phi}(\tau)\int_0^\tau \widehat{\Phi}(s)^{-1}\Big[B(s)u(s) + b(s)\Big]ds. \end{cases}$$

Note that $[\widehat{\Phi}(\tau)]^n = \widehat{\Phi}(n\tau)$ for every positive integer n. It follows that $I - \widehat{\Phi}(\tau)$ is invertible, since by (6.7),

$$\lim_{n\to\infty} \Big|[\widehat{\Phi}(\tau)]^n\Big| = \lim_{n\to\infty} \Big|\widehat{\Phi}(n\tau)\Big| \leqslant \lim_{n\to\infty} Ke^{-\lambda n\tau} = 0.$$

Thus, the ODE (6.4) has a unique solution with the initial state $x(0)$ given by

$$x(0)=\Big[I-\widehat{\Phi}(\tau)\Big]^{-1}\widehat{\Phi}(\tau)\int_0^\tau \widehat{\Phi}(s)^{-1}\Big[B(s)u(s)+b(s)\Big]ds.$$

The proof is complete. □

Remark 6.2. It is worth noting that the ODE (6.4) may have many solutions if $\widehat{A}(\cdot)$ is not exponentially stable. For example, consider the one-dimensional ODE

$$\begin{cases}\dot{x}(t)=u(t), & t\in[0,\tau],\\ x(0)=x(\tau).\end{cases}$$

If $u(\cdot)$ satisfies $\int_0^\tau u(t)dt=0$, then the above equation has infinite many (periodic) solutions. Because $A(\cdot)$ might not be stable, we take (6.4) as the state equation instead of the following ODE:

$$\begin{cases}\dot{x}(t)=A(t)x(t)+B(t)u(t)+b(t), & t\in[0,\tau],\\ x(0)=x(\tau).\end{cases}$$

We now introduce the deterministic periodic LQ optimal control problem.

Problem (DLQ)$_\tau$. Find a control $u^*(\cdot)\in L^2(0,\tau;\mathbb{R}^m)$ such that

$$J_\tau(u^*(\cdot))=\inf_{u(\cdot)\in L^2(0,\tau;\mathbb{R}^m)} J_\tau(u(\cdot)). \tag{6.8}$$

Optimal control for deterministic periodic system was studied by Colonius [5] and Da Prato–Ichikawa [6]. The following result provides a characterization for the optimal control of Problem (DLQ)$_\tau$, which can be regarded as a minimum principle for Problem (DLQ)$_\tau$. This can also be regarded as some modifications of some relevant results in [6].

Proposition 6.3. *Let* (A1)–(A3) *hold. Then* $(x^*(\cdot),u^*(\cdot))$ *is an optimal pair of Problem (DLQ)*$_\tau$ *if and only if the solution of*

$$\begin{cases}\dot{y}^*(t)=-\widehat{A}(t)^\top y^*(t)-\dfrac{1}{2}g_x^\top(t,x^*(t),u^*(t)), & t\in[0,\tau],\\ y^*(0)=y^*(\tau)\end{cases} \tag{6.9}$$

satisfies

$$B(t)^\top y^*(t)+\frac{1}{2}g_u^\top(t,x^*(t),u^*(t))=0, \quad a.e.\ t\in[0,\tau]. \tag{6.10}$$

Remark 6.4. The equation (6.9) has a unique solution. Indeed, let $\widehat{\Psi}(\cdot)$ be the solution to the matrix ODE

$$\begin{cases} d\widehat{\Psi}(t) = -\widehat{A}(t)^\top \widehat{\Psi}(t)dt, & t \geqslant 0, \\ \widehat{\Psi}(0) = I. \end{cases} \tag{6.11}$$

One can verify that

$$\widehat{\Psi}(t) = \Big[\widehat{\Phi}(t)^\top\Big]^{-1}, \quad t \geqslant 0.$$

Thus,

$$I - \widehat{\Psi}(\tau) = \Big[\widehat{\Phi}(\tau)^\top - I\Big]\widehat{\Psi}(\tau) = \Big[\widehat{\Phi}(\tau) - I\Big]^\top \widehat{\Psi}(\tau)$$

is invertible. Then we can proceed similarly to the proof of Proposition 6.1 to obtain the unique solvability of (6.9).

Proof of Proposition 6.3. By definition, a control $u^*(\cdot) \in L^2(0,\tau;\mathbb{R}^m)$ is optimal for Problem (DLQ)$_\tau$ if and only if

$$J_\tau(u^*(\cdot) + \varepsilon v(\cdot)) - J_\tau(u^*(\cdot)) \geqslant 0, \quad \forall \varepsilon \in \mathbb{R},\ \forall v(\cdot) \in L^2(0,\tau;\mathbb{R}^m). \tag{6.12}$$

Let $x^*(\cdot)$ and $x^\varepsilon(\cdot)$ be the solutions of (6.4) corresponding to $u^*(\cdot)$ and $u^\varepsilon(\cdot) \triangleq u^*(\cdot) + \varepsilon v(\cdot)$, respectively, and let $x^v(\cdot)$ be the solution of

$$\begin{cases} \dot{x}(t) = \widehat{A}(t)x(t) + B(t)v(t), & t \in [0,\tau], \\ x(0) = x(\tau). \end{cases}$$

