Mathematical Control and Related Fields
Vol. 14, No. 4, December 2024, pp. 1240-1274
doi:10.3934/mcrf.2024002

A STOCHASTIC HYBRID OPTIMAL CONTROL WITH REGIME SWITCHING AND RECURSIVE COST FUNCTIONAL

Yue Zhou[1] and Jiongmin Yong[*2]

[1]School of Mathematics and Statistics,
Central South University, Changsha, Hunan, China

[2]Department of Mathematics,
University of Central Florida, Orlando, FL 32816, USA

This is dedicated to Professor Hélène Frankowska on the occasion of her 70th birthday

Abstract. An optimal control problem for a regime switching stochastic differential equation under continuous-time, switching and impulse controls with recursive cost functional is considered. Using the approach of dynamic programming, the corresponding Hamilton-Jacoobi-Bellman quasi-variational inequality is derived, to which the value function is proved to be the unique viscosity solution. Due to the appearance of all those kinds of controls, as well as the regime switching (governed by a Markov chain) and the recursive cost functional (determined by a backward stochastic Volterra integral equation), quite a few particular technical difficulties have to be overcome under some delicate conditions to prove the continuity of the value function.

1. Introduction.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a d-dimensional standard Brownian motion $\{W(s)\,|\,0 \le s \le T\}$ and a finite state Markov chain $M(\cdot)$ are defined. We assume that $W(\cdot)$ and $M(\cdot)$ are independent and the state space of $M(\cdot)$ is $\mathbf{M} = \{1, 2, 3, \cdots, |\mathbf{M}|\}$, where $|\mathbf{M}|$ is the number of elements in $\mathbf{M}$. Also, $M(\cdot)$ is right-continuous with left-limits. It is known that $M(\cdot)$ can be regarded as the solution of the following integral equation:

$$dM(s) = \int_{\mathbb{R}} \mu(M(s-), \theta) N(d\theta, ds), \quad s \ge t \ge 0, \qquad M(t-0) = m, \tag{1.1}$$

for some Poisson random measure $N(d\theta, ds)$, whose intensity measure is assumed to be $\mathbb{E}[N(d\theta, ds)] = \pi(d\theta)ds$ for some Radon measure $\pi(d\theta)$, and a map $\mu : \mathbf{M} \times \mathbb{R} \to$

2020 *Mathematics Subject Classification.* Primary: 49N25, 60J25, 93C27, 93E20; Secondary: 49L05, 49L20, 49L25.

Key words and phrases. Continuous-time control, switching and impulse controls, regime switching, recursive cost functional, dynamic programming principle, Hamilton-Jacobi-Bellman quasi-variational inequality, viscosity solution.

The second author was partially supported by NSF DMS-2305475.

*Corresponding author: Jiongmin Yong.

$\mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, of the following form:

$$\mu(m,\theta)=\sum_{m'=1}^{|\mathbf{M}|}(m'-m)\mathbf{1}_{\Delta_{mm'}}(\theta),$$

with $\{\Delta_{mm'}|1\le m'\le|\mathbf{M}|\}$ being a partition of $\mathbb{R}$, for each $m=1,2,\cdots,|\mathbf{M}|$. See the book by Yin–Zhu [16] (p.29) for relevant details (see also, [6]).

Next, we assume that

$$\mathbb{F}=\{\mathcal{F}_s\}_{s\ge0}\equiv\mathbb{F}^W\vee\mathbb{F}^N,\qquad \mathcal{F}_s=\mathcal{F}_s^W\vee\mathcal{F}_s^N,$$

with

$$\begin{aligned}&\mathcal{F}_s^W=\sigma\Big(W(\tau),\ 0\le\tau\le s\Big)\vee\mathcal{N}_0,\qquad \mathbb{F}^W=\{\mathcal{F}_s^W\}_{s\ge0}\\&\mathcal{F}_s^N=\sigma\Big(N(B,\tau);B\in\mathcal{B}(\mathbb{R}),0\le\tau\le s\Big)\vee\mathcal{N}_0,\qquad \mathbb{F}^N=\{\mathcal{F}_s^N\}_{s\ge0},\end{aligned}$$

where $\mathcal{N}_0$ is the set of all subsets of $\mathbb{P}$-null sets. For convenience, we let $\mathcal{F}=\mathcal{F}_T$ (if necessary, one may shrink $\mathcal{F}$ to achieve this). It is standard that $s\mapsto\mathcal{F}_s$ is right-continuous with left-limit.

Let the transition probability of $M(\cdot)$ be given by the following (with $0\le t<s\le T$)

$$\mathbb{P}\Big(M(s)=m'\;\Big|\;M(t)=m\Big)=\begin{cases}q_{mm'}(s-t)+o(s-t), & m'\ne m,\\[2mm] 1+q_{mm}(s-t)+o(s-t), & m'=m.\end{cases}\tag{1.2}$$

with

$$q_{mm'}\ge0,\quad m\ne m',\qquad \sum_{m'=1}^{|\mathbf{M}|}q_{mm'}=0,\quad \forall m\in\mathbf{M}.\tag{1.3}$$

We denote $Q=\big(q_{mm'}\big)_{m,m'\in\mathbf{M}}$ which is called the *generator* of $M(\cdot)$.

Consider the following controlled stochastic differential equation (SDE, for short):

$$\begin{aligned}X(s)=x&+\int_t^s b\big(r,X(r),M(r),u(r),a(r)\big)dr\\&+\int_t^s\sigma\big(r,X(r),M(r),u(r),a(r)\big)dW(r)+\xi(s),\quad s\in[t,T]\end{aligned}\tag{1.4}$$

where $b:[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{U}\times\mathbf{A}\to\mathbb{R}^n$ and $\sigma:[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{U}\times\mathbf{A}\to\mathbb{R}^{n\times d}$ are some given maps, with $\mathbf{U}$ being a metric spaces and $\mathbf{A}=\{a_1,\cdots,a_{|\mathbf{A}|}\}$ (similar as before, $|\mathbf{A}|\in\mathbb{N}$ is the number of elements in $\mathbf{A}$, which is finite). In the above, $X(\cdot)$ is the *state process*, $M(\cdot)$ is the *regime switching process*, $u(\cdot)$ is a continuous-time control valued in $\mathbf{U}$. Process $a(\cdot)$ is a *switching control*, determined by a sequence $\{\theta_i,a_i\}_{i\ge0}$, with $t=\theta_0\le\theta_1\le\cdots$ being $\mathbb{F}$-stopping times, a_i being an $\mathcal{F}_{\theta_i}$-measurable $\mathbf{A}$-valued random variable, and the initial value is denoted by $a_0\equiv a$. For convenience, we write

$$a(\cdot)=\sum_{i\ge1}a_{i-1}\mathbf{1}_{[\theta_{i-1},\theta_i)}(\cdot).\tag{1.5}$$

With such a process appearing in the state equation (1.4), the system has the generator (b,σ) piecewise determined. This is exactly the same result as that provided by the switching control. Note that for different $\omega\in\Omega$ (almost surely), at the moment $\theta_i(\omega)$, the value $a_i(\omega) \in \mathbf{A}$ could be different. Process $\xi(\cdot)$ is an *impulse control*, of the following form:

$$\xi(\cdot) = \sum_{j \geq 1} \xi_j \mathbf{1}_{[\tau_j, T]}(\cdot), \tag{1.6}$$

determined by a sequence $\{(\tau_j, \xi_j)\}_{j \geq 1}$, with $t \leq \tau_1 \leq \tau_2 \leq \cdots$ being $\mathbb{F}$-stopping times, and ξ_j being an $\mathcal{F}_{\tau_j}$-measurable, K-valued random variable such that $\mathbb{E}|\xi_j|^p < \infty$, $p \geq 1$, where $K \subseteq \mathbb{R}^n$ is a convex and closed cone. We denote by $\mathcal{U}[t,T]$, $\mathcal{A}^a[t,T]$, and $\mathcal{K}^p[t,T]$ the sets of *feasible* (continuous-time) controls, switching controls (with the initial setting a), and impulse controls, respectively (see the next section for further details on the feasibility). Finally, we let

$$\mathcal{D}^p = \Big\{(t,x) \bigm| t \in [0,T],\ x \text{ is } \mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-measurable, } \mathbb{E}|x|^p < \infty\Big\}, \tag{1.7}$$

which is called the set of *initial pairs*.

