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Optimal controls for forward-backward stochastic differential equations: Time-inconsistency and time-consistent solutions

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A B S T R A C T

This paper is concerned with an optimal control problem for a forward-backward stochastic differential equation (FBSDE, for short) with a recursive cost functional determined by a backward stochastic Volterra integral equation (BSVIE, for short). It is found that such an optimal control problem is time-inconsistent in general, even if the cost functional is reduced to a classical Bolza type one as in Peng [47], Lim–Zhou [38], and Yong [72]. Therefore, instead of finding a global optimal control (which is time-inconsistent), we will look for a time-consistent and locally optimal equilibrium strategy, which can be constructed via the solution of an associated equilibrium Hamilton–Jacobi–Bellman (HJB, for short) equation. A verification theorem for the local optimality of the equilibrium strategy is proved by means of the generalized Feynman–Kac formula for BSVIEs and some stability estimates of the representation parabolic partial differential equations (PDEs, for short). Under certain conditions, it is proved that the equilibrium HJB equation, which is a nonlocal PDE, admits a unique classical solution. As special cases and applications, the linear-quadratic problems, a mean-variance model, a social planner problem with heterogeneous Epstein–Zin utilities, and a Stackelberg game are briefly investigated. It turns out that our framework can cover not only the optimal control problems for FBSDEs studied in [47,38,72], and so on, but also the problems of the general discounting and some nonlinear appearance of conditional expectations for the terminal state, studied in Yong [73,75] and Björk–Khapko–Murgoci [6].

R É S U M É

Cet article traite d'un problème de contrôle optimal pour une équation différentielle stochastique progressive-rétrograde (EDSP-R), avec une fonction de coût récursive déterminée par une équation intégrale stochastique rétrograde de Volterra (EISRV). Il est constaté qu'un tel problème de contrôle optimal est généralement incohérent dans le temps, même si la fonction de coût est réduite à une forme classique de type Bolza, comme observé dans les travaux de Peng [47], Lim–Zhou [38], et Yong [72]. Par conséquent, au lieu de chercher un contrôle optimal global (qui est incohérent dans le temps), nous proposons de rechercher une stratégie d'équilibre localement

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optimale et cohérente dans le temps, qui peut être construite via la solution d'une équation de Hamilton–Jacobi–Bellman (HJB) associée à l'équilibre. Un théorème de vérification pour l'optimalité locale de la stratégie d'équilibre est prouvé au moyen de la formule de Feynman–Kac généralisée pour les EISRV et de certaines estimations de stabilité de la représentation pour les équations aux dérivées partielles (EDP) paraboliques. Sous certaines conditions, il est prouvé que l'équation de HJB d'équilibre, qui est une EDP non locale, admet une solution classique unique. En tant que cas spéciaux et applications, les problèmes linéaires-quadratiques, un modèle de moyenne-variance, un problème de planificateur social avec des utilités hétérogènes d'Epstein–Zin, et un jeu de Stackelberg sont brièvement examinés. Il s'avère que notre cadre peut couvrir non seulement les problèmes de contrôle optimal pour les EDSP-R étudiés dans des travaux antérieurs, tels que ceux de Peng [47], Lim–Zhou [38], et Yong [72], mais aussi les problèmes de l'actualisation générale et certaines apparitions non linéaires des espérances conditionnelles pour l'état terminal, étudiés dans les travaux de Yong [73,75] et Björk–Khapko–Murgoci [6].

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \le t < \infty\}$ is defined. The augmented natural filtration of W is denoted by $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0}$. Let $T > 0$ be a fixed time horizon. We denote

$$\begin{aligned}
&\mathcal{X}_t = L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n) = \big\{\xi : \Omega \to \mathbb{R}^n \mid \xi \text{ is } \mathcal{F}_t\text{-measurable}, \mathbb{E}[|\xi|^2] < \infty\big\},\\
&\mathcal{D} = \big\{(t,\xi) \mid t \in [0,T),\ \xi \in \mathcal{X}_t\big\},\\
&\mathcal{U}[t,T] = \Big\{\varphi : [t,T]\times\Omega \to U \,\Big|\, \varphi \text{ is } \mathbb{F}\text{-progressively measurable},\\
&\qquad\qquad\qquad\qquad \mathbb{E}\int_t^T |\varphi(s)|^2 ds < \infty\Big\},
\end{aligned}$$

where $U \subseteq \mathbb{R}^\ell$ is a nonempty measurable set (either bounded or unbounded). For any given *initial pair* $(t,\xi) \in \mathcal{D}$ and *control process* $u \in \mathcal{U}[t,T]$, consider the following controlled (decoupled) forward-backward stochastic differential equation (FBSDE, for short) on the time horizon $[t,T]$:

$$\begin{cases} dX(s) = b(s,X(s),u(s))ds + \sigma(s,X(s),u(s))dW(s),\\ dY(s) = -g(s,X(s),u(s),Y(s),Z(s))ds + Z(s)dW(s),\\ X(t)=\xi, \quad Y(T) = h(X(T)),\end{cases} \tag{1.1}$$

where $b,\sigma : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^n$, $g : [0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^m$, and $h : \mathbb{R}^n \to \mathbb{R}^m$ are given deterministic mappings. Under certain mild conditions, for any $(t,\xi) \in \mathcal{D}$ and $u \in \mathcal{U}[t,T]$, (1.1) admits a unique adapted solution $(X,Y,Z) \equiv \big(X(\cdot\,;t,\xi,u), Y(\cdot\,;t,\xi,u), Z(\cdot\,;t,\xi,u)\big)$, which is called a *state process*. To measure the performance of the control u, we introduce the following *recursive cost functional*:

$$J(t,\xi;u) = Y^0(t), \tag{1.2}$$

where Y^0 is uniquely determined by the following backward stochastic Volterra integral equation (BSVIE, for short) over $[t,T]$:

$$Y^0(r) = h^0(r,X(r),X(T),Y(r)) - \int_r^T Z^0(r,s)dW(s)$$

$$+\int_r^T g^0\big(r,s,X(r),X(s),u(s),Y(s),Z(s),Y^0(s),Z^0(r,s)\big)ds, \tag{1.3}$$

for which (Y^0, Z^0) is the adapted solution. Here, $h^0 : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ and $g^0 : \Delta^*[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given deterministic mappings with

$$\Delta^*[0,T] = \big\{(t,s)\in[0,T]^2 \bigm| 0\le t\le s\le T\big\} \tag{1.4}$$

being the upper triangle domain in the square $[0,T]^2$. In the case that

$$h^0(r,\tilde{x},x,y) = h^0(x,y), \quad g^0(r,s,\tilde{x},x,u,y,z,y^0,z^0) = g^0(s,x,u,y,z), \tag{1.5}$$

the recursive cost functional (1.2)–(1.3) is reduced to a Bolza type cost functional for FBSDE state equation (see Peng [47] and Yong [72], for examples):

$$J(t,\xi;u) = \mathbb{E}_t\Big[h^0(X(T),Y(t)) + \int_t^T g^0(s,X(s),u(s),Y(s),Z(s))ds\Big], \tag{1.6}$$

where $\mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ is the conditional expectation operator. Further, if

$$h^0(r,\tilde{x},x,y) = h^0(x), \quad g^0(r,s,\tilde{x},x,u,y,z,y^0,z^0) = g^0(s,x,u),$$

then the cost functional is reduced to the most familiar classical Bolza functional:

$$J(t,\xi;u) = \mathbb{E}_t\Big[h^0(X(T)) + \int_t^T g^0(s,X(s),u(s))ds\Big],$$

where the two terms on the right-hand side are called *terminal* and *running* costs, respectively. Thus, our recursive cost functional is an extension of Bolza type cost functional. With the state equation (1.1) and the recursive cost functional (1.2)–(1.3), we may pose the following optimal control problem:

Problem (N). For any given initial pair $(t,\xi)\in\mathcal{D}$, find a control $\bar{u}\in\mathcal{U}[t,T]$ such that

$$J(t,\xi;\bar{u}) = \operatorname*{essinf}_{u\in\mathcal{U}[t,T]} J(t,\xi;u) = V(t,\xi). \tag{1.7}$$

Any $\bar{u}\in\mathcal{U}[t,T]$ satisfying (1.7) is called an (*open-loop*) *optimal control* of Problem (N) for the initial pair (t,ξ); the corresponding state process $(\bar{X},\bar{Y},\bar{Z}) \equiv \big(X(\cdot\,;t,\xi,\bar{u}),Y(\cdot\,;t,\xi,\bar{u}),Z(\cdot\,;t,\xi,\bar{u})\big)$ is called an (*open-loop*) *optimal state process*; and $V:\mathcal{D}\to\mathbb{R}$ is called the *value function* of Problem (N).

We now briefly illustrate the major **motivation** of the above framework as follows: The (vector-valued) process X follows a (forward) stochastic differential equation (FSDE, for short). Components of X consist of two types processes: uncontrolled ones (by the individuals), including prices of securities (such as bonds, stocks), some economic factors (such as interest rates, unemployment rates, GDP, etc.), and controlled ones (by the individuals), including market values of the investor's wealth (subject to trading strategies), inventory of commodities (subject to the ordering), amounts of goods (subject to the production), etc. On the other hand, the components of Y, following a multi-dimensional backward stochastic differential equation (BSDE, for short), could include the prices of some European type contingent claims of the underlying assets (whose prices are some components of X), and some dynamic risk measures, and so on. Therefore, it is natural to have an FBSDE as a state equation. Further, the dynamic expected utility/disutility of the total assets will be calculated in the recursive way, which can be described by the adapted solution to a BSVIE (see below). Putting all the above together, we have the framework and the formulation of the problem.

Let us now briefly illustrate the recursive cost functional of form (1.2)–(1.3). In 1992, Duffie–Epstein [15,16] introduced a stochastic differential formulation of recursive utility in the case of information generated by a Brownian motion. In 1997, El Karoui–Peng–Quenez [18] showed that such a process actually is a part of the adapted solution to a particular BSDE and then they defined a more general class of recursive utilities, through a general BSDE (see also Lazrak [34] for further developments). The main feature of such a recursive process, denoted by Y^R, is that the current value $Y^R(t)$ depends on the future values $Y^R(s)$, $t < s \le T$ of the process. Then, on top of the classical Bolza type cost functional (1.6), mimicking [18], for our FBSDE state equation, it is natural to introduce the following *recursive* cost functional:

$$J^R(t,\xi;u) = Y^R(t),$$

where Y^R is determined by the following equation over $[t,T]$:

$$Y^R(r) = \mathbb{E}_r\Big[h^0(X(T),Y(r)) + \int_r^T g^0(s,X(s),u(s),Y(s),Z(s),Y^R(s))ds\Big].$$

From the above, we see that the value $Y^R(t)$ depends on the values $Y^R(s)$ for $s \in [t,T]$, through the above equation. Hence the cost process Y^R has a recursive feature, and thus its name. By Yong [71], for some process Z^R, the pair (Y^R, Z^R) is the adapted solution to the following BSVIE:

$$\begin{aligned} Y^R(r) = h^0(X(T),Y(r)) &+ \int_r^T g^0(s,X(s),u(s),Y(s),Z(s),Y^R(s))ds \\ &- \int_r^T Z^R(r,s)dW(s), \quad r \in [t,T]. \end{aligned} \tag{1.8}$$

Note that (1.8) is not a BSDE on $[t,T]$, because the free term $h^0(X(T),Y(r))$ depends on the time variable r, which leads to the adjustment process $Z^R(r,s)$ depending on r and s. Inspired by the above, we introduce the general recursive cost functional (1.2)–(1.3). Note that in BSVIE (1.3), the *free term* h^0 and the *generator* g^0 are allowed to depend on the initial pair $(r, X(r))$ at the current time r, which is motivated by the *non-exponential discounting* [32,20,73] and the *state-dependent risk aversion* [8,27] in finance. The recursive cost functional of form (1.2)–(1.3) was introduced by Wang–Yong [63] for the first time, motivated by the recursive utility/disutility process for classical optimal control problems. Comparing with the cost functional studied in [63], the free term h^0 and the generator g^0 of BSVIE (1.3) are additionally allowed to depend on the initial state $X(r)$ and the backward process (Y, Z). Moreover, we highlight that (1.2)–(1.3) can also be regarded as a recursive version of the cost functional studied in Björk–Khapko–Murgoci [6], because $\mathbb{E}_\cdot[X(T)]$ is the backward state process Y of a trivial BSDE.

It is well-known by now that the introduction of BSDEs by Bismut [4,5] in the early 1970s was for the purpose of studying optimal control of FSDEs. The later developments of general BSDEs by Pardoux–Peng [45] (see also Duffie–Epstein [15] and El Karoui–Peng–Quenez [18]), and the extension to FBSDEs by Antonelli [1], Ma–Protter–Yong [39], Hu–Peng [28] (see also the books of Ma–Yong [40] and Zhang [78]) have been attracting many researchers' attention. Among many other publications, a big number of literature on the optimal control problems for BSDEs/FBSDEs keep appearing. See, Peng [47], Xu [69], Dokuchaev–Zhou [14], Ji–Zhou [31], Shi–Wu [52], Huang–Wang–Xiong [30], Yong [72], Wang–Wu–Xiong [59], and Hu–Ji–Xue [24] on the Pontryagin's maximum principle for controlled BSDEs/FBSDEs; Lim–Zhou [38], Wang–Wu–Xiong [60], Huang–Wang–Wu [29], Wang–Xiao–Xiong [61], Li–Sun–Xiong [37], Hu–Ji–Xue [25], Sun–Wang [55], Sun–Wu–Xiong [58], Sun–Wang–Wen [56] on the linear-quadratic (LQ, for short) optimal control problems for BSDEs/FBSDEs; and so on. It is observed that the problems investigated in the above listed works are all essentially the special cases of Problem (N), and have been treated as usual stochastic optimal control problems. There is an *essential feature* has been overlooked in all the above, which we now indicate that.

For a dynamic optimal control problem, suppose that at a given initial pair $(t,\xi)\in\mathcal{D}$, the problem has an (open-loop) optimal control $\bar u\equiv\bar u(\cdot\,;t,\xi)$ with the (open-loop) optimal state being $\bar X\equiv X(\cdot\,;t,\xi,\bar u)$. Then, we could not expect the following:

$$J\big(\tau,\bar X(\tau);\bar u\big|_{[\tau,T]}\big)=\inf_{u\in\mathcal{U}[\tau,T]}J(\tau,\bar X(\tau);u),\quad \forall\tau\in(t,T],\ \text{a.s.}\tag{1.9}$$

In other words, an optimal control selected at a given initial pair might not stay optimal thereafter. Then, we say that the optimal control problem is **time-inconsistent**. It turns out that, in general, Problem (N) is time-inconsistent, and the *dynamical programming principle* (DPP, for short) does not hold. This reveals a surprising feature of Problem (N). To see that, let us elaborate the time-inconsistency in a little more details, from which we will see how Problem (N) is generally time-inconsistent.

• *Time-preferences and discounting.* Suppose the continuously compound interest rate is a constant $\lambda>0$. Then one needs to deposit an amount $e^{-\lambda T_0}$ at τ in order to get 1 unit at $\tau+T_0$. We call $e^{-\lambda T_0}$ the *discount factor* of the time interval $[\tau,\tau+T_0]$, which could also be defined as the *value* of this time interval. Clearly, such a value $e^{-\lambda T_0}$ of $[\tau,\tau+T_0]$ is independent of the initial time τ and it is also independent of the time $t\in[0,\infty)$ at which $[\tau,\tau+T_0]$ is evaluated, either $t\le\tau$ or $t>\tau$. Because of this, such an exponential evaluation is said to be *rational*. Or equivalently, rationality can be described by the *exponential discounting*. On the other hand, it is common that most people overweight the utility of the immediate future events, which can be convinced by the fact that one often regrets the (optimal) decisions made earlier. This means that people evaluate the immediate future time period more expensively than it should be, which amounts to saying that the discount factor for that time interval is larger than the rational one. Hence, we need to replace the exponential discounting by more general ones to more precisely describe the real situations.

In the above recursive cost functional (1.2)–(1.3), if we have

$$\begin{aligned}&h^0(r,\tilde x,x,y)=e^{-\lambda(T-r)}h^0(x),\\&g^0(r,s,\tilde x,x,u,y,z,y^0,z^0)=e^{-\lambda(s-r)}g^0(s,x,u),\end{aligned}$$

for some discount rate $\lambda>0$, then the cost functional is reduced to the classical exponential discounting Bolza cost functional:

$$J(t,\xi;u)=\mathbb{E}_t\Big[e^{-\lambda(T-t)}h^0(X(T))+\int_t^T e^{-\lambda(s-t)}g^0(s,X(s),u(s))ds\Big].$$

In this case, there are no (European type) contingent claims involved, and there are no dynamic risks taken into account. Therefore, the BSDE for (Y,Z) in (1.1) is irrelevant. Also, the involved individual is completely *rational* (as far as the time-preferences are concerned). For such a case, the corresponding Problem (N) is time-consistent. Now, if $e^{-\lambda(T-t)}$ and $e^{-\lambda(s-t)}$ are replaced by some non-exponential decay functions, the cost functionals are referred to as non-exponential ones, which describe some kinds of *irrationality* of time-preferences for the involved individuals. In this case, namely, the cost functional is given by (1.2)–(1.3), our Problem (N) is time-inconsistent. The earliest mathematical consideration in this aspect was given by Strotz [54], followed by Pollak [50], and the recent works of Ekeland–Pirvu [20], Ekeland–Lazrak [19], Yong [73,75,76], Wei–Yong–Yu [68], Mei–Yong [43], Mei–Zhu [44], Wang–Yong [63], Hamaguchi [22], Hernández–Possamai [23], and Lazrak–Wang–Yong [35] for various kinds of problems relevant to non-exponential discounting.

• *Risk-preferences and nonlinear appearance of conditional expectations of the (terminal) state.* Different groups of people should have different opinions of risks on the in-coming events. This is referred to as people's *subjective* risk-preferences. One way to describe this is to allow the conditional expectation of the state to (nonlinearly) appear in the cost functional. It turns out that such a formulation will lead to time-inconsistency of the optimal control problem in general. See Basak–Chabakauri [3], Hu–Jin–Zhou [26,27], Björk–Murgoci [7], Björk–Murgoci–Zhou [8], Björk–Khapko–Murgoci [6], and Yong [76] for some relevant results.

Let us now make an observation for our Problem (N). Let $m = n$, and

$$
\begin{aligned}
&h(x) = x, \quad g(s,x,u,y,z) \equiv 0, \quad h^0(r,\tilde{x},x,y) = h^0(x,y),\\
&g^0(r,s,\tilde{x},x,u,y,z,y^0,z^0) = g^0(s,x,u,y),
\end{aligned}
$$

then

$$
Y(s) = \mathbb{E}_s[X(T)], \quad s \in [t,T],
$$

and the recursive cost functional (1.2)–(1.3) becomes

$$
J(t,\xi;u) = \mathbb{E}_t\Big[h^0\big(X(T),\mathbb{E}_t[X(T)]\big) + \int_t^T g^0\big(s,X(s),u(s),\mathbb{E}_s[X(T)]\big)ds\Big].
$$

In the above, $\mathbb{E}_t[X(T)]$ appears nonlinearly and the corresponding optimal control problem is time-inconsistent. From the above observation, we see that the state equation being an FBSDE can include many situations of nonlinear appearance of conditional expectations.

We have seen that Problem (N) is generally time-inconsistent. Therefore, we should treat it from the angle differently from the usual classical ones. Before going further, let us present the following simple example, from which we will see the more essential reason for Problem (N) to be time-inconsistent.

Example 1.1. Consider the one-dimensional (degenerate) FBSDE state equation

$$
\begin{cases}
\dot{X}(s) = 0, \quad \dot{Y}(s) = u(s), \quad s \in [t,T],\\
X(t) = x, \quad Y(T) = 0,
\end{cases}
\tag{1.10}
$$

with the cost functional

$$
J(t,x;u) = \int_t^T [Y(s) + u(s) + |u(s)|^2]ds. \tag{1.11}
$$

A straightforward calculation (see Example 3.2 for details) shows that at the initial pair (t,x), the unique optimal control $\bar{u}(\cdot\,;t,x)$ is given by

$$
\bar{u}(s) \equiv \bar{u}(s;t,x) = \frac{s-t-1}{2}, \quad s \in [t,T].
$$

Then, for any $\tau \in (t, T)$, the unique optimal control at $(\tau, \bar{X}(\tau)) \equiv (\tau, x)$ is given by

$$\tilde{u}(s) \equiv \tilde{u}(s; \tau, \bar{X}(\tau)) = \frac{s - \tau - 1}{2}, \quad s \in [\tau, T].$$

Clearly,

$$\bar{u}(s) \neq \tilde{u}(s), \quad s \in [\tau, T].$$

Thus, the problem is time-inconsistent.

It is worthy of pointing out that in the above example, (1.11) is a Bolza type cost functional for FBSDE state equations, and unlike Yong [73] and Björk–Khapko–Murgoci [6], neither non-exponential discounting nor conditional expectations (nonlinearly) appear. Furthermore, the controlled system (1.10) is a deterministic ordinary differential equation, and the terminal cost of (1.11) equals zero, due to which (1.11) is also a Lagrange type cost functional. This tells us that *an optimal control problem could be time-inconsistent solely because the state equation is a forward-backward one*. Hence, the time-inconsistency feature is intrinsically contained in the optimal control problems for FBSDEs. Such a feature distinguishes the current paper from the previous ones concerning the time-inconsistency, in other aspects.

Having the above time-inconsistent feature of the problem, we now highlight the main results of this paper.

(i) Using Pontryagin's maximum principle, we will show that Problem (N) is generically time-inconsistent. The advantage of such an approach is that we are not satisfied with just some counterexamples, instead, we will show that if $\bar{u}$ is optimal at (t, ξ), which will satisfy the Pontragin's type maximum principle (MP, for short) on $[t, T]$, then $\bar{u}|_{[\tau,T]}$ hardly satisfies the MP on $[\tau, T]$ for $\tau \in (t, T]$. Therefore, (1.9) should not be expected in general.

(ii) Since Problem (N) is time-inconsistent in general, finding an optimal control at any given initial pair (t, ξ) is not very useful. Instead, one should find an *equilibrium strategy*, which is time-consistent and possesses certain kind of local optimality. Inspired by Yong [73], we derive the *equilibrium HJB equation* associated with Problem (N), through which an equilibrium strategy can be constructed. Our equilibrium HJB equation can cover the results obtained in Yong [73] and Björk–Khapko–Murgoci [6]. In the case that the recursive cost functional is governed by a BSDE, one could apply the method of multi-person differential games, by viewing that the controller is playing a cooperative game with all his incarnations in the future. Such an idea can be traced back to the work of Pollak [50] in 1968. Later, the approach was adopted and further developed in [19,20,73,75,76,7,8,6,68,43,44,63]. We point out that the multi-person differential game approach used in [73,68] does not directly apply to Problem (N) of the current paper, because the DPP does not hold for controlled FBSDEs even if the cost functional does not depend on the initial values $(t, X(t), Y(t))$. We overcome the difficulty by making use of the Feymann–Kac formula for BSVIEs, which has been recently well-developed in our works [67,62,64]. In the proof of the verification theorem, some technical assumptions imposed in [68,63] and [6] are relaxed.

(iii) When the diffusion term of the forward state equation does not contain the control u, the equilibrium HJB equation associated with Problem (N) is a system of semi-linear parabolic partial differential equations with non-local terms. Under the non-degenerate condition, the well-posdness of the equilibrium HJB equation is established in the sense of classical solutions.

(iv) Some comparisons between our equilibrium HJB equations and those derived by Peng [49], by Yong [73,75], and by Björk–Khapko–Murgoci [6] are carefully made, respectively. We find that the backward controlled equation has a significant influence on the form of the associated equilibrium HJB equation. When Problem (N) is reduced to the problem studied by Björk–Khapko–Murgoci [6], the form of our equilibrium HJB equations is more natural than their so-called *extended HJB equation*. We note that there was no rigorous proof on the well-posedness of the extended HJB equation presented in [6].

(v) The linear-quadratic optimal control problems for FBSDEs are briefly studied and a linear equilibrium strategy is obtained, provided the associated Riccati equation is solvable. This partially covers the work of Yong [76]. Further, as applications, a mean-variance model, a social planner model of Merton's consumption–portfolio selection with heterogeneous Epstein–Zin utilities, and a Stackelberg game are investigated, which are all special cases of Problem (N). It is shown that these specific problems are all time-inconsistent, and by the theoretical results obtained in the paper, the associated equilibrium strategies can be explicitly constructed.

The rest of this paper is organized as follows. In Section 2, we state the main results of our paper, with some explanations. In Section 3, we compare the results obtained in the paper with the existing ones. The linear-quadratic problem is studied in Section 4, and three applications are presented in Section 5. In Section 6, the verification theorem is proved. Some technical and lengthy proofs are given in Section 7.

2. The main results

2.1. Preliminaries: notations and Feynman–Kac formula

Let $T > 0$ be a given time horizon and recall the upper triangle domain $\Delta^*[0,T]$ from (1.4). Let $\mathbb{S}^n$ be the subspace of $\mathbb{R}^{n\times n}$ consisting of symmetric matrices and $U \subseteq \mathbb{R}^\ell$ be a nonempty measurable set which could be bounded or unbounded. We will use $K > 0$ to represent a generic constant which could be different from line to line. For any Euclidean space $\mathbb{H}$ (as well as $\mathbb{H}_1$, $\mathbb{H}_2$), we introduce the following spaces:

$$\begin{aligned}
L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{H})) &= \Big\{\varphi : [0,T]\times\Omega \to \mathbb{H} \;\Big|\; \varphi \text{ is } \mathbb{F}\text{-adapted, pathwise} \\
&\qquad\qquad \text{continuous, } \mathbb{E}\Big[\sup_{0\le s\le T}|\varphi(s)|^2\Big] < \infty\Big\}; \\
C_{\mathbb{F}}([0,T]; L^2(\Omega;\mathbb{H})) &= \Big\{\varphi : [0,T]\times\Omega \to \mathbb{H} \;\Big|\; \varphi \text{ is } \mathbb{F}\text{-adapted, } \mathbb{E}[\varphi] \text{ is} \\
&\qquad\qquad \text{continuous, } \sup_{0\le s\le T}\mathbb{E}\big[|\varphi(s)|^2\big] < \infty\Big\}; \\
L^2_{\mathbb{F}}(0,T;\mathbb{H}) &= \Big\{\varphi : [0,T]\times\Omega \to \mathbb{H} \;\Big|\; \varphi \text{ is } \mathbb{F}\text{-progressively measurable} \\
&\qquad\qquad \text{on } [0,T],\ \mathbb{E}\int_0^T |\varphi(s)|^2 ds < \infty\Big\}; \\
C([0,T]; L^2_{\mathbb{F}}(\cdot\,,T;\mathbb{H})) &= \Big\{\varphi : \Delta^*[0,T]\times\Omega \to \mathbb{H} \;\Big|\; \varphi(t,\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{H}),\ t\in[0,T], \\
&\qquad\qquad \mathbb{E}\int_\cdot^T |\varphi(\cdot\,,s)|^2 ds \in C([0,T])\Big\}; \\
L^\infty(\mathbb{H}_1;\mathbb{H}_2) &= \big\{\varphi : \mathbb{H}_1 \to \mathbb{H}_2 \;\big|\; \varphi \text{ is essentially bounded}\big\}; \\
C^k(\mathbb{H}_1;\mathbb{H}_2) &= \big\{\varphi : \mathbb{H}_1 \to \mathbb{H}_2 \;\big|\; \varphi \text{ is } j\text{-th continuously differentiable} \\
&\qquad\qquad \text{for any } 0\le j\le k\big\}; \\
C^k_b(\mathbb{H}_1;\mathbb{H}_2) &= \big\{\varphi \in C^k(\mathbb{H}_1;\mathbb{H}_2) \mid \text{the } j\text{-th derivatives are bounded, } 0\le j\le k\big\}.
\end{aligned}$$

To guarantee the well-posedness of the controlled FBSDE (1.1) and BSVIE (1.3) governing the recursive cost functional, we introduce the following assumptions.

(H1). *Let the mappings* $b, \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $g : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, *and* $h : \mathbb{R}^n \to \mathbb{R}^m$ *be continuous. There exists a constant* $L > 0$ *such that*

$$
\begin{aligned}
&|b(s,0,u)| + |\sigma(s,0,u)| + |h(0)| + |g(s,0,u,0,0)| \le L(1+|u|),\\
&|b(s,x_1,u) - b(s,x_2,u)| + |\sigma(s,x_1,u) - \sigma(s,x_2,u)| + |h(x_1) - h(x_2)|\\
&+ |g(s,x_1,u,y_1,z_1) - g(s,x_2,u,y_2,z_2)| \le L\big[|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\big],\\
&\qquad \forall (s,u) \in [0,T] \times U,\ (x_i,y_i,z_i) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m,\ i = 1,2.
\end{aligned}
$$

(H2). *Let the mappings* $h^0 : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *and* $g^0 : \Delta^*[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be continuous. There exists a constant* $L > 0$ *such that*

$$
\begin{aligned}
&|h^0(t,0,0,0)| + |g^0(t,s,0,0,u,0,0,0,0)| \le L(1+|u|),\\
&|g^0(t_1,s,\tilde{x}_1,x_1,u,y_1,z_1,y_1^0,z_1^0) - g^0(t_2,s,\tilde{x}_2,x_2,u,y_2,z_2,y_2^0,z_2^0)|\\
&+ |h^0(t_1,\tilde{x}_1,x_1,y_1) - h^0(t_2,\tilde{x}_2,x_2,y_2)| \le L\big[|t_1 - t_2| + |\tilde{x}_1 - \tilde{x}_2| + |x_1 - x_2|\\
&+ |y_1 - y_2| + |z_1 - z_2| + |y_1^0 - y_2^0| + |z_1^0 - z_2^0|\big],\\
&\qquad \forall (t_i,s) \in \Delta^*[0,T],\ \tilde{x}_i, x_i \in \mathbb{R}^n,\ u \in U,\ y_i, z_i \in \mathbb{R}^m,\ y_i^0, z_i^0 \in \mathbb{R},\ i = 1,2.
\end{aligned}
$$

By Yong–Zhou [77, Chapter 7] and Yong [71], we have the following results about the well-posedness of (decoupled) FBSDE (1.1) and BSVIE (1.3).