Clearly, $x^\varepsilon(\cdot) = x^*(\cdot) + \varepsilon x^v(\cdot)$, which, substituting into $J_\tau(u^*(\cdot) + \varepsilon v(\cdot))$ yields

$$\begin{aligned} & J_\tau(u^*(\cdot) + \varepsilon v(\cdot)) - J_\tau(u^*(\cdot)) \\ &\quad = \int_0^\tau \Big[g(t, x^\varepsilon(t), u^\varepsilon(t)) - g(t, x^*(t), u^*(t))\Big]dt \\ &\quad = \varepsilon^2 \int_0^\tau \Big\{\Big\langle Q(t)x^v(t), x^v(t)\Big\rangle + \Big\langle R(t)\hat{v}(t), \hat{v}(t)\Big\rangle + 2\Big\langle S(t)x^v(t), \hat{v}(t)\Big\rangle \\ &\qquad + \Big\langle P(t)[C(t)x^v(t) + D(t)\hat{v}(t)], C(t)x^v(t) + D(t)\hat{v}(t)\Big\rangle\Big\}dt \\ &\qquad + 2\varepsilon \int_0^\tau \Big\{\Big\langle \hat{v}(t), \beta_1(t)\Big\rangle + \Big\langle x^v(t), \beta_2(t)\Big\rangle\Big\}dt, \end{aligned} \tag{6.13}$$

where we have used the notation

$$\hat{v}(t) \triangleq \Theta(t)x^v(t) + v(t), \quad \hat{u}^*(t) \triangleq \Theta(t)x^*(t) + u^*(t),$$
$$\beta_1(t) \triangleq R(t)\hat{u}^*(t) + S(t)x^*(t) + r(t) + D(t)^\top P(t)[C(t)x^*(t) + D(t)\hat{u}^*(t) + \sigma(t)],$$
$$\beta_2(t) \triangleq Q(t)x^*(t) + S(t)^\top \hat{u}^*(t) + q(t) + C(t)^\top P(t)[C(t)x^*(t) + D(t)\hat{u}^*(t) + \sigma(t)].$$

Note that by (A2) the first integral after the last equal sign in (6.13) is always nonnegative. Thus, (6.12) is equivalent to

$$\int_0^\tau \Big\{\Big\langle \hat{v}(t), \beta_1(t)\Big\rangle + \Big\langle x^v(t), \beta_2(t)\Big\rangle\Big\}dt = 0, \quad \forall v(\cdot) \in L^2(0,\tau;\mathbb{R}^m). \tag{6.14}$$

Observe that

$$\begin{aligned}
\frac{1}{2}g_u(t,x,u)^\top &= \frac{1}{2}f_u(t,x,\Theta(t)x+u)^\top \\
&= R(t)[\Theta(t)x+u] + S(t)x + r(t) \\
&\quad + D(t)^\top P(t)\big\{C(t)x + D(t)[\Theta(t)x+u] + \sigma(t)\big\}, \\
\frac{1}{2}g_x(t,x,u)^\top &= \frac{1}{2}f_x(t,x,\Theta(t)x+u)^\top + \frac{1}{2}\Theta(t)^\top f_u(t,x,\Theta(t)x+u)^\top \\
&= \frac{1}{2}\Theta(t)^\top g_u(t,x,u)^\top + Q(t)x + S(t)^\top[\Theta(t)x+u] + q(t) \\
&\quad + C(t)^\top P(t)\big\{C(t)x + D(t)[\Theta(t)x+u] + \sigma(t)\big\}.
\end{aligned}$$

Thus,

$$\beta_1(t) = \frac{1}{2}g_u(t,x^*(t),u^*(t))^\top, \quad \beta_2(t) = \frac{1}{2}g_x(t,x^*(t),u^*(t))^\top - \Theta(t)^\top\beta_1(t),$$

and thereby

$$\begin{aligned}
&\int_0^\tau \Big\{\Big\langle \hat{v}(t), \beta_1(t)\Big\rangle + \Big\langle x^v(t), \beta_2(t)\Big\rangle\Big\}dt \\
&\quad = \frac{1}{2}\int_0^\tau \Big[g_x(t,x^*(t),u^*(t))x^v(t) + g_u(t,x^*(t),u^*(t))v(t)\Big]dt.
\end{aligned} \tag{6.15}$$

Now applying integration by parts, we have

$$\begin{aligned}
0 &= \langle x^v(\tau), y^*(\tau)\rangle - \langle x^v(0), y^*(0)\rangle \\
&= \int_0^\tau \Big[\langle B(t)^\top y^*(t), v(t)\rangle - \frac{1}{2}g_x(t,x^*(t),u^*(t))x^v(t)\Big]dt,
\end{aligned}$$

which, together with (6.15), implies that (6.14) is equivalent to

$$\int_0^\tau \Big\langle B(t)^\top y^*(t) + \frac{1}{2} g_u^\top(t, x^*(t), u^*(t)), v(t)\Big\rangle dt = 0, \quad \forall v(\cdot) \in L^2(0, \tau; \mathbb{R}^m).$$

The above is clearly equivalent to (6.10). □

The next result establishes the solvability of Problem (DLQ)$_\tau$ and provides an explicit representation for the optimal control.

Proposition 6.5. *Let* (A1)–(A3) *hold. Then Problem (DLQ)$_\tau$ admits a unique optimal control, which is given by*

$$u_\tau^*(t) = -\Big[R(t) + D(t)^\top P(t) D(t)\Big]^{-1}\Big[B(t)^\top \eta_\tau(t) + D(t)^\top P(t)\sigma(t) + r(t)\Big], \tag{6.16}$$

where $\eta_\tau(\cdot)$ is the solution to the following ODE:

$$\begin{cases} \dot\eta_\tau(t) + \widehat A(t)^\top \eta_\tau(t) + \widehat C(t)^\top P(t)\sigma(t) + \Theta(t)^\top r(t) + P(t)b(t) + q(t) = 0, \\ \eta_\tau(0) = \eta_\tau(\tau). \end{cases} \tag{6.17}$$