Under some mild conditions, for any *initial tuple* $(t,x,m,a) \in \mathcal{D}^p \times \mathbf{M} \times \mathbf{A}$ and feasible controls $(u(\cdot), a(\cdot), \xi(\cdot)) \in \mathcal{U}[t,T] \times \mathcal{A}^a[t,T] \times \mathcal{K}^p[t,T]$, the state equation (1.4) admits a unique solution $X(\cdot) \equiv X(\cdot\,; t,x,m,a; u(\cdot), a(\cdot), \xi(\cdot))$ which is $\mathbb{F}$-adapted. Note that we will treat (t,x) and (m,a) differently: the former will be the independent variables and the latter will be the indices. Markov chain $M(\cdot)$ gives a regime switching to the system, which is mandatory (or passive to the controller). The three types controls (continuous-time, switching and impulse) can be applied by the controller actively.

Next, we consider the cost functional to be used to measure the performance of the controls. Inspired by the *stochastic differential utility* introduced by Duffie–Epstein [1, 2] (see also [3, 14]), we introduce the following *backward stochastic Volterra integral equation* (BSVIE, for short):

$$\begin{aligned} Y(s) = {} & h(X(T), M(T), a(T)) + \sum_{\theta_i \geq s} k(\theta_i, a_{i-1}, a_i) + \sum_{\tau_j \geq s} \ell(\tau_j, \xi_j) \\ & + \int_s^T g\Big(r, X(r), M(r), u(r), a(r), Y(r), Z(s,r), \int_{\mathbb{R}} \Gamma(s,r,\theta)\pi(d\theta)\Big) dr \\ & - \int_s^T Z(s,r)dW(r) - \int_s^T \int_{\mathbb{R}} \Gamma(s,r-,\theta)\widetilde{N}(d\theta,dr), \qquad s \in [t,T], \end{aligned} \tag{1.8}$$

where

$$\widetilde{N}(d\theta, ds) = N(d\theta, ds) - \pi(d\theta)ds,$$

is the *compensated random measure* of the Poisson random measure $N(d\theta, ds)$ introduced earlier. In the above, $h : \mathbb{R}^n \times \mathbf{M} \times \mathbf{A} \to [0,\infty)$, $g : [0,T] \times \mathbb{R}^n \times \mathbf{M} \times \mathbf{U} \times \mathbf{A} \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R} \to [0,\infty)$ are deterministic functions, called the *terminal cost* and the *running cost rate*, respectively; $k : [0,T] \times \mathbf{A} \times \mathbf{A} \to (0,\infty)$ and $\ell : [0,T] \times K \to (0,\infty)$ are called the *switching cost* and the *impulse cost*, respectively. We call (1.8) a BSVIE (instead of a BSDE) since the "terminal state" is

$$h(X(T), M(T), a(T)) + \sum_{\theta_i \geq s} k(\theta_i, a_{i-1}, a_i) + \sum_{\tau_j \geq s} \ell(\tau_j, \xi_j),$$

which is an $\mathcal{F}_T$-measurable process (depending on $s \in [0,T]$, not necessarily $\mathbb{F}$-adapted). Because of this, we refer to it as the "free term", instead of "terminal state". According to the general theory of BSVIEs, the above (1.8) is called a Type-I BSVIE (see [21, 22, 10, 23]). Under proper conditions, the above BSVIE admits a unique adapted solution $(Y(\cdot), Z(\cdot\,,\cdot), \Gamma(\cdot\,,\cdot\,,\cdot))$. We point out that the inclusion of the martingale term $\int_t^T \Gamma(s,r-,\theta)\widetilde N(d\theta,dr)$ in the BSVIE (1.8) affecting the cost functional (1.9) is not just for the generality of the presentation, it is needed due to the martingale representation theorem when we solve the corresponding BSVIE/BSDE to obtain the adapted solution. Also, we note that $Y(\cdot)$ is merely right-continuous with left-limit in general. Now, let us define the cost functional to be the following

$$\begin{aligned} &J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot)) = Y(t)\\ &\equiv \mathbb{E}_t\Big[h(X(T),M(T),a(T)) + \sum_{\theta_i\ge t} k(\theta_i,a_{i-1},a_i) + \sum_{\tau_j\ge t}\ell(\tau_j,\xi_j)\\ &\qquad + \int_t^T g\Big(r,X(r),M(r),u(r),a(r),Y(r),Z(t,r),\int_{\mathbb{R}}\Gamma(t,r,\theta)\pi(d\theta)\Big)dr\Big], \end{aligned} \tag{1.9}$$

where $\mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ is the conditional expectation operator. Roughly speaking, the value $Y(s)$ of $Y(\cdot)$ can be regarded as the *disutility* at s involving the terminal and running costs as well as the switching and impulse costs. By letting $Y(\cdot)$ be a part of the adapted solution to the BSVIE (1.8), we see that the current disutililty $Y(s)$ depends on the future ones, which is exactly the reason that we refer to (1.9) as a *recursive cost functional*. We also note that at time $s\in[t,T]$, only those switchings with $\theta_i \ge s$ and those impulses with $\tau_j \ge s$ are counted in the cost functional. This really matters when we derive the Bellman' dynamic programming principle. Due to this, we feel that the framework presented in [26] seems to be questionable.

Comparing (1.8) and (1.9), we see that $Z(\cdot\,,\cdot)$ and $\Gamma(\cdot\,,\cdot\,,\cdot)$ may not appear explicitly in (1.9) if

$$g(t,x,m,u,a,y,z,\gamma) = g(t,x,m,u,a,y) \qquad \text{(independent of } (z,\gamma)).$$

However, we know that for any proper functions $\widehat Z(\cdot)$ and $\widehat\Gamma(\cdot\,,\cdot)$, it holds

$$\mathbb{E}_t\Big[\int_t^T \widehat Z(t,r)dW(r) + \int_t^T\int_{\mathbb{R}}\widehat\Gamma(t,r-,\theta)\widetilde N(d\theta,dr)\Big] = 0. \tag{1.10}$$

This implies that the operator $\mathbb{E}_t$ is not injective. Thus, the determination from the adapted solution $(Y(\cdot), Z(\cdot\,,\cdot), \Gamma(\cdot\,,\cdot\,,\cdot))$ to $Y(t)$ is well-defined; whereas, for a given process of conditional expectation form, the corresponding $Z(\cdot\,,\cdot)$ and $\Gamma(\cdot\,,\cdot\,,\cdot)$ seem to be non-unique due to (1.10). To avoid this ambiguity, we specify these two processes through BSVIE (1.8), which ensures the uniqueness.

We now formulate our optimal control problem.

Problem (C). For given initial tuple $(t,x,m,a)\in\mathcal{D}^p\times\mathbf{M}\times\mathbf{A}$, find a feasible control triple $(\bar u(\cdot),\bar a(\cdot),\bar\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}^a[t,T]\times\mathcal{K}^p[t,T]$ such that

$$J^{m,a}(t,x;\bar u(\cdot),\bar a(\cdot),\bar\xi(\cdot)) = \inf_{u(\cdot),a(\cdot),\xi(\cdot)} J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot)) = V^{m,a}(t,x). \tag{1.11}$$

In the case that $g(t,x,m,u,a,y,z,\gamma)=g(t,x,m,u,a)$ (independent of (y,z,γ)) and $h(x,m,a)=h(x)$ (independent of (m,a)), (1.9) becomes

$$\begin{aligned}
&J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot))\\
&=\mathbb{E}_t\Big[h(X(T))+\int_t^T g(r,X(r),M(r),u(r),a(r))dr\\
&\quad+\sum_{i\ge1}k(\theta_i,a_{i-1},a_i)+\sum_{j\ge1}\ell(\tau_j,\xi_j)\Big].
\end{aligned}\tag{1.12}$$

The above is the usual Bolza type cost functional for optimal switching and impulse controls with regime switching; For such a case, the first and the second terms on the right-hand side of (1.12) are the usual terminal and running costs, respectively. Hence, our framework is an extension of a classical one.

The purpose of the current paper is to establish a general theory for the above Problem (C). The main feature are highlighted as follows:

• The state equation contains two types of switchings: The passive one (or mandatory one) determined by the Markov chain $M(\cdot)$ (the regime switching), and the active one determined by the switching control $a(\cdot)$. As a result, the value function is matrix valued with independent variables $(t,x)\in\mathcal{D}^p$, indexed by $(m,a)\in\mathbf{M}\times\mathbf{A}$, the initial values of the Markov chain and the switching control.