Lemma 2.1. *Let (H1) hold. Then for any initial pair* $(t,\xi) \in \mathcal{D}$ *and control* $u \in \mathcal{U}[t,T]$, *state equation* (1.1) *admits a unique adapted solution* $(X,Y,Z) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^m)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$. *Moreover, there exists a constant* $K > 0$, *independent of* (t,ξ) *and* u, *such that*

$$
\mathbb{E}_t\Big[\sup_{t\le r\le T}\big(|X(r)|^2 + |Y(r)|^2\big) + \int_t^T |Z(s)|^2 ds\Big] \le K\mathbb{E}_t\Big[1 + |\xi|^2 + \int_t^T |u(s)|^2 ds\Big].
$$

In addition, if (H2) holds, then for any initial pair $(t,\xi) \in \mathcal{D}$, *control* $u \in \mathcal{U}[t,T]$, *and the corresponding state process* (X,Y,Z), *BSVIE* (1.3) *admits a unique adapted solution* $(Y^0,Z^0) \in C_{\mathbb{F}}([t,T];L^2(\Omega;\mathbb{R})) \times C([t,T];L^2_{\mathbb{F}}(\cdot,T;\mathbb{R}))$. *Moreover, there exists a constant* $K > 0$, *independent of* (t,ξ) *and* u, *such that*

$$
\sup_{t\le r\le T} \mathbb{E}_t\Big[|Y^0(r)|^2 + \int_r^T |Z^0(r,s)|^2 ds\Big] \le K\mathbb{E}_t\Big[1 + |\xi|^2 + \int_t^T |u(s)|^2 ds\Big].
$$

As another preparation, we consider the following system of FBSDEs and BSVIEs without controls over $[t,T]$:

$$\begin{cases} X(r) = \xi + \int_t^r b(s, X(s))ds + \int_t^r \sigma(s, X(s))dW(s), \\ Y(r) = h(X(T)) + \int_r^T g(s, X(s), Y(s), Z(s))ds - \int_r^T Z(s)dW(s), \\ Y^0(r) = h^0(r, X(r), X(T), Y(r)) + \int_r^T g^0(r, s, X(r), X(s), Y(s), Z(s), Y^0(s), Z^0(r, s))ds \\ \qquad - \int_r^T Z^0(r, s)dW(s), \end{cases} \tag{2.1}$$

where the coefficients $b, \sigma, h, g, h^0, g^0$ satisfy (H1)–(H2) (independent of the control u). Suggested by Wang–Yong [67] and Wang–Yong–Zhang [64], we introduce the following system of semi-linear PDEs:

$$\begin{cases} \Theta^k_s(s, x) + \frac{1}{2}\mathrm{tr}\,[\Theta^k_{xx}(s, x)\sigma(s, x)\sigma(s, x)^\top] + \Theta^k_x(s, x)b(s, x) \\ \quad + g\big(s, x, \Theta(s, x), \Theta_x(s, x)\sigma(s, x)\big) = 0, \quad \forall (s, x) \in [t, T] \times \mathbb{R}^n,\ 1 \le k \le m, \\ \Theta^0_s(r, s, \tilde{x}, x, y) + \frac{1}{2}\mathrm{tr}\,[\Theta^0_{xx}(r, s, \tilde{x}, x, y)\sigma(s, x)\sigma(s, x)^\top] + \Theta^0_x(r, s, \tilde{x}, x, y)b(s, x) \\ \quad + g^0\big(r, s, \tilde{x}, x, \Theta(s, x), \Theta_x(s, x)\sigma(s, x), \Theta^0(s, s, x, x, \Theta(s, x)), \Theta^0_x(r, s, \tilde{x}, x, y)\sigma(s, x)\big) = 0, \\ \qquad \forall (r, s, \tilde{x}, x, y) \in \Delta^*[t, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m, \\ \Theta(T, x) = h(x), \quad \Theta^0(r, T, \tilde{x}, x, y) = h^0(r, \tilde{x}, x, y), \end{cases} \tag{2.2}$$

with $\Theta = (\Theta^1, \cdots, \Theta^m)^\top$. Note that Θ^0 is a function of $(r, s, \tilde{x}, x, y)$, and $\Theta^0_x, \Theta^0_{xx}$ are the derivatives with respect to the 4th argument. We have the following representation theorem.

Proposition 2.2. *Suppose that the PDE (2.2) admits a classical solution (Θ, Θ^0). Assume that the system of FBSDEs and BSVIEs (2.1) admits a unique adapted solution (X, Y, Z, Y^0, Z^0). Then the following representation holds:*

$$\begin{aligned} &Y(r) = \Theta(r, X(r)), \quad Z(r) = \Theta_x(r, X(r))\sigma(r, X(r)), \quad r \in [t, T],\ a.s., \\ &Y^0(r) = \Theta^0\big(r, r, X(r), X(r), \Theta(r, X(r))\big), \quad r \in [t, T],\ a.s., \\ &Z^0(r, s) = \Theta^0_x\big(r, s, X(r), X(s), \Theta(r, X(r))\big)\sigma(s, X(s)), \quad (r, s) \in \Delta^*[t, T],\ a.s. \end{aligned}$$

The proof of Proposition 2.2 is given in [65].

Remark 2.3. Proposition 2.2 is a generalization of the Feynman–Kac formula for Markovian BSVIEs, which was established by Wang–Yong [67] and Wang [62], in the sense of classical solutions. Under the non-degenerate assumption, the well-posedness of PDE (2.2) will be established by an analytic method, as a byproduct of Theorem 2.10. The probabilistic approach, without the non-degenerate assumption, can be also obtained by the arguments employed by Wang–Yong–Zhang [64].

2.2. Time-inconsistency analysis of problem (N)

In this subsection, we shall discuss the time-inconsistency of Problem (N) from the Pontryagin's maximum principle viewpoint. For simplicity, we consider the case that (1.5) holds so that the cost functional reads as (1.6) (of Bolza type, without involving BSVIEs). Also, we suppose that the control domain $U \equiv \mathbb{R}^\ell$ and all involved functions are continuously differentiable. Let $\big(\bar{X}^{t,\xi}, \bar{u}^{t,\xi}, \bar{Y}^{t,\xi}, \bar{Z}^{t,\xi}\big)$ be an optimal 4-tuple (supposing it exists) of Problem (N) on $[t,T]$ with a given initial pair $(t,\xi) \in \mathcal{D}$, for which we assume to be time-consistent. Then, for any $\tau \in (t,T]$,

$$J(\tau, \bar{X}^{t,\xi}(\tau); \bar{u}^{t,\xi}|_{[\tau,T]}) = \inf_{u\in\mathcal{U}[\tau,T]} J(\tau, \bar{X}^{t,\xi}(\tau); u),$$

and

$$\begin{aligned}&\big(\bar{X}^{t,\xi}(s),\, \bar{u}^{t,\xi}(s),\, \bar{Y}^{t,\xi}(s),\, \bar{Z}^{t,\xi}(s)\big)\\&= \big(\bar{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(s),\, \bar{u}^{\tau,\bar{X}^{t,\xi}(\tau)}(s),\, \bar{Y}^{\tau,\bar{X}^{t,\xi}(\tau}(s),\, \bar{Z}^{\tau,\bar{X}^{t,\xi}(\tau)}(s)\big), \quad s \in [\tau,T],\ \text{a.s.}\end{aligned}$$

Now, we denote

$$\begin{aligned}
&\bar{b}_x^{t,\xi}(s) = b_x(s, \bar{X}^{t,\xi}(s), \bar{u}^{t,\xi}(s)), && \bar{b}_u^{t,\xi}(s) = b_u(s, \bar{X}^{t,\xi}(s), \bar{u}^{t,\xi}(s)),\\
&\bar{g}_x^{t,\xi}(s) = g_x(s, \bar{X}^{t,\xi}(s), \bar{u}^{t,\xi}(s), \bar{Y}^{t,\xi}(s), \bar{Z}^{t,\xi}(s)), && \bar{g}_u^{t,\xi}(s) = g_u(s, \cdots, \bar{Z}^{t,\xi}(s)),\\
&\bar{g}_y^{t,\xi}(s) = g_y(s, \cdots, \bar{Z}^{t,\xi}(s)), && \bar{g}_z^{t,\xi}(s) = g_z(s, \cdots, \bar{Z}^{t,\xi}(s)),\\
&\bar{h}_x^{t,\xi}(T) = h_x(\bar{X}^{t,\xi}(T)), && \bar{h}_x^{0,t,\xi}(t) = h_x^0(\bar{X}^{t,\xi}(T), \bar{Y}^{t,\xi}(t)),
\end{aligned}$$

and $\bar{\sigma}_x^{t,\xi}(s)$, $\bar{\sigma}_u^{t,\xi}(s)$, $\bar{g}_x^{0,t,\xi}(s)$, $\bar{g}_u^{0,t,\xi}(s)$, $\bar{g}_y^{0,t,\xi}(s)$, $\bar{g}_z^{0,t,\xi}(s)$, $\bar{h}_y^{0,t,\xi}(t)$ are defined similarly. Then by applying the Pontryagin's maximum principle (see [47,24,25], for examples), to the optimal 4-tuple on $[t,T]$ and $[\tau,T]$, respectively, we get the following *stationarity conditions*:

$$\begin{aligned}&\bar{g}_u^{0,t,\xi}(s)^\top + \bar{g}_u^{t,\xi}(s)^\top \mathcal{X}^{t,\xi}(s) + \bar{b}_u^{t,\xi}(s)^\top \mathcal{Y}^{t,\xi}(s)\\&+ \bar{\sigma}_u^{t,\xi}(s)^\top \mathcal{Z}^{t,\xi}(s) = 0, \quad s \in [t,T],\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\bar{g}_u^{0,t,\xi}(s)^\top + \bar{g}_u^{t,\xi}(s)^\top \mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) + \bar{b}_u^{t,\xi}(s)^\top \mathcal{Y}^{\tau,\bar{X}^{t,\xi}(\tau)}(s)\\&+ \bar{\sigma}_u^{t,\xi}(s)^\top \mathcal{Z}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) = 0, \quad s \in [\tau,T],\end{aligned} \tag{2.4}$$

where $(\mathcal{Y}^{t,\xi}, \mathcal{Z}^{t,\xi})$ is the co-state process pair of $\bar{X}^{t,\xi}$, and $\mathcal{X}^{t,\xi}$ is the co-state process of $(\bar{Y}^{t,\xi}, \bar{Z}^{t,\xi})$, for which the following holds on $[t,T]$ almost surely:

$$\begin{cases}d\mathcal{Y}^{t,\xi}(s) = -\big[\bar{g}_x^{t,\xi}(s)^\top \mathcal{X}^{t,\xi}(s) + \bar{b}_x^{t,\xi}(s)^\top \mathcal{Y}^{t,\xi}(s) + \bar{\sigma}_x^{t,\xi}(s)^\top \mathcal{Z}^{t,\xi}(s) + \bar{g}_x^{0,t,\xi}(s)^\top\big]ds\\ \qquad\qquad\quad + \mathcal{Z}^{t,\xi}(s)dW(s),\\ d\mathcal{X}^{t,\xi}(s) = \big[\bar{g}_y^{t,\xi}(s)^\top \mathcal{X}^{t,\xi}(s) + \bar{g}_y^{0,t,\xi}(s)^\top\big]ds + \big[\bar{g}_z^{t,\xi}(s)^\top \mathcal{X}^{t,\xi}(s) + \bar{g}_z^{0,t,\xi}(s)^\top\big]dW(s),\\ \mathcal{Y}^{t,\xi}(T) = \bar{h}_x^{t,\xi}(T)^\top \mathcal{X}^{t,\xi}(T) + \bar{h}_x^{0,t,\xi}(t)^\top, \quad \mathcal{X}^{t,\xi}(t) = \mathbb{E}_t[\bar{h}_y^{0,t,\xi}(t)^\top],\end{cases} \tag{2.5}$$

and $(\mathcal{Y}^{\tau,\bar{X}^{t,\xi}(\tau)}, \mathcal{Z}^{\tau,\bar{X}^{t,\xi}(\tau)})$ is the co-state process pair of $\bar{X}^{\tau,\bar{X}^{t,\xi}(\tau)}$, and $\mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}$ is the co-state process of $(\bar{Y}^{\tau,\bar{X}^{t,\xi}(\tau)}, \bar{Z}^{\tau,\bar{X}^{t,\xi}(\tau)})$, for which the following holds on $[\tau, T]$ almost surely:

$$
\begin{cases}
d\mathcal{Y}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) = -\big[\bar{g}_x^{t,\xi}(s)^\top \mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) + \bar{b}_x^{t,\xi}(s)^\top \mathcal{Y}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) \\
\qquad\qquad + \bar{\sigma}_x^{t,\xi}(s)^\top \mathcal{Z}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) + \bar{g}_x^{0,t,\xi}(s)^\top\big]ds + \mathcal{Z}^{\tau,\bar{X}^{t,\xi}(\tau)}(s)dW(s), \\
d\mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) = \big[\bar{g}_y^{t,\xi}(s)^\top \mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) + \bar{g}_y^{0,t,\xi}(s)^\top\big]ds \\
\qquad\qquad + \big[\bar{g}_z^{t,\xi}(s)^\top \mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(s) + \bar{g}_z^{0,t,\xi}(s)^\top\big]dW(s), \\
\mathcal{Y}^{\tau,\bar{X}^{t,\xi}(\tau)}(T) = \bar{h}_x^{t,\xi}(T)^\top \mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(T) + \bar{h}_x^{0,t,\xi}(\tau)^\top, \\
\mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(\tau) = \mathbb{E}_\tau[\bar{h}_y^{0,t,\xi}(\tau)^\top].
\end{cases}
\tag{2.6}
$$

We conclude the results as follows.

Proposition 2.4. *If the optimal 4-tuple $(\bar{X}^{t,\xi}, \bar{u}^{t,\xi}, \bar{Y}^{t,\xi}, \bar{Z}^{t,\xi})$ is time-consistent, then (2.4) holds for any $\tau \in (t, T]$, subject to (2.6).*

The necessary condition (2.4) with $\tau \in [t, T]$ can be regarded as a *dynamic* version of the famous Pontryagin's maximum principle. Interestingly, we can use it to characterize the time-consistency of the optimal controls.

Now, let us make a careful comparison between (2.5) and (2.6). First of all, these decoupled FBSDEs have exactly the same coefficients. If we restrict (2.5) on $[\tau, T]$, then it has the initial condition $\mathcal{X}^{t,\xi}(\tau)$, and we do not expect the following:

$$
\mathcal{X}^{t,\xi}(\tau) = \mathbb{E}_\tau[\bar{h}_y^{0,t,\xi}(\tau)^\top] \equiv \mathbb{E}_\tau\big[h_y^0(\bar{X}^{t,\xi}(T), \bar{Y}^{t,\xi}(t))^\top\big], \quad \forall \tau \in (t, T].
$$

Hence, in general, the following cannot be guaranteed:

$$
\begin{aligned}
&\big(\mathcal{X}^{\tau,\bar{X}^{t,\xi}(\tau)}(s),\, \bar{u}^{\tau,\bar{X}^{t,\xi}(\tau)}(s),\, \mathcal{Y}^{\tau,\bar{X}^{t,\xi}(\tau)}(s),\, \mathcal{Z}^{\tau,\bar{X}^{t,\xi}(\tau)}(s)\big) \\
&= \big(\mathcal{X}^{t,\xi}(s),\, \bar{u}^{t,\xi}(s),\, \mathcal{Y}^{t,\xi}(s),\, \mathcal{Z}^{t,\xi}(s)\big), \quad s \in [\tau, T], \text{ a.s., } \forall \tau \in [t, T].
\end{aligned}
$$

Consequently, having (2.3), it is too much to request (2.4). From this, we see that Problem (N) is intrinsically time-inconsistent.

2.3. Equilibrium strategy and equilibrium HJB equation

Since Problem (N) is time-inconsistent in general, we shall find the equilibrium strategy, whose definition is given as follows.

Definition 2.5. A mapping $\Psi : [0, T] \times \mathbb{R}^n \to U$ is called a *feedback strategy* (of state equation (1.1)) on $[0, T]$ if for every $(t, \xi) \in \mathcal{D}$, the following *closed-loop* system:

$$
\begin{cases}
dX(s) = b(s, X(s), \Psi(s, X(s)))ds + \sigma(s, X(s), \Psi(s, X(s)))dW(s), \\
dY(s) = -g(s, X(s), \Psi(s, X(s)), Y(s), Z(s))ds + Z(s)dW(s), \\
X(t) = \xi, \quad Y(T) = h(X(T)),
\end{cases}
$$

admits a unique adapted solution $(X, Y, Z) \equiv \big(X(\cdot\,; t, \xi, \Psi),\, Y(\cdot\,; t, \xi, \Psi),\, Z(\cdot\,; t, \xi, \Psi)\big) \equiv (X^\Psi, Y^\Psi, Z^\Psi)$ and the *outcome* $u^\Psi \equiv \Psi(\cdot\,, X^\Psi(\cdot))$ of Ψ belongs to $\mathcal{U}[t, T]$.

We now introduce the following definition.

Definition 2.6. A feedback strategy $\bar{\Psi}$, with $\bar{X}$ being the forward component of the corresponding state process, is called an *equilibrium strategy* if

$$\liminf_{\varepsilon\to 0^+}\frac{J(t,\bar{X}(t);\Psi^\varepsilon)-J(t,\bar{X}(t);\bar{\Psi})}{\varepsilon}\geq 0,$$

for any $t\in[0,T)$ and $u\in L^2_{\mathcal{F}_t}(\Omega;U)$, where

$$\Psi^\varepsilon(s):=\begin{cases}\bar{\Psi}(s,X^\varepsilon(s)), & s\in[t+\varepsilon,T];\\ u, & s\in[t,t+\varepsilon),\end{cases}\tag{2.7}$$

with $X^\varepsilon:=X^{\Psi^\varepsilon}\equiv X(\cdot\,;t,\bar{X}(t),\Psi^\varepsilon)$ being the forward component of the state process corresponding to Ψ^ε.

The intuition behind Definition 2.6 is similar to that in [73,26,6,63]. At any given time t, the controller is playing a game with all his/her incarnations in the future by minimizing his/her cost functional on $[t,t+\varepsilon)$, and knowing that he/she will lose the control of the system beyond $t+\varepsilon$. We now briefly list our main results as follows.

To find an equilibrium strategy of Problem (N), we introduce the following spaces:

$$\begin{aligned}&\mathcal{A}[0,T]:=[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^m\times\mathbb{R}^{m\times n},\\&\mathcal{A}^0[0,T]:=\Delta^*[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\times\mathbb{R}\times\mathbb{R}^{1\times n}.\end{aligned}$$

For simplicity, we denote

$$\boldsymbol{\Theta}=(\theta,p)\in\mathbb{R}^m\times\mathbb{R}^{m\times n},\quad \boldsymbol{\Theta}^0=(\theta^0,p^0)\in\mathbb{R}\times\mathbb{R}^{1\times n}.$$

Now, we define the following Hamiltonians:

$$\begin{aligned}&\boldsymbol{H}(s,x,u,\boldsymbol{\Theta},P)=\mathrm{tr}[Pa(s,x,u)]+pb(s,x,u)+g\big(s,x,u,\theta,p\sigma(s,x,u)\big),\\&\boldsymbol{H}^0(t,s,\tilde{x},x,u,\boldsymbol{\Theta},\boldsymbol{\Theta}^0,P^0)\\&=\mathrm{tr}[P^0a(s,x,u)]+p^0b(s,x,u)+g^0(t,s,\tilde{x},x,u,\theta,p\sigma(s,x,u),\theta^0,p^0\sigma(s,x,u)),\\&\hat{\boldsymbol{H}}^0(t,s,\tilde{x},x,u,\boldsymbol{\Theta},P,\boldsymbol{\Theta}^0,q^0,P^0)\\&=\boldsymbol{H}^0(t,s,\tilde{x},x,u,\boldsymbol{H},\boldsymbol{H}^0,P^0)+q^0\boldsymbol{H}(s,x,u,\boldsymbol{\Theta},P),\\&\quad\forall(t,s,\tilde{x},x,u,\boldsymbol{\Theta},\boldsymbol{\Theta}^0)\in\mathcal{A}^0[0,T],\ P\in\big[\mathbb{S}^n\big]^m,\ q^0\in\mathbb{R}^{1\times m},\ P^0\in\mathbb{S}^n,\end{aligned}\tag{2.8}$$

where a is defined by

$$a(s,x,u)=\frac{1}{2}\sigma(s,x,u)\sigma(s,x,u)^\top,\quad (s,x,u)\in[0,T]\times\mathbb{R}^n\times U,$$

and, with $P=(P^1,P^2,\cdots,P^m)^\top\in[\mathbb{S}^n]^m$,

$$\mathrm{tr}\left[Pa(s,x,u)\right]=\begin{pmatrix}\mathrm{tr}\left[P^1a(s,x,u)\right]\\\mathrm{tr}\left[P^2a(s,x,u)\right]\\\vdots\\\mathrm{tr}\left[P^ma(s,x,u)\right]\end{pmatrix}.$$

In what follows, we will use the following hypothesis.

(H3). *Suppose that there exists a unique mapping ψ such that*

$$\begin{aligned}
&\hat{\boldsymbol{H}}^0(t,s,\tilde{x},x,\psi(t,s,\tilde{x},x,\boldsymbol{\Theta},P,\boldsymbol{\Theta}^0,q^0,P^0),\boldsymbol{\Theta},P,\boldsymbol{\Theta}^0,q^0,P^0)\\
&=\inf_{u\in U}\hat{\boldsymbol{H}}^0(t,s,\tilde{x},x,u,\boldsymbol{\Theta},P,\boldsymbol{\Theta}^0,q^0,P^0),\\
&\quad\forall(t,s,\tilde{x},x,u,\boldsymbol{\Theta},\boldsymbol{\Theta}^0)\in\mathcal{A}^0[0,T],\ P\in\big[\mathbb{S}^n\big]^m,\ q^0\in\mathbb{R}^{1\times m},\ P^0\in\mathbb{S}^n.
\end{aligned}\tag{2.9}$$

Moreover, we suppose that ψ is smooth enough with bounded derivatives.

We now introduce the following *equilibrium HJB equation*:

$$\begin{cases}\Theta_s(s,x)+\boldsymbol{H}\big(s,x,\bar{\Psi}(s,x),\Theta(s,x),\Theta_x(s,x),\Theta_{xx}(s,x)\big)=0,\\
\Theta_s^0(t,s,\tilde{x},x,y)+\boldsymbol{H}^0\big(t,s,\tilde{x},x,\bar{\Psi}(s,x),\Theta(s,x),\Theta_x(s,x),\\
\quad\Theta^0(s,s,x,x,\Theta(s,x)),\Theta_x^0(t,s,\tilde{x},x,y),\Theta_{xx}^0(t,s,\tilde{x},x,y)\big)=0,\\
\qquad(t,s,\tilde{x},x,y)\in\Delta^*[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m,\\
\Theta(T,x)=h(x),\quad\Theta^0(t,T,\tilde{x},x,y)=h^0(t,\tilde{x},x,y),\\
\qquad(t,\tilde{x},x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m,\end{cases}\tag{2.10}$$

where for any $(s,x)\in[0,T]\times\mathbb{R}^n$,

$$\begin{aligned}\bar{\Psi}(s,x)=\psi\big(&s,s,x,x,\Theta(s,x),\Theta_x(s,x),\Theta_{xx}(s,x),\Theta^0(s,s,x,x,\Theta(s,x)),\\
&\Theta_x^0(s,s,x,x,\Theta(s,x)),\Theta_y^0(s,s,x,x,\Theta(s,x)),\Theta_{xx}^0(s,s,x,x,\Theta(s,x))\big).\end{aligned}\tag{2.11}$$

We have the following result.

Theorem 2.7. *Let $\bar{\Psi}:[0,T]\times\mathbb{R}^n\to U$ be defined by (2.11), with (Θ,Θ^0) being the classical solution to the equilibrium HJB equation (2.10). Let $\bar{\Psi}$ be a feedback strategy. Then it is an equilibrium strategy of* Problem (N).

Remark 2.8. Theorem 2.7 is a verification theorem for Problem (N), whose proof is given in Section 6. Taking the equilibrium strategy $\bar{\Psi}$ in (1.1) and (1.3), we get the following equilibrium system on $[0,T]$:

$$\begin{cases}d\bar{X}(s)=b(s,\bar{X}(s),\bar{\Psi}(s,\bar{X}(s)))ds+\sigma(s,\bar{X}(s),\bar{\Psi}(s,\bar{X}(s)))dW(s),\\
d\bar{Y}(s)=-g(s,\bar{X}(s),\bar{\Psi}(s,\bar{X}(s)),\bar{Y}(s),\bar{Z}(s))ds+\bar{Z}(s)dW(s),\\
\bar{X}(0)=\xi,\quad\bar{Y}(T)=h(\bar{X}(T)),\end{cases}$$

and

$$\begin{aligned}\bar{Y}^0(r)=h^0(r,\bar{X}(r),\bar{X}(T),\bar{Y}(r))-\int_r^T\bar{Z}^0(r,s)dW(s)\\
+\int_r^T g^0\big(r,s,\bar{X}(r),\bar{X}(s),\bar{\Psi}(s,\bar{X}(s)),\bar{Y}(s),\bar{Z}(s),\bar{Y}^0(s),\bar{Z}^0(r,s)\big)ds.\end{aligned}$$

Then by Proposition 2.2, we have the following representation formula:

$$\begin{aligned}
&\bar{Y}(r) = \Theta(r, \bar{X}(r)), \quad \bar{Z}(r) = \Theta_x(r, \bar{X}(r))\sigma(r, \bar{X}(r), \bar{\Psi}(r, \bar{X}(r))), \\
&\bar{Y}^0(r) = \Theta^0\big(r, r, \bar{X}(r), \bar{X}(r), \Theta(r, \bar{X}(r))\big), \\
&\bar{Z}^0(r, s) = \Theta^0_x\big(r, s, \bar{X}(r), \bar{X}(s), \Theta(r, \bar{X}(r))\big)\sigma(s, \bar{X}(s), \bar{\Psi}(s, \bar{X}(s))),
\end{aligned}$$

provided the equilibrium HJB equation (2.10) admits a classical solution (Θ, Θ^0). Thus, the form of the equilibrium HJB equation (2.10) is very natural, though it seems a little bit complicated. Using the local optimality condition (2.9), the equilibrium strategy value $\bar{\Psi}(s, x)$ is determined by $\Theta(s, x)$ and the diagonal value $\Theta^0(s, s, x, x, \Theta(s, x))$.

Remark 2.9. If the cost functional reads as (1.6), then we have

$$\begin{aligned}
&\boldsymbol{H}^0(s, x, u, \boldsymbol{\Theta}, \boldsymbol{\Theta}^0, P^0) = \mathrm{tr}[P^0 a(s, x, u)] + p^0 b(s, x, u) + g^0(s, x, u, \theta, p\sigma(s, x, u)), \\
&\hat{\boldsymbol{H}}^0(s, x, u, \boldsymbol{\Theta}, P, \boldsymbol{\Theta}^0, q^0, P^0) = \mathrm{tr}[P^0 a(s, x, u)] + p^0 b(s, x, u) + g^0(s, x, u, \theta, p\sigma(s, x, u)) \\
&\quad + q^0\big(\mathrm{tr}[P a(s, u, x)] + p b(s, x, u) + g(s, x, u, \theta, p\sigma(s, x, u))\big), \\
&\qquad \forall (s, x, u, \boldsymbol{\Theta}, P, \boldsymbol{\Theta}^0, q^0, P^0) \in \mathcal{A}[0, T] \times [\mathbb{S}^n]^m \times \mathbb{R} \times \mathbb{R}^{1\times n} \times \mathbb{R}^{1\times m} \times \mathbb{S}^n.
\end{aligned} \tag{2.12}$$

2.4. Well-posedness of the equilibrium HJB equation

In this subsection, we will present the well-posedness of equation (2.10) to some extent. Note that (2.10) is a coupled system of fully nonlinear parabolic PDEs with a non-local feature, whose well-posedness is a very challenging problem. Indeed, even for the equilibrium HJB equation associated with the time-inconsistent problems for SDEs (see Yong [73]), the well-posedness is still widely open, except when the time horizon is small enough (see Lei–Pun [36]). For the small time case, one can construct a contraction mapping in a Banach space depending on the terminal conditions and does not need to establish a prior estimate, which is exactly the main difficulty in establishing the well-posedness.

We now assume that

$$\sigma(s, x, u) = \sigma(s, x), \quad (s, x, u) \in [0, T] \times \mathbb{R}^n \times U. \tag{2.13}$$

In this case, we denote

$$\begin{aligned}
&\boldsymbol{H}(s, x, u, \boldsymbol{\Theta}) = p b(s, x, u) + g(s, x, u, \theta, p\sigma(s, x)), \\
&\boldsymbol{H}^0(t, s, \tilde{x}, x, u, \boldsymbol{\Theta}, \boldsymbol{\Theta}^0) = p^0 b(s, x, u) + g^0(t, s, \tilde{x}, x, u, \theta, p\sigma(s, x), \theta^0, p^0\sigma(s, x)), \\
&\hat{\boldsymbol{H}}^0(t, s, \tilde{x}, x, u, \boldsymbol{\Theta}, \boldsymbol{\Theta}^0, q^0) = \boldsymbol{H}^0(t, s, \tilde{x}, x, u, \boldsymbol{\Theta}, \boldsymbol{\Theta}^0) + q^0 \boldsymbol{H}(s, x, u, \boldsymbol{\Theta}), \\
&\qquad \forall (t, s, \tilde{x}, x, u, \boldsymbol{\Theta}, \boldsymbol{\Theta}^0) \in \mathcal{A}^0[0, T],\ q^0 \in \mathbb{R}^{1\times m}.
\end{aligned}$$

Then the mapping ψ determined by (H3) can be determined by the following:

$$\begin{aligned}
&\hat{\boldsymbol{H}}^0(t, s, \tilde{x}, x, \psi(t, s, \tilde{x}, x, \boldsymbol{\Theta}, \boldsymbol{\Theta}^0, q^0), \boldsymbol{\Theta}, \boldsymbol{\Theta}^0, q^0) = \inf_{u\in U} \hat{\boldsymbol{H}}^0(t, s, \tilde{x}, x, u, \boldsymbol{\Theta}, \boldsymbol{\Theta}^0, q^0), \\
&\qquad \forall (t, s, \tilde{x}, x, u, \boldsymbol{\Theta}, \boldsymbol{\Theta}^0) \in \mathcal{A}^0[0, T],\ q^0 \in \mathbb{R}^{1\times m}.
\end{aligned} \tag{2.14}$$

Namely, in the current case, ψ is independent of P and P^0. Then (2.10) is reduced to the following system of semilinear PDEs:

$$\begin{cases}\Theta_s(s,x)+\mathrm{tr}[\Theta_{xx}(s,x)a(s,x)]+\Theta_x(s,x)b(s,x,\bar\Psi(s,x))\\ \quad +g(s,x,\bar\Psi(s,x),\Theta(s,x),\Theta_x(s,x)\sigma(s,x))=0,\\ \Theta^0_s(t,s,\tilde x,x,y)+\mathrm{tr}[\Theta^0_{xx}(t,s,\tilde x,x,y)a(s,x)]+\Theta^0_x(t,s,\tilde x,x,y)b(s,x,\bar\Psi(s,x))\\ \quad +g^0\big(t,s,\tilde x,x,\bar\Psi(s,x),\Theta(s,x),\Theta_x(s,x)\sigma(s,x),\Theta^0(s,s,x,x,\Theta(s,x)),\\ \qquad\quad \Theta^0_x(t,s,\tilde x,x,y)\sigma(s,x)\big)=0,\\ \Theta(T,x)=h(x),\quad \Theta^0(t,T,\tilde x,x,y)=h^0(t,\tilde x,x,y),\end{cases}\tag{2.15}$$

with

$$\begin{aligned}\bar\Psi(s,x)=\psi\big(&s,s,x,x,\Theta(s,x),\Theta_x(s,x),\Theta^0(s,s,x,x,\Theta(s,x)),\\ &\Theta^0_x(s,s,x,x,\Theta(s,x)),\Theta^0_y(s,s,x,x,\Theta(s,x))\big).\end{aligned}$$

Now, we denote

$$\begin{aligned}&\tilde b\big(s,x,\Theta(s,x),\Theta_x(s,x),\Theta^0(s,s,x,x,\Theta(s,x)),\Theta^0_x(s,s,x,x,\Theta(s,x)),\Theta^0_y(s,s,x,x,\Theta(s,x))\big)\\ &:=b\big(s,x,\psi\big(s,s,x,x,\Theta(s,x),\Theta_x(s,x),\Theta^0(s,s,x,x,\Theta(s,x)),\Theta^0_x(s,s,x,x,\Theta(s,x)),\\ &\qquad \Theta^0_y(s,s,x,x,\Theta(s,x))\big)\big).\end{aligned}$$

Further, $\tilde g$ and $\tilde g^0$ can be defined similarly. Because of the above dependence, we may write the above (2.15) as follows:

$$\begin{cases}\Theta_s(s,x)+\mathrm{tr}[\Theta_{xx}(s,x)a(s,x)]+\Theta_x(s,x)\tilde b\big(s,x,\Theta(s,x),\Theta_x(s,x),\\ \quad \Theta^0(s,s,x,x,\Theta(s,x)),\Theta^0_x(s,s,x,x,\Theta(s,x)),\Theta^0_y(s,s,x,x,\Theta(s,x))\big)\\ \quad +\tilde g\big(s,x,\Theta(s,x),\Theta_x(s,x),\Theta^0(s,s,x,x,\Theta(s,x)),\\ \quad \Theta^0_x(s,s,x,x,\Theta(s,x)),\Theta^0_y(s,s,x,x,\Theta(s,x))\big)=0,\\ \Theta^0_s(t,s,\tilde x,x,y)+\mathrm{tr}[\Theta^0_{xx}(t,s,\tilde x,x,y)a(s,x)]+\Theta^0_x(t,s,\tilde x,x,y)\\ \quad\times\tilde b\big(s,x,\Theta(s,x),\Theta_x(s,x),\Theta^0(s,s,x,x,\Theta(s,x)),\Theta^0_x(s,s,x,x,\Theta(s,x)),\\ \quad \Theta^0_y(s,s,x,x,\Theta(s,x))\big)+\tilde g^0\big(t,s,\tilde x,x,\Theta(s,x),\Theta_x(s,x),\Theta^0(s,s,x,x,\Theta(s,x)),\\ \quad \Theta^0_x(s,s,x,x,\Theta(s,x)),\Theta^0_y(s,s,x,x,\Theta(s,x)),\Theta^0_x(t,s,\tilde x,x,y)\big)=0,\\ \Theta(T,x)=h(x),\quad \Theta^0(t,T,\tilde x,x,y)=h^0(t,\tilde x,x,y).\end{cases}\tag{2.16}$$

Although the above looks complicated, it actually has a usual HJB equation form. For the above system, we introduce the following assumption.