Proof. Take a control $u(\cdot) \in L^2(0, \tau; \mathbb{R}^m)$ and let $x(\cdot)$ be the corresponding solution of (6.4). Let $y(\cdot)$ be the solution of

$$\begin{cases} \dot y(t) = -\widehat A(t)^\top y(t) - \frac{1}{2} g_x^\top(t, x(t), u(t)), \quad t \in [0, \tau], \\ y(0) = y(\tau), \end{cases}$$

and define

$$\eta_\tau(t) \triangleq y(t) - P(t)x(t), \quad t \in [0, \tau].$$

Then $\eta_\tau(0) = \eta_\tau(\tau)$ and

$$\begin{aligned} -\dot\eta_\tau(t) &= -\dot y(t) + P(t)\dot x(t) + \dot P(t)x(t) \\ &= \widehat A(t)^\top y(t) + \frac{1}{2} g_x^\top(t, x(t), u(t)) + P(t)\widehat A(t)x(t) \\ &\quad + P(t)B(t)u(t) + P(t)b(t) + \dot P(t)x(t) \\ &= \widehat A(t)^\top P(t)x(t) + \widehat A(t)^\top \eta_\tau(t) + \frac{1}{2} g_x^\top(t, x(t), u(t)) + P(t)\widehat A(t)x(t) \\ &\quad + P(t)B(t)u(t) + P(t)b(t) + \dot P(t)x(t) \\ &= \Big[\dot P(t) + P(t)\widehat A(t) + \widehat A(t)^\top P(t) + \widehat C(t)^\top P(t)\widehat C(t) + Q(t) \\ &\quad + \Theta(t)^\top R(t)\Theta(t) + S(t)^\top \Theta(t) + \Theta(t)^\top S(t)\Big]x(t) \\ &\quad + \Big[P(t)B(t) + C(t)^\top P(t)D(t) + S(t)^\top + \Theta(t)^\top \Big(R(t) + D(t)^\top P(t)D(t)\Big)\Big]u(t) \end{aligned}$$

$$
\begin{aligned}
&\quad + \widehat{A}(t)^\top \eta_\tau(t) + \widehat{C}(t)^\top P(t)\sigma(t) + \Theta(t)^\top r(t) + P(t)b(t) + q(t)\\
&= \widehat{A}(t)^\top \eta_\tau(t) + \widehat{C}(t)^\top P(t)\sigma(t) + \Theta(t)^\top r(t) + P(t)b(t) + q(t).
\end{aligned}
$$

By Proposition 6.3, $u(\cdot)$ is optimal if and only if

$$
\begin{aligned}
0 &= B(t)^\top y(t) + \frac{1}{2} g_u^\top(t, x(t), u(t))\\
&= B(t)^\top P(t)x(t) + B(t)^\top \eta_\tau(t) + \frac{1}{2} g_u^\top(t, x(t), u(t))\\
&= \Big[R(t) + D(t)^\top P(t)D(t)\Big]u(t) + \Big[B(t)^\top \eta_\tau(t) + D(t)^\top P(t)\sigma(t) + r(t)\Big],
\end{aligned}
$$

or equivalently, if and only if

$$
u(t) = -\Big[R(t) + D(t)^\top P(t)D(t)\Big]^{-1}\Big[B(t)^\top \eta_\tau(t) + D(t)^\top P(t)\sigma(t) + r(t)\Big].
$$

The proof is complete since the ODE (6.17) is uniquely solvable. □

7. The turnpike property

In this section we establish the turnpike property for Problem $(\text{SLQ})_T$. Let $\eta_\tau(\cdot)$ be the solution to the ODE (6.17), and let $(x_\tau^*(\cdot), u_\tau^*(\cdot))$ be the unique optimal pair of Problem $(\text{DLQ})_\tau$. We extend $\eta_\tau(\cdot)$, $x_\tau^*(\cdot)$, and $u_\tau^*(\cdot)$ to $[0,\infty)$ by defining them to be τ-periodic, that is, we define for $t \in [0,\infty)$,

$$
\eta(t) \triangleq \eta_\tau(t - k\tau), \quad x^*(t) \triangleq x_\tau^*(t - k\tau), \quad u^*(t) \triangleq u_\tau^*(t - k\tau), \tag{7.1}
$$

if $t \in [k\tau, (k+1)\tau)$ for some integer $k \geqslant 0$. Recall the $\widehat{A}(\cdot)$ and $\widehat{C}(\cdot)$ defined by (6.1). It is easily seen:

$$
\dot{\eta}(t) + \widehat{A}(t)^\top \eta(t) + \widehat{C}(t)^\top P(t)\sigma(t) + \Theta(t)^\top r(t) + P(t)b(t) + q(t) = 0, \quad t \geqslant 0, \tag{7.2}
$$

$$
\dot{x}^*(t) = \widehat{A}(t)x^*(t) + B(t)u^*(t) + b(t), \quad t \geqslant 0. \tag{7.3}
$$

Further, we let

$$
\rho(t) \triangleq \widehat{C}(t)x^*(t) + D(t)u^*(t) + \sigma(t), \tag{7.4}
$$

and consider the following SDE:

$$
\begin{cases}
dX(t) = \widehat{A}(t)X(t)dt + [\widehat{C}(t)X(t) + \rho(t)]dW(t), & t \geqslant 0,\\
X(0) = \xi,
\end{cases} \tag{7.5}
$$

where ξ is an $\mathbb{R}^n$-valued, square-integrable random vector independent of the Brownian motion W. By Proposition 5.3, $\Theta(\cdot)$ is a stabilizer of $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$. Then, according to Proposition 4.2, the initial state ξ can be selected so as to the solution of (7.5) is τ-periodic.

Proposition 7.1. *Let* (A1)–(A3) *hold. Let ξ be an initial state such that the solution $X(\cdot)$ of* (7.5) *is τ-periodic. Then $\mathbb{E}[X(t)] = \mathbb{E}[\xi] = 0$ for all $t \geqslant 0$, and the covariance matrix $\Sigma(t) \triangleq Cov\,(X(t), X(t))$ is the τ-periodic solution of the following ODE:*

$$\dot{\Sigma}(t) = \widehat{A}(t)\Sigma(t) + \Sigma(t)\widehat{A}(t)^\top + \widehat{C}(t)\Sigma(t)\widehat{C}(t)^\top + \rho(t)\rho(t)^\top.$$

Proof. The proof is trivial and we omit it here. $\square$

Now, we are ready to state the turnpike property of Problem (SLQ)$_T$.