• The problem involves regime switching governed by a Markov chain, and the cost functional is recursive, determined by a special type BSVIE driven by Brownian motion $W(\cdot)$ and Poisson process $N(\cdot)$, with discontinuous free term. Thus, the cost functional is discontinuous in t, so that the continuity of the value function becomes non-trivial. Some very careful estimation involving both SDEs with impulses and BSDEs/BSVIEs is needed to achieve the goal. The compatibility of certain growth assumptions, as well as monotonicity play crucial roles.

• It turns out that the Hamiltonian in the Hamilton-Jacobi-Bellman (HJB, for short) quasi-variational inequality (QVI, for short) explicitly depends on the value function $V^{m,a}(\cdot\,,\cdot)$, in a coupled fashion, besides the decoupled appearance of its gradient $V_x^{m,a}(\cdot\,,\cdot)$ and its Hessian $V_{xx}^{m,a}(\cdot\,,\cdot)$, which leads to some technical difficulty to be overcome in the viscosity solution characterization of the value function.

The rest of the paper is organized as follows. Some preliminary results will be collected in Section 2. In Section 3, we will present some basic properties of the value function, including the boundedness and the continuity of the value function. Dynamic programming principle will be established in Section 4. Then, the value function will be characterized as the unique viscosity solution of the corresponding HJB-QVI in Section 5. Finally, in Section 6, we collect some conclusion remarks.

2. **Preliminaries.**

2.1. **Some spaces.** Let us begin with the introduction of some spaces. For any $p, q \geq 1$ and $t \in [0,T]$, let

$$
\begin{aligned}
L^p_{\mathcal{F}_s}(\Omega;\mathbb{R}^n) &= \Big\{\eta : \Omega \to \mathbb{R}^n \;\Big|\; \eta \text{ is } \mathcal{F}_s\text{-measurable and } \mathbb{E}|\eta|^p < \infty\Big\},\\
L^p_{\mathbb{F}}(\Omega;L^q([t,T];\mathbb{R}^n)) &= \Big\{\varphi : [t,T]\times\Omega \to \mathbb{R}^n \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,}\\
&\qquad \mathbb{E}\Big(\int_t^T |\varphi(s)|^q ds\Big)^{\frac{p}{q}} < \infty\Big\},\\
L^p_{\mathbb{F}}(\Omega;C_+([t,T];\mathbb{R}^n)) &= \Big\{\varphi : [t,T]\times\Omega \to \mathbb{R}^n \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, has}\\
&\qquad \text{right-continuous paths with left-limits,}\\
&\qquad \mathbb{E}\Big[\sup_{s\in[t,T]} |\varphi(s)|^p\Big] < \infty\Big\},\\
L^p_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n)) &= \Big\{\varphi : [t,T]\times\Omega \to \mathbb{R}^n \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, has continuous}\\
&\qquad \text{paths, and } \mathbb{E}\Big[\sup_{s\in[t,T]} |\varphi(s)|^p\Big] < \infty\Big\},\\
L^p_{\mathbb{F}}(\Omega;L^2([t,T];L^2_\pi(\mathbb{R}))) &= \Big\{\varphi : [t,T]\times\mathbb{R}\times\Omega \to \mathbb{R} \;\Big|\; \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and}\\
&\qquad \mathbb{E}\Big(\int_t^T \int_{\mathbb{R}} |\varphi(s,\theta)|^2 \pi(d\theta) ds\Big)^{\frac{p}{2}} < \infty\Big\}.
\end{aligned}
$$

Now, we look at three control sets. First, let

$$
\mathcal{U}[t,T] = \Big\{u : [t,T]\times\Omega \to \mathbf{U} \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-proressively measurable }\Big\}.
$$

which is a set of usual (continuous-time) controls valued in a metric space $\mathbf{U}$.

Next, we precisely define the set of feasible switching controls. A switching control is uniquely determined by a sequence $\{(\theta_i, a_i) \;\big|\; i \geq 0\}$ with $\mathbb{F}$-stopping times $t = \theta_0 \leq \theta_1 \leq \theta_2 \leq \cdots$, and $\mathcal{F}_{\theta_i}$-measurable $\mathbf{A}$-valued random variables a_i. We point out that for different $\omega \in \Omega$, at a switching moment $\theta_i(\omega)$, both the positions $a_{i-1}(\omega)$ switching from and $a_i(\omega)$ switching to could be different. Also, for different ω, the number of switchings within $[t,T]$ could be different. To describe this, we allow $\mathbb{P}\big(\theta_i \in (T, T+1]\big) > 0$, for some i, and define the switching number of $a(\cdot)$ on $[t,T]$ to be

$$
N^S(a(\cdot)) = \max\{i \geq 0 \;\big|\; \theta_i \leq T\}. \tag{2.1}
$$

Since every switching is made, a strictly positive fixed cost will be paid. Thus, if a switching control $a(\cdot)$ whose $N^S(a(\cdot))$ is infinite over a set of positive probability, then the switching cost will be infinite. Hence, such a switching control will not be selected in the process of optimization. Then we define a switching control $a(\cdot)$ to be *feasible* if $N^S(a(\cdot))$ is finite almost surely. Therefore, a feasible switching control $\{(\theta_i, a_i) \;\big|\; i \geq 0\}$ can be identified as follows:

$$
a(s) = \sum_{i=1}^{N^S(a(\cdot))} a_{i-1}\mathbf{1}_{(\theta_{i-1},\theta_i]}(s), \qquad s \in [t,T]. \tag{2.2}
$$

Consequently, we define the set of feasible switching controls by

$$\begin{aligned}\mathcal{A}^a[t,T]=\Big\{a(\cdot)=\sum_{i=1}^{N^S(a(\cdot))}a_{i-1}\mathbf{1}_{(\theta_{i-1},\theta_i]}(\cdot)\;\Big|\;t=\theta_0\le\theta_1\le\theta_2\le\cdots\le T+1\\ \text{are }\mathbb{F}\text{-stopping times, } a_i \text{ is } \mathcal{F}_{\theta_i}\text{-measurable,}\\ \mathbf{A}\text{-valued, } a_0=a\Big\},\qquad a\in\mathbf{A},\ t\in[0,T].\end{aligned}\tag{2.3}$$

The switching control that does not contain any switching is called a *trivial switching control*, denoted by $a_0(\cdot)$, which is completely determined by its initial value. Note that any $a(\cdot)\in\mathcal{A}^a[t,T]$ is always bounded.

Finally, let us describe impulse controls. An impulse control $\xi(\cdot)$ is identified with a sequence $\{(\tau_j,\xi_j)\bigm| j\ge 1\}$ where $t\le\tau_0\le\tau_1\le\cdots\le T+1$ are $\mathbb{F}$-stopping times and each ξ_j is an $\mathcal{F}_{\tau_j}$-measurable random variable taking values in the closed convex cone $K\subseteq\mathbb{R}^n$. Note that, for given $\omega\in\Omega$, the number of impulses within $[t,T]$ could be different. Therefore, we allow $\mathbb{P}\big(\tau_j\in(T,T+1]\big)>0$, for some $j\ge 1$, and define the impulse number of $\xi(\cdot)$ within $[t,T]$ to be

$$N^I(\xi(\cdot))=\max\{j\ge 1\bigm|\tau_j\le T\}.\tag{2.4}$$

Similar to the switching controls, an impulse control $\xi(\cdot)$ is *feasible* if $N^I(\xi(\cdot))$ is finite almost surely. Clearly, a feasible impulse control $\{(\tau_j,\xi_j)\bigm| j\ge 1\}$ can be identified as follows:

$$\xi(s)=\sum_{j=1}^{N^I(\xi(\cdot))}\xi_j\mathbf{1}_{[\tau_j,T]}(s),\qquad s\in[t,T].\tag{2.5}$$

Then, we define the set of feasible impulse controls by $(p\ge 1)$

$$\begin{aligned}\mathcal{K}^p[t,T]=\Big\{\xi(\cdot)=\sum_{j=1}^{N^I(\xi(\cdot))}\xi_j\mathbf{1}_{[\tau_j,T]}(\cdot)\;\Big|\;t\le\tau_1\le\tau_2\le\cdots\le T+1\\ \text{are }\mathbb{F}\text{-stopping times, } \xi_j \text{ is } \mathcal{F}_{\tau_j}\text{-measurable,}\\ K\text{-valued, and } \mathbb{E}\Big[\sup_{s\in[t,T]}|\xi(s)|^p\Big]<\infty\Big\}.\end{aligned}\tag{2.6}$$

In what follows, the impulse control that does not contain any impulse in $[t,T]$ is called the *trivial impulse control*, and will be denoted by $\xi_0(\cdot)$.