(H4). *The mappings*

$$\begin{cases}(s,x)\mapsto a(s,x),\quad x\mapsto h(x),\quad (t,\tilde x,x,y)\mapsto h^0(t,\tilde x,x,y),\\ (s,x,\boldsymbol\Theta,\boldsymbol\Theta^0,q^0)\mapsto\tilde b(s,x,\boldsymbol\Theta,\boldsymbol\Theta^0,q^0),\quad (s,x,\boldsymbol\Theta,\boldsymbol\Theta^0,q^0)\mapsto\tilde g(s,x,\boldsymbol\Theta,\boldsymbol\Theta^0,q^0),\\ (t,s,\tilde x,x,\boldsymbol\Theta,\boldsymbol\Theta^0,q^0,\hat p^0)\mapsto\tilde g^0(t,s,\tilde x,x,\boldsymbol\Theta,\boldsymbol\Theta^0,q^0,\hat p^0)\end{cases}$$

are bounded, have all required differentiability with bounded derivatives. Moreover, there exist two constants $\lambda_0,\lambda_1>0$ *such that*

$$\lambda_0 I\le a(t,x)\le\lambda_1 I,\quad \forall(t,x)\in[0,T]\times\mathbb R^n.$$

Theorem 2.10. *Let (H4) hold. Then the equilibrium HJB equation* (2.16) *admits a unique classical solution.*

The proof of Theorem 2.10 is technical and lengthy, which will be given in Section 7. Note that (2.16) contains the diagonal term $\Theta_y^0(s,s,x,x,\Theta(s,x))$. To our best knowledge, it is the first time that such type of equilibrium HJB equations is derived.

3. Comparison with the existing results

3.1. Comparison with Peng [49]

In [49], Peng established the well-known dynamic programming principle (DPP, for short) for the recursive optimal control problem with the state equation

$$\begin{cases} dX(s) = b(s,X(s),u(s))ds + \sigma(s,X(s),u(s))dW(s), \quad s\in[t,T],\\ dY(s) = -g(s,X(s),u(s),Y(s),Z(s))ds + Z(s)dW(s), \quad s\in[t,T],\\ X(t)=\xi, \quad Y(T)=h(X(T)), \end{cases} \tag{3.1}$$

and the cost functional

$$J(t,\xi;u) = Y(t), \tag{3.2}$$

with the backward process Y being one-dimensional. Such a problem is denoted by Problem (R). It turns out that in this case, the optimal control problem is time-consistent. The following provides a time-consistency analysis of Problem (R) from the viewpoint of Pontryagin's maximum principle.

Proposition 3.1. *Suppose that $\bar u^{t,\xi}$ is an optimal control of* Problem (R) *with the initial pair $(t,\xi)\in\mathcal{D}$. Then $\bar u^{t,\xi}$ satisfies the necessary condition* (2.4) *for any $\tau\in(t,T]$.*

Proposition 3.1 can be proved by comparing (2.3)–(2.4), and we give the proof in [65].

We remark that Problem (R) is a very special case of Problem (N). When Y is multi-dimensional, the problem is time-inconsistent in general, even if $J(t,\xi;u)$ is a linear function of $Y(t)$. Here is a simple example.

Example 3.2. Consider the (degenerate) FBSDE state equation

$$\begin{cases} \dot X(s)=0, \quad \dot Y_1(s)=u(s), \quad \dot Y_2(s) = -Y_1(s)-u(s)-|u(s)|^2, \quad s\in[t,T],\\ X(t)=x, \quad Y_1(T)=0, \quad Y_2(T)=0, \end{cases}$$

with the cost functional

$$J(t,x;u) = Y_2(t).$$

Then

$$J(t,x;u) = \int_t^T \Big[(1+t-s)u(s)+|u(s)|^2\Big]ds.$$

Thus, the unique optimal control $\bar u(\cdot\,;t,x)$ for initial pair (t,x) is given by

$$\bar{u}(s) \equiv \bar{u}(s; t, x) = \frac{s - t - 1}{2}, \quad s \in [t, T].$$

And for any $0 \leq t < \tau \leq T$, the optimal control at $(\tau, \bar{X}(\tau)) \equiv (\tau, x)$ is given by

$$\tilde{u}(s) \equiv \tilde{u}(s; \tau, \bar{X}(\tau)) = \frac{s - \tau - 1}{2}, \quad s \in [\tau, T].$$

Clearly,

$$\bar{u}(s) \neq \tilde{u}(s), \quad s \in [\tau, T],$$

which implies that the problem is time-inconsistent.

We now show that in the case of (3.1)–(3.2) (with $m = 1$), the equilibrium HJB equation (2.10) is reduced to the classical HJB equation associated with recursive stochastic optimal control problems. In fact, the associated equilibrium HJB equation is given as follows:

$$\begin{cases} \Theta_s(s,x) + \Theta_x(s,x) b(s,x,\bar{\Psi}(s,x)) + \mathrm{tr}[\Theta_{xx}(s,x) a(s,x,\bar{\Psi}(s,x))] \\ \quad + g(s,x,\bar{\Psi}(s,x),\Theta(s,x),\Theta_x(s,x)\sigma(s,x,\bar{\Psi}(s,x))) = 0, \\ \Theta^0_s(t,s,\tilde{x},x,y) + \Theta^0_x(t,s,\tilde{x},x,y) b(s,x,\bar{\Psi}(s,x)) \\ \quad + \mathrm{tr}[\Theta^0_{xx}(t,s,\tilde{x},x,y) a(s,x,\bar{\Psi}(s,x))] = 0, \\ \Theta(T,x) = h(x), \quad \Theta^0(t,y,z,T,x) = y, \end{cases} \tag{3.3}$$

where $\bar{\Psi}$ satisfies the local optimality condition (2.9). Clearly, $\Theta^0 \equiv y$ is a classical solution to the second PDE in (3.3). Thus, the local optimality condition (2.9) can be rewritten as follows: For any $(s,x) \in [0,T] \times \mathbb{R}^n$,

$$\begin{aligned} &\boldsymbol{H}(s,x,\bar{\Psi}(s,x),\Theta(s,x),\Theta_x(s,x),\Theta_{xx}(s,x)) \\ &= \inf_{u \in U} \boldsymbol{H}(s,x,u,\Theta(s,x),\Theta_x(s,x),\Theta_{xx}(s,x)), \end{aligned}$$

where $\boldsymbol{H}$ is defined by (2.8). Then the equilibrium value function is given by

$$\Theta^0(s,s,x,x,\Theta(s,x)) = \Theta(s,x), \quad (s,x) \in [0,T] \times \mathbb{R}^n,$$

with Θ being uniquely determined by

$$\begin{cases} \Theta_s(s,x) + \inf\limits_{u \in U} \Big\{ \Theta_x(s,x) b(s,x,u) + \mathrm{tr}[\Theta_{xx}(s,x) a(s,x,u)] \\ \quad + g(s,x,u,\Theta(s,x),\Theta_x(s,x)\sigma(s,x,u)) \Big\} = 0, \quad (s,x) \in [0,T] \times \mathbb{R}^n, \\ \Theta(T,x) = h(x), \quad x \in \mathbb{R}^n, \end{cases}$$

which is exactly the HJB equation derived by Peng [49].

3.2. Comparison with Yong [73,75] and Wang–Yong [63]

As an equilibrium recursive version of [73,75,68], Wang–Yong [63] considered the optimal control problems with the state equation given by the forward equation in (1.1), and the cost functional given by

$$J(t,\xi;u) = Y^0(t),$$

where Y^0 is uniquely determined by the following BSVIE:

$$Y^0(r) = h^0(r, X(T)) + \int_r^T g^0(r, s, X(s), u(s), Y^0(s), Z^0(r, s))ds$$
$$- \int_r^T Z^0(r, s)dW(s), \quad r \in [t, T].$$

Then by comparing the above with (1.2) and (1.3), we see that in our problem, the cost functional can additionally depend on the initial state $X(r)$ and the backward process (Y, Z). If the diffusion term of the state equation does not depend on the control u, the associated equilibrium HJB equation admits the following form:

$$\begin{cases} \Theta^0_s(t, s, x) + \mathrm{tr}[\Theta^0_{xx}(t, s, x)a(s, x)] + \Theta^0_x(t, s, x)\tilde{b}\big(s, x, \Theta^0(s, s, x), \Theta^0_x(s, s, x)\big) \\ \quad + \tilde{g}^0\big(t, s, x, \Theta^0(s, s, x), \Theta^0_x(s, s, x), \Theta^0_x(t, s, x)\big) = 0, \quad (t, s) \in \Delta^*[0, T], \ x \in \mathbb{R}^n, \\ \Theta^0(t, T, x) = h^0(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^n. \end{cases} \tag{3.4}$$

Compared with the equilibrium HJB equation (3.4) derived in [73,68,63], (2.16) has the following new features:

- Equilibrium HJB equation (2.16) is a coupled system of parabolic PDEs. It is interesting that the last PDE in (2.16) is coupled with the first m equations not only through the appearance of $\Theta(s, x)$ and $\Theta_x(s, x)$ in the function g^0, but also through the non-local terms $\Theta^0(s, s, x, x, \Theta(s, x))$, $\Theta^0_x(s, s, x, x, \Theta(s, x))$, and $\Theta^0_y(s, s, x, x, \Theta(s, x))$ of the unknown function Θ^0.
- Equilibrium HJB equation (2.16) depends on the partial derivative Θ^0_y along the "diagonal" points $(s, s, x, x, \Theta(s, x))$, by which we see that the backward controlled equation has a significant influence on deducing the equilibrium HJB equation. To be more clear, we take a look at this from a probabilistic viewpoint. By the Itô's formula, the stochastic system associated with (2.16) reads

$$\begin{cases} X(t) = \xi + \displaystyle\int_0^t \tilde{b}\big(s, X(s), Y(s), Z(s)\sigma(s, X(s))^{-1}, Y^0(s), \\ \qquad Z^0(s, s)\sigma(s, X(s))^{-1}, \hat{Y}^0(s)\big)ds + \displaystyle\int_0^t \sigma(s, X(s))dW(s), \\ Y(t) = h(X(T)) + \displaystyle\int_t^T \tilde{g}\big(s, X(s), Y(s), Z(s)\sigma(s, X(s))^{-1}, Y^0(s), \\ \qquad Z^0(s, s)\sigma(s, X(s))^{-1}, \hat{Y}^0(s)\big)ds - \displaystyle\int_t^T Z(s)dW(s), \end{cases}$$

and

$$\begin{cases} Y^0(t) = h^0(t, X(t), X(T), Y(t)) + \int_t^T \tilde{g}^0\big(t, s, X(t), X(s), Y(s), Z(s)\sigma(s, X(s))^{-1}, Y^0(s), \\ \qquad Z^0(s,s)\sigma(s, X(s))^{-1}, \hat{Y}^0(s), Z^0(t,s)\sigma(s, X(s))^{-1}\big)ds - \int_t^T Z^0(t,s)dW(s), \\ \hat{Y}^0(t) = h_y^0(t, X(t), X(T), Y(t)) + \int_t^T \tilde{g}^0_{p^0}\big(t, s, X(t), X(s), Y(s), Z(s)\sigma(s, X(s))^{-1}, Y^0(s), \\ \qquad Z^0(s,s)\sigma(s, X(s))^{-1}, \hat{Y}^0(s), Z^0(t,s)\sigma(s, X(s))^{-1}\big)\hat{Z}^0(t,s)\sigma(s, X(s))^{-1}ds \\ \qquad - \int_t^T \hat{Z}^0(t,s)dW(s). \end{cases}$$

Compared with [63, Theorem 5.1], the first backward equation and the third backward equation are new. The appearance of the first backward equation is natural, because the state system (1.1) is a controlled FBSDE. However, the appearance of the third backward equation is surprising. Indeed, the process $\hat{Y}^0$ is introduced for providing a probabilistic representation for $\Theta^0_y(s, x, s, x, \Theta(s, x))$, which comes from the local optimality condition of the Hamiltonian (2.14).

3.3. *Comparison with Björk–Khapko–Murgoci [6]*

In [6], Björk, Khapko, and Murgoci considered the optimal control problems with the state equation

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \\ X(t) = \xi, \end{cases} \tag{3.5}$$

and the cost functional

$$J(t, x; u) = \mathbb{E}_t[\hat{F}(X(t), X(T))] + \hat{G}(X(t)\mathbb{E}_t[X(T)]),$$

where $\hat{F}$ and $\hat{G}$ are given deterministic functions. The so-called *extended HJB equation* derived by Björk–Khapko–Murgoci [6] reads

$$\begin{cases} \inf_{u\in U}\Big((\boldsymbol{A}^u\hat{V})(t,x) - (\boldsymbol{A}^u\hat{f})(t,x,x) + (\boldsymbol{A}^u\hat{f}^x)(t,x) - \boldsymbol{A}^u(\hat{G}\diamond\hat{g})(t,x) \\ \quad + (\boldsymbol{H}^u\hat{g})(t,x)\Big) = 0, \quad \boldsymbol{A}^{\hat{u}}\hat{f}^{\tilde{x}}(t,x) = 0, \quad \boldsymbol{A}^{\hat{u}}\hat{g}(t,x) = 0, \\ \hat{V}(T,x) = \hat{F}(x,x) + \hat{G}(x,x), \quad \hat{f}^{\tilde{x}}(T,x) = \hat{F}(\tilde{x},x), \quad \hat{g}(T,x) = x, \end{cases} \tag{3.6}$$

where $\hat{u}(t,x); (t,x) \in [0,T]\times\mathbb{R}^n$ denotes the strategy which realizes the infimum in the first equation; that is

$$\begin{aligned} &(\boldsymbol{A}^{\hat{u}}\hat{V})(t,x) - (\boldsymbol{A}^{\hat{u}}\hat{f})(t,x,x) + (\boldsymbol{A}^{\hat{u}}\hat{f}^x)(t,x) - \boldsymbol{A}^{\hat{u}}(\hat{G}\diamond\hat{g})(t,x) + (\boldsymbol{H}^{\hat{u}}\hat{g})(t,x) \\ &= \inf_{u\in U}\Big((\boldsymbol{A}^u\hat{V})(t,x) - (\boldsymbol{A}^u\hat{f})(t,x,x) + (\boldsymbol{A}^u\hat{f}^x)(t,x) - \boldsymbol{A}^u(\hat{G}\diamond\hat{g})(t,x) + (\boldsymbol{H}^u\hat{g})(t,x)\Big). \end{aligned}$$

In the above, the following notations are used

$$\hat{f}(t,x,\tilde{x}) = \hat{f}^{\tilde{x}}(t,x), \quad (\hat{G} \diamond \hat{g})(t,x) = \hat{G}(x,\hat{g}(t,x)),$$
$$\boldsymbol{H}^u \hat{g}(t,x) = \hat{G}_y(s,\hat{g}(t,x))\boldsymbol{A}^u \hat{g}(t,x),$$

and the operator $\boldsymbol{A}^u$ is determined by

$$\mathbb{E}_t[k(t+h,X(t+h))] = k(t,x) + h\boldsymbol{A}^u k(t,x) + o(h), \quad \forall k \in C^{1,2},$$

where $X = X(\cdot;t,x,u)$ is the unique solution to (3.5).

The associated equilibrium HJB equation (2.10) reads

$$\begin{cases} \Theta_s(s,x) + \Theta_x(s,x)b(s,x,\bar{\Psi}(s,x)) + \operatorname{tr}[\Theta_{xx}(s,x)a(s,x,\bar{\Psi}(s,x))] = 0, \\ \Theta^0_s(s,\tilde{x},x,y) + \operatorname{tr}[\Theta^0_{xx}(s,\tilde{x},x,y)a(s,x,\bar{\Psi}(s,x))] + \Theta^0_x(s,\tilde{x},x,y)b(s,x,\bar{\Psi}(s,x)) = 0, \\ \Theta(T,x) = x, \quad \Theta^0(T,\tilde{x},x,y) = \widehat{F}(\tilde{x},x) + \widehat{G}(\tilde{x},y), \end{cases} \tag{3.7}$$

where $\bar{\Psi}$ satisfies the *local optimality condition*:

$$\begin{aligned} &\Theta^0_x(t,x,x,\Theta(t,x))b(t,x,\bar{\Psi}(t,x)) + \operatorname{tr}[\Theta^0_{xx}(t,x,x,\Theta(t,x))a(t,x,\bar{\Psi}(t,x))] \\ &+ \Theta^0_y(t,x,x,\Theta(t,x))\big\{\Theta_x(t,x)b(t,x,\bar{\Psi}(t,x)) + \operatorname{tr}[\Theta_{xx}(t,x)a(t,x,\bar{\Psi}(t,x))]\big\} \\ &= \inf_{u\in U}\Big\{\Theta^0_x(t,x,x,\Theta(t,x))b(t,x,u) + \operatorname{tr}[\Theta^0_{xx}(t,x,x,\Theta(t,x))a(t,x,u)] \\ &\qquad + \Theta^0_y(t,x,x,\Theta(t,x))\big\{\Theta_x(t,x)b(t,x,u) + \operatorname{tr}[\Theta_{xx}(t,x)a(t,x,u)]\big\}\Big\}. \end{aligned}$$

Now we compare the equilibrium HJB equation (3.7) with the extended HJB equation (3.6) derived in [6] carefully.

Proposition 3.3. *Suppose that the equilibrium HJB equation* (3.7) *admits a classical solution* (Θ,Θ^0)*. Then the solution* $\widehat{V}$ *of the extended HJB equation* (3.6) *and the equilibrium control law* $\widehat{u}$ *can be given by*

$$\widehat{V}(t,x) = \Theta^0(t,x,x,\Theta(t,x)), \quad \widehat{u}(t,x) = \bar{\Psi}(t,x), \quad (t,x) \in [0,T]\times\mathbb{R}^n.$$

The proof of Proposition 3.3 is given in [65]. Compared with Björk–Khapko–Murgoci [6], our approach has the following advantages.

- The equilibrium value function is given by $\widehat{V}(t,x) \equiv \Theta^0(t,x,x,\Theta(t,x))$, in which $\Theta^0(\cdot,\tilde{x},\cdot,y)$ can be regarded as an auxiliary function with parameters $(\tilde{x},y)$. By introducing this auxiliary function, the structure of equilibrium HJB equations is much clearer than that of extended HJB equations (compare (3.7) with (3.6), for example), and the meaning of the two PDEs in equilibrium HJB equations is also very clear (see Remark 2.8).
- The state term $X(T)$ and the conditional expectation term $\mathbb{E}_t[X(T)]$ in the terminal cost of (1.3) could be inseparable, while in [6], they are required to be separable. The reason is that in our approach, we do not need to introduce a PDE to give an additional representation for $\widehat{G}(X(t),\mathbb{E}_t[X(T)])$. Thus, there are only two PDEs in the equilibrium HJB equation, while the extended HJB equation (3.6) is involved with three PDEs.
- More importantly, by Theorem 2.10 the well-posedness of equilibrium HJB equations is established under Assumption (H4), while there is no rigorous argument about the well-posedness of the extended HJB equation (3.6) given in [6]. More generally, the problem studied in the paper can depend on a controlled backward process and have a recursive cost functional, which is determined by a BSVIE. In Subsections 5.2 and 5.3, two examples are presented to show that the introduction of backward controlled processes is necessary in some applications.

4. Linear-quadratic problems

Consider the controlled linear FBSDEs:

$$\begin{cases} dX(s) = [A(s)X(s) + B(s)u(s)]ds + [C(s)X(s) + D(s)u(s)]dW(s), \\ dY(s) = -\big[\widehat{A}(s)X(s) + \widehat{B}(s)u(s) + \widehat{C}(s)Y(s) + \widehat{D}(s)Z(s)\big]ds + Z(s)dW(s), \\ X(t) = x, \quad Y(T) = HX(T). \end{cases} \tag{4.1}$$

We introduce the following cost functional:

$$\begin{aligned} \mathcal{J}(t,x;u) = \frac{1}{2}\mathbb{E}_t\Big\{ \int_t^T \big[\langle Q(s)X(s),X(s)\rangle + \langle M(s)Y(s),Y(s)\rangle + \langle N(s)Z(s),Z(s)\rangle + \langle R(s)u(s),u(s)\rangle\big]ds \\ + \langle G_1X(T),X(T)\rangle + \langle G_2Y(t),Y(t)\rangle + \langle G_3X(t),Y(t)\rangle + 2\langle g,X(T)\rangle\Big\}. \end{aligned} \tag{4.2}$$

The above problem is referred to as a *linear-quadratic* (LQ, for short) optimal control problem for FBSDEs, due to the linearity of the state equation (4.1) and the quadratic form of the cost functional (4.2). For simplicity, we shall denote the optimal control problem with state equation (4.1) and cost functional (4.2) by Problem (FBLQ). We refer [38,60,29,61,37,25,55,58] again for some related results of the LQ control problems for FBSDEs/BSDEs.

Remark 4.1. Note that in the cost functional (4.2), we introduce a cross term $\langle G_3X(t),Y(t)\rangle$. In the literature, the dependence of initial states is motivated by the so-called *state-dependent risk aversions* in finance (see Björk–Murgoci–Zhou [8]). Indeed, the initial state $X(t)$, with a form of $\langle X(t),Y(t)\rangle$, will also arise naturally when we study the leader's problem of an LQ Stackelberg game (see [56, Subsection 3.2]).

Let us take a look at a special case of the above LQ problem.

Example 4.2. Let $m = n = 1$; $A,D \equiv 0$, $C,B \equiv 1$; $\widehat{A},\widehat{B},\widehat{C},\widehat{D} \equiv 0$, $H = 1$; and $Q,M,N \equiv 0$, $R \equiv 2I$, $G_1 = 0$, $G_2 = 2I$, $G_3 = 0$, $g = 0$. Note that $Y(t) = \mathbb{E}_t[X(T)]$. Then the state equation (4.1) and the cost functional (4.2) are reduced to

$$dX(s) = u(s)ds + X(s)dW(s), \quad s \in [t,T], \quad X(t) = x,$$

and

$$J(t,x;u) = \mathbb{E}_t\Big[\int_t^T |u(s)|^2ds + |\mathbb{E}_t[X(T)]|^2\Big].$$

By Yong [74], the unique optimal control $\bar{u}(\cdot;t,x)$ with initial pair (t,x) is given by

$$\bar{u}(s;t,x) = -\frac{x}{T-t+1}, \quad s \in [t,T].$$

Then the optimal state process $\bar{X} \equiv \bar{X}(\cdot\,;t,x)$ is given by

$$\bar{X}(s) = e^{-\frac{1}{2}(s-t)+W(s)-W(t)}x + \frac{x}{T-t+1}\int_t^s e^{-\frac{1}{2}(s-r)+W(s)-W(r)}dr, \quad s \in [t,T].$$

For any $\tau \in (t, T)$, the optimal control with initial pair $(\tau, \bar{X}(\tau))$ is given by

$$\bar{u}(s; \tau, \bar{X}(\tau)) = -\frac{\bar{X}(\tau)}{T - \tau + 1}, \quad s \in [\tau, T].$$

Thus, on $[\tau, T]$,

$$\bar{u}(\cdot\,; t, x) \neq \bar{u}(\cdot\,; \tau, \bar{X}(\tau)),$$

which implies that the problem is time-inconsistent.

From the above example and Example 1.1, we see that the LQ optimal control problem for FBSDEs is also time-inconsistent in general. Recently, an LQ problem for coupled FBSDEs was studied by Hu–Ji–Xue [25], in which, however, the time-consistency was not considered. Thus the optimal control obtained in [25] is a pre-committed optimal control.

In the following, we will mainly look at the corresponding forms of our equilibrium HJB equations. The well-posedness of the associated Riccati equation is left for our future research. The associated equilibrium HJB equation reads

$$\begin{cases} \Theta_s(s,x) + \Theta_x(s,x)[A(s)x + B(s)\bar{\Psi}(s,x)] \\ \quad + \frac{1}{2}\langle \Theta_{xx}(s,x)[C(s)x + D(s)\bar{\Psi}(s,x)], C(s)x + D(s)\bar{\Psi}(s,x)\rangle \\ \quad + \widehat{A}(s)x + \widehat{B}(s)\bar{\Psi}(s,x) + \widehat{C}(s)\Theta(s,x) + \widehat{D}(s)\Theta_x(s,x)[C(s)x + D(s)\bar{\Psi}(s,x)] = 0, \\ \Theta^0_s(s,\tilde{x},x,y) + \Theta^0_x(s,\tilde{x},x,y)[A(s)x + B(s)\bar{\Psi}(s,x)] \\ \quad + \frac{1}{2}\langle \Theta^0_{xx}(s,\tilde{x},x,y)[C(s)x + D(s)\bar{\Psi}(s,x)],\ C(s)x + D(s)\bar{\Psi}(s,x)\rangle \\ \quad + \frac{1}{2}\big\{\langle Q(s)x, x\rangle + \langle M(s)\Theta(s,x), \Theta(s,x)\rangle + \langle R(s)\bar{\Psi}(s,x), \bar{\Psi}(s,x)\rangle \\ \quad + \langle N(s)\Theta_x(s,x)[C(s)x + D(s)\bar{\Psi}(s,x)], \Theta_x(s,x)[C(s)x + D(s)\bar{\Psi}(s,x)]\rangle\big\} = 0, \\ \Theta(T,x) = Hx, \quad \Theta^0(T,\tilde{x},x,y) = \frac{1}{2}\langle G_1 x, x\rangle + \frac{1}{2}\langle G_2 y, y\rangle + \frac{1}{2}\langle G_3\tilde{x}, y\rangle + \langle g, x\rangle, \end{cases}$$

where

$$\begin{aligned} \bar{\Psi}(s,x) = &-[D^\top \Theta^0_{xx}(s,x,x,\Theta(s,x))D + R + D^\top \Theta_x(s,x)^\top N \Theta_x(s,x) D]^{-1} \\ &\times \big\{[D^\top \Theta^0_{xx}(s,x,x,\Theta(s,x))C + D^\top \Theta_x(s,x)^\top N\Theta_x(s,x)C]x + B^\top \Theta^0_x(s,x,x,\Theta(s,x)) \\ &+ [B^\top \Theta_x(s,x)^\top + \widehat{B}^\top + D^\top \Theta_x(s,x)^\top \widehat{D}^\top]\Theta^0_y(s,x,x,\Theta(s,x))\big\}. \end{aligned}$$

In the above, we have taken the ansatz $\Theta_{xx} \equiv 0$. Now let us take the following ansatz for Θ^0 and Θ:

$$\begin{aligned} &\Theta^0(s,\tilde{x},x,y) = \frac{1}{2}\langle \Phi_1(s)x, x\rangle + \frac{1}{2}\langle \Phi_2(s)y, y\rangle + \frac{1}{2}\langle \Phi_3(s)\tilde{x}, y\rangle + \Phi_4(s)x + \Phi_5(s), \\ &\Theta(s,x) = \Phi_6(s)x + \Phi_7(s), \end{aligned}$$

where Φ_i $(i = 1, ..., 7)$ are undetermined functions (of proper dimensions). Then the equilibrium strategy is given by

$$
\begin{aligned}
\bar{\Psi}(s,x) = &-[D^\top\Phi_1 D + R + D^\top\Phi_6^\top N\Phi_6 D]^{-1}\Big\{D^\top\Phi_1 C + D^\top\Phi_6^\top N\Phi_6 C \\
&+ B^\top\Phi_1 + B^\top\Phi_6^\top\Phi_2\Phi_6 + \widehat{B}^\top\Phi_2\Phi_6 + D^\top\Phi_6^\top\widehat{D}^\top\Phi_2\Phi_6 + \frac{1}{2}B^\top\Phi_6^\top\Phi_3 \\
&+ \frac{1}{2}\widehat{B}^\top\Phi_3 + \frac{1}{2}D^\top\Phi_6^\top\widehat{D}^\top\Phi_3\Big\}x - [D^\top\Phi_1 D + R + D^\top\Phi_6^\top N\Phi_6 D]^{-1} \\
&\times\big\{B^\top\Phi_4 + [B^\top\Phi_6^\top + \widehat{B}^\top + D^\top\Phi_6^\top\widehat{D}^\top]\Phi_2\Phi_7\big\} \\
=: &\ \bar{\Psi}(s)x + \bar{v}(s),
\end{aligned}
\tag{4.3}
$$

where Φ_i is determined by the following system of *Riccati-type* ordinary differential equations (ODEs, for short):

$$
\begin{cases}
\dot{\Phi}_1 + \Phi_1(A+B\bar{\Psi}) + (A+B\bar{\Psi})^\top\Phi_1 + (C+D\bar{\Psi})^\top\Phi_1(C+D\bar{\Psi}) \\
\quad + Q + \Phi_6^\top M\Phi_6 + (C+D\bar{\Psi})^\top\Phi_6^\top N\Phi_6(C+D\bar{\Psi}) + \bar{\Psi}^\top R\bar{\Psi} = 0, \\
\dot{\Phi}_2 = 0, \quad \dot{\Phi}_3 = 0, \\
\dot{\Phi}_4 + \bar{v}^\top B^\top\Phi_1 + \Phi_4(A+B\bar{\Psi}) + \bar{v}^\top D^\top\Phi_1(C+D\bar{\Psi}) + \Phi_7^\top M\Phi_6 \\
\quad + \bar{v}^\top D^\top\Phi_6^\top N\Phi_6(C+D\bar{\Psi}) + \bar{v}^\top R\bar{\Psi} = 0, \\
\dot{\Phi}_5 + \Phi_4 B\bar{v} + \frac{1}{2}\bar{v}^\top D^\top\Phi_1 D\bar{v} + \frac{1}{2}\bar{v}^\top D^\top\Phi_6^\top N\Phi_6 D\bar{v} + \frac{1}{2}\bar{v}^\top R\bar{v} + \frac{1}{2}\Phi_7^\top M\Phi_7 = 0, \\
\dot{\Phi}_6 + \Phi_6(A+B\bar{\Psi}) + \widehat{A} + \widehat{B}\bar{\Psi} + \widehat{C}\Phi_6 + \widehat{D}\Phi_6(C+D\bar{\Psi}) = 0, \\
\dot{\Phi}_7 + \Phi_6 B\bar{v} + \widehat{B}\bar{v} + \widehat{C}\Phi_7 + \widehat{D}\Phi_6 D\bar{v} = 0, \\
\Phi_1(T) = G_1, \quad \Phi_2(T) = G_2, \quad \Phi_3(T) = G_3, \quad \Phi_4(T) = g, \\
\Phi_5(T) = 0, \quad \Phi_6(T) = H, \quad \Phi_7(T) = 0.
\end{cases}
\tag{4.4}
$$

Proposition 4.3. *Suppose that the Riccati equation* (4.4) *admits a unique solution. Then the strategy* $\bar{\Psi}$ *given by* (4.3) *is an equilibrium strategy of* Problem (FBLQ).