Theorem 7.2. *Let* (A1)–(A3) *hold. Let $(\bar{X}_T(\cdot), \bar{u}_T(\cdot))$ be the optimal pair of Problem (SLQ)$_T$ for the initial state $x \in \mathbb{R}^n$ (which could be arbitrary). Let $(x^*(\cdot), u^*(\cdot))$ be defined as in* (7.1), *and let $X(\cdot)$ be the solution of* (7.5). *Define*

$$\boldsymbol{X}^*(t) \triangleq X(t) + x^*(t), \quad \boldsymbol{u}^*(t) \triangleq \Theta(t)\boldsymbol{X}^*(t) + u^*(t), \quad t \geqslant 0. \tag{7.6}$$

Then there exist constants $K, \lambda > 0$, independent of T, such that

$$\mathbb{E}\Big[|\bar{X}_T(t) - \boldsymbol{X}^*(t)|^2 + |\bar{u}_T(t) - \boldsymbol{u}^*(t)|^2\Big] \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big], \quad \forall t \in [0, T]. \tag{7.7}$$

In order to prove Theorem 7.2, we need the following result.

Proposition 7.3. *Let* (A1)–(A3) *hold. Let $\varphi_T(\cdot)$ be the solution of* (5.5). *Then there exist constants $K, \lambda > 0$, independent of T, such that*

$$|\eta(t) - \varphi_T(t)| \leqslant K e^{-\lambda(T-t)}, \quad \forall t \in [0, T]. \tag{7.8}$$

Consequently, the function $\phi_T(\cdot)$ defined by (5.4) *satisfies*

$$|\phi_T(t) - u^*(t)| \leqslant K e^{-\lambda(T-t)}, \quad \forall t \in [0, T] \tag{7.9}$$

with possibly a different constant $K > 0$ that is independent of T.

Proof. Recall the notation of (6.1). Also, let

$$\widehat{A}_T(t) \triangleq A(t) + B(t)\Theta_T(t), \quad \widehat{C}_T(t) \triangleq C(t) + D(t)\Theta_T(t); \quad t \in [0, T]. \tag{7.10}$$

Then $h_T(\cdot) \triangleq \eta(\cdot) - \varphi_T(\cdot)$ satisfies $h_T(T) = \eta(T)$, and

$$\begin{aligned}
0 &= \dot{h}_T(t) + \widehat{A}_T(t)^\top h_T(t) + [\widehat{A}(t) - \widehat{A}_T(t)]^\top \eta(t) + [P(t)\widehat{C}(t) - P_T(t)\widehat{C}_T(t)]^\top \sigma(t) \\
&\quad + [\Theta(t) - \Theta_T(t)]^\top r(t) + [P(t) - P_T(t)]b(t) \\
&= \dot{h}_T(t) + \widehat{A}(t)^\top h_T(t) + [\widehat{A}_T(t) - \widehat{A}(t)]^\top h_T(t) + [\widehat{A}(t) - \widehat{A}_T(t)]^\top \eta(t) \\
&\quad + [P(t)\widehat{C}(t) - P_T(t)\widehat{C}_T(t)]^\top \sigma(t) + [\Theta(t) - \Theta_T(t)]^\top r(t) + [P(t) - P_T(t)]b(t) \\
&\equiv \dot{h}_T(t) + \widehat{A}(t)^\top h_T(t) + l_T(t).
\end{aligned}$$

Let $\widehat{\Psi}(\cdot)$ be the solution of (6.11). Then

$$h_T(t) = \widehat{\Psi}(t)\widehat{\Psi}(T)^{-1}\eta(T) + \int_t^T \widehat{\Psi}(t)\widehat{\Psi}(s)^{-1} l_T(s)ds.$$

From Theorem 5.6, Corollary 5.7, and Corollary 3.4, we know that there are constants $K, \lambda > 0$ such that for any $0 \leqslant t \leqslant s \leqslant T < \infty$,

$$\begin{aligned}
&|P(t) - P_T(t)| + |\Theta(t) - \Theta_T(t)| + |\widehat{A}(t) - \widehat{A}_T(t)| + |P(t)\widehat{C}(t) - P_T(t)\widehat{C}_T(t)| \\
&\quad \leqslant K e^{-\lambda(T-t)}, \\
&|\widehat{\Psi}(t)\widehat{\Psi}(s)^{-1}| = |\widehat{\Phi}(s)\widehat{\Phi}(t)^{-1}| \leqslant K e^{-\lambda(s-t)}.
\end{aligned}$$

Also, note that $\eta(\cdot)$, $\sigma(\cdot)$, $r(\cdot)$, and $b(\cdot)$ are all bounded. Thus,

$$\begin{aligned}
|h_T(t)| &\leqslant K e^{-\lambda(T-t)} + K \int_t^T e^{-\lambda(s-t)} |l_T(s)| dt \\
&\leqslant K e^{-\lambda(T-t)} + K \int_t^T e^{-\lambda(s-t)} e^{-\lambda(T-s)} \Big[|h_T(s)| + 1 \Big] ds \\
&= K e^{-\lambda(T-t)} + K e^{-\lambda(T-t)} \int_t^T \Big[|h_T(s)| + 1 \Big] ds,
\end{aligned}$$

with possibly a different constant K. For convenience, hereafter we shall use K and λ to denote two generic positive constants which do not depend on T and may vary from line to line. Set

$$\beta_T(t) \triangleq |h_T(t)| e^{\lambda(T-t)}.$$

Then the above can be written as

$$\beta_T(t) \leqslant K + K \int_t^T \Big[e^{-\lambda(T-s)} \beta_T(s) + 1 \Big] ds, \quad \forall t \in [0, T].$$