We point out that in the above, $T+1$ could be replaced by any $T'>T$,

2.2. **The state equation.** Now, let us introduce the following hypothesis on the coefficients of the state equation.

(H1)$_\delta$ Let $b:[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{U}\times\mathbf{A}\to\mathbb{R}^n$, and $\sigma:[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{U}\times\mathbf{A}\to\mathbb{R}^{n\times d}$ be continuous. There exist $L>0$ and $\delta\in(0,1]$ such that

$$\begin{aligned}&|b(t,x,m,u,a)|+|\sigma(t,x,m,u,a)|^2\le L(1+|x|^\delta),\\ &(t,x,m,u,a)\in[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{U}\times\mathbf{A},\\ &|b(t,x,m,u,a)-b(t,x',m,u,a)|+|\sigma(t,x,m,u,a)-\sigma(t,x',m,u,a)|\le L|x-x'|,\\ &(t,m,u,a)\in[0,T]\times\mathbf{M}\times\mathbf{U}\times\mathbf{A},\quad x,x'\in\mathbb{R}^n.\end{aligned}$$

The sublinearity condition on b and σ with respect to x (in (H1)$_\delta$) will play an essential role later (similar to [13]). Now we present a result concerning the well-posedness of state equation (1.4). The results also includes the integrability of the state process, as well as the stability estimates of the state process with respect to the perturbations of the initial state and the initial time. For convenience, we will identify (H1)$_1$ with (H1) below.

Proposition 2.1. *Let* (H1) *hold and* $p \geq 1$. *Then, the following hold:*

(i) *For any* $(t, x, m, a) \in \mathcal{D}^p \times \mathbf{M} \times \mathbf{A}$, *and* $(u(\cdot), a(\cdot), \xi(\cdot)) \in \mathcal{U}[t,T] \times \mathcal{A}^a[t,T] \times \mathcal{K}^p[t,T]$, *the state equation* (1.4) *admits a unique solution* $X(\cdot) \equiv X(\cdot\,; t, x, m, a; u(\cdot), a(\cdot), \xi(\cdot)) \in L^p_{\mathbb{F}}(\Omega; C_+([t,T]; \mathbb{R}^n))$, *and for any* $0 < q \leq p$,

$$\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}|X(\tau)|^q\Big] \leq C\Big[1+|x|^q+\mathbb{E}_t\Big(\sup_{\tau\in[t,s]}|\xi(\tau)|^q\Big)\Big], \qquad s\in[t,T]. \tag{2.7}$$

Hereafter, $C > 0$ *will be a generic constant which could be different from line to line.*

(ii) *If* $(t, x') \in \mathcal{D}^p$ *is another initial pair, and* $X'(\cdot) \equiv X(\cdot\,; t, x', m, a; u(\cdot), a(\cdot), \xi(\cdot))$ *is the corresponding state process (under the same controls), then for any* $0 < q \leq p$,

$$\mathbb{E}_t\Big[\sup_{s\in[t,T]}|X(s)-X'(s)|^q\Big] \leq C|x-x'|^q. \tag{2.8}$$

(iii) *Let* (H1)$_\delta$ *hold with* $\delta \in (0,1)$. *Let* $0 \leq t < t' \leq T$, $(t, x, m, a) \in \mathcal{D}^p \times \mathbf{M} \times \mathbf{A}$, $(u(\cdot), a(\cdot), \xi(\cdot)) \in \mathcal{U}[t,T] \times \mathcal{A}^a[t,T] \times \mathcal{K}^p[t,T]$. *Let* $u'(\cdot) = u(\cdot)\big|_{[t',T]}$, *and*

$$\begin{aligned} a'(s) &= \sum_{i=1}^{N^S(a(\cdot))} a_i \mathbf{1}_{[\theta_{i-1}\vee t', \theta_i \vee t')}(s), \\ \xi'(s) &= \sum_{j=1}^{N^I(\xi(\cdot))} \xi_j \mathbf{1}_{[\tau_j \vee t', T]}(s), \end{aligned} \qquad s \in [t', T], \tag{2.9}$$

i.e., all the switchings and impulses on $[t, t')$ *are moved to* t'. *Let* $X'(\cdot) = X(\cdot\,; t', x', M(t'), a(t'); u'(\cdot), a'(\cdot), \xi'(\cdot) - \xi(t'))$, *with* $x' = x + \sum_{t\leq\tau_j\leq t'} \xi_j$. *Then, for any* $0 < q \leq p$,

$$\mathbb{E}_t\Big[\sup_{s\in[t',T]}|X(s)-X'(s)|^q\Big] \leq C\Big[1+|x|^{\delta q}+\mathbb{E}_t\Big(\sup_{s\in[t,t']}|\xi(s)|^{\delta q}\Big)\Big](t'-t)^{\frac{q}{2}}. \tag{2.10}$$

In particular, at t',

$$\mathbb{E}_t\Big[|X(t')-x'|^q\Big] \leq C\Big[1+|x|^{\delta q}+\mathbb{E}_t\Big(\sup_{s\in[t,t']}|\xi(s)|^{\delta q}\Big)\Big](t'-t)^{\frac{q}{2}}. \tag{2.11}$$

Consequently, in the case that $a(\cdot)$ *and* $\xi(\cdot)$ *are trivial on* $[t, t']$, *it holds*

$$\mathbb{E}_t\Big[\sup_{s\in[t,t']}|X(s)-x|^q\Big] \leq C(1+|x|^{\delta q})(t'-t)^{\frac{q}{2}}. \tag{2.12}$$

Proof. (i) First of all, for any $(t,x,m,a)\in\mathcal{D}^p\times\mathbf{M}\times\mathbf{A}$, and $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}^a[t,T]\times\mathcal{K}^p[t,T]$, with $p\geq 1$, we define a map Φ as follows:

$$\begin{aligned}\Phi[X(\cdot)](s)=x+\int_t^s b(r,X(r),M(r),u(r),a(r))dr\\ +\int_t^s\sigma(r,X(r),M(r),u(r),a(r))dW(r)+\xi(s),\\ \forall X(\cdot)\in L^p_{\mathbb{F}}(\Omega;C_+([t,T];\mathbb{R}^n)).\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}|\Phi[X(\cdot)](\tau)|^p\Big]\\ &\leq C\Big\{|x|^p+\mathbb{E}_t\Big(\textstyle\int_t^s|b(r,X(r),M(r),u(r),a(r))|dr\Big)^p\\ &\qquad+\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}\Big|\int_t^\tau\sigma(r,X(r),M(r),u(r),a(r))dW(r)\Big|^p\Big]+\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}|\xi(\tau)|^p\Big]\Big\}\\ &\leq C\Big\{|x|^p+\mathbb{E}_t\Big(\int_t^s\big(1+|X(r)|\big)dr\Big)^p+\mathbb{E}_t\Big(\int_t^s\big(1+|X(r)|^2\big)dr\Big)^{\frac{p}{2}}\\ &\qquad+\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}|\xi(\tau)|^p\Big]\Big\}\\ &\leq C\mathbb{E}_t\Big[1+|x|^p+\sup_{\tau\in[t,s]}|X(\tau)|^p+\sup_{\tau\in[t,s]}|\xi(\tau)|^p\Big].\end{aligned}$$

Thus, Φ maps $L^p_{\mathbb{F}}(\Omega;C_+([t,T];\mathbb{R}^n))$ into itself. Next, pick $X(\cdot),X'(\cdot)\in L^p_{\mathbb{F}}(\Omega;C_+([t,T];\mathbb{R}^n))$, and estimate the following:

$$\begin{aligned}&\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}|\Phi(X(\cdot)](\tau)-\Phi[X'(\cdot)](\tau)|^p\Big]\\ &=\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}\Big|\int_t^\tau\Big(b(r,X(r),M(r),u(r),a(r))-b(r,X'(r),M(r),a(r))\Big)dr\\ &\quad+\int_t^\tau\Big(\sigma(r,X(r),M(r),u(r),a(r))-\sigma(r,X'(r),M(r),u(r),a(r))\Big)dW(r)\Big|^p\Big]\\ &\leq C\mathbb{E}_t\Big[\Big(\int_t^s|X(r)-X'(r)|dr\Big)^p+\Big(\int_t^s|X(r)-X'(r)|^2dr\Big)^{\frac{p}{2}}\Big]\\ &\leq C(s-t)^{\frac{p}{2}}\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}|X(\tau)-X'(\tau)|^p\Big].\end{aligned}$$

Then by choosing $s-t>0$ small, we see that the map Φ is a contraction on $L^p_{\mathbb{F}}(\Omega;C_+([t,s];\mathbb{R}^n))$ and it has a unique fixed point. Following a standard augment, one gets the existence and uniqueness of the solution to the state equation on the space $L^p_{\mathbb{F}}(\Omega;C_+([t,T];\mathbb{R}^n))$.