When the weighting matrices $G_3 = 0$ and $g = 0$, then $\Phi_i \equiv 0$ for $i = 3,4,5,7$, and the Riccati equation (4.4) can be simplified. The following result shows that the LQ problem for FBSDEs is closely related to the so-called *mean-field LQ optimal control problems* (see [74,76,57], for example) as well. Let $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{D}$, M, $N \equiv 0$, $G_3 = 0$, $g = 0$ and $H = I_n$. Then the state equation (4.1) and the cost functional (4.2) are reduced to

$$
\begin{cases}
dX(s) = [A(s)X(s) + B(s)u(s)]ds + [C(s)X(s) + D(s)u(s)]dW(s), \\
\ X(t) = x,
\end{cases}
$$

and

$$
\begin{aligned}
J(t,x;u) = \frac{1}{2}\mathbb{E}_t\Big\{\int_t^T \big[\langle Q(s)X(s), X(s)\rangle + \langle R(s)u(s), u(s)\rangle\big]ds \\
+ \langle G_1 X(T), X(T)\rangle + \langle G_2\mathbb{E}_t[X(T)], \mathbb{E}_t[X(T)]\rangle\Big\},
\end{aligned}
$$

respectively. The associated Riccati equation (4.4) reads

$$
\begin{cases}
\dot{\Phi}_1 + \Phi_1(A + B\bar{\Psi}) + (A + B\bar{\Psi})^\top \Phi_1 + (C + D\bar{\Psi})^\top \Phi_1(C + D\bar{\Psi}) + Q + \bar{\Psi}^\top R\bar{\Psi} = 0, \\
\dot{\Phi}_2 = 0, \quad \dot{\Phi}_6 + \Phi_6(A + B\bar{\Psi}) = 0, \\
\Phi_1(T) = G_1, \quad \Phi_2(T) = G_2, \quad \Phi_6(T) = I_n,
\end{cases}
\tag{4.5}
$$

with

$$
\bar{\Psi} = -[D^\top \Phi_1 D + R]^{-1}[D^\top \Phi_1 C + B^\top \Phi_1 + B^\top \Phi_6^\top \Phi_2 \Phi_6]. \tag{4.6}
$$

Denote $\Phi = \Phi_1$ and $\widehat{\Phi} = \Phi_1 + \Phi_6^\top \Phi_2 \Phi_6$. Then we can rewrite (4.5)–(4.6) as follows:

$$
\begin{cases}
\dot{\Phi} + \Phi(A + B\bar{\Psi}) + (A + B\bar{\Psi})^\top \Phi + (C + D\bar{\Psi})^\top \Phi(C + D\bar{\Psi}) + Q + \bar{\Psi}^\top R\bar{\Psi} = 0, \\
\dot{\widehat{\Phi}} + \widehat{\Phi}(A + B\bar{\Psi}) + (A + B\bar{\Psi})^\top \widehat{\Phi} + (C + D\bar{\Psi})^\top \Phi(C + D\bar{\Psi}) + Q + \bar{\Psi}^\top R\bar{\Psi} = 0, \\
\Phi(T) = G_1, \quad \widehat{\Phi}(T) = G_1 + G_2,
\end{cases}
\tag{4.7}
$$

with

$$
\bar{\Psi} = -[D^\top \Phi D + R]^{-1}[D^\top \Phi C + B^\top \widehat{\Phi}].
$$

We emphasize that (4.7) is exactly a special case of the Riccati-type equation derived by Yong [76]. Thus, under some positivity conditions, one can obtain the well-posedness of (4.7) from [76, Theorem 4.6] directly. The general well-posedness of Riccati equation (4.4) is still under study and we hope to report it in the future.

5. Applications

In this section, we shall investigate three important applications, which are also the main motivations of studying forward-backward optimal control problems mentioned in Introduction.

5.1. Dynamic mean-variance models

Consider a Black–Scholes market model in which there is one bond with the riskless interest rate $r > 0$ and one stock with the appreciation rate $\mu > 0$ and volatility $\sigma > 0$. Then a standard argument leads to the following SDE for the wealth process X:

$$
\begin{cases}
dX(s) = [rX(s) + (\mu - r)u(s)]ds + \sigma u(s)dW(s), \quad s \in [t, T], \\
X(t) = \xi,
\end{cases}
$$

where u is the dollar amount invested in the stock. The investor wishes to minimize the following functional:

$$
J(t, \xi; u) = -\mathbb{E}_t[X(T)] + \frac{\gamma}{2}\mathbb{E}_t[|X(T)|^2] - \frac{\gamma}{2}|\mathbb{E}_t[X(T)]|^2. \tag{5.1}
$$

It is known (see Basak–Chabakauri [3]) that the optimal control of the above mean-variance model is time-inconsistent. We shall apply Theorem 2.7 to find a time-consistent equilibrium. Note that the cost functional (5.1) can be rewritten as

$$
J(t, \xi; u) = -\mathbb{E}_t[X(T)] + \frac{\gamma}{2}\mathbb{E}_t[|X(T)|^2] - \frac{\gamma}{2}|Y(t)|^2,
$$

with

$$\begin{cases} dX(s) = [rX(s) + (\mu - r)u(s)]ds + \sigma u(s)dW(s), \quad s \in [t,T], \\ dY(s) = Z(s)dW(s), \quad s \in [t,T], \quad X(t) = \xi, \quad Y(T) = X(T). \end{cases}$$

Thus, the mean-variance model is a special case of the linear-quadratic problems for FBSDEs. By Proposition 4.3, the equilibrium strategy $\bar{\Psi}$ can be given by

$$\bar{\Psi}(t,x) = \bar{\Psi}(s)x + \bar{v}(s), \quad (t,x) \in [0,T] \times \mathbb{R},$$

where

$$\begin{aligned} \bar{\Psi} &= -[\sigma^2\Phi_1]^{-1}\big[(\mu - r)\Phi_1 + (\mu - r)\Phi_2\Phi_6^2 + \frac{1}{2}(\mu - r)\Phi_3\Phi_6\big], \\ \bar{v} &= -[\sigma^2\Phi_1]^{-1}\big[(\mu - r)\Phi_4 + (\mu - r)\Phi_6\Phi_2\Phi_7\big], \end{aligned}$$

with

$$\begin{cases} \dot{\Phi}_1 + 2\Phi_1(r + (\mu - r)\bar{\Psi}) + \sigma\bar{\Psi}\Phi_1\sigma\bar{\Psi} = 0, \quad \dot{\Phi}_2 = 0, \quad \dot{\Phi}_3 = 0, \\ \dot{\Phi}_4 + \bar{v}(\mu - r)\Phi_1 + \Phi_4(r + (\mu - r)\bar{\Psi}) + \bar{v}\sigma\Phi_1\sigma\bar{\Psi} = 0, \\ \dot{\Phi}_5 + \Phi_4(\mu - r)\bar{v} + \dfrac{1}{2}\bar{v}\sigma\Phi_1\sigma\bar{v} = 0, \\ \dot{\Phi}_6 + \Phi_6(r + (\mu - r)\bar{\Psi}) = 0, \quad \dot{\Phi}_7 + \Phi_6(\mu - r)\bar{v} = 0, \\ \Phi_1(T) = \gamma, \quad \Phi_2(T) = -\gamma, \quad \Phi_3(T) = 0, \quad \Phi_4(T) = -1, \\ \Phi_5(T) = 0, \quad \Phi_6(T) = 1, \quad \Phi_7(T) = 0. \end{cases}$$

From the above, it is easily seen that $\Phi_2 \equiv -\gamma$, $\Phi_3 \equiv 0$ and $\Phi_1 + \Phi_6\Phi_2\Phi_6 \equiv 0$. Thus,

$$\bar{\Psi} = 0, \quad \bar{v} = -[\sigma^2\Phi_1]^{-1}\big[(\mu - r)\Phi_4 - (\mu - r)\Phi_6\gamma\Phi_7\big], \tag{5.2}$$

and

$$\begin{cases} \dot{\Phi}_1 + 2r\Phi_1 = 0, \quad \dot{\Phi}_4 - \sigma^{-2}(\mu - r)^2\big[\Phi_4 - \Phi_6\gamma\Phi_7\big]\Phi_1 + r\Phi_4 = 0, \\ \dot{\Phi}_6 + r\Phi_6 = 0, \quad \dot{\Phi}_7 - \Phi_6(\mu - r)[\sigma^2\Phi_1]^{-1}\big[(\mu - r)\Phi_4 - (\mu - r)\Phi_6\gamma\Phi_7\big] = 0, \\ \Phi_1(T) = \gamma, \quad \Phi_4(T) = -1, \quad \Phi_6(T) = 1, \quad \Phi_7(T) = 0. \end{cases} \tag{5.3}$$

By first solving the unknown variables Φ_1 and Φ_6, equation (5.3) becomes a linear equation. By the variation of constants formula, the unique solution $(\Phi_1, \Phi_4, \Phi_6, \Phi_7)$ of equation (5.3) can be explicitly solved. Then the equilibrium strategy can be given by (5.2). Indeed, we can observe that

$$\frac{d[\Phi_4 - \Phi_6\gamma\Phi_7]}{dt} = -\gamma[\Phi_4 - \Phi_6\gamma\Phi_7], \quad [\Phi_4 - \Phi_6\gamma\Phi_7](T) = -1,$$

which implies that

$$[\Phi_4 - \Phi_6\gamma\Phi_7](t) = -e^{-\gamma(t-T)}.$$

Substituting the above and $\Phi_1(t) = e^{-2\gamma(t-T)}$ into (5.2), the equilibrium strategy $\bar{\Psi}$ is explicitly given by

$$\bar{\Psi}(t,x) = \frac{\mu - r}{\gamma\sigma^2}e^{-r(t-T)}, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

From the above, we see that the optimal control problem of FBSDEs is a natural extension of the conditional mean-variance problem, with the dynamic risk measure $\mathrm{var}_t[X(T)]$ replaced by some more general ones, which can be described by some process Y satisfying a BSDE. We refer the reader to Riedel [51], Barrieu–El Karoui [2] and Detlefsen–Scandolo [13] for the theory of risk measures. Moreover, Problem (N) can also be regarded as an extension of the dynamic mean-variance models with the conditional expectation operator $\mathbb{E}_t[\,\cdot\,]$ replaced by the so-called *g-expectation* operator $\mathcal{E}_{g,t}[\,\cdot\,]$, which was introduced by Peng [48] and has been widely applied in finance; see Chen–Epstein [10], Coquet et al. [11] and Chen–Chen–Davison [9], for example.

5.2. Social planner problems with heterogeneous Epstein-Zin utilities

In this subsection, we shall consider a social planner problem for Merton's investment-consumption models, in which each agent's objective is given by an Epstein–Zin utility. The social planner would like to maximize the utility of the coalition, which is a convex combination of each agent's utility. The main feature of our model is that the discount rate in each agent's utility can be different. We will reveal two interesting facts: (i) the model is time-inconsistent; (ii) the situation of controlled backward state equations is not avoidable in this model.

Consider the following SDE for the wealth process X:

$$\begin{cases} dX(s) = \big\{rX(s) + (\mu - r)[u_1(s) + u_2(s)] - [c_1(s) + c_2(s)]\big\}ds \\ \qquad\qquad\quad + \sigma\big[u_1(s) + u_2(s)\big]dW(s), \\ X(t) = \xi, \end{cases}$$

where u_i and c_i are the dollar amount invested in the stock and the consumption of agent i (i=1,2), respectively. Naturally, agent i wants to maximize his/her utility functional

$$J_i(t, \xi; u_1, u_2, c_1, c_2) = Y_i(t),$$

where Y_i, called an Epstein–Zin utility (see [15,18], for example), is determined by

$$Y_i(s) = \mathbb{E}_s\Big[\int_s^T g_i(c_1(r) + c_2(r), Y_i(r))dr + h_i(X(T))\Big], \quad s \in [t, T],$$

with

$$g_i(c, y) = \alpha^{-1}((1-\gamma)y)^{1-\frac{\alpha}{1-\gamma}}[c^\alpha - \rho_i((1-\gamma)y)^{\frac{\alpha}{1-\gamma}}], \quad h_i(x) = \frac{x^{1-\gamma}}{1-\gamma}.$$

The parameter $\gamma > 0$ controls the risk aversion of the agents, $\frac{\alpha}{1-\gamma} > 0$ gives the agents' IES, and ρ_i is the discount rate of agent i (which could be different for different i).

Such type of models was initially studied by Duffie–Geoffard–Skiadas [17] (also see Ma–Yong [40, Page 6]), however, the time-inconsistency issue was not realized. If the agents decide to cooperate, then the social planer would try to maximize

$$J^\lambda(t, \xi; u_1, u_2, c_1, c_2) = \lambda J_1(t, \xi; u_1, u_2, c_1, c_2) + (1-\lambda)J_2(t, \xi; u_1, u_2, c_1, c_2),$$

where $\lambda \in (0, 1)$ is a weighting parameter of the two agents. Denote $c = c_1 + c_2$ and $u = u_1 + u_2$. Then the state equation and the utility functional of the social planner (or called a group decision-maker) become

$$\begin{cases} dX(s) = [rX(s) + (\mu - r)u(s) - c(s)]ds + \sigma u(s)dW(s), \\ dY_i(s) = -g_i(c(s), Y_i(s))ds + Z_i(s)dW(s); \quad i = 1, 2, \\ X(t) = \xi, \quad Y_i(T) = h_i(X(T)), \end{cases} \tag{5.4}$$

and

$$\begin{aligned} J^\lambda(t, \xi; u, c) = \mathbb{E}_t\Big\{ \frac{X(T)^{1-\gamma}}{1-\gamma} + \int_t^T \Big[&\lambda\alpha^{-1}((1-\gamma)Y_1(r))^{1-\frac{\alpha}{1-\gamma}}\big(c(r)^\alpha - \rho_1((1-\gamma)Y_1(r))^{\frac{\alpha}{1-\gamma}}\big) \\ &+ (1-\lambda)\alpha^{-1}((1-\gamma)Y_2(r))^{1-\frac{\alpha}{1-\gamma}}\big(c(r)^\alpha - \rho_2((1-\gamma)Y_2(r))^{\frac{\alpha}{1-\gamma}}\big)\Big]dr\Big\}. \end{aligned} \tag{5.5}$$

Remark 5.1. Note that when $\alpha = 1 - \gamma$, the Epstein–Zin utility Y_i is reduced to the standard constant relative risk aversion (CRRA, for short) utility case. Then the corresponding utility (5.5) becomes

$$\begin{aligned} J^\lambda(t, \xi; u, c) = \mathbb{E}_t\Big\{ &\big[\lambda e^{-\rho_1(T-t)} + (1-\lambda)e^{-\rho_2(T-t)}\big]\frac{X(T)^\alpha}{\alpha} \\ &+ \int_t^T \big[\lambda e^{-\rho_1(r-t)} + (1-\lambda)e^{-\rho_2(r-t)}\big]c(r)^\alpha dr\Big\}. \end{aligned} \tag{5.6}$$

The control problem with state equation (5.4) and utility functional (5.6) is exactly the Merton's problem with a quasi-exponential discounting function $\lambda e^{-\rho_1(s-t)} + (1-\lambda)e^{-\rho_2(s-t)}$. We refer the reader to [20,19,41,42,73] for more results on this special case. In the general case, that is α could not equal $1 - \gamma$, the Epstein–Zin utility Y_i is described by the solution to a nonlinear BSDE. Then the situation of controlled BSDEs is not avoidable.

It is clearly seen that the control problem with state equation (5.4) and utility functional (5.5) is time-inconsistent. Thus, the group decision-maker should look for an equilibrium strategy for the coalition. The associated equilibrium HJB equation reads

$$\begin{cases} \Theta^1_t(t,x) + \Theta^1_x(t,x)[rx + (\mu - r)\mathbb{U}(t,x) - \mathbb{C}(t,x)] + \frac{1}{2}\Theta^1_{xx}(t,x)[\sigma\mathbb{U}(t,x)]^2 \\ \quad + \alpha^{-1}((1-\gamma)\Theta^1(t,x))^{1-\frac{\alpha}{1-\gamma}}[\mathbb{C}(t,x)^\alpha - \rho_1((1-\gamma)\Theta^1(t,x))^{\frac{\alpha}{1-\gamma}}] = 0, \\ \Theta^2_t(t,x) + \Theta^2_x(t,x)[rx + (\mu - r)\mathbb{U}(t,x) - \mathbb{C}(t,x)] + \frac{1}{2}\Theta^2_{xx}(t,x)[\sigma\mathbb{U}(t,x)]^2 \\ \quad + \alpha^{-1}((1-\gamma)\Theta^2(t,x))^{1-\frac{\alpha}{1-\gamma}}[\mathbb{C}(t,x)^\alpha - \rho_2((1-\gamma)\Theta^2(t,x))^{\frac{\alpha}{1-\gamma}}] = 0, \\ \Theta^0_t(t,x) + \Theta^0_x(t,x)[rx + (\mu - r)\mathbb{U}(t,x) - \mathbb{C}(t,x)] + \frac{1}{2}\Theta^0_{xx}(t,x)[\sigma\mathbb{U}(t,x)]^2 \\ \quad + \lambda\alpha^{-1}((1-\gamma)\Theta^1(t,x))^{1-\frac{\alpha}{1-\gamma}}[\mathbb{C}(t,x)^\alpha - \rho_1((1-\gamma)\Theta^1(t,x))^{\frac{\alpha}{1-\gamma}}] \\ \quad + (1-\lambda)\alpha^{-1}((1-\gamma)\Theta^2(t,x))^{1-\frac{\alpha}{1-\gamma}}[\mathbb{C}(t,x)^\alpha - \rho_2((1-\gamma)\Theta^2(t,x))^{\frac{\alpha}{1-\gamma}}] = 0, \\ \Theta^1(T,x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \Theta^2(T,x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \Theta^0(T,x) = \frac{x^{1-\gamma}}{1-\gamma}, \end{cases}$$

with the equilibrium investment strategy:

$$\mathbb{U}(t,x) = \frac{(r-\mu)\Theta^0_x(t,x)}{\sigma^2\Theta^0_{xx}(t,x)}, \tag{5.7}$$

and the equilibrium consumption strategy:

$$\mathbb{C}(t,x) = \frac{(1-\gamma)^{\frac{1-\gamma-\alpha}{(1-\gamma)(1-\alpha)}}\Theta^0_x(t,x)^{\frac{1}{\alpha-1}}}{[\lambda\Theta^1(t,x)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\Theta^2(t,x)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}}. \tag{5.8}$$

Let us make the ansatz:

$$\begin{aligned}
&\Theta^1(t,x) = \frac{1}{1-\gamma}x^{1-\gamma}\theta_1(t), \quad \Theta^2(t,x) = \frac{1}{1-\gamma}x^{1-\gamma}\theta_2(t),\\
&\Theta^0(t,x) = \frac{\theta_0(t)}{1-\gamma}x^{1-\gamma} = \frac{\lambda\theta_1(t)+(1-\lambda)\theta_2(t)}{1-\gamma}x^{1-\gamma}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Theta^i_x(t,x) = \theta_i(t)x^{-\gamma}, \quad \Theta^i_{xx}(x) = -\gamma\theta_i(t)x^{-\gamma-1},\\
&\Theta^0_x(t,x) = \lambda\theta_1(t)x^{-\gamma} + (1-\lambda)\theta_2(t)x^{-\gamma},\\
&\Theta^0_{xx}(x) = -\lambda\gamma\theta_1(t)x^{-\gamma-1} - (1-\lambda)\gamma\theta_2(t)x^{-\gamma-1}.
\end{aligned}$$

The equilibrium investment strategy (5.7) and the equilibrium consumption strategy (5.8) become

$$\mathbb{U}(t,x) = \frac{(\mu-r)}{\gamma\sigma^2}x, \quad x\in\mathbb{R}, \tag{5.9}$$

and

$$\mathbb{C}(t,x) = \frac{[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}}x, \quad x\in\mathbb{R}^n, \tag{5.10}$$

with

$$\begin{cases}
\dot\theta_1(t) + (1-\gamma)\theta_1(t)\Big[r + \dfrac{(\mu-r)^2}{2\gamma\sigma^2} - \dfrac{[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}}\Big]\\
\qquad -(1-\gamma)\rho_1\alpha^{-1}\theta_1(t) + \dfrac{\alpha^{-1}(1-\gamma)\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{\alpha}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{\alpha}{\alpha-1}}} = 0,\\
\dot\theta_2(t) + (1-\gamma)\theta_2(t)\Big[r + \dfrac{(\mu-r)^2}{2\gamma\sigma^2} - \dfrac{[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}}\Big]\\
\qquad -(1-\gamma)\rho_2\alpha^{-1}\theta_2(t) + \dfrac{\alpha^{-1}(1-\gamma)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{\alpha}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{\alpha}{\alpha-1}}} = 0,\\
\theta_1(T) = \theta_2(T) = 1.
\end{cases} \tag{5.11}$$

Proposition 5.2. *Let $\gamma\in[1-\alpha,1)$. Then the system of ODEs (5.11) admits a unique solution (θ_1,θ_2), and the strategies $\mathbb{U}$ and $\mathbb{C}$, given by (5.9)–(5.10), is an equilibrium investment–consumption strategy pair.*

Proof. It suffices to show that if (θ_1,θ_2) is a postive solution of (5.11) on $[t_0,T]$, then

$$\delta \le \theta_i(s) \le \kappa, \quad s\in[t_0,T]; \quad i=1,2,$$

for some positive constants $\delta, \kappa > 0$ independent of t_0. Without loss of generality, let $\rho_1 \geq \rho_2$. Denote $\Delta\theta = \theta_1 - \theta_2$. Note that

$$\begin{cases} \Delta\dot{\theta}(t) + (1-\gamma)\Delta\theta(t)\Big[r + \dfrac{(\mu-r)^2}{2\gamma\sigma^2} - \dfrac{[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}}\Big] \\ \quad - (1-\gamma)\rho_1\alpha^{-1}\Delta\theta(t) + (1-\gamma)(\rho_2-\rho_1)\alpha^{-1}\theta_2(t) \\ \quad + \dfrac{\alpha^{-1}(1-\gamma-\alpha)[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{\alpha}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{\alpha}{\alpha-1}}} \displaystyle\int_0^1 [l\theta_1(t) + (1-l)\theta_2(t)]^{\frac{-\alpha}{1-\gamma}} dl \Delta\theta(t) = 0, \\ \Delta\theta(T) = 0, \end{cases}$$

and

$$(1-\gamma)(\rho_2-\rho_1)\alpha^{-1}\theta_2(t) \leq 0.$$

Then, $\theta_1 \leq \theta_2$. Thus,

$$\begin{aligned} &\frac{\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{\alpha}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{\alpha}{\alpha-1}}} \\ &= \frac{\theta_1(t)[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_1(t)^{\frac{-\alpha}{1-\gamma}}\theta_2(t)][\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}][\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}} \\ &\geq \frac{\theta_1(t)[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}][\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}][\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}} \\ &= \frac{\theta_1(t)[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}}, \end{aligned}$$

which implies

$$\begin{aligned} &(1-\gamma)\theta_1(t)\Big[-\frac{[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}}\Big] \\ &+ \frac{\alpha^{-1}(1-\gamma)\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{\alpha}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{\alpha}{\alpha-1}}} \\ &\geq \frac{(\alpha^{-1}-1)(1-\gamma)\theta_1(t)[\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{1}{\alpha-1}}}{[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{\alpha-1}}} \geq 0. \end{aligned}$$

It follows that

$$\theta_1(t) \geq e^{\int_t^T (1-\gamma)[r-\rho_1\alpha^{-1}+\frac{(\mu-r)^2}{2\gamma\sigma^2}]ds} \geq e^{-T|1-\gamma||r-\rho_1\alpha^{-1}+\frac{(\mu-r)^2}{2\gamma\sigma^2}|} =: \delta > 0.$$

By $\theta_2 \geq \theta_1$, we get $\theta_2 \geq \delta$. Then

$$\begin{aligned} &\theta_i(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} \leq \delta^{\frac{1-\gamma-\alpha}{1-\gamma}}; \, i = 1,2, \quad [\lambda\theta_1(t)+(1-\lambda)\theta_2(t)]^{\frac{\alpha}{\alpha-1}} \leq \delta^{\frac{\alpha}{\alpha-1}}, \\ &[\lambda\theta_1(t)^{\frac{1-\gamma-\alpha}{1-\gamma}} + (1-\lambda)\theta_2(t)^{\frac{1-\gamma-\alpha}{1-\gamma}}]^{\frac{1}{1-\alpha}} \leq \delta^{\frac{1-\gamma-\alpha}{(1-\alpha)(1-\gamma)}}. \end{aligned}$$

From the above, we get that there exists a constant $\kappa > 0$, independent of t_0, such that

$$\theta_i \le \kappa.$$

Then the well-posedness of (5.11) can be proved by routine arguments. □

For more details of this type of models, we refer the reader to Wang–Zhou [66]. In particular, [66] showed that the Epstein-Zin utility is much more effective than the CRRA utility in the social planner problem.

5.3. Stackelberg games

In this subsection, we consider a specific Stackelberg game (also called a leader-follower game). We will show that the leader's problem in this Stackelberg game is an optimal control problem for FBSDEs, whose optimal control is time-inconsistent. By applying Theorem 2.7, we can find a time-consistent equilibrium for the leader. This will give a very good illustration.

Example 5.3. Consider the following one-dimensional state equation

$$\dot{X}(s) = u_1(s) - u_2(s), \quad s \in [t,1], \quad X(t) = x,$$

and the cost functionals

$$J_1(t,x;u_1,u_2) = |X(1)|^2 + \int_t^1 \big[|u_1(s)|^2 - |u_2(s)|^2\big] ds,$$

$$J_2(t,x;u_1,u_2) = \int_t^1 \Big[- u_1(s) + \frac{X(s)}{s-2} + u_2(s) + |u_2(s)|^2 \Big] ds.$$

In the above, Player 2 is the leader (or the principal), who announces his/her control u_2 first, and Player 1 is the follower (or the agent), who chooses his/her control u_1 accordingly. Whatever the leader announces, the follower will select a control $\bar{u}_1(\cdot\,;t,x,u_2)$ (depending the control u_2 announced by the leader as well as the initial pair (t,x)) such that $u_1 \mapsto J_1(t,x;u_1,u_2)$ is minimized. Knowing this, the leader will choose a $\bar{u}_2$ a priori so that $u_2 \mapsto J_2(t,x;\bar{u}_1(\cdot\,;t,x,u_2),u_2)$ is minimized. For any given initial pair (t,x) and control u_2 of the leader, by the standard results of LQ control problems (see [77, Chapter 6]), the follower admits a unique optimal strategy $\bar{u}_1(\cdot\,;t,x,u_2)$. Then by some straightforward calculations, the leader's problem can be stated as follows: Find a control u_2 to minimize

$$J_2(t,x;\bar{u}_1(\cdot\,;t,x,u_2),u_2) = \int_t^1 \big[Y(s) + u_2(s) + |u_2(s)|^2\big] ds,$$

with the backward state equation

$$\dot{Y}(s) = \frac{1}{2-s} Y(s) + \frac{1}{2-s} u_2(s), \quad s \in [t,1], \quad Y(1) = 0.$$

Note that

$$Y(s) = \frac{1}{s-2} \int_s^1 u_2(r) dr.$$

Then

$$J_2(t,x;\bar u_1(\cdot\,;t,x,u_2),u_2)=\int_t^1\Big\{\big[\ln(2-s)-\ln(2-t)+1\big]u_2(s)+|u_2(s)|^2\Big\}ds.$$

It follows that the unique optimal control of the leader is given by

$$\bar u_2(s;t,x)=\frac{\ln(2-t)-\ln(2-s)-1}{2},\quad s\in[t,1].$$

In particular, at the initial pair $(0,x)$, the unique optimal control of the leader is

$$\bar u_2(s;0,x)=\frac{\ln 2-\ln(2-s)-1}{2},\quad s\in[0,1].$$

Let $\bar X\equiv\bar X(\cdot;0,x)$ be the state process with initial pair $(0,x)$ and optimal controls $(\bar u_1(\cdot;t,x,\bar u_2),\,\bar u_2)$. For any given $t\in(0,1)$, at the initial pair $(t,\bar X(t;0,x))$, the unique optimal control of the leader is

$$\bar u_2(s;t,\bar X(t;0,x))=\frac{\ln(2-t)-\ln(2-s)-1}{2},\quad s\in[t,1].$$

Thus, for $t\in(0,1)$, on the time interval $[t,1]$,

$$\bar u_2(\cdot;0,x)\neq\bar u_2(\cdot;t,\bar X(t;0,x)),$$

which implies that the leader's problem is time-inconsistent. By Theorem 2.7, we can easily obtain the time-consistent equilibrium strategy of the leader, which is given by

$$\bar\Psi(s,x)=-\frac{1}{2},\quad (s,x)\in[0,1]\times\mathbb{R}^n.$$

Remark 5.4. We refer the reader to [53,70,12,56] for some theoretical results and financial applications of Stackelberg games. It is worthy of pointing out that the well-known *principal-agent model* can be regarded as a special case.