Applying Gronwall's inequality, we obtain

$$\beta_T(t) \leqslant K + K(T - t), \quad \forall t \in [0, T],$$

from which it follows that

$$|h_T(t)| \leqslant K e^{-\frac{\lambda}{2}(T-t)}, \quad \forall t \in [0, T].$$

This completes the proof. □

Proof of Theorem 7.2. Clearly, $\boldsymbol{X}^*(\cdot)$ satisfies the following SDE (recalling $\widehat{A}(\cdot)$ and $\widehat{C}(\cdot)$ defined in (6.1)):

$$\begin{cases} d\boldsymbol{X}^*(t) = [\widehat{A}(t)\boldsymbol{X}^*(t) + B(t)u^*(t) + b(t)]dt \\ \qquad\qquad + [\widehat{C}(t)\boldsymbol{X}^*(t) + D(t)u^*(t) + \sigma(t)]dW(t), \quad t \geqslant 0, \\ \boldsymbol{X}^*(0) = \xi + x^*(0). \end{cases} \tag{7.11}$$

On the other hand, substituting the closed-loop representation (5.2) into the optimal state equation for Problem (SLQ)$_T$, we obtain (recalling the notation (7.10))

$$\begin{cases} d\bar{X}_T(t) = [\widehat{A}_T(t)\bar{X}_T(t) + B(t)\phi_T(t) + b(t)]dt \\ \qquad\qquad + [\widehat{C}_T(t)\bar{X}_T(t) + D(t)\phi_T(t) + \sigma(t)]dW(t), \quad t \in [0, T], \\ \bar{X}_T(0) = x. \end{cases} \tag{7.12}$$

Define

$$H_T(t) \triangleq \bar{X}_T(t) - \boldsymbol{X}^*(t), \quad t \in [0, T].$$

Subtracting (7.11) from (7.12) yields $H_T(0) = x - x^*(0) - \xi$ and

$$\begin{aligned} dH_T(t) = &\Big\{\widehat{A}_T(t)H_T(t) + [\widehat{A}_T(t) - \widehat{A}(t)]\boldsymbol{X}^*(t) + B(t)[\phi_T(t) - u^*(t)]\Big\}dt \\ &+ \Big\{\widehat{C}_T(t)H_T(t) + [\widehat{C}_T(t) - \widehat{C}(t)]\boldsymbol{X}^*(t) + D(t)[\phi_T(t) - u^*(t)]\Big\}dW(t). \end{aligned}$$

Recall that $P(\cdot) \in C([0, \infty); \mathbb{S}^n)$ is the unique τ-periodic, uniformly positive definite solution to the Riccati equation (5.9). By Itô's rule,

$$\begin{aligned} &\mathbb{E}\langle P(t)H_T(t), H_T(t)\rangle - \mathbb{E}\langle P(0)H_T(0), H_T(0)\rangle \\ &= \mathbb{E}\int_0^t \Big\{\langle \dot{P}H_T, H_T\rangle + 2\langle PH_T, \widehat{A}_T H_T + (\widehat{A}_T - \widehat{A})\boldsymbol{X}^* + B(\phi_T - u^*)\rangle \\ &\quad + \langle P[\widehat{C}_T H_T + (\widehat{C}_T - \widehat{C})\boldsymbol{X}^* + D(\phi_T - u^*)], \widehat{C}_T H_T + (\widehat{C}_T - \widehat{C})\boldsymbol{X}^* + D(\phi_T - u^*)\rangle\Big\}ds \\ &= \mathbb{E}\int_0^t \Big\{\langle(\dot{P} + P\widehat{A}_T + \widehat{A}_T^\top P + \widehat{C}_T^\top P\widehat{C}_T)H_T, H_T\rangle \\ &\quad + 2\langle H_T, P[(\widehat{A}_T - \widehat{A})\boldsymbol{X}^* + B(\phi_T - u^*)] + \widehat{C}_T^\top P[(\widehat{C}_T - \widehat{C})\boldsymbol{X}^* + D(\phi_T - u^*)]\rangle \\ &\quad + \langle P[(\widehat{C}_T - \widehat{C})\boldsymbol{X}^* + D(\phi_T - u^*)], (\widehat{C}_T - \widehat{C})\boldsymbol{X}^* + D(\phi_T - u^*)\rangle\Big\}ds. \end{aligned} \tag{7.13}$$

Note that

$$
\begin{aligned}
&\dot{P}+P\widehat{A}_T+\widehat{A}_T^\top P+\widehat{C}_T^\top P\widehat{C}_T\\
&\quad=\dot{P}+P\widehat{A}+\widehat{A}^\top P+\widehat{C}^\top P\widehat{C}+P(\widehat{A}_T-\widehat{A})+(\widehat{A}_T-\widehat{A})^\top P\\
&\qquad+(\widehat{C}_T-\widehat{C})^\top P\widehat{C}+\widehat{C}_T^\top P(\widehat{C}_T-\widehat{C})\\
&\quad=-(Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S)+P(\widehat{A}_T-\widehat{A})+(\widehat{A}_T-\widehat{A})^\top P\\
&\qquad+(\widehat{C}_T-\widehat{C})^\top P\widehat{C}+\widehat{C}_T^\top P(\widehat{C}_T-\widehat{C}),
\end{aligned}
\tag{7.14}
$$

and that by (A2), there exists a constant $\delta>0$ such that on the interval $[0,\infty)$,

$$
\begin{aligned}
&Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S\\
&\quad=(\Theta+R^{-1}S)^\top R(\Theta+R^{-1}S)+Q-S^\top R^{-1}S>2\delta I.
\end{aligned}
\tag{7.15}
$$