Now, for any $0<q\leq p$, and $t\leq t_1<t_2\leq s$, by the state equation, we estimate:

$$
\begin{aligned}
&\mathbb{E}_{t_1}\Big[\sup_{\tau\in[t_1,t_2]}|X(\tau)|^q\Big]\le C\mathbb{E}_{t_1}\Big\{|X(t_1)|^q+\Big(\int_{t_1}^{t_2}|b(r,X(r),M(r),u(r),a(r))|dr\Big)^q\\
&\qquad+\sup_{\tau\in[t_1,t_2]}\Big|\int_{t_1}^{t_2}\sigma(r,X(r),M(r),u(r),a(r))dW(r)\Big|^q+\sup_{\tau\in[t_1,t_2]}|\xi(\tau)|^q\Big\}\\
&\le C\mathbb{E}_{t_1}\Big[|X(t_1)|^q+\Big(\int_{t_1}^{t_2}(1+|X(r)|)dr\Big)^q+\Big(\int_{t_1}^{t_2}(1+|X(r)|^2)dr\Big)^{\frac{q}{2}}+\sup_{\tau\in[t_1,t_2]}|\xi(\tau)|^q\Big]\\
&\le C\Big\{1+|X(t_1)|^q+(t_2-t_1)^{\frac{q}{2}}\mathbb{E}_{t_1}\Big[\sup_{\tau\in[t_1,t_2]}|X(\tau)|^q\Big]+\mathbb{E}_{t_1}\Big[\sup_{\tau\in[t_1,t_2]}|\xi(\tau)|^q\Big]\Big\},
\end{aligned}
$$

where $C>0$ is an absolute constant, independent of t_1,t_2. Let $\delta_0=\Big(\frac{1}{2C}\Big)^{\frac{2}{q}}$. Then there exists a natural number $i_0\ge 1$ such that $i_0\delta_0\ge T-t$. We denote $t_i=t+i\delta_0$, $i=0,1,2,\cdots,i_0$, and

$$
\varphi_{i+1}=\mathbb{E}_t\Big[\sup_{\tau\in[t_i,t_{i+1}]}|X(\tau)|^q\Big],\qquad i\ge 0,\qquad \varphi_0\equiv|x|^q.
$$

The above estimate implies

$$
\varphi_{i+1}\le C\Big\{1+\varphi_i+\delta_0^{\frac{q}{2}}\varphi_{i+1}+\mathbb{E}\Big[\sup_{\tau\in[t_i,t_{i+1}]}|\xi(\tau)|^q\Big]\Big\},\qquad i\ge 0.
$$

Hence, for a larger absolute constant $C>0$,

$$
\varphi_{i+1}\le C\Big\{1+\varphi_i+\mathbb{E}\Big[\sup_{\tau\in[t_1,t_2]}|\xi(\tau)|^q\Big]\Big\},\qquad i\ge 0.
$$

By solving the above difference inequlity, we obtain (2.7).

(ii) Let $X'(\cdot)\equiv X(\cdot\,;t,x',m,a;u(\cdot),a(\cdot),\xi(\cdot))$. Then

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}|X(\tau)-X'(\tau)|^p\Big]\\
&\le C\mathbb{E}_t\Big[|x-x'|^p+\Big(\int_t^s|X(r)-X'(r)|dr\Big)^p+\Big(\int_t^s|X(r)-X'(r)|^2dr\Big)^{\frac{p}{2}}\Big]\\
&\le C\mathbb{E}_t\Big[|x-x'|^p+\int_t^s\sup_{\tau\in[t,r]}|X(\tau)-X'(\tau)|^pdr\Big].
\end{aligned}
$$

Hence, by Gronwall's inequality, we obtain (2.8), for $q=p$. Now, for $0<q\le p$,

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sup_{\tau\in[t,s]}|X(\tau)-X'(\tau)|^q\Big]\le\Big[\mathbb{E}_t\Big(\sup_{\tau\in[t,s]}|X(\tau)-X'(\tau)|^p\Big)\Big]^{\frac{q}{p}}\\
&\le\Big(C|x-x'|^p\Big)^{\frac{q}{p}}=C|x-x'|^q.
\end{aligned}
$$

This proves (2.8) for general case.

(iii) Let $(\mathrm{H})_\delta$ hold with $\delta\in(0,1)$. Let $u'(\cdot)$, $a'(\cdot)$ and $\xi'(\cdot)$ be constructed as in the statement of proposition. Then,

$$
u'(s)=u(s),\quad a'(s)=a(s),\quad \xi'(s)=\xi(s),\qquad s\in[t',T].
$$

With $x' = x + \sum_{t \le \tau_j \le t'} \xi_j$, for $s \in [t', T]$, we have

$$
\begin{aligned}
X(s) &= X(s; t, x, m, a; u(\cdot), a(\cdot), \xi(\cdot)) \\
&= X(s; t', X(t'), M(t'), a(t'); u(\cdot), a(\cdot), \xi(\cdot) - \xi(t')) \\
X'(s) &= X(s; t', x', M(t'), a(t'); u'(\cdot), a'(\cdot), \xi'(\cdot) - \xi(t')).
\end{aligned}
$$

Note that

$$
\begin{aligned}
X(t') - x' = &\int_t^{t'} b(s, X(s), M(s), u(s), a(s)) ds \\
&+ \int_t^{t'} \sigma(s, X(s), M(s), u(s), a(s)) dW(s).
\end{aligned}
$$

Thus, using (2.7) and the sublinearity of b and σ in x, one has

$$
\begin{aligned}
\mathbb{E}_t\Big[|X(t') - x'|^q\Big] &\le C\mathbb{E}_t\Big[\Big(\int_t^{t'} (1 + |X(s)|^\delta) ds\Big)^q + \Big(\int_t^{t'} (1 + |X(s)|^\delta) ds\Big)^{\frac{q}{2}}\Big] \\
&\le C\mathbb{E}_t\Big(1 + \sup_{r \in [t,t']} |X(r)|^{\delta q}\Big)(t' - t)^{\frac{q}{2}} \\
&\le C\mathbb{E}_t\Big(1 + |x|^{\delta q} + \sup_{s \in [t,t']} |\xi(s)|^{\delta q}\Big)(t' - t)^{\frac{q}{2}}.
\end{aligned}
$$

This proves (2.11). We further have

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sup_{s \in [t',T]} |X(s) - X'(s)|^q\Big] \le C\mathbb{E}_t|X(t') - x'|^q \\
&\le C\mathbb{E}_t\Big(1 + |x|^{\delta q} + \sup_{s \in [t,t']} |\xi(s)|^{\delta q}\Big)(t' - t)^{\frac{q}{2}}.
\end{aligned}
$$

This proves (2.10). Now, when $a(\cdot)$ and $\xi(\cdot)$ are trivial on $[t, t']$, then $x' = x$, and (2.11) reads (2.12). This completes the proof. $\square$

Note that estimate (2.7)–(2.8) as well as (2.10)–(2.12), are standard for $1 \le q \le p$ and $\delta = 1$. However, the case of $0 < q < 1$ and $\delta \in (0, 1)$ are new here. They will play crucial roles later. This is one particular feature of the current paper.