6. Verification theorem

In this section, we shall show that the function $\bar\Psi$, determined by (2.11), is an equilibrium strategy of Problem (N). In other words, we would like to rigorously prove the verification theorem (i.e., Theorem 2.7). To do this, we assume that the equilibrium HJB equation (2.10) admits a classical solution and the function $\bar\Psi$ defined by (2.11) is a feedback strategy. We also assume that all the involved functions are bounded and differentiable with bounded derivatives.

Let $(\bar X,\bar Y,\bar Z)$ and $(\bar Y^0,\bar Z^0)$ be the solutions to FBSDE (1.1) and BSVIE (1.3), respectively, corresponding to the strategy $\bar\Psi$ and the initial pair $(0,\xi)$. For any $t\in[0,T)$, $u\in L^2_{\mathcal F_t}(\Omega;U)$ and $\varepsilon\in[0,T-t)$, define the strategy Ψ^ε by (2.7). With the initial pair $(t,\bar X(t))\in\mathcal D$, take the strategy Ψ^ε, then the corresponding state equation (1.1) and cost functional (1.2)–(1.3) become

$$\begin{cases}dX^\varepsilon(s)=b(s,X^\varepsilon(s),\bar\Psi(s,X^\varepsilon(s)))ds\\ \qquad\qquad +\sigma(s,X^\varepsilon(s),\bar\Psi(s,X^\varepsilon(s)))dW(s),\quad s\in[t+\varepsilon,T];\\ dX^\varepsilon(s)=b(s,X^\varepsilon(s),u)ds+\sigma(s,X^\varepsilon(s),u)dW(s),\quad s\in[t,t+\varepsilon),\\ dY^\varepsilon(s)=-g(s,X^\varepsilon(s),\bar\Psi(s,X^\varepsilon(s)),Y^\varepsilon(s),Z^\varepsilon(s))ds+Z^\varepsilon(s)dW(s),\quad s\in[t+\varepsilon,T];\\ dY^\varepsilon(s)=-g(s,X^\varepsilon(s),u,Y^\varepsilon(s),Z^\varepsilon(s))ds+Z^\varepsilon(s)dW(s),\quad s\in[t,t+\varepsilon),\\ X^\varepsilon(t)=\bar X(t),\quad Y^\varepsilon(T)=h(X^\varepsilon(T)),\end{cases}\tag{6.1}$$

and

$$J(t,\bar{X}(t);\Psi^{\varepsilon})=Y^{0,\varepsilon}(t),$$

respectively, with

$$\begin{aligned}Y^{0,\varepsilon}(r)&=h^0(r,X^{\varepsilon}(r),X^{\varepsilon}(T),Y^{\varepsilon}(r))-\int_r^T Z^{0,\varepsilon}(r,s)dW(s)\\&+\int_{(t+\varepsilon)\vee r}^{T} g^0\big(r,s,X^{\varepsilon}(r),X^{\varepsilon}(s),\bar{\Psi}(s,X^{\varepsilon}(s)),Y^{\varepsilon}(s),Z^{\varepsilon}(s),Y^{0,\varepsilon}(s),Z^{0,\varepsilon}(r,s)\big)ds\\&+\int_{(t+\varepsilon)\wedge r}^{t+\varepsilon} g^0\big(r,s,X^{\varepsilon}(r),X^{\varepsilon}(s),u,Y^{\varepsilon}(s),Z^{\varepsilon}(s),Y^{0,\varepsilon}(s),Z^{0,\varepsilon}(r,s)\big)ds,\quad r\in[t,t+\varepsilon).\end{aligned}\tag{6.2}$$

Next, let us deduce the PDEs associated with the FBSDE (6.1) and the BSVIE (6.2).

By the Feynman–Kac formula for BSDEs (see Pardoux–Peng [46], for example), we get

$$Y^{\varepsilon}(s)=\Theta(s,X^{\varepsilon}(s)),\quad Z^{\varepsilon}(s)=\Theta_x(s,X^{\varepsilon}(s))\sigma(s,X^{\varepsilon}(s),\bar{\Psi}(s,X^{\varepsilon}(s))),\quad s\in[t+\varepsilon,T],$$

where Θ is the unique solution to the first PDE in (2.10). Then on the time interval $[t,t+\varepsilon]$, we can rewrite (6.1) as follows:

$$\begin{cases}dX^{\varepsilon}(s)=b(s,X^{\varepsilon}(s),u)ds+\sigma(s,X^{\varepsilon}(s),u)dW(s),\quad s\in[t,t+\varepsilon],\\ dY^{\varepsilon}(s)=-g(s,X^{\varepsilon}(s),u,Y^{\varepsilon}(s),Z^{\varepsilon}(s))ds+Z^{\varepsilon}(s)dW(s),\quad s\in[t,t+\varepsilon],\\ X^{\varepsilon}(t)=\bar{X}(t),\quad Y^{\varepsilon}(t+\varepsilon)=\Theta(t+\varepsilon,X^{\varepsilon}(t+\varepsilon)).\end{cases}$$

Note that the control $u\in L^2_{\mathcal{F}_t}(\Omega;U)$ is $\mathcal{F}_t$-measurable. Then by the Feynman–Kac formula for BSDEs again, we get

$$Y^{\varepsilon}(s)=\Theta^{\varepsilon}(s,X^{\varepsilon}(s)),\quad Z^{\varepsilon}(s)=\Theta^{\varepsilon}_x(s,X^{\varepsilon}(s))\sigma(s,X^{\varepsilon}(s),u),\quad s\in[t,t+\varepsilon],\tag{6.3}$$

where Θ^{ε} is the unique classical solution to the following *perturbation* PDE:

$$\begin{cases}\Theta^{\varepsilon}_s(s,x)+\Theta^{\varepsilon}_x(s,x)b(s,x,u)+\mathrm{tr}\,[\Theta^{\varepsilon}_{xx}(s,x)a(s,x,u)]\\ \quad +g(s,x,u,\Theta^{\varepsilon}(s,x),\Theta^{\varepsilon}_x(s,x)\sigma(s,x,u))=0,\quad s\in[t,t+\varepsilon],\\ \Theta^{\varepsilon}(t+\varepsilon,x)=\Theta(t+\varepsilon,x).\end{cases}\tag{6.4}$$

Remark 6.1. Indeed, (6.4) is a PDE with random parameters, because $u\in L^2_{\mathcal{F}_t}(\Omega;U)$ is a random variable. However, note that u is $\mathcal{F}_t$-measurable and (6.4) is considered on $[t,t+\varepsilon]$. The random PDE (6.4) can be treated as a deterministic one.

By Proposition 2.2, on the time interval $[t+\varepsilon,T]$, we get

$$Y^{0,\varepsilon}(s)=\Theta^0(s,s,x,x,\Theta(s,X^{\varepsilon}(s))),\tag{6.5}$$

where Θ^0 is the solution of the second PDE in (2.10). Motivated by [64], we introduce the following auxiliary processes with two time variables:

$$\begin{cases} dY^{0,\varepsilon}(r;s) = -g^0\big(r,s,X^\varepsilon(r),X^\varepsilon(s),\bar\Psi(s,X^\varepsilon(s)),Y^\varepsilon(s),Z^\varepsilon(s),Y^{0,\varepsilon}(s),Z^{0,\varepsilon}(r;s)\big)ds \\ \qquad\qquad + Z^{0,\varepsilon}(r;s)dW(s), \quad s\in[(t+\varepsilon)\vee r,T],\ r\in[t,T]; \\ dY^{0,\varepsilon}(r;s) = -g^0\big(r,s,X^\varepsilon(r),X^\varepsilon(s),u,Y^\varepsilon(s),Z^\varepsilon(s),Y^{0,\varepsilon}(s),Z^{0,\varepsilon}(r;s)\big)ds \\ \qquad\qquad + Z^{0,\varepsilon}(r;s)dW(s), \quad (r,s)\in\Delta^*[t,t+\varepsilon], \\ Y^{0,\varepsilon}(r;T) = h^0(r,X^\varepsilon(r),X^\varepsilon(T),Y^\varepsilon(r)), \quad r\in[t,T], \end{cases} \tag{6.6}$$

which can give the unique solution of BSVIE (6.2) by

$$Y^{0,\varepsilon}(s) = Y^{0,\varepsilon}(s;s), \quad Z^{0,\varepsilon}(r,s) = Z^{0,\varepsilon}(r;s), \quad (r,s)\in\Delta^*[t,T]. \tag{6.7}$$

Notice that for any fixed $r\in[t,T]$, (6.6) is a BSDE. Recall the representations (6.3) and (6.5). Then by the Feynman–Kac formula for BSDEs again, we get that for any $r\in[t,t+\varepsilon]$ and $s\in[t+\varepsilon,T]$,

$$\begin{aligned} &Y^{0,\varepsilon}(r;s) = \Theta^0(r,s,X^\varepsilon(r),X^\varepsilon(s),\Theta^\varepsilon(r,X^\varepsilon(r))), \\ &Z^{0,\varepsilon}(r;s) = \Theta^0_x(r,s,X^\varepsilon(r),X^\varepsilon(s),\Theta^\varepsilon(r,X^\varepsilon(r)))\sigma(s,X^\varepsilon(s),\bar\Psi(s,X^\varepsilon(s))). \end{aligned} \tag{6.8}$$

On the other hand, by the flow property of the auxiliary process $Y^{0,\varepsilon}$, we have

$$\begin{aligned} Y^{0,\varepsilon}(r;r) = Y^{0,\varepsilon}(r;t+\varepsilon) &+ \int_r^{t+\varepsilon} g^0\big(r,s,X^\varepsilon(r),X^\varepsilon(s),u,Y^\varepsilon(s),Z^\varepsilon(s),Y^{0,\varepsilon}(s),Z^{0,\varepsilon}(r;s)\big)ds \\ &- \int_r^{t+\varepsilon} Z^{0,\varepsilon}(r;s)dW(s), \quad r\in[t,t+\varepsilon]. \end{aligned}$$

Substituting (6.8) into the above and noting (6.7), we get

$$\begin{aligned} Y^{0,\varepsilon}(r) &= \Theta^0(r,t+\varepsilon,X^\varepsilon(r),X^\varepsilon(t+\varepsilon),\Theta^\varepsilon(r,X^\varepsilon(r))) \\ &\quad + \int_r^{t+\varepsilon} g^0(r,s,X^\varepsilon(r),X^\varepsilon(s),u,Y^\varepsilon(s),Z^\varepsilon(s),Y^{0,\varepsilon}(s),Z^{0,\varepsilon}(r,s))ds \\ &\quad - \int_r^{t+\varepsilon} Z^{0,\varepsilon}(r,s)dW(s), \quad r\in[t,t+\varepsilon]. \end{aligned} \tag{6.9}$$

Then by Proposition 2.2 (recalling (6.3)–(6.4)), we have the following representation:

$$\begin{aligned} &Y^{0,\varepsilon}(r) = \Theta^{0,\varepsilon}(r,r,X^\varepsilon(r),X^\varepsilon(r),\Theta^\varepsilon(r,X^\varepsilon(r))), \\ &Z^{0,\varepsilon}(r,s) = \Theta^{0,\varepsilon}_x(r,s,X^\varepsilon(r),X^\varepsilon(s),\Theta^\varepsilon(r,X^\varepsilon(r)))\sigma(s,X^\varepsilon(s),u), \quad (r,s)\in\Delta^*[t,t+\varepsilon], \end{aligned} \tag{6.10}$$

where $\Theta^{0,\varepsilon}$ is the unique solution to the following perturbation PDE:

$$\begin{cases} \Theta^{0,\varepsilon}_s(r,s,\tilde{x},x,y)+\Theta^{0,\varepsilon}_x(r,s,\tilde{x},x,y)b(s,x,u)+\operatorname{tr}[\Theta^{0,\varepsilon}_{xx}(r,s,\tilde{x},x,y)a(s,x,u)] \\ \quad +g^0\big(r,s,\tilde{x},x,u,\Theta^\varepsilon(s,x),\Theta^\varepsilon_x(s,x)\sigma(s,x,u),\Theta^{0,\varepsilon}(s,s,x,x,\Theta^\varepsilon(s,x)), \\ \qquad \Theta^{0,\varepsilon}_x(r,s,\tilde{x},x,y)\sigma(s,x,u)\big)=0, \quad (r,s)\in\Delta^*[t,t+\varepsilon], \\ \Theta^{0,\varepsilon}(r,t+\varepsilon,\tilde{x},x,y)=\Theta^0(r,t+\varepsilon,\tilde{x},x,y), \quad r\in[t,t+\varepsilon]. \end{cases} \tag{6.11}$$

Remark 6.2. Note that both (6.4) and (6.11) are semilinear parabolic equations. To guarantee the well-posedness of PDEs (6.4) and (6.11), we assume that the following non-degenerate condition holds: There exist two constants $\lambda_0,\lambda_1>0$ such that

$$\lambda_0 I\le a(t,x,u)\le\lambda_1 I, \quad \forall(t,x,u)\in[0,T]\times\mathbb{R}^n\times U. \tag{6.12}$$

Under the assumption (6.12), we have the following convergence result of the families $\{\Theta^\varepsilon\}_{\varepsilon>0}$ and $\{\Theta^{0,\varepsilon}\}_{\varepsilon>0}$.

Proposition 6.3. *Let (6.12) hold. Then the PDEs (6.4) and (6.11) admit unique classical solutions Θ^ε and $\Theta^{0,\varepsilon}$, respectively. Moreover, there exists a constant $K>0$, only depending on $\|\Theta\|_{C^{\frac{\alpha}{2},2+\alpha}}$ and $\|\Theta^0\|_{C^{\frac{\alpha}{2},\frac{\alpha}{2},\alpha,1+\alpha,\alpha}}$, such that*

$$\|\Theta^\varepsilon-\Theta\|_{C^{0,2}[t,t+\varepsilon]}+\|\Theta^{0,\varepsilon}-\Theta^0\|_{C^{0,0,0,1,0}[t,t+\varepsilon]}\le K\varepsilon^{\frac{\alpha}{2}}, \tag{6.13}$$

where $\alpha\in(0,1)$ is a constant.

Proposition 6.3 can be obtained by modifying [67, Theorem 5.2] immediately. We emphasize that for Proposition 6.3, the assumption (6.12) should not be necessary, because one could replace the analytic approach by a probabilistic argument (see [46,64]).

Remark 6.4. The estimate (6.13) plays the same role as the convergence assumption (H3) in Wei–Yong–Yu [68], which was proved only for some special cases (see [68, Theorem 6.2]). In Proposition 6.3, we can show that (6.13) holds in general. The key point is that (6.13) is only a byproduct of the regualirty of semi-linear parabolic equations, while the assumption (H3) in [68] is concerned with the fully nonlinear PDEs.

6.1. Proof of Theorem 2.7

For any fixed $t\in[0,T)$, $\varepsilon\in[0,T-t]$ and $u\in L^2_{\mathcal{F}_t}(\Omega;U)$, let Θ^ε and $\Theta^{0,\varepsilon}$ be the unique classical solution to PDEs (6.4) and (6.11), respectively. With the representations (6.3) and (6.10), by (6.9) we can represent $Y^{0,\varepsilon}(t)$ as follows:

$$Y^{0,\varepsilon}(t)=\mathbb{E}_t\Big[\Theta^0(t,t+\varepsilon,X^\varepsilon(t),X^\varepsilon(t+\varepsilon),\Theta^\varepsilon(t,X^\varepsilon(t)))+\int_t^{t+\varepsilon}g^{0,\varepsilon}(t,s,u)ds\Big],$$

where

$$\begin{aligned} g^{0,\varepsilon}(t,s,u):=g^0\Big(&t,\,s,\,X^\varepsilon(t),\,X^\varepsilon(s),\,u,\,\Theta^\varepsilon(X^\varepsilon(s),s),\Theta^\varepsilon_x(s,X^\varepsilon(s))\sigma(s,X^\varepsilon(s),u), \\ &\Theta^{0,\varepsilon}(s,s,X^\varepsilon(s),X^\varepsilon(s),\Theta^\varepsilon(s,X^\varepsilon(s))),\Theta^{0,\varepsilon}_x(t,s,X^\varepsilon(t),X^\varepsilon(s),\Theta^\varepsilon(t,X^\varepsilon(t))) \\ &\times\sigma(s,X^\varepsilon(s),u)\Big). \end{aligned} \tag{6.14}$$

Note that $X^\varepsilon(t) = \bar{X}(t)$. Applying Itô's formula to the mapping $s \mapsto \Theta^0(t, s, X^\varepsilon(t), X^\varepsilon(s), \Theta^\varepsilon(t, X^\varepsilon(t)))$ yields that

$$\begin{aligned} Y^{0,\varepsilon}(t) = \mathbb{E}_t\Big\{&\Theta^0(t,t,\bar{X}(t),\bar{X}(t),\Theta^\varepsilon(t,\bar{X}(t))) + \int_t^{t+\varepsilon} \Big[\Theta^0_s(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t))) \\ &+ \Theta^0_x(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))b(s,X^\varepsilon(s),u) \\ &+ \operatorname{tr}\left[\Theta^0_{xx}(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))a(s,X^\varepsilon(s),u)\right] + g^{0,\varepsilon}(t,s,u)\Big]ds\Big\}. \end{aligned}$$

Using the fact $\Theta^\varepsilon(t+\varepsilon,\cdot) = \Theta(t+\varepsilon,\cdot)$ and then by applying the Itô's formula to the mapping $s \mapsto \Theta^0(t,t,\bar{X}(t),\bar{X}(t),\Theta^\varepsilon(s,\bar{X}(s)))$, we have

$$\begin{aligned} Y^{0,\varepsilon}(t) = \mathbb{E}_t\Big\{&\Theta^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(t+\varepsilon,\bar{X}(t+\varepsilon))) \\ &- \frac{1}{2}\int_t^{t+\varepsilon} \operatorname{tr}\Big[\Theta^0_{yy}(t,t,\bar{X}(t),\bar{X}(t),\Theta^\varepsilon(s,\bar{X}(s)))\Theta^\varepsilon_x(s,\bar{X}(s))\bar{\sigma}(s)[\Theta^\varepsilon_x(s,\bar{X}(s))\bar{\sigma}(s)]^\top\Big]ds \\ &- \int_t^{t+\varepsilon} \Theta^0_y(t,t,\bar{X}(t),\bar{X}(t),\Theta^\varepsilon(s,\bar{X}(s)))\Big[\Theta^\varepsilon_s(s,\bar{X}(s)) + \Theta^\varepsilon_x(s,\bar{X}(s))\bar{b}(s) \\ &+ \operatorname{tr}\left[\Theta^\varepsilon_{xx}(s,\bar{X}(s))\bar{a}(s)\right]\Big]ds + \int_t^{t+\varepsilon}\Big[\Theta^0_s(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t))) \\ &+ \Theta^0_x(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))b(s,X^\varepsilon(s),u) + g^{0,\varepsilon}(t,s,u) \\ &+ \operatorname{tr}\left[\Theta^0_{xx}(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))a(s,X^\varepsilon(s),u)\right]\Big]ds\Big\}, \end{aligned} \tag{6.15}$$

where

$$\bar{\varphi}(s) := \varphi(s,\bar{X}(s),\bar{\Psi}(s,\bar{X}(s))),\ \ s \in [t,t+\varepsilon], \tag{6.16}$$

for $\varphi = b, \sigma, a$. Recalling (6.4) and (2.10), we get that on $[t, t+\varepsilon]$,

$$\begin{aligned} &\Theta^\varepsilon_s(s,\bar{X}(s)) + \Theta^\varepsilon_x(s,\bar{X}(s))\bar{b}(s) + \operatorname{tr}\left[\Theta^\varepsilon_{xx}(s,\bar{X}(s))\bar{a}(s)\right] \\ &= \Theta^\varepsilon_x(s,\bar{X}(s))[\bar{b}(s) - b(s,\bar{X}(s),u)] - g^\varepsilon(s,u) + \operatorname{tr}\left\{\Theta^\varepsilon_{xx}(s,\bar{X}(s))[\bar{a}(s) - a(s,\bar{X}(s),u)]\right\}, \end{aligned} \tag{6.17}$$

and

$$\begin{aligned} &\Theta^0_s(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t))) + \Theta^0_x(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))b(s,X^\varepsilon(s),u) \\ &+ \operatorname{tr}\left[\Theta^0_{xx}(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))a(s,X^\varepsilon(s),u)\right] \\ &= \Theta^0_x(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))[b(s,X^\varepsilon(s),u) - \bar{b}^\varepsilon(s)] - \bar{g}^{0,\varepsilon}(t,s) \\ &\quad + \operatorname{tr}\left\{\Theta^0_{xx}(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))[a(s,X^\varepsilon(s),u) - \bar{a}^\varepsilon(s)]\right\}, \end{aligned} \tag{6.18}$$

where

$$\begin{aligned}
&\bar{\varphi}^{\varepsilon}(s) := \varphi(s, X^{\varepsilon}(s), \bar{\Psi}(s, X^{\varepsilon}(s))), \quad \text{for} \quad \varphi = b,\, \sigma,\, a,\\
&g^{\varepsilon}(s,u) := g\big(s, \bar{X}(s), u, \Theta^{\varepsilon}(s,\bar{X}(s)), \Theta^{\varepsilon}_x(s,\bar{X}(s))\sigma(s,\bar{X}(s),u)\big),\\
&\bar{g}^{0,\varepsilon}(t,s) := g^0\big(t, s, \bar{X}(t), X^{\varepsilon}(s), \bar{\Psi}(s, X^{\varepsilon}(s)), \Theta(s, X^{\varepsilon}(s)), \Theta_x(s, X^{\varepsilon}(s))\bar{\sigma}^{\varepsilon}(s),\\
&\qquad \Theta^0(s,s,X^{\varepsilon}(s),X^{\varepsilon}(s),\Theta(s,X^{\varepsilon}(s))), \Theta^0_x(t,s,\bar{X}(t),X^{\varepsilon}(s),\Theta(t,\bar{X}(t)))\bar{\sigma}^{\varepsilon}(s)\big). \qquad (6.19)
\end{aligned}$$

Substituting (6.17) and (6.18) into (6.15) yields that

$$\begin{aligned}
Y^{0,\varepsilon}(t) = \mathbb{E}_t\Big\{&\Theta^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(t+\varepsilon,\bar{X}(t+\varepsilon)))\\
&+ \int_t^{t+\varepsilon} \Theta^0_y(t,t,\bar{X}(t),\bar{X}(t),\Theta^{\varepsilon}(s,\bar{X}(s)))\Big[g^{\varepsilon}(s,u) + \Theta^{\varepsilon}_x(s,\bar{X}(s))\\
&\times [b(s,\bar{X}(s),u) - \bar{b}(s)] + \mathrm{tr}\left\{\Theta^{\varepsilon}_{xx}(s,\bar{X}(s))[a(s,\bar{X}(s),u) - \bar{a}(s)]\right\}\Big]ds\\
&- \frac{1}{2}\int_t^{t+\varepsilon} \mathrm{tr}\Big[\Theta^0_{yy}(t,t,\bar{X}(t),\bar{X}(t),\Theta^{\varepsilon}(s,\bar{X}(s)))\Theta^{\varepsilon}_x(s,\bar{X}(s))\bar{\sigma}(s)[\Theta^{\varepsilon}_x(s,\bar{X}(s))\bar{\sigma}(s)]^{\top}\Big]ds\\
&+ \int_t^{t+\varepsilon} \Big[g^{0,\varepsilon}(t,s,u) - \bar{g}^{0,\varepsilon}(t,s) + \Theta^0_x(t,s,\bar{X}(t),X^{\varepsilon}(s),\Theta^{\varepsilon}(t,\bar{X}(t)))[b(s,X^{\varepsilon}(s),u) - \bar{b}^{\varepsilon}(s)]\\
&+ \mathrm{tr}\left\{\Theta^0_{xx}(t,s,\bar{X}(t),X^{\varepsilon}(s),\Theta^{\varepsilon}(t,\bar{X}(t)))[a(s,X^{\varepsilon}(s),u) - \bar{a}^{\varepsilon}(s)]\right\}\Big]ds\Big\}. \qquad (6.20)
\end{aligned}$$

Applying the above arguments to $\bar{Y}^0(t)$, we have

$$\begin{aligned}
J(t,\bar{X}(t);\bar{\Psi}) \equiv \bar{Y}^0(t) = \mathbb{E}_t\Big\{&\Theta^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(t+\varepsilon,\bar{X}(t+\varepsilon)))\\
&- \frac{1}{2}\int_t^{t+\varepsilon} \mathrm{tr}\left\{\Theta^0_{yy}(t,t,\bar{X}(t),\bar{X}(t),\Theta(s,\bar{X}(s)))\Theta_x(s,\bar{X}(s))\bar{\sigma}(s)[\Theta_x(s,\bar{X}(s))\bar{\sigma}(s)]^{\top}\right\}ds\\
&+ \int_t^{t+\varepsilon} \Theta^0_y(t,t,\bar{X}(t),\bar{X}(t),\Theta(s,\bar{X}(s)))\bar{g}(s,\bar{\Psi}(s,\bar{X}(s)))ds\Big\}, \qquad (6.21)
\end{aligned}$$

where

$$\bar{g}(s,u) := g\big(s,\bar{X}(s),u,\Theta(s,\bar{X}(s)),\Theta_x(s,\bar{X}(s))\sigma(s,\bar{X}(s),u)\big). \qquad (6.22)$$

Combining (6.20) with (6.21) together, we get

$$\begin{aligned}
Y^{0,\varepsilon}(t) - \bar{Y}^0(t) = \mathbb{E}_t\Big\{&\int_t^{t+\varepsilon} \Theta^0_y(t,t,\bar{X}(t),\bar{X}(t),\Theta^{\varepsilon}(s,\bar{X}(s)))\Big[g^{\varepsilon}(s,u)\\
&+ \Theta^{\varepsilon}_x(s,\bar{X}(s))[b(s,\bar{X}(s),u) - \bar{b}(s)] + \mathrm{tr}\left[\Theta^{\varepsilon}_{xx}(s,\bar{X}(s))[a(s,\bar{X}(s),u) - \bar{a}(s)]\right]\Big]ds\\
&- \frac{1}{2}\int_t^{t+\varepsilon} \mathrm{tr}\Big[\Theta^0_{yy}(t,t,\bar{X}(t),\bar{X}(t),\Theta^{\varepsilon}(s,\bar{X}(s)))\Theta^{\varepsilon}_x(s,\bar{X}(s))\bar{\sigma}(s)[\Theta^{\varepsilon}_x(s,\bar{X}(s))\bar{\sigma}(s)]^{\top}\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_t^{t+\varepsilon} \Big[g^{0,\varepsilon}(t,s,u) - \bar{g}^{0,\varepsilon}(t,s) + \Theta_x^0(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))[b(s,X^\varepsilon(s),u) \\
&- \bar{b}^\varepsilon(s)] + \operatorname{tr}\left\{ \Theta_{xx}^0(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t)))[a(s,X^\varepsilon(s),u) - \bar{a}^\varepsilon(s)] \right\} \Big] ds \\
&- \int_t^{t+\varepsilon} \Theta_y^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(s,\bar{X}(s)))\bar{g}(s,\bar{\Psi}(s,\bar{X}(s)))ds \\
&+ \frac{1}{2}\int_t^{t+\varepsilon} \operatorname{tr}\big[\Theta_{yy}^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(s,\bar{X}(s)))\Theta_x(s,\bar{X}(s))\bar{\sigma}(s) \\
&\times [\Theta_x(s,\bar{X}(s))\bar{\sigma}(s)]^\top \big] ds \Big\}. \qquad (6.23)
\end{aligned}
$$

By the standard results of SDEs, we get

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sup_{s\in[t,t+\varepsilon]} \big(|\bar{X}(s)|^2 + |X^\varepsilon(s)|^2 \big) \Big] \le K(1+|\bar{X}(t)|^2), \\
&\mathbb{E}_t\Big[\sup_{s\in[t,t+\varepsilon]} |\bar{X}(s) - X^\varepsilon(s)|^2 \Big] \le K(1+|\bar{X}(t)|^2)\varepsilon. \qquad (6.24)
\end{aligned}
$$

By Proposition 6.3, we have

$$
\begin{aligned}
&\mathbb{E}_t\big[|\Theta^\varepsilon(s,\bar{X}(s)) - \Theta(s,\bar{X}(s))| + |\Theta_x^\varepsilon(s,\bar{X}(s)) - \Theta_x(s,\bar{X}(s))| \\
&+ |\Theta_{xx}^\varepsilon(s,\bar{X}(s)) - \Theta_{xx}(s,\bar{X}(s))| \big] \le K\varepsilon^{\frac{\alpha}{2}}. \qquad (6.25)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mathbb{E}_t\Big[\big| \Theta_y^0(t,t,\bar{X}(t),\bar{X}(t),\Theta^\varepsilon(s,\bar{X}(s))) - \Theta_y^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(s,\bar{X}(s))) \big| \\
&\quad + \big| \Theta_{yy}^0(t,t,\bar{X}(t),\bar{X}(t),\Theta^\varepsilon(s,\bar{X}(s))) - \Theta_{yy}^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(s,\bar{X}(s))) \big| \\
&\quad + \big| \Theta_x^0(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t))) - \Theta_x^0(t,s,\bar{X}(t),\bar{X}(s),\Theta(t,\bar{X}(t))) \big| \\
&\quad + \big| \Theta_{xx}^0(t,s,\bar{X}(t),X^\varepsilon(s),\Theta^\varepsilon(t,\bar{X}(t))) - \Theta_{xx}^0(t,s,\bar{X}(t),\bar{X}(s),\Theta(t,\bar{X}(t))) \big| \Big] \\
&\le K\varepsilon^{\frac{\alpha}{2}} + K\varepsilon^{\frac{1}{2}}(1+|\bar{X}(t)|) \le K\varepsilon^{\frac{\alpha}{2}}(1+|\bar{X}(t)|), \qquad (6.26)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_t\Big[|g^\varepsilon(s,u) - \bar{g}(s,u)| + |\bar{g}^{0,\varepsilon}(t,s) - \bar{g}^0(t,s,\bar{\Psi}(s,\bar{X}(s)))| \\
&+ |\bar{b}^\varepsilon(s) - \bar{b}(s)| + |\bar{\sigma}^\varepsilon(s) - \bar{\sigma}(s)| \Big] \le K\varepsilon^{\frac{\alpha}{2}}(1+|\bar{X}(t)|), \qquad (6.27)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{g}^0(t,s,u) := g^0\Big(&t,\ s,\ \bar{X}(t),\ \bar{X}(s),\ u,\ \Theta(s,\bar{X}(s)), \Theta_x(s,\bar{X}(s))\sigma(s,\bar{X}(s),u), \\
&\Theta^0(s,s,\bar{X}(s),\bar{X}(s),\Theta(s,\bar{X}(s))), \Theta_x^0(t,s,\bar{X}(t),\bar{X}(s),\Theta(t,\bar{X}(t)))\sigma(s,\bar{X}(s),u) \Big), \quad (6.28)
\end{aligned}
$$

and the form of other functions is given in (6.16), (6.19), and (6.22). Moreover, by Proposition 6.3 and (6.24), we have

$$\mathbb{E}_t\Big[|g^{0,\varepsilon}(t,s,u)-\bar{g}^0(t,s,u)|\Big] \le K\varepsilon^{\frac{\alpha}{2}}(1+|\bar{X}(t)|), \tag{6.29}$$

where $g^{0,\varepsilon}(t,s,u)$ is given by (6.14). With the above estimates (6.24)–(6.29), from (6.23) we have

$$\begin{aligned}
Y^{0,\varepsilon}(t)-\bar{Y}^0(t) = \mathbb{E}_t\Big\{ \int_t^{t+\varepsilon} & \Theta_y^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(s,\bar{X}(s)))\Big[\bar{g}(s,u)-\bar{g}(s,\bar{\Psi}(s,\bar{X}(s))) \\
& + \Theta_x(s,\bar{X}(s))[b(s,\bar{X}(s),u)-\bar{b}(s)] + \mathrm{tr}\left\{\Theta_{xx}(s,\bar{X}(s))[a(s,\bar{X}(s),u)-\bar{a}(s)]\right\}\Big]ds \\
& + \int_t^{t+\varepsilon} \Big[\bar{g}^0(t,s,u)-\bar{g}^0(t,s,\bar{\Psi}(s,\bar{X}(s))) + \Theta_x^0(t,s,\bar{X}(t),\bar{X}(s),\Theta(s,\bar{X}(s))) \\
& \times [b(s,\bar{X}(s),u)-\bar{b}(s)] + \mathrm{tr}\left\{\Theta_{xx}^0(t,s,\bar{X}(t),\bar{X}(s),\Theta(s,\bar{X}(s)))[a(s,\bar{X}(s),u)\right. \\
& \left. -\bar{a}(s)]\right\}\Big]ds\Big\} + o(\varepsilon)(1+|\bar{X}(t)|).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0^+}\frac{J(t,\bar{X}(t);\Psi^\varepsilon)-J(t,\bar{X}(t);\bar{\Psi})}{\varepsilon} = \liminf_{\varepsilon\to 0^+}\frac{Y^{0,\varepsilon}(t)-\bar{Y}^0(t)}{\varepsilon} \\
&= \bar{\Theta}_y^0(t)\Big\{\bar{\Theta}_x(t)[b(t,\bar{X}(t),u)-\bar{b}(t)] + \mathrm{tr}\left\{\bar{\Theta}_{xx}(t)[a(t,\bar{X}(t),u)-\bar{a}(t)]\right\} \\
&\quad + \bar{g}(t,u)-\bar{g}(t,\bar{\Psi}(t,\bar{X}(t)))\Big\} + \bar{\Theta}_x^0(t)[b(t,\bar{X}(t),u)-\bar{b}(t)] \\
&\quad + \mathrm{tr}\left\{\bar{\Theta}_{xx}^0(t)[a(t,\bar{X}(t),u)-\bar{a}(t)]\right\} + \bar{g}^0(t,t,u)-\bar{g}^0(t,t,\bar{\Psi}(t,\bar{X}(t))),
\end{aligned}$$

where

$$\bar{\Theta}(t) := \Theta(t,\bar{X}(t)), \quad \bar{\Theta}^0(t) := \Theta^0(t,t,\bar{X}(t),\bar{X}(t),\Theta(t,\bar{X}(t))), \quad t\in[0,T].$$

Then by the local optimality condition (2.9) of $\bar{\Psi}$, we have

$$\liminf_{\varepsilon\to 0^+}\frac{J(t,\bar{X}(t);\Psi^\varepsilon)-J(t,\bar{X}(t);\bar{\Psi})}{\varepsilon} \ge 0,$$

which completes the proof.