Recalling Corollary 5.7 and Proposition 7.3, we know that there exist constants $K,\lambda>0$, independent of T, such that

$$
|\widehat{A}_T(t)-\widehat{A}(t)|+|\widehat{C}_T(t)-\widehat{C}(t)|+|\phi_T(t)-u^*(t)|\leqslant Ke^{-\lambda(T-t)},\quad \forall t\in[0,T]. \tag{7.16}
$$

Also, by Proposition 7.1, the function $\mathbb{E}|\boldsymbol{X}^*(\cdot)|^2$ is bounded. Thus,

$$
\mathbb{E}\langle(\dot{P}+P\widehat{A}_T+\widehat{A}_T^\top P+\widehat{C}_T^\top P\widehat{C}_T)H_T,H_T\rangle(t)\leqslant[-2\delta+Ke^{-\lambda(T-t)}]\mathbb{E}|H_T(t)|^2, \tag{7.17}
$$

$$
\begin{aligned}
&2\mathbb{E}\langle H_T,P[(\widehat{A}_T-\widehat{A})\boldsymbol{X}^*+B(\phi_T-u^*)]+\widehat{C}_T^\top P[(\widehat{C}_T-\widehat{C})\boldsymbol{X}^*+D(\phi_T-u^*)]\rangle(t)\\
&\quad\leqslant\delta\mathbb{E}|H_T(t)|^2+2\delta^{-1}\mathbb{E}\big|P[(\widehat{A}_T-\widehat{A})\boldsymbol{X}^*+B(\phi_T-u^*)]\big|^2(t)\\
&\qquad+2\delta^{-1}\mathbb{E}\big|\widehat{C}_T^\top P[(\widehat{C}_T-\widehat{C})\boldsymbol{X}^*+D(\phi_T-u^*)]\big|^2(t)\\
&\quad\leqslant\delta\mathbb{E}|H_T(t)|^2+Ke^{-\lambda(T-t)},
\end{aligned}
\tag{7.18}
$$

$$
\begin{aligned}
&\mathbb{E}\langle P[(\widehat{C}_T-\widehat{C})\boldsymbol{X}^*+D(\phi_T-u^*)],(\widehat{C}_T-\widehat{C})\boldsymbol{X}^*+D(\phi_T-u^*)\rangle(t)\\
&\quad\leqslant Ke^{-\lambda(T-t)}.
\end{aligned}
\tag{7.19}
$$

Differentiating both sides of (7.13) with respect to t and taking into account (7.17)–(7.19), we obtain

$$
\frac{d}{dt}\mathbb{E}\langle P(t)H_T(t),H_T(t)\rangle\leqslant\big[-\delta+Ke^{-\lambda(T-t)}\big]\mathbb{E}|H_T(t)|^2+Ke^{-\lambda(T-t)}. \tag{7.20}
$$

Let $\alpha,\beta>0$ be such that

$$
\alpha^{-1}I\leqslant P(t)\leqslant\beta^{-1}I,\quad \forall t\in[0,\infty).
$$

Then (7.20) implies

$$
\frac{d}{dt}\mathbb{E}\langle P(t)H_T(t),H_T(t)\rangle\leqslant\big[-\delta\beta+\alpha Ke^{-\lambda(T-t)}\big]\mathbb{E}\langle P(t)H_T(t),H_T(t)\rangle+Ke^{-\lambda(T-t)}.
$$

Using the Gronwall inequality, we get

$$\mathbb{E}|\bar{X}_T(t) - \boldsymbol{X}^*(t)|^2 = \mathbb{E}|H_T(t)|^2 \leqslant \alpha\mathbb{E}\langle P(t)H_T(t), H_T(t)\rangle \\ \leqslant K\big[e^{-\lambda t} + e^{-\lambda(T-t)}\big], \quad \forall 0 \leqslant t \leqslant T < \infty,$$

for possibly different K and λ. Using the above inequality and the fact

$$\bar{u}_T(t) - \boldsymbol{u}^*(t) = \Theta_T(t)\big[\bar{X}_T(t) - \boldsymbol{X}^*(t)\big] + \big[\Theta_T(t) - \Theta(t)\big]\boldsymbol{X}^*(t) + \big[\phi_T(t) - u^*(t)\big],$$

we further obtain

$$\mathbb{E}|\bar{u}_T(t) - \boldsymbol{u}^*(t)|^2 \leqslant K\big[e^{-\lambda t} + e^{-\lambda(T-t)}\big], \quad \forall 0 \leqslant t \leqslant T < \infty,$$

and therefore (7.7). □

The construction of the turnpike limit $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ in Theorem 7.2 reveals that both processes have τ-periodic distributions. We now establish the uniqueness of the turnpike limit with this characteristic in the distribution sense.

Theorem 7.4. *Suppose that $\bar{\boldsymbol{X}}^*(\cdot)$ and $\bar{\boldsymbol{u}}^*(\cdot)$ are two processes with τ-periodic distributions such that*

$$\mathbb{E}\Big[|\bar{X}_T(t) - \bar{\boldsymbol{X}}^*(t)|^2 + |\bar{u}_T(t) - \bar{\boldsymbol{u}}^*(t)|^2\Big] \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big], \quad \forall t \in [0, T]$$

holds for some constants $K, \lambda > 0$ independent of T. Then

$$\boldsymbol{X}^*(t) \overset{d}{=} \bar{\boldsymbol{X}}^*(t), \quad \boldsymbol{u}^*(t) \overset{d}{=} \bar{\boldsymbol{u}}^*(t); \quad \forall t \geqslant 0. \tag{7.21}$$

That is, $\boldsymbol{X}^*(t)$ and $\bar{\boldsymbol{X}}^*(t)$ have the same distribution for all $t \geqslant 0$, and similarly, $\boldsymbol{u}^*(t)$ and $\bar{\boldsymbol{u}}^*(t)$ have the same distribution for all $t \geqslant 0$.