2.3. **The recursive cost functional.** Now, we look at BSVIE (1.8). For given initial tuple $(t, x, m, a) \in \mathcal{D}^p \times \mathbf{M} \times \mathbf{A}$, and feasible controls $(u(\cdot), a(\cdot), \xi(\cdot)) \in \mathcal{U}[t, T] \times \mathcal{A}^a[t, T] \times \mathcal{K}^p[t, T]$, let $X(\cdot)$ be the corresponding state process. Define

$$
\widehat{Y}(s) \equiv Y(s) + \sum_{t \le \theta_i < s} k(\theta_i, a_{i-1}, a_i) + \sum_{t \le \tau_j < s} \ell(\tau_j, \xi_j), \qquad s \in [t, T]. \tag{2.13}
$$

Then, $\widehat{Y}(\cdot)$ is $\mathbb{F}$-adapted, and

$$
\begin{aligned}
\widehat{Y}(s) = &\, h(X(T), M(T), a(T)) + \sum_{i=1}^{N^S(a(\cdot))} k(\theta_i, a_{i-1}, a_i) + \sum_{j=1}^{N^I(\xi(\cdot))} \ell(\tau_j, \xi_j) \\
&+ \int_s^T \widehat{g}\Big(r, X(r), M(r), u(r), a_r, \xi_r, \widehat{Y}(r), Z(r), \int_{\mathbb{R}} \Gamma(r, \theta)\pi(d\theta)\Big) dr \\
&- \int_s^T Z(r) dW(r) - \int_s^T \int_{\mathbb{R}} \Gamma(r-, \theta)\widetilde{N}(d\theta, dr), \qquad s \in [t, T],
\end{aligned} \tag{2.14}
$$

where

$$\begin{aligned}
&\widehat{g}(r,x,m,u,a_r,\xi_r,\widehat{y},z,\gamma)\\
&=g\Big(r,x,m,u,a(r),\widehat{y}-\sum_{t\le\theta_i<r}k(\theta_i,a_{i-1},a_i)-\sum_{t\le\tau_j<r}\ell(\tau_j,\xi_j),z,\gamma\Big),
\end{aligned}$$

with a_r standing for the path $\{a(\tau)\bigm| t\le\tau<r\}$ and ξ_r standing for the path $\{\xi(\tau)\bigm| t\le\tau<r\}$. Clearly, $\widehat{g}$ is Lipschitz continuous in $(\widehat{y},z,\gamma)$, for given $(u(\cdot),a(\cdot),\xi(\cdot))$. The above (2.14) is a BSDE on $[t,T]$, as the terminal state is a given time-independent $\mathcal{F}_T$-measurable random variable, and thus $Z(s,r)$, $\Gamma(s,r,\theta)$ have been already changed to $Z(r)$, $\Gamma(r,\theta)$. Due to the fact that in the definition of $\widehat{g}$, the summation terms involving $a(\cdot)$ and $\xi(\cdot)$ starting from t, $\widehat{g}$ contains some memory with respect to $a(\cdot)$ and $\xi(\cdot)$. Although such kind of memories will not affect the well-posedness of the BSDE (2.14), they will bring some technical difficulties in some other aspects later. We also note that (see (2.13))

$$\widehat{Y}(t)=Y(t)\ge 0,$$

and

$$\widehat{Y}(s)\ge Y(s)\ge 0,\quad s\in(t,t'],$$

with the both inequalities being strict if either $a(\cdot)$ or $\xi(\cdot)$ is non-trivial on $[t,t']$. Further, from the general BSDE theory, in order the BSDE (2.14) to admit an adapted solution, we need to impose some conditions on the switching and impulse controls through their costs so that the terminal state of (2.14) has good enough integrability. Also if the BSDE (2.14) is solvable, then $\widehat{Y}(\cdot)$ has continuous path almost surely on $[t,T]$, whereas $Y(\cdot)$ might have jumps. Now, we introduce the following hypothesis.

(H2) The maps $h:\mathbb{R}^n\times\mathbf{M}\times\mathbf{A}\to[0,L]$, and $g:[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{U}\times\mathbf{A}\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{R}\to[0,L]$ are continuous for some $L>0$. Moreover,

$$\begin{aligned}
&|h(x,m,a)-h(x',m,a)|+|g(t,x,m,u,a,y,z,\gamma)-g(t,x',m,u,a,y',z',\gamma')|\\
&\le L\Big(|x-x'|+|y-y'|+|z-z'|+|\gamma-\gamma'|\Big),\\
&0\le h(x,m,a),\quad g(t,x,m,u,a,0,0,0)\le T,\\
&\qquad(t,m,u,a)\in[0,T]\times\mathbf{M}\times\mathbf{U}\times\mathbf{A},\\
&\qquad(x,y,z,\gamma),(x',y',z',\gamma')\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{R}.
\end{aligned}\tag{2.15}$$

The nonnegativity condition of h and g can be relaxed to the boundedness from below of these functions. However, we impose the boundedness of these functions from the above as well, which avoids some technical difficulties. Next, we introduce the following hypotheses for the switching and impulse costs.

(H3) The map $k:[0,T]\times\mathbf{A}\times\mathbf{A}\to[0,\infty)$ is continuous and differentiable in t. such that

$$\begin{aligned}
k(t,a,a)=0,\quad &k(t,a',a)=k(t,a,a')>0,\\
&\forall t\in[0,T],\ a,a'\in\mathbf{A},a\ne a'.
\end{aligned}\tag{2.16}$$

Moreover,

$$\begin{aligned}
&k(t,a_1,a_3)<k(t,a_1,a_2)+k(t,a_2,a_3),\\
&\qquad\forall t\in[0,T],\ a_1,a_2,a_3\in\mathbf{A},\ a_1\ne a_2\ne a_3,
\end{aligned}\tag{2.17}$$

and

$$-L \le k_t(t,a,a') \le 0, \qquad \forall 0 \le t \le T,\ a,a' \in \mathbf{A}. \tag{2.18}$$

Note that under (H3), we may assume that for some $k_1 > k_0 > 0$,

$$0 < k_0 \le k(t,a,a') \le k_1, \qquad \forall t \in [0,T],\ a,a' \in \mathbf{A},\ a \neq a'. \tag{2.19}$$

(H4) The map $\ell : [0,T] \times K \to [\ell_0, \infty)$ is continuous and differentiable in t, with $\ell_0 > 0$ and for some $L > 0$ and $\nu \in [\delta, 1]$,

$$\ell_0(1 + |\xi|^\nu) \le \ell(t,\xi) \le L(1 + |\xi|^\nu), \qquad \forall (t,\xi) \in [0,T] \times K. \tag{2.20}$$

Moreover,

$$\ell(t,\xi+\xi') < \ell(t,\xi) + \ell(t,\xi'), \qquad \forall t \in [0,T],\ \xi,\xi' \in K, \tag{2.21}$$

and

$$-L \le \ell_t(t,\xi) \le 0, \qquad \forall 0 \le t \le T,\ \xi \in K. \tag{2.22}$$

The strict subadditivity conditions (2.17) and (2.21) are very crucial in proving the uniqueness of the viscosity solution to the HJB-QVI. Also, in (2.18) and (2.22), we have assumed a little more than the monotonicity conditions which are commonly assumed for switching and/or impulse problems (see [19, 13, 20]). We indicate that (2.22) can be relaxed to the following:

$$-L - \ell_1|\xi|^\nu \le \ell_t(t,\xi) \le 0, \qquad \forall (t,\xi) \in [0,T] \times K,$$

for some $\ell_1 \ge 0$. Conditions (2.19) and (2.20) will lead to the coercivity of the cost functional with respect to the size of the impulse, and the numbers of switchings and impulses; Due to such two conditions, we see that it makes sense to define feasible switching/impulse controls (for which the numbers of switchings and impulses are finite almost surely on $[t,T]$).

Before going further, let us introduce the following smaller control sets: For $q > 1$,

$$\mathcal{A}^{a,q}[t,T] = \Big\{ a(\cdot) \in \mathcal{A}^a[t,T] \mid \mathbb{E}\Big[\Big(\sum_{i=1}^{N^S(a(\cdot))} k(\theta_i, a_{i-1}, a_i) \Big)^q\Big] < \infty \Big\},$$

$$\mathcal{K}^{p,q}[t,T] = \Big\{ \xi(\cdot) \in \mathcal{K}^p[t,T] \mid \mathbb{E}\Big[\Big(\sum_{j=1}^{M^I(\xi(\cdot))} \ell(\tau_j, \xi_j) \Big)^q\Big] < \infty \Big\}.$$

We further let

$$\mathcal{A}^{a,1+}[t,T] = \bigcup_{q>1} \mathcal{A}^{a,q}[t,T], \qquad \mathcal{K}^{p,1+}[t,T] = \bigcup_{q>1} \mathcal{K}^{p,q}[t,T].$$

Concerning BSDE (2.14), we have the following well-posedness result, which also includes the integrability of the adapted solution, the stability estimates of the adapted solution with respect to the perturbation of the initial state x and the initial time t.