7. Some proofs

For the ease of presentation, in the rest of the paper we restrict to the case with $m=1$ only. However, all our results hold true in the multiple dimensional situation. To begin with, let us first adopt some notations.

Some Notations: For any functions $\varsigma:[S,T]\to\mathbb{R}$ and $\nu:\mathbb{R}^n\to\mathbb{R}$, with $\alpha\in(0,1)$ and $S\in[0,T)$, let

$$\begin{aligned}
\|\varsigma\|_{\frac{\alpha}{2}} &= \sup_{s_1,s_2\in[S,T],\, s_1\neq s_2}\frac{|\varsigma(s_1)-\varsigma(s_2)|}{|s_1-s_2|^{\frac{\alpha}{2}}}, \\
\|\nu\|_\alpha &= \sup_{x_1,x_2\in\mathbb{R}^n,\, 0<|x_1-x_2|\le 1}\frac{|\nu(x_1)-\nu(x_2)|}{|x_1-x_2|^\alpha}.
\end{aligned}$$

For any $\varphi:[S,T]\times\mathbb{R}^n\to\mathbb{R}$, let

$$\|\varphi\|_{C^{\frac{\alpha}{2},\alpha}([S,T]\times\mathbb{R}^n;\mathbb{R})} = \|\varphi\|_{L^\infty([S,T]\times\mathbb{R}^n;\mathbb{R})} + \sup_{x\in\mathbb{R}^n}\|\varphi(\cdot\,,x)\|_{\frac{\alpha}{2}} + \sup_{s\in[S,T]}\|\varphi(s,\cdot)\|_\alpha,$$

$$\|\varphi\|_{C^{\frac{\alpha}{2},1+\alpha}([S,T]\times\mathbb{R}^n;\mathbb{R})} = \|\varphi\|_{C^{0,1}([S,T]\times\mathbb{R}^n;\mathbb{R})} + \|\varphi\|_{C^{\frac{\alpha}{2},\alpha}([S,T]\times\mathbb{R}^n;\mathbb{R})} + \|\varphi_x\|_{C^{\frac{\alpha}{2},\alpha}([S,T]\times\mathbb{R}^n;\mathbb{R})}.$$

We will often simply write $C^{\frac{\alpha}{2},1+\alpha}([S,T]\times\mathbb{R}^n;\mathbb{R})$ as $C^{\frac{\alpha}{2},1+\alpha}$ when there is no confusion. Similarly, we can define $C^{\frac{\alpha}{2},\frac{\alpha}{2},\alpha,1+\alpha,2}([S,T]\times[S,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R};\mathbb{R})$, etc. For any $\theta\in C^{0,1+\alpha}$ and $\theta^0\in C^{0,0,\alpha,1+\alpha,2}$, let us consider the following PDE:

$$\begin{cases}\mathcal{L}\Theta(s,x)+\Theta_x(s,x)\tilde{b}\big(s,x;\theta,\theta^0\big)+\tilde{g}(s,x;\theta,\theta^0\big)=0,\\ \mathcal{L}\Theta^0(t,s,\tilde{x},x,y)+\Theta^0_x(t,s,\tilde{x},x,y)\tilde{b}\big(s,x;\theta,\theta^0\big)+\tilde{g}^0(t,s,\tilde{x},x,y;\theta,\theta^0)=0,\\ \Theta(T,x)=h(x),\quad \Theta^0_s(t,T,\tilde{x},x,y)=h^0(t,\tilde{x},x,y),\end{cases} \tag{7.1}$$

where the differential operator $\mathcal{L}$ is defined by the following:

$$\mathcal{L}\varphi(s,x)=\varphi_s(s,x)+\operatorname{tr}\left[\varphi_{xx}(x)a(s,x)\right],\quad \forall\varphi\in C^{1,2}, \tag{7.2}$$

and

$$\begin{aligned}\tilde{\varphi}(s,x;\theta,\theta^0):=\tilde{\varphi}\big(&s,\,x,\,\theta(s,x),\,\theta_x(s,x),\,\theta^0(s,x,s,x,\theta(s,x)),\\ &\theta^0_x(s,x,s,x,\theta(s,x)),\,\theta^0_y(s,x,s,x,\theta(s,x))\big),\end{aligned}$$

for $\varphi=\tilde{b},\tilde{g}$, and

$$\begin{aligned}\tilde{g}^0(t,s,\tilde{x},x,y;\theta,\theta^0):=\tilde{g}^0\big(&t,\,s,\,\tilde{x},\,x,\,\theta(s,x),\,\theta_x(s,x),\,\theta^0(s,x,s,x,\theta(s,x)),\\ &\theta^0_x(s,x,s,x,\theta(s,x)),\theta^0_y(s,x,s,x,\theta(s,x)),\,\theta^0_x(t,s,\tilde{x},x,y)\big).\end{aligned}$$

From [21, Chapter 1], the fundamental solution associated with the differential operator $\mathcal{L}$ is given by

$$\Xi(s,x,r,\mu)=\Gamma(s,x,r,\mu)+\widehat{\Gamma}(s,x,r,\mu),\quad (s,x),(r,\mu)\in[0,T]\times\mathbb{R}^n,$$

with

$$\Gamma(s,x,r,\mu)=\frac{1}{(4\pi(r-s))^{\frac{n}{2}}(\det[a(r,\mu)])^{\frac{1}{2}}}e^{-\frac{\langle a(r,\mu)^{-1}(x-\mu),(x-\mu)\rangle}{4(r-s)}},$$

$$\widehat{\Gamma}(s,x,r,\mu)=\int_s^r\int_{\mathbb{R}^n}\Gamma_x(s,x,\tau,\eta)\Upsilon(\tau,\eta,r,\mu)d\eta d\tau,$$

and Υ is the unique solution to the following Volterra integral equation:

$$\Upsilon(s,x,r,\mu)=\mathcal{L}\Gamma(s,x,r,\mu)+\int_s^r\int_{\mathbb{R}^n}\mathcal{L}\Gamma(s,x,\tau,\nu)\Upsilon(\tau,\nu,r,\mu)d\tau d\nu.$$

Moreover, from [21, Chapter 1] we have

$$\begin{cases} |\Gamma(s,x,r,\mu)|, |\Xi(s,x,r,\mu)| \leq K \dfrac{1}{(r-s)^{\frac{n}{2}}} e^{\frac{-\lambda|x-\mu|^2}{4(r-s)}}, \\ |\Gamma_x(s,x,r,\mu)|, |\Xi_x(s,x,r,\mu)| \leq K \dfrac{1}{(r-s)^{\frac{n+1}{2}}} e^{\frac{-\lambda|x-\mu|^2}{4(r-s)}}, \\ |\Upsilon(s,x,r,\mu)| \leq K \dfrac{1}{(r-s)^{\frac{n+1+\epsilon}{2}}} e^{\frac{-\lambda|x-\mu|^2}{4(r-s)}}, \end{cases} \tag{7.3}$$

for some $0 < \lambda < \lambda_0$ and some small enough $0 < \epsilon < 1$.

Lemma 7.1. *Fix a $(t, \tilde{x}, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}$. Then for any $\theta \in C^{0,1+\alpha}$ and $\theta^0 \in C^{0,0,\alpha,1+\alpha,2}$, the PDE (7.1) admits a unique classical solution $(\Theta, \Theta^0(t, \cdot, \tilde{x}, \cdot, y)) \in C^{1,2} \times C^{1,2}$ with the following relationship:*

$$\Theta(s,x) = \int_{\mathbb{R}^n} \Xi(s,x,T,\mu)h(\mu)d\mu + \int_s^T \int_{\mathbb{R}^n} \Xi(s,x,r,\mu) \times \Big[\Theta_x(r,\mu)\tilde{b}\big(r,\mu;\theta,\theta^0\big) + \tilde{g}\big(r,\mu;\theta,\theta^0\big)\Big] d\mu dr, \tag{7.4}$$

$$\Theta^0(t,s,\tilde{x},x,y) = \int_{\mathbb{R}^n} \Xi(s,x,T,\mu)h^0(t,\tilde{x},\mu,y)d\mu + \int_s^T \int_{\mathbb{R}^n} \Xi(s,x,r,\mu) \times \Big[\Theta_x^0(t,r,\tilde{x},\mu,y)\tilde{b}\big(r,\mu;\theta,\theta^0\big) + \tilde{g}^0\big(t,r,\tilde{x},\mu,y;\theta,\theta^0\big)\Big] d\mu dr. \tag{7.5}$$

Proof. For any fixed $\theta \in C^{0,1+\alpha}$ and $\theta^0 \in C^{0,0,\alpha,1+\alpha,2}$, denote

$$\nu_1(s,x) = \theta(s,x), \quad \nu_2(s,x) = \theta_x(s,x), \quad \nu_3(s,x) = \theta^0(s,s,x,x,\theta(s,x)),$$
$$\nu_4(s,x) = \theta_x^0(s,s,x,x,\theta(s,x)), \quad \nu_5(s,x) = \theta_y^0(s,s,x,x,\theta(s,x)).$$

Then we have $\nu_i \in C^{0,\alpha}$, for $i = 1, ..., 5$. Taking $(t, \tilde{x}, y)$ as parameters, by [21, Chapter 1, Theorem 12], we get that PDE (7.1) admits a classical solution $(\check{\Theta}(\cdot,\cdot), \check{\Theta}^0(t, \cdot, \tilde{x}, \cdot, y)) \in C^{1,2}$, which is given by

$$\check{\Theta}(s,x) = \int_{\mathbb{R}^n} \check{\Xi}(s,x,T,\mu)h(\mu)d\mu + \int_s^T \int_{\mathbb{R}^n} \check{\Xi}(s,x,r,\mu)\tilde{g}\big(r,\mu;\theta,\theta^0\big)d\mu dr,$$

$$\check{\Theta}^0(t,s,\tilde{x},x,y) = \int_{\mathbb{R}^n} \check{\Xi}(s,x,T,\mu)h^0(t,\tilde{x},\mu,y)d\mu + \int_s^T \int_{\mathbb{R}^n} \check{\Xi}(s,x,r,\mu)\tilde{g}^0\big(t,r,\tilde{x},\mu,y;\theta,\theta^0\big)d\mu dr,$$

where $\check{\Xi}$ is the fundamental solution associated with the following operator:

$$\check{\mathcal{L}}\varphi(s,x) = \varphi_s(s,x) + \operatorname{tr}\left[\varphi_{xx}(x)a(s,x)\right] + \varphi_x(s,x)\tilde{b}\big(r,\mu;\theta,\theta^0\big), \quad \forall \varphi \in C^{1,2}.$$

Now we consider the following equation with the unknown $(\Theta, \Theta^0(t, \cdot, \tilde{x}, \cdot, y))$:

$$\begin{cases} \mathcal{L}\Theta(s,x) + \check{\Theta}_x(s,x)\tilde{b}\big(s,x;\theta,\theta^0\big) + \tilde{g}\big(s,x;\theta,\theta^0\big) = 0, \\ \mathcal{L}\Theta^0(t,s,\tilde{x},x,y) + \check{\Theta}_x^0(t,s,\tilde{x},x,y)\tilde{b}\big(s,x;\theta,\theta^0\big) + \tilde{g}^0\big(t,s,\tilde{x},x,y;\theta,\theta^0\big) = 0, \\ \Theta(T,x) = h(x), \quad \Theta_s^0(t,T,\tilde{x},x,y) = h^0(t,\tilde{x},x,y), \end{cases} \tag{7.6}$$

with the operator $\mathcal{L}$ is defined by (7.2). By [21, Chapter 1, Theorems 12 and 16], (7.6) admits a unique classical solution $(\Theta, \Theta^0(t,\cdot,\tilde{x},\cdot,y)) \in C^{1,2}$, which is given by

$$\begin{aligned}\Theta(s,x) = &\int_{\mathbb{R}^n} \Xi(s,x,T,\mu)h(\mu)d\mu + \int_s^T \int_{\mathbb{R}^n} \Xi(s,x,r,\mu)\\ &\times \Big[\check{\Theta}_x(r,\mu)\tilde{b}\big(r,\mu;\theta,\theta^0\big) + \tilde{g}\big(r,\mu;\theta,\theta^0\big)\Big] d\mu dr, \end{aligned}\tag{7.7}$$

$$\begin{aligned}\Theta^0(t,s,\tilde{x},x,y) = &\int_{\mathbb{R}^n} \Xi(s,x,T,\mu)h^0(t,\tilde{x},\mu,y)d\mu + \int_s^T \int_{\mathbb{R}^n} \Xi(s,x,r,\mu)\\ &\times \Big[\check{\Theta}^0_x(t,r,\tilde{x},\mu,y)\tilde{b}\big(r,\mu;\theta,\theta^0\big) + \tilde{g}^0\big(t,r,\tilde{x},\mu,y;\theta,\theta^0\big)\Big] d\mu dr. \end{aligned}\tag{7.8}$$

Note that $(\check{\Theta}, \check{\Theta}^0(t,\cdot,\tilde{x},\cdot,y)) \in C^{1,2}$ also satisfies (7.6), which, together with the uniqueness of solutions to (7.7)– (7.8), implies that $(\Theta,\Theta^0) = (\check{\Theta},\check{\Theta}^0)$. Thus, (7.4) and (7.5) hold. The other results can be obtained easily. □

Define

$$\Theta(s,x;T) := \int_{\mathbb{R}^n} \Xi(s,x,T,\mu)h(\mu)d\mu,$$

$$\Theta^0(t,s,\tilde{x},x,y;T) := \int_{\mathbb{R}^n} \Xi(s,x,T,\mu)h^0(t,\tilde{x},\mu,y)d\mu.$$

Then $(\Theta(\cdot\,;T), \Theta^0(\cdot\,;T))$ satisfies the following equation:

$$\begin{cases}\mathcal{L}\Theta(s,x;T) = 0, \quad \mathcal{L}\Theta^0(t,s,\tilde{x},x,y;T) = 0,\\ \Theta(T,x;T) = h(x), \quad \Theta^0_s(t,T,\tilde{x},x,y;T) = h^0(t,\tilde{x},x,y).\end{cases}\tag{7.9}$$

Note that $(\Theta(\cdot\,;T), \Theta^0(\cdot\,;T))$ is independent of (θ,θ^0) and by taking $(t,\tilde{x},y)$ as parameters, (7.9) is a classical linear parabolic equation. Then by the standard estimates of PDEs (see [33, Chapter IV]), we have the following results.

Lemma 7.2. *There exists a constant $\kappa > 0$ such that*

$$\|\Theta(\cdot;T)\|_{C^{1+\frac{\alpha}{2},2+\alpha}} + \|\Theta^0(\cdot;T)\|_{C^{\frac{\alpha}{2},1+\frac{\alpha}{2},\alpha,2+\alpha,2}} \le \kappa[\|h\|_{C^{2+\alpha}} + \|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}].$$

We now establish a (global) $C^{0,1}$-norm estimate for $(\Theta, \Theta^0(t,\cdot,\tilde{x},\cdot,y))$.

Lemma 7.3. *There exists a constant $\kappa > 0$, independent of θ and θ^0, such that*

$$\begin{aligned}&\|\Theta\|_{C^{0,1}} + \sup_{t,\tilde{x},y\in[0,T]\times\mathbb{R}^n\times\mathbb{R}} \|\Theta^0(t,\cdot,\tilde{x},\cdot,y))\|_{C^{0,1}}\\ &\le \kappa\Big[1 + \|h\|_{C^{2+\alpha}} + \sup_{t,\tilde{x},y\in[0,T]\times\mathbb{R}^n\times\mathbb{R}} \|h^0(t,\tilde{x},\cdot,y)\|_{C^{2+\alpha}}\Big].\end{aligned}\tag{7.10}$$

Proof. By (7.4), we have

$$\Theta_x(s,x) = \Theta_x(s,x;T) + \int_s^T \int_{\mathbb{R}^n} \Xi_x(s,x,r,\mu)\Big[\Theta_x(r,\mu)\tilde{b}(r,\mu;\theta,\theta^0) + \tilde{g}(r,\mu;\theta,\theta^0)\Big]d\mu dr. \tag{7.11}$$

Then by Lemma 7.2 and the estimate (7.3), we get

$$\begin{aligned}|\Theta_x(s,x)| &\leq |\Theta_x(s,x;T)| + \int_s^T \int_{\mathbb{R}^n} K\frac{1}{(r-s)^{\frac{n+1}{2}}} e^{\frac{-\lambda|x-\eta|^2}{4(r-s)}}(1+|\Theta_x(r,\mu)|)d\mu dr \\ &\leq K(1+\|h\|_{C^{2+\alpha}}) + \int_s^T \int_{\mathbb{R}^n} K\frac{1}{(r-s)^{\frac{n+1}{2}}} e^{\frac{-\lambda|x-\mu|^2}{4(r-s)}}|\Theta_x(r,\mu)|d\mu dr.\end{aligned}$$

By Grönwall's inequality, we obtain

$$|\Theta_x(s,x)| \leq K(1+\|h\|_{C^{2+\alpha}}), \quad \forall (s,x) \in [0,T]\times\mathbb{R}^n.$$

Substituting the above into (7.4) and then by (7.3) again, we have

$$|\Theta(s,x)| \leq K(1+\|h\|_{C^{2+\alpha}}), \quad \forall (s,x) \in [0,T]\times\mathbb{R}^n.$$

It follows that

$$\|\Theta\|_{C^{0,1}} \leq K(1+\|h\|_{C^{2+\alpha}}). \tag{7.12}$$

By (7.5), we get

$$\begin{aligned}\Theta^0_x(t,s,\tilde{x},x,y) = \Theta^0_x(t,s,\tilde{x},x,y;T) + \int_s^T \int_{\mathbb{R}^n} \Xi_x(s,x,r,\mu)\Big[&\Theta^0_x(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) \\ &+ \tilde{g}^0(t,r,\tilde{x},\mu,y;\theta,\theta^0)\Big]d\mu dr.\end{aligned} \tag{7.13}$$

By the same argument as the above, we get

$$\|\Theta^0(t,\cdot,\tilde{x},\cdot,y))\|_{C^{0,1}} \leq K \sup_{t,\tilde{x},y\in[0,T]\times\mathbb{R}^n\times\mathbb{R}} \big[1+\|h^0(t,\tilde{x},\cdot,y)\|_{C^{2+\alpha}}\big].$$

Combining the above with (7.12) implies the estimate (7.10) holds. □

The following gives the (local) regularity estimate of $\Theta^0(t,s,\tilde{x},x,y)$ with respect to the parameters t, $\tilde{x}$, and y.

Lemma 7.4. *There exist two constants $0 < \tilde{\varepsilon} \leq T$ and $\kappa > 0$ such for any $\theta \in C^{0,1+\alpha}$ and $\theta^0 \in C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}$ with*

$$\|\theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1,2}([T-\tilde{\varepsilon},T])} \leq \kappa\big[1+\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}\big], \tag{7.14}$$

the unique solution (Θ,Θ^0) of PDE (7.1) satisfies

$$\|\Theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1,2}([T-\tilde{\varepsilon},T])} \leq \kappa\big[1+\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}\big]. \tag{7.15}$$

Proof. From (7.5) and (7.13), it is easily seen that both $\Theta^0(t,s,\tilde{x},x,y)$ and $\Theta^0_x(t,s,\tilde{x},x,y)$ are differentiable with respect to the parameter y. Moreover, the derivatives are given by

$$\begin{aligned}\Theta^0_y(t,s,\tilde{x},x,y) = \Theta^0_y(t,s,\tilde{x},x,y;T) + \int_s^T\int_{\mathbb{R}^n} \Xi(s,x,r,\mu)\Big[\Theta^0_{xy}(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) \\ + \tilde{g}^0_{p^0}(t,r,\tilde{x},\mu,y;\theta,\theta^0_x)\theta^0_{xy}(t,r,\tilde{x},\mu,y)\Big]d\mu dr,\end{aligned} \tag{7.16}$$

$$\begin{aligned}\Theta^0_{xy}(t,s,\tilde{x},x,y) = \Theta^0_{xy}(t,s,\tilde{x},x,y;T) + \int_s^T\int_{\mathbb{R}^n} \Xi_x(s,x,r,\mu)\Big[\Theta^0_{xy}(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) \\ + \tilde{g}^0_{p^0}(t,r,\tilde{x},\mu,y;\theta,\theta^0_x)\theta^0_{xy}(t,r,\tilde{x},\mu,y)\Big]d\mu dr.\end{aligned} \tag{7.17}$$

Applying the arguments employed in the proof of Lemma 7.3, we have

$$\|\Theta^0_y\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \|\Theta^0_{xy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} \le \bar{K}\big(1+\|h^0_y\|_{C^{0,0,2+\alpha,0}}\big) + \bar{K}\sqrt{\tilde{\varepsilon}}\|\theta^0_{xy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}}. \tag{7.18}$$

Let $\kappa = 10\bar{K}+10$ and $\tilde{\varepsilon}$ be small enough such that $\bar{K}\sqrt{\tilde{\varepsilon}} \le \frac{1}{10}$, then

$$\|\Theta^0_y\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \|\Theta^0_{xy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} \le (2\bar{K}+1)\big(1+\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}\big), \tag{7.19}$$

for θ^0 satisfying (7.14). Note that

$$\begin{aligned}\Theta^0_{yy}(t,s,\tilde{x},x,y) = \Theta^0_{yy}(t,s,\tilde{x},x,y;T) + \int_s^T\int_{\mathbb{R}^n} \Xi(s,x,r,\mu)\Big[\Theta^0_{xyy}(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) \\ + \big\langle\tilde{g}^0_{p^0p^0}(t,r,\tilde{x},\mu,y;\theta,\theta^0)\theta^0_{xy}(t,r,\tilde{x},\mu,y),\ \theta^0_{xy}(t,r,\tilde{x},\mu,y)\big\rangle \\ + \tilde{g}^0_{p^0}(t,r,\tilde{x},\mu,y;\theta,\theta^0_x)\theta^0_{xyy}(t,r,\tilde{x},\mu,y)\Big]d\mu dr,\end{aligned} \tag{7.20}$$

$$\begin{aligned}\Theta^0_{xyy}(t,s,\tilde{x},x,y) = \Theta^0_{xyy}(t,s,\tilde{x},x,y;T) + \int_s^T\int_{\mathbb{R}^n} \Xi_x(s,x,r,\mu)\Big[\Theta^0_{xyy}(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) \\ + \big\langle\tilde{g}^0_{p^0p^0}(t,r,\tilde{x},\mu,y;\theta,\theta^0)\theta^0_{xy}(t,r,\tilde{x},\mu,y),\ \theta^0_{xy}(t,r,\tilde{x},\mu,y)\big\rangle \\ + \tilde{g}^0_{p^0}(t,r,\tilde{x},\mu,y;\theta,\theta^0_x)\theta^0_{xyy}(t,r,\tilde{x},\mu,y)\Big]d\mu dr.\end{aligned} \tag{7.21}$$

Then by the arguments employed in the proof of Lemma 7.3 again, we get

$$\begin{aligned}&\|\Theta^0_{yy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \|\Theta^0_{xyy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} \\ &\le \bar{K}\big(1+\|h^0_{yy}\|_{C^{0,0,2+\alpha,0}}\big) + \bar{K}\sqrt{\varepsilon}\|\theta^0_{xyy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \bar{K}\sqrt{\varepsilon}\|\theta^0_{xy}\|^2_{L^\infty_{[T-\tilde{\varepsilon},T]}},\end{aligned}$$

where $\bar{K}>0$ can be same as that in (7.18). Further, we let $\tilde{\varepsilon}>0$ be small enough such that $\bar{K}\sqrt{\tilde{\varepsilon}} \le \frac{1}{40}$ and $\bar{K}\sqrt{\tilde{\varepsilon}}\|\theta^0_{xy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} \le \bar{K}\sqrt{\tilde{\varepsilon}}2\kappa(1+\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}) \le \frac{1}{40}$, then

$$\begin{aligned}&\|\Theta^0_{yy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \|\Theta^0_{xyy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} \\ &\le \bar{K}\big(1+\|h^0_{yy}\|_{C^{0,0,2+\alpha,0}}\big) + \frac{1}{40}\|\theta^0_{xyy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \frac{1}{40}\|\theta^0_{xy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} \\ &\le (2\bar{K}+1)\big(1+\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}\big).\end{aligned}$$

For any $\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^n$, denote

$$\delta\Theta^0(t,s,x,y) := \Theta^{0,1}(t,s,x,y) - \Theta^{0,2}(t,s,x,y),$$

with $\Theta^{0,i}(t,s,x,y) := \Theta^0(t,s,\tilde{x}_i,x,y)$, $i=1,2$. Similarly, we define $\delta\theta^0$ and $\theta^{0,i}$. Then, we have

$$\begin{aligned}\delta\Theta^0(t,s,x,y) &= \Theta^0(t,s,\tilde{x}_1,x,y;T) - \Theta^0(t,s,\tilde{x}_2,x,y;T) + \int_s^T\int_{\mathbb{R}^n} \Xi(s,x,r,\mu) \\ &\quad\times \Big[\delta\Theta^0_x(t,r,\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) + \tilde{g}^0(t,r,\tilde{x}_1,\mu,y;\theta,\theta^0,\theta^{0,1}) \\ &\quad - \tilde{g}^0(t,r,\tilde{x}_2,\mu,y;\theta,\theta^0,\theta^{0,2})\Big] d\mu dr, \end{aligned} \tag{7.22}$$

$$\begin{aligned}\delta\Theta^0_y(t,s,x,y) &= \Theta^0_y(t,s,\tilde{x}_1,x,y;T) - \Theta^0_y(t,s,\tilde{x}_2,x,y;T) + \int_s^T\int_{\mathbb{R}^n} \Xi(s,x,r,\mu) \\ &\quad\times \Big\{\delta\Theta^0_{xy}(t,r,\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) + \delta\theta^0_{xy}(t,r,\mu,y)\tilde{g}^0_{p^0}(t,r,\tilde{x}_1,\mu,y;\theta,\theta^0,\theta^{0,1}) \\ &\quad + \theta^{0,2}_{xy}(t,r,\mu,y)\big[\tilde{g}^0_{p^0}(t,r,\tilde{x}_1,\mu,y;\theta,\theta^0,\theta^{0,1}) - \tilde{g}^0_{p^0}(t,r,\tilde{x}_2,\mu,y;\theta,\theta^0,\theta^{0,2})\big]\Big\} d\mu dr, \end{aligned} \tag{7.23}$$

$$\begin{aligned}\delta\Theta^0_x(t,s,x,y) &= \Theta^0_x(t,s,\tilde{x}_1,x,y;T) - \Theta^0_x(t,s,\tilde{x}_2,x,y;T) + \int_s^T\int_{\mathbb{R}^n} \Xi_x(s,x,r,\mu) \\ &\quad\times \Big[\delta\Theta^0_x(t,r,\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) + \tilde{g}^0(t,r,\tilde{x}_1,\mu,y;\theta,\theta^0,\theta^{0,1}) \\ &\quad - \tilde{g}^0(t,r,\tilde{x}_2,\mu,y;\theta,\theta^0,\theta^{0,2})\Big] d\mu dr, \end{aligned} \tag{7.24}$$

and

$$\begin{aligned}\delta\Theta^0_{xy}(t,s,x,y) &= \Theta^0_{xy}(t,s,\tilde{x}_1,x,y;T) - \Theta^0_{xy}(t,s,\tilde{x}_2,x,y;T) + \int_s^T\int_{\mathbb{R}^n} \Xi_x(s,x,r,\mu) \\ &\quad\times \Big\{\delta\Theta^0_{xy}(t,r,\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) + \delta\theta^0_{xy}(t,r,\mu,y)\tilde{g}^0_{p^0}(t,r,\tilde{x}_1,\mu,y;\theta,\theta^0,\theta^{0,1}) \\ &\quad + \theta^{0,2}_{xy}(t,r,\mu,y)\big[\tilde{g}^0_{p^0}(t,r,\tilde{x}_1,\mu,y;\theta,\theta^0,\theta^{0,1}) - \tilde{g}^0_{p^0}(t,r,\tilde{x}_2,\mu,y;\theta,\theta^0,\theta^{0,2})\big]\Big\} d\mu dr. \end{aligned} \tag{7.25}$$

By Lemma 7.2, (7.3), and (7.18), we get

$$\begin{aligned}&\|\delta\Theta^0\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \|\delta\Theta^0_y\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \|\delta\Theta^0_x\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \|\delta\Theta^0_{xy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} \\ &\le \bar{K}\big(1 + \|h^0\|_{C^{0,\alpha,2+\alpha,0}} + \|h^0_y\|_{C^{0,\alpha,2+\alpha,0}} + \|h^0_{yy}\|_{C^{0,\alpha,2+\alpha,0}}\big)|\tilde{x}_1 - \tilde{x}_2|^\alpha + \bar{K}\sqrt{\tilde{\varepsilon}}\|\delta\theta^0_x\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} \\ &\quad + \bar{K}\sqrt{\tilde{\varepsilon}}\|\delta\theta^0_{xy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + \bar{K}\sqrt{\tilde{\varepsilon}}\|\theta^{0,2}_{xy}\|_{L^\infty_{[T-\tilde{\varepsilon},T]}}[\|\delta\theta^0_x\|_{L^\infty_{[T-\tilde{\varepsilon},T]}} + |\tilde{x}_1 - \tilde{x}_2|^\alpha],\end{aligned}$$

which implies that

$$\begin{aligned}&\|\Theta^0\|_{C^{0,0,\alpha,0,0}_{[T-\tilde{\varepsilon},T]}} + \|\Theta^0_y\|_{C^{0,0,\alpha,0,0}_{[T-\tilde{\varepsilon},T]}} + \|\Theta^0_x\|_{C^{0,0,\alpha,0,0}_{[T-\tilde{\varepsilon},T]}} + \|\Theta^0_{xy}\|_{C^{0,0,\alpha,0,0}_{[T-\tilde{\varepsilon},T]}} \\ &\le (2\bar{K}+1)\big(1 + \|h^0\|_{C^{0,\alpha,2+\alpha,1}}\big),\end{aligned}$$

by choosing a proper $\tilde{\varepsilon} > 0$. By continuing the above arguments, we can also have

$$\|\Theta^0_{yy}\|_{C^{0,0,\alpha,0,0}_{[T-\tilde{\varepsilon},T]}} + \|\Theta^0_{xyy}\|_{C^{0,0,\alpha,0,0}_{[T-\tilde{\varepsilon},T]}} \leq (2\bar{K}+1)\big(1+\|h^0\|_{C^{0,\alpha,2+\alpha,2}}\big).$$

The $C^{\frac{\alpha}{2}}$-estimate for the parameter t can be obtained by the same arguments as above. Combining these estimates together, we get (7.15). □

Next, we are going to establish the $C^{1+\alpha}$-norm estimate for $\Theta^0(t,s,\tilde{x},\cdot,y)$ and $\Theta(s,\cdot)$. To achieve this, we need to make some preparations.