Proof. We provide the proof for

$$\boldsymbol{X}^*(t) \overset{d}{=} \bar{\boldsymbol{X}}^*(t), \quad \forall t \geqslant 0.$$

The proof for $\boldsymbol{u}^*(t) \overset{d}{=} \bar{\boldsymbol{u}}^*(t)$ follows a similar argument.

Step 1: Let F be an arbitrary closed set in $\mathbb{R}^n$. Define for $k = 1, 2, \ldots,$

$$f_k(x) \triangleq (1 - k|x - F|)^+ = \max\{1 - k|x - F|, 0\}, \quad x \in \mathbb{R}^n,$$

where $|x - F|$ denotes the distance between x and F. It is clear that

$$0 \leqslant f_k(x) \leqslant 1, \quad \forall x \in \mathbb{R}^n, \ \forall k \geqslant 1, \tag{7.22}$$

$$\lim_{k\to\infty} f_k(x) = \mathbf{1}_F(x), \quad \forall x \in \mathbb{R}^n. \tag{7.23}$$

Furthermore, we conclude that

$$|f_k(x) - f_k(y)| \leqslant k|x-y|, \quad \forall x, y \in \mathbb{R}^n. \tag{7.24}$$

Indeed, if $|x-F| < 1/k$ and $|y-F| < 1/k$, then (7.24) follows easily from the triangle inequality. Now suppose $|x-F| \geqslant 1/k$. Then it suffices to show that $1 - k|y-F| \leqslant k|x-y|$, or equivalently,

$$\frac{1}{k} \leqslant |x-y| + |y-F|. \tag{7.25}$$

For this, take any $z \in F$ and note that

$$\frac{1}{k} \leqslant |x-F| \leqslant |x-z| \leqslant |x-y| + |y-z|.$$

Taking the infimum over $z \in F$ yields (7.25). By a similar argument we can show that (7.24) also holds for the case $|y-F| \geqslant 1/k$.

Step 2: We now show that for each integer $k \geqslant 1$,

$$\mathbb{E}[f_k(\bar{\boldsymbol{X}}^*(t))] = \mathbb{E}[f_k(\boldsymbol{X}^*(t))], \quad \forall t \geqslant 0. \tag{7.26}$$

To prove the above, we first observe that for any integer $j \geqslant 1$,

$$\begin{aligned}|\mathbb{E}[f_k(\bar{\boldsymbol{X}}^*(t))] - \mathbb{E}[f_k(\boldsymbol{X}^*(t))]| &= |\mathbb{E}[f_k(\bar{\boldsymbol{X}}^*(t+j\tau))] - \mathbb{E}[f_k(\boldsymbol{X}^*(t+j\tau))]| \\ &\leqslant \mathbb{E}|f_k(\bar{\boldsymbol{X}}^*(t+j\tau)) - f_k(\boldsymbol{X}^*(t+j\tau))|,\end{aligned}$$

since both $\bar{\boldsymbol{X}}^*(\cdot)$ and $\boldsymbol{X}^*(\cdot)$ have τ-periodic distributions. Using (7.24) we obtain

$$\begin{aligned}&|\mathbb{E}[f_k(\bar{\boldsymbol{X}}^*(t))] - \mathbb{E}[f_k(\boldsymbol{X}^*(t))]| \leqslant k\mathbb{E}|\bar{\boldsymbol{X}}^*(t+j\tau) - \boldsymbol{X}^*(t+j\tau)| \\ &\quad \leqslant k\mathbb{E}|\bar{\boldsymbol{X}}^*(t+j\tau) - \boldsymbol{X}^*(t+j\tau)|^2 \\ &\quad \leqslant 2k\mathbb{E}|\bar{X}_T(t+j\tau) - \bar{\boldsymbol{X}}^*(t+j\tau)|^2 + 2k\mathbb{E}|\bar{X}_T(t+j\tau) - \boldsymbol{X}^*(t+j\tau)|^2 \\ &\quad \leqslant 4Kk\Big[e^{-\lambda(t+j\tau)} + e^{-\lambda(T-t-j\tau)}\Big], \quad \forall T \geqslant 0, \ \forall j \geqslant 1.\end{aligned}$$

Letting $T \to \infty$ first, then $j \to \infty$ yields (7.26).

Step 3: Letting $k \to \infty$ in (7.26), we obtain from (7.22)–(7.23) and the bounded convergence theorem that

$$\mathbb{P}[\bar{\boldsymbol{X}}^*(t) \in F] = \mathbb{P}[\boldsymbol{X}^*(t) \in F], \quad \forall t \geqslant 0.$$

The desired result follows, since the closed set F is arbitrary. □

To conclude this section, let us look at the case where all the coefficients are time invariant. In this case, the unique positive definite solution $P(\cdot)$ of (5.9) is constant-valued, satisfying the following algebraic Riccati equation (ARE, for short):

$$\begin{aligned} &PA + A^\top P + C^\top PC + Q \\ &\quad - (PB + C^\top PD + S^\top)(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S) = 0, \end{aligned} \tag{7.27}$$

and (5.10) becomes

$$\Theta \triangleq -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S). \tag{7.28}$$

The solution to the ODE (6.17) is also constant-valued, given by

$$\eta_\tau(t) = -(\widehat{A}^\top)^{-1}(\widehat{C}^\top P\sigma + \Theta^\top r + Pb + q), \quad \forall t \in [0, \tau],$$

where

$$\widehat{A} \triangleq A + B\Theta, \quad \widehat{C} \triangleq C + D\Theta.$$

Consequently, (7.1) defines three constant vectors:

$$\begin{cases} \eta \triangleq -(\widehat{A}^\top)^{-1}(\widehat{C}^\top P\sigma + \Theta^\top r + Pb + q), \\ u^* \triangleq -(R + D^\top PD)^{-1}(B^\top \eta + D^\top P\sigma + r), \\ x^* \triangleq -\widehat{A}^{-1}(Bu^* + b). \end{cases} \tag{7.29}$$

Let $v^* \triangleq \Theta x^* + u^*$. We have the following result.