Proposition 2.2. *Let* (H1)–(H4) *hold and* $p > 1$. *Then the following hold:*

(i) *For any given* $(t,x,m,a) \in \mathcal{D}^p \times \mathbf{M} \times \mathbf{A}$ *and* $(u(\cdot), a(\cdot), \xi(\cdot)) \in \mathcal{U}[t,T] \times \mathcal{A}^{a,1+}[t,T] \times \mathcal{K}^{p,1+}[t,T]$, *with* $X(\cdot)$ *being the corresponding solution to* (1.4), *the BSDE* (2.14) *admits a unique adapted solution* $(\widehat{Y}(\cdot), Z(\cdot), \Gamma(\cdot\,,\cdot)) \in L^q_{\mathbb{F}}(\Omega; C([t,T];$

$\mathbb{R})) \times L^q_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^{1\times d})) \times L^q_{\mathbb{F}}(\Omega; L^2(t,T; L^2_\pi(\mathbb{R})))$ *for some* $q \in (1,p]$. *Further, for such a* q,

$$\begin{aligned}&\mathbb{E}_t\Big[\sup_{s\in[t,T]}|\widehat{Y}(s)|^q + \Big(\int_t^T |Z(s)|^2 ds\Big)^{\frac{q}{2}} + \Big(\int_t^T\!\!\int_{\mathbb{R}} |\Gamma(t,\theta)|^2\pi(d\theta)dt\Big)^{\frac{q}{2}}\Big]\\ &\le C\Big\{1+\mathbb{E}_t\Big[\Big(\sum_{i=1}^{N^S(a(\cdot))} k(\theta_i,a_{i-1},a_i)\Big)^q + \Big(\sum_{j-1}^{N^I(\xi(\cdot))}\ell(\tau_j,\xi_j)\Big)^q\Big]\Big\}.\end{aligned}\tag{2.23}$$

Consequently,

$$\begin{aligned}&\mathbb{E}_t\Big[\sup_{s\in[t,T]}|Y(s)|^q\Big]\\ &\le C\Big\{1+\mathbb{E}_t\Big[\Big(\sum_{i=1}^{N^S(a(\cdot))} k(\theta_i,a_{i-1},a_i)\Big)^q + \Big(\sum_{j-1}^{N^I(\xi(\cdot))}\ell(\tau_j,\xi_j)\Big)^q\Big]\Big\}.\end{aligned}\tag{2.24}$$

(ii) *Let* $(t,x),(t,x') \in \mathcal{D}^q$, $(m,a) \in \mathbf{M}\times\mathbf{A}$, *and* $(u(\cdot),a(\cdot),\xi(\cdot)),(u'(\cdot),a'(\cdot),\xi'(\cdot)) \in \mathcal{U}[t,T]\times\mathcal{A}^{a,1+}[t.T]\times\mathcal{K}^{p,1+}[t,T]$. *Let* $X(\cdot)$ *and* $X'(\cdot)$ *be the corresponding solution of the state equation* (1.4) *(with the same controls), and* $(\widehat{Y}(\cdot),Z(\cdot),\Gamma(\cdot\,,\cdot))$ *and* $(\widehat{Y}'(\cdot),Z'(\cdot),\Gamma'(\cdot\,,\cdot))$ *be the corresponding adapted solutions to the BSDE* (2.14). *Then, for some* $1<q\le p$, *and any* $0<q'\le q$, *it holds*

$$\begin{aligned}&\mathbb{E}_t\Big[\sup_{r\in[t,T]}|\widehat{Y}(r)-\widehat{Y}'(r)|^{q'} + \Big(\int_t^T|Z(r)-Z'(r)|^2dr\Big)^{\frac{q'}{2}}\\ &\quad+\Big(\int_t^T\int_{\mathbb{R}}|\Gamma(r,\theta)-\Gamma'(r,\theta)|^2\pi(d\theta)dr\Big)^{\frac{q'}{2}}\Big]\\ &\le C\Big\{\mathbb{E}_t|h(X(T),M(T),a(T))-h(X'(T),M(T),a'(T))|^q\\ &\quad+\mathbb{E}_t\Big[\int_t^T\big|g\Big(r,X(r),M(r),u(r),a(r),\\ &\quad\widehat{Y}(r)-\sum_{t\le\theta_i<r}k(\theta_i,a_{i-1},a_i)-\sum_{t\le\tau_j<r}\ell(\tau_j,\xi_j),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta)\pi(d\theta)\Big)\\ &\quad-g\Big(r,X'(r),M(r),u'(r),a'(r),\\ &\quad\widehat{Y}(r)-\sum_{t\le\theta'_i<r}k(\theta'_i,a'_{i-1},a'_i)-\sum_{t\le\tau'_j<r}\ell(\tau'_j,\xi'_j),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta)\pi(d\theta)\Big)\big|dr\Big]^q\Big\}^{\frac{q'}{q}},\end{aligned}\tag{2.25}$$

(iii) *If* $0\le t<t'\le T$, $(t,x,m,a)\in\mathcal{D}^p\times\mathbf{M}\times\mathbf{A}$, *and* $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}^{a,1+}[t,T]\times\mathcal{K}^{p,1+}[t,T]$, *then*

$$\Big|\mathbb{E}_t\big[\widehat{Y}(t)-\widehat{Y}(t')\big]\Big|\le C(t'-t).\tag{2.26}$$

Further, let $(\mathrm{H1})_\delta$ *hold for some* $0<\delta\le\nu$. *Let* $u'(\cdot)$, $a'(\cdot)$ *and* $\xi'(\cdot)$ *as in* (iii) *of Proposition 2.1,* $X'(\cdot)=X(\cdot\,;t',x,M(t'),a(t');u'(\cdot),a'(\cdot),\xi'(\cdot))$ *and* $(\widehat{Y}'(\cdot),Z'(\cdot),\Gamma'(\cdot\,,\cdot))$ *be the adapted solution of the corresponding BSDE* (2.14), *then for some*

$1 < q \le p$, *and any* $0 < q' \le q$,

$$\mathbb{E}_t\Big[\sup_{s\in[t',T]}|\widehat Y(s)-\widehat Y'(s)|^{q'}+\Big(\int_{t'}^T|Z(s)-Z'(s)|^2ds\Big)^{\frac{q'}{2}}$$
$$+\Big(\int_t^T\!\!\int_{\mathbb{R}}|\Gamma(t,\theta)-\Gamma'(t,\theta)|^2\pi(d\theta)dt\Big)^{\frac{q'}{2}}\Big] \tag{2.27}$$
$$\le C\mathbb{E}_t\Big(1+|x|^{\delta q'}+\sup_{r\in[t,t']}|\xi(r)|^{\delta q'}+N^S(a(\cdot))^{q'}+N^I(\xi(\cdot))^{q'}\Big)(t'-t)^{\frac{q'}{2}}$$

Proof. (i) For any $(t,x,m,a)\in\mathcal{D}^p\times\mathbf{M}\times\mathbf{A}$ and $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}^{a,q}[t,T]\times\mathcal{K}^{p,q}[t,T]$, with some $q>1$, we have

$$\mathbb{E}|h(X(T),M(T),a(T))|^q+\mathbb{E}\Big[\Big(\sum_{i\ge1}k(\theta_i,a_{i-1},a_i)\Big)^q+\Big(\sum_{j\ge1}\ell(\tau_j,\xi_j)\Big)^q\Big]<\infty.$$

Thus, BSDE (2.14) is well-posed with the adapted solution $(\widehat Y(\cdot),Z(\cdot),\Gamma(\cdot\,,\cdot))$ in $L^q_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}))\times L^q_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^{1\times d}))\times L^q_{\mathbb{F}}(\Omega;L^2(t,T;L^2_\pi(\mathbb{R}))$. Then, by a standard estimate of BSDEs (see [15], and also [12]), we have (by the boundedness of h and g; Otherwise the estimate will be much more complicated)

$$\mathbb{E}_t\Big[\sup_{s\in[t,T]}|\widehat Y(s)|^q+\Big(\int_t^T|Z(s)|^2ds\Big)^{\frac q2}+\Big(\int_t^T\!\!\int_{\mathbb{R}}|\Gamma(t,\theta)|^2\pi(d\theta)dt\Big)^{\frac q2}\Big]$$
$$\le C\mathbb{E}_t\Big[1+\Big(\sum_{i=1}^{N^S(a(\cdot))}k(\theta_i,a_{i-1},a_i)\Big)^q+\Big(\sum_{j-1}^{N^I(\xi(\cdot))}\ell(\tau_j,\xi_j)\Big)^q\Big].$$

This proves (2.23). Consequently, (2.24) follows easily.