By making the transform $x-\mu = (\sqrt{r-s})\tilde{\mu}$ in the integral term of (7.11)–(7.13), respectively, we have

$$\begin{aligned}\Theta_x(s,x) = \Theta_x(s,x;T) &+ \int_s^T\int_{\mathbb{R}^n} \tilde{\Gamma}_x(s,x,r,x-\sqrt{r-s}\tilde{\mu})\Big[\Theta_x(r,x-\sqrt{r-s}\tilde{\mu}) \\ &\times \tilde{b}(r,x-\sqrt{r-s}\tilde{\mu};\theta,\theta^0) + \tilde{g}(r,x-\sqrt{r-s}\tilde{\mu};\theta,\theta^0)\Big]d\tilde{\mu}dr \\ &+ \int_s^T\int_{\mathbb{R}^n} \widehat{\Gamma}_x(s,x,r,\mu)\Big[\Theta_x(r,\mu)\tilde{b}(r,\mu;\theta,\theta^0) + \tilde{g}(r,\mu;\theta,\theta^0)\Big]d\mu dr,\end{aligned} \tag{7.26}$$

and

$$\begin{aligned}\Theta^0_x(t,s,\tilde{x},x,y) = \Theta^0_x(t,s,\tilde{x},x,y;T) &+ \int_s^T\int_{\mathbb{R}^n} \tilde{\Gamma}_x(s,x,r,x-\sqrt{r-s}\tilde{\mu}) \\ &\times \Big[\Theta^0_x(t,r,\tilde{x},x-\sqrt{r-s}\tilde{\mu},y)\tilde{b}(r,x-\sqrt{r-s}\tilde{\mu};\theta,\theta^0) \\ &+ \tilde{g}^0(t,r,\tilde{x},x-\sqrt{r-s}\tilde{\mu};\theta,\theta^0)\Big]d\tilde{\mu}dr + \int_s^T\int_{\mathbb{R}^n} \widehat{\Gamma}_x(s,x,r,\mu) \\ &\times \Big[\Theta^0_x(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta,\theta^0) + \tilde{g}^0(t,r,\tilde{x},\mu,y;\theta,\theta^0)\Big]d\mu dr,\end{aligned} \tag{7.27}$$

where

$$\begin{aligned}\tilde{\Gamma}_x(s,x,r,x-\sqrt{r-s}\tilde{\mu} &:= -\Gamma_x(s,x,r,x-\sqrt{r-s}\tilde{\mu})(r-s)^{\frac{n}{2}} \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}(\det[a(r,x-\sqrt{r-s}\tilde{\mu})])^{\frac{1}{2}}} \\ &\quad\times e^{-\frac{\langle a(r,x-\sqrt{r-s}\tilde{\mu})^{-1}\tilde{\mu},\tilde{\mu}\rangle}{4}} \frac{a(r,x-\sqrt{r-s}\tilde{\mu})^{-1}}{2\sqrt{r-s}}\tilde{\mu}.\end{aligned} \tag{7.28}$$

By some straightforward calculations, it is clearly seen that

$$\begin{aligned}|\tilde{\Gamma}_x(s,x,r,x-\sqrt{r-s}\tilde{\mu})| &\leq \frac{K}{\sqrt{r-s}}e^{-\lambda|\tilde{\mu}|^2}, \\ \|\tilde{\Gamma}_x(s,\cdot,r,\cdot-\sqrt{r-s}\tilde{\mu})\|_\alpha &\leq \frac{K}{\sqrt{r-s}}e^{-\lambda|\tilde{\mu}|^2},\end{aligned} \tag{7.29}$$

for some $0<\lambda<\lambda_0$. Moreover, by [21, Chapter 1, Lemma 3 and Theorem 7], we have

$$|\widehat{\Gamma}_x(s,x,r,\mu)| \le K\frac{e^{\frac{-\lambda|x-\mu|^2}{r-s}}}{(r-s)^{\frac{n+\epsilon}{2}}}, \quad |\widehat{\Gamma}_x(s,x_1,r,\mu)-\widehat{\Gamma}_x(s,x_2,r,\mu)| \tag{7.30}$$

$$\le K\Big[\frac{e^{\frac{-\lambda|x_1-\mu|^2}{r-s}}}{(r-s)^{\frac{n+\epsilon}{2}}}+\frac{e^{\frac{-\lambda|x_2-\mu|^2}{r-s}}}{(r-s)^{\frac{n+\epsilon}{2}}}\Big]|x_1-x_2|^\alpha, \quad \forall x_1,x_2\in\mathbb{R}^n, \tag{7.31}$$

for some small enough $0<\epsilon<1$.

Lemma 7.5. *There exist two constants $0<\bar{\varepsilon}\le\tilde{\varepsilon}\le T$ and $\bar{\kappa}>0$ such for any $\theta\in C^{0,1+\alpha}$ and $\theta^0\in C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}$ with satisfying (7.14) and*

$$\|\theta_x\|_{C^{0,\alpha}([T-\bar{\varepsilon},T])}+\|\theta^0_x\|_{C^{0,0,0,\alpha,0}([T-\bar{\varepsilon},T])}\le 2\bar{\kappa}\big[1+\|h\|_{C^{2+\alpha}}+\|h^0\|_{C^{0,0,2+\alpha,0}}\big], \tag{7.32}$$

the unique solution (Θ,Θ^0) of PDE (7.1) satisfies

$$\|\Theta_x\|_{C^{0,\alpha}([T-\bar{\varepsilon},T])}+\|\Theta^0_x\|_{C^{0,0,0,\alpha,0}([T-\bar{\varepsilon},T])}\le 2\bar{\kappa}\big[1+\|h\|_{C^{2+\alpha}}+\|h^0\|_{C^{0,0,2+\alpha,0}}\big]. \tag{7.33}$$

Moreover, there exists a constant $\widehat{\kappa}>0$, which depends on $\bar{\kappa}$, such that

$$\|\Theta^0_{xy}\|_{C^{0,0,0,\alpha,0}([T-\bar{\varepsilon},T])}\le \widehat{\kappa}\big[1+\|h\|_{C^{2+\alpha}}+\|h^0\|_{C^{0,0,2+\alpha,1}}\big]. \tag{7.34}$$

Proof. For any x_1, $x_2\in\mathbb{R}^n$, from (7.26), by the estimate (7.31) we have

$$\begin{aligned}
&|\Theta_x(s,x_1)-\Theta_x(s,x_2)|\\
&\le K\|h\|_{C^{2+\alpha}}|x_1-x_2|^\alpha+\int_s^T\int_{\mathbb{R}^n}\frac{K}{\sqrt{r-s}}e^{-\lambda|\tilde{\mu}|^2}d\tilde{\mu}dr\big[1+\|\Theta_x\|_{L^\infty}\big]|x_1-x_2|^\alpha\\
&\quad+\int_s^T\int_{\mathbb{R}^n}\frac{K}{\sqrt{r-s}}e^{-\lambda|\tilde{\mu}|^2}\Big[\big|\Theta_x(r,x_1-\sqrt{r-s}\tilde{\mu})-\Theta_x(r,x_2-\sqrt{r-s}\tilde{\mu})\big|\\
&\quad+\big|\tilde{b}(r,x_1-\sqrt{r-s}\tilde{\mu};\theta,\theta^0)-\tilde{b}(r,x_2-\sqrt{r-s}\tilde{\mu};\theta,\theta^0)\big|\|\Theta_x\|_{L^\infty}\\
&\quad+\big|\tilde{g}(r,x_1-\sqrt{r-s}\tilde{\mu};\theta,\theta^0)-\tilde{g}(r,x_2-\sqrt{r-s}\tilde{\mu};\theta,\theta^0)\big|\Big]d\tilde{\mu}dr\\
&\quad+\int_s^T\int_{\mathbb{R}^n}\Big[\frac{e^{\frac{-\lambda|x_1-\mu|^2}{r-s}}}{(r-s)^{\frac{n+\epsilon}{2}}}+\frac{e^{\frac{-\lambda|x_2-\mu|^2}{r-s}}}{(r-s)^{\frac{n+\epsilon}{2}}}\Big]d\mu dr\big[1+\|\Theta_x\|_{L^\infty}\big]|x_1-x_2|^\alpha.
\end{aligned} \tag{7.35}$$

Note that for $\tilde{\varphi}=\tilde{b},\tilde{g}$,

$$\begin{aligned}
&\tilde{\varphi}(r,x_1-\sqrt{r-s}\tilde{\mu};\theta,\theta^0)-\tilde{\varphi}(r,x_2-\sqrt{r-s}\tilde{\mu};\theta,\theta^0)\\
&=\tilde{\varphi}\Big(r,x_1-\sqrt{r-s}\tilde{\mu},\theta(r,x_1-\sqrt{r-s}\tilde{\mu}),\theta_x(r,x_1-\sqrt{r-s}\tilde{\mu}),\\
&\qquad\theta^0(r,x_1-\sqrt{r-s}\tilde{\mu},r,x_1-\sqrt{r-s}\tilde{\mu},\theta(r,x_1-\sqrt{r-s}\tilde{\mu})),\\
&\qquad\theta^0_x(r,x_1-\sqrt{r-s}\tilde{\mu},r,x_1-\sqrt{r-s}\tilde{\mu},\theta(r,x_1-\sqrt{r-s}\tilde{\mu})),\\
&\qquad\theta^0_y(r,x_1-\sqrt{r-s}\tilde{\mu},r,x_1-\sqrt{r-s}\tilde{\mu},\theta(r,x_1-\sqrt{r-s}\tilde{\mu}))\Big)\\
&\quad-\tilde{\varphi}\Big(r,x_2-\sqrt{r-s}\tilde{\mu},\theta(r,x_2-\sqrt{r-s}\tilde{\mu}),\theta_x(r,x_2-\sqrt{r-s}\tilde{\mu}),
\end{aligned}$$

$$\theta^0(r, x_2 - \sqrt{r-s}\tilde{\mu}, r, x_2 - \sqrt{r-s}\tilde{\mu}, \theta(r, x_2 - \sqrt{r-s}\tilde{\mu})),$$
$$\theta_x^0(r, x_2 - \sqrt{r-s}\tilde{\mu}, r, x_2 - \sqrt{r-s}\tilde{\mu}, \theta(r, x_2 - \sqrt{r-s}\tilde{\mu})),$$
$$\theta_y^0(r, x_2 - \sqrt{r-s}\tilde{\mu}, r, x_2 - \sqrt{r-s}\tilde{\mu}, \theta(r, x_2 - \sqrt{r-s}\tilde{\mu}))\Big).$$

Then by Lemma 7.4, we get

$$\begin{aligned}&|\tilde{\varphi}(r, x_1 - \sqrt{r-s}\tilde{\mu}; \theta, \theta^0) - \tilde{\varphi}(r, x_2 - \sqrt{r-s}\tilde{\mu}; \theta, \theta^0)|\\ &\le K\big(1 + \|\theta_x(r,\cdot)\|_{C^\alpha} + \|\theta_x^0(r,\cdot,r,\cdot,\cdot)\|_{C^{0,\alpha,0}}\big)|x_1 - x_2|^\alpha,\end{aligned}$$

where $K > 0$ depends on h and h^0. Substituting the above into (7.35) and then by Lemma 7.3, we have

$$\begin{aligned}|\Theta_x(s, x_1) - \Theta_x(s, x_2)| \le K\big(1 + \|h\|_{C^{2+\alpha}}\big)|x_1 - x_2|^\alpha + \int_s^T \frac{K}{\sqrt{r-s}}\Big[\|\Theta_x(r,\cdot)\|_\alpha\\ + \|\theta_x(r,\cdot)\|_{C^\alpha} + \|\theta_x^0(r,\cdot,r,\cdot,\cdot)\|_{C^{0,\alpha,0}}\Big]dr|x_1 - x_2|^\alpha,\end{aligned}$$

which implies that

$$\begin{aligned}\|\Theta_x(s,\cdot)\|_\alpha \le K\big(1 + \|h\|_{C^{2+\alpha}}\big) + \int_s^T \frac{K}{\sqrt{r-s}}\Big[\|\Theta_x(r,\cdot)\|_\alpha\\ + \|\theta_x(r,\cdot)\|_{C^\alpha} + \|\theta_x^0(r,\cdot,r,\cdot,\cdot)\|_{C^{0,\alpha,0}}\Big]dr.\end{aligned} \tag{7.36}$$

By the same argument as the above (noting (7.27)), we also have

$$\begin{aligned}\|\Theta_x^0(t,s,\tilde{x},\cdot,y)\|_\alpha \le K\big(1 + \|h^0(t,\tilde{x},\cdot,y)\|_{C^{2+\alpha}}\big) + \int_s^T \frac{K}{\sqrt{r-s}}\Big[\|\Theta_x^0(t,s,\tilde{x},\cdot,y)\|_\alpha\\ + \|\theta_x(r,\cdot)\|_\alpha + \|\theta_x^0(r,\cdot,r,\cdot,\cdot)\|_{C^{0,\alpha,0}} + \|\theta_x^0(t,r,\tilde{x},\cdot,y)\|_\alpha\Big]dr,\end{aligned} \tag{7.37}$$

$$\begin{aligned}\|\Theta_{xy}^0(t,s,\tilde{x},\cdot,y)\|_\alpha \le K\big(1 + \|h^0(t,\tilde{x},\cdot,\cdot)\|_{C^{2+\alpha,1}}\big) + \int_s^T \frac{K}{\sqrt{r-s}}\Big[\|\Theta_{xy}^0(t,s,\tilde{x},\cdot,y)\|_\alpha\\ + \|\theta_x(r,\cdot)\|_\alpha + \|\theta_x^0(r,\cdot,r,\cdot,\cdot)\|_{C^{0,\alpha,0}} + \|\theta_{xy}^0(t,r,\tilde{x},\cdot,y)\|_\alpha\Big]dr.\end{aligned} \tag{7.38}$$

Combining (7.36) and (7.37) yields that

$$\begin{aligned}&\|\Theta_x\|_{C^{0,\alpha}([T-\bar{\varepsilon},T])} + \|\Theta_x^0\|_{C^{0,0,0,\alpha,0}([T-\bar{\varepsilon},T])}\\ &\le \bar{\kappa}\big[1 + \|h\|_{C^{2+\alpha}} + \|h^0\|_{C^{0,0,2+\alpha,0}}\big] + \bar{\kappa}\sqrt{\bar{\varepsilon}}\big[\|\Theta_x\|_{C^{0,\alpha}([T-\bar{\varepsilon},T])}\\ &\quad + \|\Theta_x^0\|_{C^{0,0,0,\alpha,0}([T-\bar{\varepsilon},T])} + \|\theta_x\|_{C^{0,\alpha}([T-\bar{\varepsilon},T])} + \|\theta_x^0\|_{C^{0,0,0,\alpha,0}([T-\bar{\varepsilon},T])}\big],\end{aligned} \tag{7.39}$$

where $\bar{\kappa} > \kappa$, only depending on $(h, \tilde{g}, h^0, \tilde{g}^0)$, is a fixed constant. Let $0 < \bar{\varepsilon} \le \tilde{\varepsilon}$ be small enough such that $\bar{\kappa}\sqrt{\bar{\varepsilon}} \le \frac{1}{4}$ and

$$\|\theta_x\|_{C^{0,\alpha}([T-\bar{\varepsilon},T])} + \|\theta_x^0\|_{C^{0,0,0,\alpha,0}([T-\bar{\varepsilon},T])} \le 2\bar{\kappa}\big[1 + \|h\|_{C^{2+\alpha}} + \|h^0\|_{C^{0,0,2+\alpha,0}}\big]. \tag{7.40}$$

Then from (7.39), we get

$$\|\Theta_x\|_{C^{0,\alpha}([T-\bar{\varepsilon},T])}+\|\Theta_x^0\|_{C^{0,0,0,\alpha,0}([T-\bar{\varepsilon},T])}\le 2\bar{\kappa}\big[1+\|h\|_{C^{2+\alpha}}+\|h^0\|_{C^{0,0,2+\alpha,0}}\big].$$

Substituting (7.40) into (7.38) also yields (7.34) immediately. The proof is complete. □

The following is concerned with the local solvability of the equilibrium HJB equation (2.16).

Proposition 7.6. *There exists a constant $\widehat{\varepsilon}\in(0,\bar{\varepsilon}]$ such that the equilibrium HJB equation (2.16) admits a unique classical solution on the time interval $[T-\widehat{\varepsilon},T]$.*

Proof. Denote

$$\mathcal{B}_{\bar{\varepsilon}}=\big\{(\theta,\theta^0)\in C^{0,1+\alpha}\times C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}\;\big|\;(\theta,\theta^0)\text{ satisfies (7.14) and (7.32)}\big\}.$$

For any $(\theta,\theta^0)\in\mathcal{B}_{\bar{\varepsilon}}$, by Lemma 7.1, PDE (7.1) admits a unique classical solution (Θ,Θ^0). Moreover, from Lemmas 7.3, 7.4, and 7.5, we know that $(\Theta,\Theta^0)\in\mathcal{B}_{\bar{\varepsilon}}$. Thus, the mapping $\Gamma:\mathcal{B}_{\bar{\varepsilon}}\to\mathcal{B}_{\bar{\varepsilon}}$, given by

$$\Gamma(\theta,\theta^0)=(\Theta,\Theta^0),$$

is well-defined. For any $(\theta^i,\theta^{0,i})\in\mathcal{B}_{\bar{\varepsilon}}$ $(i=1,2)$, let

$$(\Theta^i,\Theta^{0,i})=\Gamma(\theta^i,\theta^{0,i}),\quad i=1,2.$$

Denote

$$\delta\theta=\theta^1-\theta^2,\quad \delta\Theta=\Theta^1-\Theta^2,\quad \delta\theta^0=\theta^{0,1}-\theta^{0,2},\quad \delta\Theta^0=\Theta^{0,1}-\Theta^{0,2},$$

and

$$\begin{aligned}&\delta\tilde{\varphi}(s,x)=\tilde{\varphi}(s,x;\theta^1,\theta^{0,1})-\tilde{\varphi}(s,x;\theta^2,\theta^{0,2}),\quad\text{for}\quad\varphi=b,g,\\&\delta\tilde{g}^0(t,s,\tilde{x},x,y)=\tilde{g}^0(t,s,\tilde{x},x,y;\theta^1,\theta^{0,1})-\tilde{g}^0(t,s,\tilde{x},x,y;\theta^1,\theta^{0,2}).\end{aligned}$$

We hope to show that

$$\|\delta\Theta\|_{C^{0,1+\alpha}}+\|\delta\Theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}}\le\frac{1}{2}\big[\|\delta\theta\|_{C^{0,1+\alpha}}+\|\delta\theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}}\big],\tag{7.41}$$

on some time interval $[T-\widehat{\varepsilon},T]\subseteq[T-\bar{\varepsilon},T]$. Thus, Γ is a contraction mapping and then it admits a unique fixed point $(\Theta,\Theta^0)\in C^{0,1+\alpha}\times C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}$. By Lemma 7.1, (Θ,Θ^0) is the unique classical solution of equilibrium HJB equation (2.16) on $[T-\widehat{\varepsilon},T]$. Further, we can show that $(\Theta,\Theta^0)\in C^{1+\frac{\alpha}{2},2+\alpha}\times C^{\frac{\alpha}{2},1+\frac{\alpha}{2},\alpha,2+\alpha,2}$. Indeed, with $(\Theta,\Theta^0)\in C^{0,1+\alpha}\times C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}$, by the classical $C^{\frac{\alpha}{2}}$-estimates for Θ_x with respect to the time variable of linear parabolic equations (see [33, Chapter IV]), we know that $(\Theta,\Theta^0)\in C^{\frac{\alpha}{2},1}\times C^{0,\frac{\alpha}{2},0,1,1}$ on $[T-\widehat{\varepsilon},T]$. Combining this with the fact that $(\Theta,\Theta^0)\in C^{0,1+\alpha}\times C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}$, we get that the values $\Theta(s,x),\Theta_x(s,x)$ and $\Theta^0(s,s,x,x,\Theta(s,x))$, $\Theta^0_x(s,x,s,x,\Theta(s,x))$, $\Theta^0_y(s,x,s,x,\Theta(s,x))$, $\Theta^0_x(t,s,\tilde{x},x,y)$ are $\frac{\alpha}{2}$-Hölder and α-Hölder continuous with respect to s and x, respectively. It follows [33, Chapter IV] that $(\Theta,\Theta^0)\in C^{1+\frac{\alpha}{2},2+\alpha}\times C^{\frac{\alpha}{2},1+\frac{\alpha}{2},\alpha,2+\alpha,2}$.

In the following, let us show that (7.41) really holds for some $\widehat{\varepsilon}>0$.

Step 1. From (7.4)–(7.5), we have

$$\delta\Theta(s,x)=\int_s^T\int_{\mathbb{R}^n}\Xi(s,x,r,\mu)\Big[\Theta_x^1(r,\mu)\delta\tilde{b}(r,\mu)+\delta\Theta_x(r,\mu)\tilde{b}(r,\mu;\theta^2,\theta^{0,2})+\delta\tilde{g}(r,\mu)\Big]d\mu dr, \tag{7.42}$$

$$\begin{aligned}\delta\Theta^0(t,s,\tilde{x},x,y)=&\int_s^T\int_{\mathbb{R}^n}\Xi(s,x,r,\mu)\Big[\Theta_x^{0,1}(t,r,\tilde{x},\mu,y)\delta\tilde{b}(r,\mu)\\&+\delta\Theta_x^0(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta^2,\theta^{0,2})+\delta\tilde{g}^0(t,r,\tilde{x},\mu,y)\Big]d\mu dr.\end{aligned} \tag{7.43}$$

By Lemma 7.3 and the estimate (7.3), from (7.42) we get

$$|\delta\Theta(s,x)|\le K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n}{2}}}\Big[|\delta\tilde{b}(r,\mu)|+|\delta\Theta_x(r,\mu)|+|\delta\tilde{g}(r,\mu)|\Big]d\mu dr. \tag{7.44}$$

For $\varphi=b,g$, by Lemma 7.4 we have

$$\begin{aligned}|\delta\tilde{\varphi}(s,x)|\le K\Big[&|\delta\theta(s,x)|+|\delta\theta_x(s,x)|+|\delta\theta^0(s,x,s,x,\theta^1(s,x))|\\&+|\delta\theta_x^0(s,x,s,x,\theta^1(s,x))|+|\delta\theta_y^0(s,x,s,x,\theta^1(s,x))|\Big].\end{aligned} \tag{7.45}$$

Substituting the above into (7.44) yields that

$$\begin{aligned}|\delta\Theta(s,x)|\le&K(T-s)\big[\|\delta\theta\|_{L^\infty}+\|\delta\theta_x\|_{L^\infty}+\|\delta\theta^0\|_{L^\infty}+\|\delta\theta_x^0\|_{L^\infty}+\|\delta\theta_y^0\|_{L^\infty}\big]\\&+K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n}{2}}}|\delta\Theta_x(r,\mu)|d\mu dr.\end{aligned} \tag{7.46}$$

Similar to (7.45), using Lemma 7.4 again, we have

$$\begin{aligned}|\delta\tilde{g}^0(t,s,\tilde{x},x,y)|\le K\Big[&|\delta\theta(s,x)|+|\delta\theta_x(s,x)|+|\delta\theta^0(s,x,s,x,\theta^1(s,x))|\\&+|\delta\theta_x^0(s,x,s,x,\theta^1(s,x))|+|\delta\theta_y^0(s,x,s,x,\theta^1(s,x))|\\&+|\delta\theta_x^0(t,s,\tilde{x},x,y)|\Big].\end{aligned} \tag{7.47}$$

Substituting the above into (7.43), we get

$$\begin{aligned}|\delta\Theta^0(t,s,\tilde{x},x,y)|\le&K(T-s)\big[\|\delta\theta\|_{L^\infty}+\|\delta\theta_x\|_{L^\infty}+\|\delta\theta^0\|_{L^\infty}+\|\delta\theta_x^0\|_{L^\infty}+\|\delta\theta_y^0\|_{L^\infty}\big]\\&+K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n}{2}}}|\delta\Theta_x^0(t,r,\tilde{x},\mu,y)|d\mu dr.\end{aligned} \tag{7.48}$$

From (7.11) and (7.13), we have

$$\delta\Theta_x(s,x)=\int_s^T\int_{\mathbb{R}^n}\Xi_x(s,x,r,\mu)\Big[\Theta_x^1(r,\mu)\delta\tilde{b}(r,\mu)+\delta\Theta_x(r,\mu)\tilde{b}(r,\mu;\theta^2,\theta^{0,2})+\delta\tilde{g}(r,\mu)\Big]d\mu dr, \tag{7.49}$$

$$\delta\Theta_x^0(t,s,\tilde{x},x,y)=\int_s^T\int_{\mathbb{R}^n}\Xi_x(s,x,r,\mu)\Big[\Theta_x^{0,1}(t,r,\tilde{x},\mu,y)\delta\tilde{b}(r,\mu)$$
$$+\delta\Theta_x^0(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta^2,\theta^{0,2})+\delta\tilde{g}^0(t,r,\tilde{x},\mu,y)\Big]d\mu dr. \tag{7.50}$$

By Lemma 7.3 and the estimates (7.3), (7.45) and (7.47), we get

$$|\delta\Theta_x(s,x)|\le K\sqrt{T-s}\big[\|\delta\theta\|_{L^\infty}+\|\delta\theta_x\|_{L^\infty}+\|\delta\theta^0\|_{L^\infty}+\|\delta\theta_x^0\|_{L^\infty}+\|\delta\theta_y^0\|_{L^\infty}\big]$$
$$+K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n+1}{2}}}|\delta\Theta_x(r,\mu)|d\mu dr, \tag{7.51}$$
$$|\delta\Theta_x^0(t,s,\tilde{x},x,y)|\le K\sqrt{T-s}\big[\|\delta\theta\|_{L^\infty}+\|\delta\theta_x\|_{L^\infty}+\|\delta\theta^0\|_{L^\infty}+\|\delta\theta_x^0\|_{L^\infty}+\|\delta\theta_y^0\|_{L^\infty}\big]$$
$$+K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n+1}{2}}}|\delta\Theta_x^0(t,r,\tilde{x},\mu,y)|d\mu dr. \tag{7.52}$$

Combining (7.46), (7.48), (7.51) and (7.52) together, and then by the Grönwall's inequality, we get

$$\|\delta\Theta\|_{C^{0,1}([T-\varepsilon,T])}+\|\delta\Theta^0\|_{C^{0,0,0,1,0}([T-\varepsilon,T])}$$
$$\le K\sqrt{\varepsilon}\big[\|\delta\theta\|_{C^{0,1}([T-\varepsilon,T])}+\|\delta\theta^0\|_{C^{0,0,0,1,1}([T-\varepsilon,T])}\big]. \tag{7.53}$$