Proposition 7.5. (x^*, v^*) *is the unique minimum point of*

$$\begin{aligned} F(x, v) \triangleq &\langle Qx, x\rangle + \langle Rv, v\rangle + 2\langle Sx, v\rangle + 2\langle q, x\rangle + 2\langle r, v\rangle \\ &+ \langle P(Cx + Dv + \sigma), Cx + Dv + \sigma\rangle \end{aligned} \tag{7.30}$$

on the space

$$\mathscr{V} \triangleq \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ax + Bv + b = 0\}. \tag{7.31}$$

Proof. Clearly, $Ax^* + Bv^* + b = 0$. By an argument similar to the proof of [20, Proposition 3.2], it suffices to show that there exists a $\lambda^* \in \mathbb{R}^n$ such that

$$\begin{cases} A^\top \lambda^* + Qx^* + C^\top P(Cx^* + Dv^* + \sigma) + S^\top v^* + q = 0, \\ B^\top \lambda^* + Rv^* + D^\top P(Cx^* + Dv^* + \sigma) + Sx^* + r = 0. \end{cases}$$

One can easily verify that the vector $\lambda^* \triangleq Px^* + \eta$ satisfies the above requirements. □

Now let us look at (7.5). In the time invariant case, (7.5) becomes

$$\begin{cases} dX(t) = \widehat{A}X(t)dt + [\widehat{C}X(t) + \rho]dW(t), \quad t \geqslant 0, \\ X(0) = \xi, \end{cases} \tag{7.32}$$

where $\rho \triangleq \widehat{C}x^* + Du^* + \sigma$. By Proposition 4.2, we can choose an initial distribution such that the solution $X(\cdot)$ of (7.32) is stationary. The following result is the stationary version of Proposition 7.1.

Proposition 7.6. *Let ξ be an initial state such that the solution of (7.5) is stationary. Then $\mathbb{E}[X(t)] = 0$, and the covariance matrix $\Sigma \triangleq Cov(X(t), X(t))$ is the solution of the following ARE:*

$$\widehat{A}\Sigma + \Sigma\widehat{A}^\top + \widehat{C}\Sigma\widehat{C}^\top + \rho\rho^\top = 0.$$

In light of Proposition 7.5 and Proposition 7.6, we see that in the time invariant case, the turnpike limit $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ defined by (7.6) has the following properties:

(i) the expectation and covariance of $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ are time invariant;
(ii) $(\mathbb{E}[\boldsymbol{X}^*(t)], \mathbb{E}[\boldsymbol{u}^*(t)]) = (x^*, v^*)$ is the unique minimum point of the optimization problem (7.30)–(7.31).

8. Concluding remarks

In this paper, we have established the exponential turnpike property (7.7) for a class of stochastic LQ optimal control problems with periodic coefficients. The turnpike limit $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ consists of a pair of τ-periodic stochastic processes and is unique in the distribution sense (see Theorem 7.4). As indicated in (7.6), the turnpike limit $(\boldsymbol{X}^*(\cdot), \boldsymbol{u}^*(\cdot))$ can be decomposed into two components. The first component is the periodic extension of the optimal pair for the deterministic Problem (DLQ)$_\tau$. The second component comprises a pair of stochastic processes with τ-periodic distribution, primarily determined by the correction process $X(\cdot)$, i.e., the τ-periodic solution of the SDE (7.5).

The heuristic approach for obtaining Problem (DLQ)$_\tau$ and the correction process can be outlined as follows. The existing result presented in Lemma 5.1 shows that the optimal control for Problem (SLQ)$_T$ follows the closed-loop representation given in (5.2). This representation is essentially determined by the solution $P_T(\cdot)$ to the differential Riccati equation (5.1). As T tends to infinity, $P_T(t)$ exponentially converges to $P(t)$, the unique τ-periodic solution to the same Riccati equation. Consequently, the $\Theta_T(t)$ defined in (5.3) converges to the stabilizer $\Theta(t)$ of $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$ defined in (5.10). By letting $T \to \infty$, it is formally expected that the solution $\varphi_T(\cdot)$ of (5.5) converges to the τ-periodic solution of (6.17). This implies that $\phi_T(t)$ converges to the τ-extension $u^*(t)$ of $u^*_\tau(t)$ defined in (6.16). Substituting (5.2) into the state equation (1.1) and letting $T \to \infty$, a similar argument shows that the optimal state process $\bar{X}_T(\cdot)$ of Problem (SLQ)$_T$ also converges to the τ-periodic solution $\boldsymbol{X}^*(\cdot)$ of the following SDE:

$$\begin{aligned} dX(t) = &\big\{[A(t) + B(t)\Theta(t)]X(t) + B(t)u^*(t) + b(t)\big\}dt \\ &+ \big\{[C(t) + D(t)\Theta(t)]X(t) + D(t)u^*(t) + \sigma(t)\big\}dW(t). \end{aligned}$$

Upon taking expectations, it is evident that $\mathbb{E}[\boldsymbol{X}^*(\cdot)]$ is exactly the function $x^*(\cdot)$ defined in (7.1) (since $x^*(\cdot)$ satisfies the same ODE as $\mathbb{E}[\boldsymbol{X}^*(\cdot)]$), and that $X(\cdot) \triangleq \boldsymbol{X}^*(\cdot) - x^*(\cdot)$ is the correction process (i.e., the periodic solution of (7.5)). Upon closer examination of the pair $(x^*(\cdot), u^*(\cdot))$, it appears to constitute the optimal pair for a deterministic LQ optimal control problem. As this pair is τ-periodic, it is necessary to adjust the formulation of the deterministic LQ problem to also incorporate periodicity. This adjustment leads to the formulation of Problem $(\mathrm{DLQ})_\tau$.

Data availability

No data was used for the research described in the article.

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