Step 2. From the proof of Lemma 7.4, by some direct computations, it is easily seen that

$$\delta\Theta_y^0(t,s,\tilde{x},x,y)=\int_s^T\int_{\mathbb{R}^n}\Xi(s,x,r,\mu)\Big[\Theta_{xy}^{0,1}(t,r,\tilde{x},\mu,y)\delta\tilde{b}(r,\mu)$$
$$+\delta\Theta_{xy}^0(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta^2,\theta^{0,2})+\delta\tilde{g}_{p^0}^0(t,r,\tilde{x},\mu,y)\theta_{xy}^{0,1}(t,r,\tilde{x},\mu,y)$$
$$+\tilde{g}_{p^0}^0(t,r,\tilde{x},\mu,y;\theta^1,\theta^{0,1},\Theta^{0,1})\delta\theta_{xy}^0(t,r,\tilde{x},\mu,y)\Big]d\mu dr,$$

$$\delta\Theta_{xy}^0(t,s,\tilde{x},x,y)=\int_s^T\int_{\mathbb{R}^n}\Xi_x(s,x,r,\mu)\Big[\Theta_{xy}^{0,1}(t,r,\tilde{x},\mu,y)\delta\tilde{b}(r,\mu)$$
$$+\delta\Theta_{xy}^0(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta^2,\theta^{0,2})+\delta\tilde{g}_{p^0}^0(t,r,\tilde{x},\mu,y)\theta_{xy}^{0,1}(t,r,\tilde{x},\mu,y)$$
$$+\tilde{g}_{p^0}^0(t,r,\tilde{x},\mu,y;\theta^1,\theta^{0,1},\Theta^{0,1})\delta\theta_{xy}^0(t,r,\tilde{x},\mu,y)\Big]d\mu dr,$$

$$\delta\Theta_{yy}^0(t,s,\tilde{x},x,y)=\int_s^T\int_{\mathbb{R}^n}\Xi(s,x,r,\mu)\Big[\Theta_{xyy}^{0,1}(t,r,\tilde{x},\mu,y)\delta\tilde{b}(r,\mu)$$
$$+\delta\Theta_{xyy}^0(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta^2,\theta^{0,2})+\delta\tilde{g}_{p^0}^0(t,r,\tilde{x},\mu,y)\Theta_{xyy}^{0,1}(t,r,\tilde{x},\mu,y)$$
$$+\tilde{g}_{p^0}^0(t,r,\tilde{x},\mu,y;\theta^1,\theta^{0,1},\Theta^{0,1})\delta\theta_{xyy}^0(t,r,\tilde{x},\mu,y)$$
$$+\langle\delta\tilde{g}_{p^0p^0}^0(t,r,\tilde{x},\mu,y)\theta_{xy}^{0,1}(t,r,\tilde{x},\mu,y),\ \theta_{xy}^{0,1}(t,r,\tilde{x},\mu,y)\rangle$$
$$+\langle\tilde{g}_{p^0p^0}^0(t,r,\tilde{x},\mu,y;\theta^2,\theta^{0,2},\Theta^{0,2})\delta\theta_{xy}^0(t,r,\tilde{x},\mu,y),\ \theta_{xy}^{0,1}(t,r,\tilde{x},\mu,y)\rangle$$
$$+\langle\tilde{g}_{p^0p^0}^0(t,r,\tilde{x},\mu,y;\theta^2,\theta^{0,2},\Theta^{0,2})\theta_{xy}^{0,2}(t,r,\tilde{x},\mu,y),\ \delta\theta_{xy}^0(t,r,\tilde{x},\mu,y)\rangle\Big]d\mu dr,$$

and

$$\delta\Theta^0_{xyy}(t,s,\tilde{x},x,y)=\int_s^T\int_{\mathbb{R}^n}\Xi_x(s,x,r,\mu)\Big[\Theta^{0,1}_{xyy}(t,r,\tilde{x},\mu,y)\delta\tilde{b}(r,\mu)$$
$$+\delta\Theta^0_{xyy}(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu;\theta^2,\theta^{0,2})+\delta\tilde{g}^0_{p^0}(t,r,\tilde{x},\mu,y)\Theta^{0,1}_{xyy}(t,r,\tilde{x},\mu,y)$$
$$+\tilde{g}^0_{p^0}(t,r,\tilde{x},\mu,y;\theta^1,\theta^{0,1},\Theta^{0,1})\delta\theta^0_{xyy}(t,r,\tilde{x},\mu,y)$$
$$+\langle\delta\tilde{g}^0_{p^0p^0}(t,r,\tilde{x},\mu,y)\theta^{0,1}_{xy}(t,r,\tilde{x},\mu,y),\ \theta^{0,1}_{xy}(t,r,\tilde{x},\mu,y)\rangle$$
$$+\langle\tilde{g}^0_{p^0p^0}(t,r,\tilde{x},\mu,y;\theta^2,\theta^{0,2},\Theta^{0,2})\delta\theta^0_{xy}(t,r,\tilde{x},\mu,y),\ \theta^{0,1}_{xy}(t,r,\tilde{x},\mu,y)\rangle$$
$$+\langle\tilde{g}^0_{p^0p^0}(t,r,\tilde{x},\mu,y;\theta^2,\theta^{0,2},\Theta^{0,2})\theta^{0,2}_{xy}(t,r,\tilde{x},\mu,y),\ \delta\theta^0_{xy}(t,r,\tilde{x},\mu,y)\rangle\Big]d\mu dr.$$

Then by Lemma 7.4 and the estimate (7.3), we get

$$|\delta\Theta^0_y(t,s,\tilde{x},x,y)|\le K(T-s)\Big[\|\delta\theta\|_{L^\infty}+\|\delta\theta_x\|_{L^\infty}+\|\delta\theta^0\|_{L^\infty}+\|\delta\theta^0_x\|_{L^\infty}+\|\delta\theta^0_y\|_{L^\infty}\Big]$$
$$+K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n}{2}}}\Big[|\delta\Theta^0_x(t,r,\tilde{x},\mu,y)|+|\delta\theta^0_{xy}(t,r,\tilde{x},\mu,y)|\Big]d\mu dr,$$
$$|\delta\Theta^0_{xy}(t,s,\tilde{x},x,y)|\le K\sqrt{T-s}\Big[\|\delta\theta\|_{L^\infty}+\|\delta\theta_x\|_{L^\infty}+\|\delta\theta^0\|_{L^\infty}+\|\delta\theta^0_x\|_{L^\infty}+\|\delta\theta^0_y\|_{L^\infty}\Big]$$
$$+K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n+1}{2}}}\Big[|\delta\Theta^0_{xy}(t,r,\tilde{x},\mu,y)|+|\delta\theta^0_{xy}(t,r,\tilde{x},\mu,y)|\Big]d\mu dr,$$
$$|\delta\Theta^0_{yy}(t,s,\tilde{x},x,y)|\le K(T-s)\Big[\|\delta\theta\|_{L^\infty}+\|\delta\theta_x\|_{L^\infty}+\|\delta\theta^0\|_{L^\infty}+\|\delta\theta^0_x\|_{L^\infty}+\|\delta\theta^0_y\|_{L^\infty}\Big]$$
$$+K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n}{2}}}\Big[|\delta\Theta^0_{xyy}(t,r,\tilde{x},\mu,y)|+|\delta\theta^0_{xy}(t,r,\tilde{x},\mu,y)|$$
$$+|\delta\theta^0_{xyy}(t,r,\tilde{x},\mu,y)|\Big]d\mu dr,$$

and

$$|\delta\Theta^0_{xyy}(t,y,\tilde{x},s,x)|\le K\sqrt{T-s}\Big[\|\delta\theta\|_{L^\infty}+\|\delta\theta_x\|_{L^\infty}+\|\delta\theta^0\|_{L^\infty}+\|\delta\theta^0_x\|_{L^\infty}+\|\delta\theta^0_y\|_{L^\infty}\Big]$$
$$+K\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n+1}{2}}}\Big[|\delta\Theta^0_{xyy}(t,r,\tilde{x},\mu,y)|+|\delta\theta^0_{xy}(t,r,\tilde{x},\mu,y)|$$
$$+|\delta\theta^0_{xyy}(t,r,\tilde{x},\mu,y)|\Big]d\mu dr.$$

Combining the above with the estimate (7.53) together, by Grönwall's inequality again we get

$$\|\delta\Theta\|_{C^{0,1}([T-\varepsilon,T])}+\|\delta\Theta^0\|_{C^{0,0,1,2}([T-\varepsilon,T])}$$
$$\le\sqrt{\varepsilon}K\big[\|\delta\theta\|_{C^{0,1}([T-\varepsilon,T])}+\|\delta\theta^0\|_{C^{0,0,0,1,2}([T-\varepsilon,T])}\big].$$

For any $\tilde{x}_1,\tilde{x}_2\in\mathbb{R}^n$, denote

$$\delta\widehat{\Theta}^0(t,s,x,y) = \delta\Theta^0(t,s,\tilde{x}_1,x,y) - \delta\Theta^0(t,s,\tilde{x}_2,x,y).$$

Then from (7.50), we have

$$\begin{aligned}\delta\widehat{\Theta}^0_x(t,s,x,y) = &\int_s^T\int_{\mathbb{R}^n}\Xi_x(s,x,r,\mu)\Big\{[\Theta^{0,1}_x(t,r,\tilde{x}_1,\mu,y) - \Theta^{0,1}_x(t,r,\tilde{x}_2,\mu,y)]\delta\tilde{b}(r,\mu)\\ &+ \delta\widehat{\Theta}^0_x(t,r,\mu,y)\tilde{b}(r,\mu;\theta^2,\theta^{0,2}) + \delta\tilde{g}^0(t,r,\tilde{x}_1,\mu,y) - \delta\tilde{g}^0(t,r,\tilde{x}_2,\mu,y)\Big\}d\mu dr.\end{aligned}$$

Note that

$$\begin{aligned}&|\delta\tilde{g}^0(t,r,\tilde{x}_1,\mu,y) - \delta\tilde{g}^0(t,r,\tilde{x}_2,\mu,y)|\\ &\le K|\delta\widehat{\theta}^0_x(t,s,x,y)| + K|\delta\theta^0_x(t,s,\tilde{x}_1,x,y)|\big[1 + \|\theta^{0,1}_x(t,s,\cdot,x,y)\|_\alpha\\ &\quad + \|\theta^{0,2}_x(t,s,\cdot,x,y)\|_\alpha\big]|\tilde{x}_1 - \tilde{x}_2|^\alpha.\end{aligned}$$

Then by the fact that $\|\theta^{0,i}_x(t,s,\cdot,x,y)\|_\alpha$ and $\|\Theta^{0,i}_x(t,s,\cdot,x,y)\|_\alpha$ are bounded over $[T-\bar{\varepsilon},T]$ (see Lemma 7.4), we get

$$\begin{aligned}|\delta\widehat{\Theta}^0_x(t,s,x,y)| \le K&\int_s^T\int_{\mathbb{R}^n}\frac{e^{-\frac{\lambda|x-\mu|^2}{4(r-s)}}}{(r-s)^{\frac{n+1}{2}}}\Big\{|\delta\widehat{\Theta}^0_x(t,r,\mu,y)| + |\delta\widehat{\theta}^0_x(t,r,\mu,y)|\\ &+ \big[\|\delta\theta\|_{L^\infty} + \|\delta\theta_x\|_{L^\infty} + \|\delta\theta^0\|_{L^\infty} + \|\delta\theta^0_x\|_{L^\infty} + \|\delta\theta^0_y\|_{L^\infty}\big]|\tilde{x}_1 - \tilde{x}_2|^\alpha\Big\}d\mu dr.\end{aligned}$$

It follows that

$$\|\delta\Theta^0\|_{C^{0,0,\alpha,1,0}([T-\varepsilon,T])} \le \sqrt{\varepsilon}K\big[\|\delta\theta\|_{C^{0,1}([T-\varepsilon,T])} + \|\delta\theta^0\|_{C^{0,0,\alpha,1,1}([T-\varepsilon,T])}\big].$$

By continuing the above arguments, we have

$$\begin{aligned}&\|\delta\Theta\|_{C^{0,1}([T-\varepsilon,T])} + \|\delta\Theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1,2}([T-\varepsilon,T])}\\ &\le \sqrt{\varepsilon}K\big[\|\delta\theta\|_{C^{0,1}([T-\varepsilon,T])} + \|\delta\theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1,1}([T-\varepsilon,T])}\big].\end{aligned}$$

Step 3. Recalling (7.26)–(7.27), similar to (7.49)–(7.50), we have

$$\begin{aligned}\delta\Theta_x(s,x) = &\int_s^T\int_{\mathbb{R}^n}\tilde{\Gamma}_x(s,x,r,x-\sqrt{r-s}\mu)\Big[\delta\tilde{g}(r,x-\sqrt{r-s}\mu)\\ &+ \Theta^1_x(r,x-\sqrt{r-s}\mu)\delta\tilde{b}(r,x-\sqrt{r-s}\mu) + \delta\Theta_x(r,x-\sqrt{r-s}\mu)\\ &\times\tilde{b}(r,x-\sqrt{r-s}\mu;\theta^2,\theta^{0,2})\Big]d\mu dr + \int_s^T\int_{\mathbb{R}^n}\widehat{\Gamma}_x(s,x,r,\mu)\\ &\times\Big[\delta\tilde{g}(r,\mu) + \Theta^1_x(r,\mu)\delta\tilde{b}(r,\mu) + \delta\Theta_x(r,\mu)\tilde{b}(r,\mu)\Big]d\mu dr,\end{aligned}$$

$$\delta\Theta^0_x(t,s,\tilde{x},x,y) = \int_s^T\int_{\mathbb{R}^n}\tilde{\Gamma}_x(s,x,r,x-\sqrt{r-s}\mu)\Big[\delta\tilde{g}^0(t,r,\tilde{x},x-\sqrt{r-s}\mu,y)$$

$$
\begin{aligned}
&+ \Theta_x^{0,1}(t,r,\tilde{x},x-\sqrt{r-s}\mu,y)\delta\tilde{b}(r,x-\sqrt{r-s}\mu)\\
&+ \delta\Theta_x^{0}(t,r,\tilde{x},x-\sqrt{r-s}\mu,y)\tilde{b}(r,x-\sqrt{r-s}\mu;\theta^2,\theta^{0,2})\Big]d\mu dr\\
&+ \int_s^T\int_{\mathbb{R}^n}\widehat{\Gamma}_x(s,x,r,\mu)\Big[\delta\tilde{g}^0(t,r,\tilde{x},\mu,y)+\Theta_x^{0,1}(t,r,\tilde{x},\mu,y)\delta\tilde{b}(r,\mu)\\
&+ \delta\Theta_x^{0}(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu)\Big]d\mu dr,
\end{aligned}
$$

where $\tilde{\Gamma}$ is defined by (7.28). For any $x_1, x_2 \in \mathbb{R}^n$, we have

$$
\begin{aligned}
&\delta\Theta_x^0(t,s,\tilde{x},x_1,y)-\delta\Theta_x^0(t,s,\tilde{x},x_2,y)\\
&= \int_s^T\int_{\mathbb{R}^n}\big[\tilde{\Gamma}_x(s,x_1,r,x_1-\sqrt{r-s}\mu)-\tilde{\Gamma}_x(s,x_2,r,x_2-\sqrt{r-s}\mu)\big]\\
&\times\Big[\delta\tilde{g}^0(t,r,\tilde{x},x_1-\sqrt{r-s}\mu,y)+\Theta_x^{0,1}(t,r,\tilde{x},x_1-\sqrt{r-s}\mu,y)\delta\tilde{b}(r,x_1-\sqrt{r-s}\mu)\\
&+\delta\Theta_x^0(t,r,\tilde{x},x_1-\sqrt{r-s}\mu,y)\tilde{b}(r,x_1-\sqrt{r-s}\mu;\theta^2,\theta^{0,2})\Big]d\mu dr\\
&+\int_s^T\int_{\mathbb{R}^n}\tilde{\Gamma}_x(s,x_2,r,x_2-\sqrt{r-s}\mu)\Big\{\big[\delta\tilde{g}^0(t,r,\tilde{x},x_1-\sqrt{r-s}\mu,y)\\
&-\delta\tilde{g}^0(t,r,\tilde{x},x_2-\sqrt{r-s}\mu,y)\big]+\big[\Theta_x^{0,1}(t,r,\tilde{x},x_1-\sqrt{r-s}\mu,y)\\
&\times\delta\tilde{b}(r,x_1-\sqrt{r-s}\mu)-\Theta_x^{0,1}(t,r,\tilde{x},x_2-\sqrt{r-s}\mu,y)\delta\tilde{b}(r,x_2-\sqrt{r-s}\mu)\big]\\
&+\big[\delta\Theta_x^0(t,r,\tilde{x},x_1-\sqrt{r-s}\mu,y)\tilde{b}(r,x_1-\sqrt{r-s}\mu;\theta^2,\theta^{0,2})\\
&-\delta\Theta_x^0(t,r,\tilde{x},x_2-\sqrt{r-s}\mu,y)\tilde{b}(r,x_2-\sqrt{r-s}\mu;\theta^2,\theta^{0,2})\big]\Big\}d\mu dr\\
&+\int_s^T\int_{\mathbb{R}^n}[\widehat{\Gamma}_x(s,x_1,r,\mu)-\widehat{\Gamma}_x(s,x_2,r,\mu)]\Big[\delta\tilde{g}^0(t,r,\tilde{x},\mu,y)\\
&+\Theta_x^{0,1}(t,r,\tilde{x},\mu,y)\delta\tilde{b}(r,\mu)+\delta\Theta_x^0(t,r,\tilde{x},\mu,y)\tilde{b}(r,\mu)\Big]d\mu dr.
\end{aligned}
\tag{7.54}
$$

Note that on $[T-\varepsilon,T]$, by Lemmas 7.4 and 7.5 we have

$$
\begin{aligned}
&\big|[\theta_x^{0,1}(r,x_1,r,x_1,\theta^1(r,x_1))-\theta_x^{0,2}(r,x_1,r,x_1,\theta^2(r,x_1))]\\
&-[\theta_x^{0,1}(r,x_2,r,x_2,\theta^1(r,x_2))-\theta_x^{0,2}(r,x_2,r,x_2,\theta^2(r,x_2))]\big|\\
&\le\Big\{\|\delta\theta_{xy}^0\|_{L^\infty}\|\theta^1\|_{C^{0,\alpha}}+\|\delta\theta_x^0\|_{C^{0,0,\alpha,\alpha,0}}+\|\theta_{xy}^{0,2}\|_{L^\infty}\|\delta\theta\|_{C^{0,\alpha}}\\
&\quad+\big[\|\theta_{xyy}^{0,2}\|_{L^\infty}(\|\theta^1\|_{C^{0,\alpha}}+\|\theta^2\|_{C^{0,\alpha}})+\|\theta_{xy}^{0,2}\|_{C^{0,0,\alpha,\alpha,0}}\big]\|\delta\theta\|_{L^\infty}\Big\}|x_1-x_2|^\alpha\\
&\le K\big[\|\delta\theta_x^0\|_{C^{0,0,\alpha,\alpha,1}}+\|\delta\theta\|_{C^{0,\alpha}}\big]|x_1-x_2|^\alpha.
\end{aligned}
\tag{7.55}
$$

By the same arguments as the above, we have

$$
\begin{aligned}
&\big|[\theta^{0,1}(r,x_1,r,x_1,\theta^1(r,x_1))-\theta^{0,2}(r,x_1,r,x_1,\theta^2(r,x_1))]\\
&-[\theta^{0,1}(r,x_2,r,x_2,\theta^1(r,x_2))-\theta^{0,2}(r,x_2,r,x_2,\theta^2(r,x_2))]\big|
\end{aligned}
$$

$$\leq K\big[\|\delta\theta^0\|_{C^{0,0,\alpha,\alpha,1}}+\|\delta\theta\|_{C^{0,\alpha}}\big]|x_1-x_2|^\alpha, \tag{7.56}$$

and

$$\begin{aligned}&\big|[\theta_y^{0,1}(r,x_1,r,x_1,\theta^1(r,x_1))-\theta_y^{0,2}(r,x_1,r,x_1,\theta^2(r,x_1))]\\&-[\theta_y^{0,1}(r,x_2,r,x_2,\theta^1(r,x_2))-\theta_y^{0,2}(r,x_2,r,x_2,\theta^2(r,x_2))]\big|\\&\leq K\big[\|\delta\theta_y^0\|_{C^{0,0,\alpha,\alpha,1}}+\|\delta\theta\|_{C^{0,\alpha}}\big]|x_1-x_2|^\alpha.\end{aligned} \tag{7.57}$$

For any $\varphi\in C^{1,2}$, denote

$$\begin{aligned}\delta\varphi(x_1)-\delta\varphi(x_2):=&\big[\varphi(x_1,\theta_x^{0,1}(r,x_1,r,x_1,\theta^1(r,x_1)))-\varphi(x_1,\theta_x^{0,2}(r,x_1,r,x_1,\theta^2(r,x_1)))\big]\\&-\big[\varphi(x_2,\theta_x^{0,1}(r,x_2,r,x_2,\theta^1(r,x_2)))-\varphi(x_2,\theta_x^{0,2}(r,x_2,r,x_2,\theta^2(r,x_2)))\big].\end{aligned}$$

From (7.55), we have the following estimate:

$$|\delta\varphi(x_1)-\delta\varphi(x_2)|\leq K\big[\|\delta\theta_x^0\|_{C^{0,0,\alpha,\alpha,1}}+\|\delta\theta\|_{C^{0,\alpha}}\big]|x_1-x_2|^\alpha,$$

on $[T-\varepsilon,T]$, where K depends on $\|\varphi\|_{C^{0,1}}$. Thus, by the estimates (7.55)–(7.57), we get

$$\begin{aligned}&|\delta\tilde g^0(t,r,\tilde x,x_1,y)-\delta\tilde g^0(t,r,\tilde x,x_2,y)|\\&\leq K\big[\|\delta\theta^0\|_{C^{0,0,\alpha,1+\alpha,2}}+\|\delta\theta\|_{C^{0,1+\alpha}}+\|\delta\theta_x^0(t,r,\tilde x,\cdot,y)\|_\alpha\big]|x_1-x_2|^\alpha.\end{aligned} \tag{7.58}$$

Similarly,

$$|\delta\tilde b(r,x_1)-\delta\tilde b(r,x_2)|\leq K\big[\|\delta\theta^0\|_{C^{0,0,\alpha,1+\alpha,2}}+\|\delta\theta\|_{C^{0,1+\alpha}}\big]|x_1-x_2|^\alpha. \tag{7.59}$$

By (7.29), (7.31), Lemmas 7.3 and 7.5, from (7.54) we obtain

$$\begin{aligned}\|\delta\Theta_x^0(t,s,\tilde x,\cdot,y)\|_\alpha\leq&\int_s^T\int_{\mathbb{R}^n}\frac{K}{\sqrt{r-s}}e^{-\lambda|\mu|^2}\Big[|\delta\tilde b(r,x-\sqrt{r-s}\mu)|+|\delta\tilde g^0(t,r,\tilde x,x-\sqrt{r-s}\mu,y)|\\&+|\delta\Theta_x^0(t,r,\tilde x,x-\sqrt{r-s}\mu,y)|\Big]d\mu dr+\int_s^T\int_{\mathbb{R}^n}\frac{K}{\sqrt{r-s}}e^{-\lambda|\mu|^2}\\&\times\Big[|\delta\tilde b(r,x_1-\sqrt{r-s}\mu)|+\|\delta\tilde b(r,\cdot)\|_\alpha+|\delta\Theta_x^0(t,r,\tilde x,x_1-\sqrt{r-s}\mu,y)|\\&+\|\delta\Theta_x^0(t,r,\tilde x,\cdot,y)\|_\alpha+\|\delta\tilde g^0(t,r,\tilde x,\cdot,y)\|_\alpha\Big]d\mu dr\\&+\int_s^T\int_{\mathbb{R}^n}\Big[\frac{e^{\frac{-\lambda|x_1-\mu|^2}{r-s}}}{(r-s)^{\frac{n+\epsilon}{2}}}+\frac{e^{\frac{-\lambda|x_2-\mu|^2}{r-s}}}{(r-s)^{\frac{n+\epsilon}{2}}}\Big]\Big[\delta\tilde g^0(t,r,\tilde x,\mu,y)\\&+\Theta_x^{0,1}(t,r,\tilde x,\mu,y)\delta\tilde b(r,\mu)+\delta\Theta_x^0(t,r,\tilde x,\mu,y)\tilde b(r,\mu)\Big]d\mu dr.\end{aligned}$$

Substituting the estimates (7.45), (7.47), (7.58) and (7.59) into the above, and then by Grönwall's inequality, we get

$$\|\delta\Theta_x^0(t,s,\tilde x,\cdot,y)\|_\alpha\leq K\sqrt{\varepsilon}\big[\|\delta\theta^0\|_{C^{0,0,\alpha,1+\alpha,2}}+\|\delta\theta\|_{C^{0,1+\alpha}}\big].$$

By the same arguments as the above, we have

$$\|\delta\Theta_x(s,\cdot)\|_\alpha \le K\sqrt{\varepsilon}\big[\|\delta\theta^0\|_{C^{0,0,\alpha,1+\alpha,2}} + \|\delta\theta\|_{C^{0,1+\alpha}}\big].$$

By continuing the above arguments, we get

$$\|\delta\Theta_x\|_{C^{0,\alpha}} + \|\delta\Theta_x^0\|_{C^{\frac{\alpha}{2},0,\alpha,\alpha,2}} \le K\sqrt{\varepsilon}\big[\|\delta\theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}} + \|\delta\theta\|_{C^{0,1+\alpha}}\big].$$

Step 4. Combining the estimates in Steps 1–3 together, we get

$$\|\delta\Theta\|_{C^{0,1+\alpha}} + \|\delta\Theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}} \le K\sqrt{\varepsilon}\big[\|\delta\theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}} + \|\delta\theta\|_{C^{0,1+\alpha}}\big].$$

Then by choosing a $0 < \widehat{\varepsilon} \le \bar{\varepsilon}$ small enough, we get that on $[T-\widehat{\varepsilon}, T]$,

$$\|\delta\Theta\|_{C^{0,1+\alpha}} + \|\delta\Theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}} \le \frac{1}{2}\big[\|\delta\theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}} + \|\delta\theta\|_{C^{0,1+\alpha}}\big].$$

Thus, (7.41) holds and this completes the proof. □

Complete the proof of Theorem 2.10. We have proved that equilibrium HJB equation (2.16) admits a unique classical solution (Θ, Θ^0) on $[T-\widehat{\varepsilon}, T]$, where $\widehat{\varepsilon}$ is given by Proposition 7.6. By a routine argument, we can prove that equilibrium HJB equation (2.16) admits a unique classical solution (Θ, Θ^0) on $[T-\bar{\varepsilon}, T]$, where $\bar{\varepsilon}$ is given by Lemma 7.5. Thus, to extend the solution to the whole time interval $[0, T]$, it suffices to prove a global prior estimate for $\|\Theta\|_{C^{0,1+\alpha}}$ and $\|\Theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1+\alpha,2}}$.

By (7.16) and (7.17), we have

$$|\Theta_y^0(t,s,\tilde{x},x,y)| \le K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}} + K\int_s^T \|\Theta_{xy}^0(t,r,\tilde{x},\cdot,y)\|_{C^0}\,d\mu dr,$$

$$|\Theta_{xy}^0(t,s,\tilde{x},x,y)| \le K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}} + \int_s^T \frac{K}{\sqrt{r-s}}\|\Theta_{xy}^0(t,r,\tilde{x},\cdot,y)\|_{C^0}\,d\mu dr,$$

which implies that

$$\|\Theta_y^0\|_{L^\infty} + \|\Theta_{xy}^0\|_{L^\infty} \le K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}. \tag{7.60}$$

Next, from (7.20) and (7.21), we have

$$\begin{aligned}|\Theta_{yy}^0(t,s,\tilde{x},x,y)| \le K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}} + K\int_s^T \Big[\|\Theta_{xyy}^0(t,r,\tilde{x},\cdot,y)\|_{C^0}\\ + \|\Theta_{xy}^0(t,r,\tilde{x},\cdot,y)\|_{C^0}^2\Big]d\mu dr,\end{aligned}$$

$$\begin{aligned}|\Theta_{xyy}^0(t,s,\tilde{x},x,y)| \le K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}} + \int_s^T \frac{K}{\sqrt{r-s}}\Big[\|\Theta_{xyy}^0(t,r,\tilde{x},\cdot,y)\|_{C^0}\\ + \|\Theta_{xy}^0(t,r,\tilde{x},\cdot,y)\|_{C^0}^2\Big]d\mu dr,\end{aligned}$$

which, together with the global prior estimate (7.60), yields that

$$\|\Theta^0_{yy}\|_{L^\infty} + \|\Theta^0_{xyy}\|_{L^\infty} \le K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}.$$

By (7.22)–(7.25), we get

$$\|\Theta^0(t,s,\cdot,x,y)\|_\alpha \le K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}} + K\int_s^T \Big[\|\Theta^0_x(t,r,\cdot,\cdot,y)\|_{C^{\alpha,0}} + 1\Big]dr,$$

$$\begin{aligned}\|\Theta^0_y(t,s,\cdot,x,y)\|_\alpha \le{}& K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}} + K\int_s^T \Big[\|\Theta^0_{xy}(t,r,\cdot,\cdot,y)\|_{C^{\alpha,0}} \\ &+ \|\Theta^0_{xy}(t,r,\cdot,\cdot,y)\|_{C^{0,0}}(1+\|\Theta^0_x(t,r,\cdot,\cdot,y)\|_{C^{\alpha,0}})\Big]dr,\end{aligned}$$

and

$$\|\Theta^0_x(t,s,\cdot,x,y)\|_\alpha \le K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}} + \int_s^T \frac{K}{\sqrt{r-s}}\Big[\|\Theta^0_x(t,r,\cdot,\cdot,y)\|_{C^{\alpha,0}} + 1\Big]dr,$$

$$\begin{aligned}\|\Theta^0_{xy}(t,s,\cdot,x,y)\|_\alpha \le{}& K\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}} + \int_s^T \frac{K}{\sqrt{r-s}}\Big[\|\Theta^0_{xy}(t,r,\cdot,\cdot,y)\|_{C^{\alpha,0}} \\ &+ \|\Theta^0_{xy}(t,r,\cdot,\cdot,y)\|_{C^{0,0}}(1+\|\Theta^0_x(t,r,\cdot,\cdot,y)\|_{C^{\alpha,0}})\Big]dr.\end{aligned}$$

With the global estimate (7.60), the above implies that

$$\begin{aligned}&\|\Theta^0(t,s,\cdot,x,y)\|_{C^\alpha} + \|\Theta^0_y(t,s,\cdot,x,y)\|_{C^\alpha} + \|\Theta^0_x(t,s,\cdot,x,y)\|_{C^\alpha} \\ &+ \|\Theta^0_{xy}(t,s,\cdot,x,y)\|_{C^\alpha} \le K(1+\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}).\end{aligned}$$

By continuing the above, we have

$$\|\Theta^0\|_{C^{\frac{\alpha}{2},0,\alpha,1,2}} \le K(1+\|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}). \tag{7.61}$$

By the arguments employed in the proof of Lemma 7.5 (see (7.36)–(7.37)), with the global estimate (7.61), we get

$$\begin{aligned}&\|\Theta^0_x(t,s,\tilde{x},\cdot,y)\|_\alpha + \|\Theta_x(s,\cdot)\|_\alpha \\ &\le K\big[1+\|h\|_{C^{2+\alpha}} + \|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}\big] + \int_s^T \frac{K}{\sqrt{r-s}}\big[\|\Theta^0_x(t,r,\tilde{x},\cdot,y)\|_\alpha + \|\Theta_x(r,\cdot)\|_\alpha\big]dr.\end{aligned}$$

Thus,

$$\|\Theta_x\|_{C^{0,0,0,\alpha,0}} + \|\Theta_x\|_{C^{0,\alpha}} \le K\big[1+\|h\|_{C^{2+\alpha}} + \|h^0\|_{C^{\frac{\alpha}{2},\alpha,2+\alpha,2}}\big].$$

This completes the proof.

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