(ii) Next, if $q\in(1,p]$, for any $(t,x),(t,x')\in\mathcal{D}^q$ and feasible controls $(u(\cdot),a(\cdot),\xi(\cdot))$, $(u'(\cdot),a'(\cdot),\,\xi'(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}^{a,q}[t,T]\times\mathcal{K}^{p,q}[t,T]$, we let $X(\cdot)$ and $X'(\cdot)$ be the corresponding state processes. Denote

$$h=h(X(T),M(T),a(T)),\quad h'=h(X'(T),M(T),a'(T)),$$
$$g(r,\widehat y,z,\gamma)=g\Big(r,X(r),M(r),u(r),a(r),$$
$$\widehat y-\sum_{t\le\theta_i<r}k(\theta_i,a_{i-1},a_i)-\sum_{t\le\tau_j<r}\ell(\tau_j,\xi_j),z,\gamma\Big),$$
$$g'(r,\widehat y',z',\gamma')=g\Big(r,X'(r),M(r),u'(r),a'(r),$$
$$\widehat y'-\sum_{t\le\theta_i'<r}k(\theta_i',a_{i-1}',a_i')-\sum_{t\le\tau_j'<r}\ell(\tau_j',\xi_j'),z',\gamma'\Big),$$
$$g''(r,\widehat y,z,\gamma)=g\Big(r,X'(r),M(r),u'(r),a'(r),$$
$$\widehat y-\sum_{t\le\theta_i'<r}k(\theta_i',a_{i-1}',a_i')-\sum_{t\le\tau_j'<r}\ell(\tau_j',\xi_j'),z,\gamma\Big).$$

Then

$$
\begin{aligned}
\widehat{Y}(s) &= h + \int_s^T g\Big(r, \widehat{Y}(r), Z(r), \int_{\mathbb{R}} \Gamma(r,\theta)\pi(d\theta)\Big) dr - \int_s^T Z(r) dW(r) \\
&\quad - \int_s^T \int_{\mathbb{R}} \Gamma(r-,\theta) \widetilde{N}(d\theta, dr), \\
\widehat{Y}'(s) &= h' + \int_s^T g'\Big(r, \widehat{Y}'(r), Z'(r), \int_{\mathbb{R}} \Gamma'(r,\theta)\pi(d\theta)\Big) dr - \int_s^T Z'(r) dW(r) \\
&\quad - \int_s^T \int_{\mathbb{R}} \Gamma'(r-,\theta) \widetilde{N}(d\theta, dr),
\end{aligned}
$$

Hence, by the standard estimate of BSDEs, we have, for any $\tau \in [t,T]$,

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sup_{s\in[\tau,T]} |\widehat{Y}(s) - \widehat{Y}'(s)|^q + \Big(\int_\tau^T |Z(r) - Z'(r)|^2 dr \Big)^{\frac{q}{2}} \\
&\quad + \Big(\int_\tau^T \int_{\mathbb{R}} |\Gamma(r,\theta) - \Gamma(r,\theta)|^2 \pi(d\theta) dr \Big)^{\frac{q}{2}} \Big] \\
&\leq C\mathbb{E}_t\Big\{ |h - h'|^q + \Big[\int_\tau^T \Big| g\Big(r, \widehat{Y}(r), Z(r), \int_{\mathbb{R}} \Gamma(r,\theta)\pi(d\theta)\Big) \\
&\quad - g''\Big(r, \widehat{Y}(r), Z(r), \int_{\mathbb{R}} \Gamma(r,\theta)\pi(d\theta)\Big) \Big| dr \Big]^q \Big\} \\
&= C\mathbb{E}_t\Big\{ |h(X(T), M(T), a(T)) - h(X'(T), M(T), a'(T))|^q \\
&\quad + \Big[\int_\tau^T \Big| g\Big(r, X(r), M(r), u(r), a(r), \\
&\quad \widehat{Y}(r) - \sum_{t\leq\theta_i<r} k(\theta_i, a_{i-1}, a_i) - \sum_{t\leq\tau_j<r} \ell(\tau_j, \xi_j), Z(r), \int_{\mathbb{R}} \Gamma(r,\theta)\pi(d\theta)\Big), \\
&\quad - g\Big(r, X'(r), M(r), u'(r), a'(r), \\
&\quad \widehat{Y}(r) - \sum_{t\leq\theta_i'<r} k(\theta_i', a_{i-1}', a_i') - \sum_{t\leq\tau_j'<r} \ell(\tau_j', \xi_j), Z(r), \int_{\mathbb{R}} \Gamma(r,\theta)\pi(d\theta)\Big) \Big| dr \Big]^q \Big\},
\end{aligned}
$$

Now, for $0 < q' \leq q$,

$$
\begin{aligned}
&\mathbb{E}_t\Big[\sup_{s\in[t,T]} |\widehat{Y}(s) - \widehat{Y}'(s)|^{q'} + \Big(\int_t^T |Z(r) - Z'(r)|^2 dr \Big)^{\frac{q'}{2}} \\
&\quad + \Big(\int_t^T \int_{\mathbb{R}} |\Gamma(r,\theta) - \Gamma'(r,\theta)|^2 \pi(d\theta) dr \Big)^{\frac{q'}{2}} \Big] \\
&\leq \Big[\mathbb{E}_t\Big(\sup_{s\in[t,T]} |\widehat{Y}(s) - \widehat{Y}'(s)|^q \Big) \Big]^{\frac{q'}{q}} + \Big[\mathbb{E}_t\Big(\int_t^T |Z(r) - Z'(r)|^2 ds \Big)^{\frac{q}{2}} \Big]^{\frac{q'}{q}} \\
&\quad + \Big[\mathbb{E}_t\Big(\int_t^T \int_{\mathbb{R}} |\Gamma(r,\theta) - \Gamma'(r,\theta)|^2 \pi(d\theta) dr \Big)^{\frac{q}{2}} \Big]^{\frac{q'}{q}} \\
&\leq C\Big\{ \mathbb{E}_t\Big[\Big(\sup_{s\in[t,T]} |\widehat{Y}(s) - \widehat{Y}'(s)|^q \Big) + \mathbb{E}_t\Big(\int_t^T |Z(r) - Z'(r)|^2 dr \Big)^{\frac{q}{2}} \\
&\quad + \mathbb{E}_t\Big(\int_t^T \int_{\mathbb{R}} |\Gamma(r,\theta) - \Gamma'(r,\theta)|^2 \pi(d\theta) dt \Big)^{\frac{q}{2}} \Big] \Big\}^{\frac{q'}{q}}.
\end{aligned}
$$

Then (2.25) follows.

(iii) Under our assumptions, one has

$$\widehat{Y}(t)=\widehat{Y}(s)+\int_t^s\widehat{g}\Big(r,X(r),M(r),u(r),a_r,\xi_r,\widehat{Y}(r),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta)\pi(d\theta)\Big)dr$$
$$-\int_t^s Z(r)dW(r)-\int_t^s\!\int_{\mathbb{R}}\Gamma(r-,\theta)\widetilde{N}(d\theta,dr).$$

Thus,

$$\begin{aligned}&\Big|\mathbb{E}_t\big[\widehat{Y}(t)-\widehat{Y}(t')\big]\Big|\\&=\Big|\mathbb{E}_t\int_t^{t'}\widehat{g}\big(r,X(r),M(r),u(r),a_r,\xi_r,\widehat{Y}(r),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta)\pi(d\theta)\big)dr\Big|\le C(t'-t).\end{aligned}$$

This completes the proof (2.26).

Next, let $u'(\cdot),a'(\cdot),\xi'(\cdot)$ be constructed through $u(\cdot),a(\cdot),\xi(\cdot)$, as (iii) of Proposition 2.1, and $X'(\cdot)$ be the corresponding state process on $[t',T]$ with the initial state x. Denoting $x'=x+\sum_{t\le\tau_j\le t'}\xi_j$. Clearly,

$$(u'(r),a'(r),\xi'(r))=(u(r),a(r),\xi(r)),\qquad r\in[t',T].$$

Also, let $(Y'(\cdot),Z'(\cdot),\Gamma'(\cdot\,,\cdot))$ be the adapted solution to the corresponding BSDE on $[t',T]$. Then, by the proved (ii), and Proposition 2.1, for $t\le t'\le T$, and $0<q'\le q$, for some $1<q\le p$, we have

$$\begin{aligned}&\mathbb{E}_{t'}\Big[\sup_{s\in[t',T]}|\widehat{Y}(s)-\widehat{Y}'(s)|^{q'}+\Big(\int_{t'}^T|Z(r)-Z'(r)|^2dr\Big)^{\frac{q'}{2}}\\&\quad+\Big(\int_{t'}^T\!\int_{\mathbb{R}}|\Gamma(r,\theta)-\Gamma'(r,\theta)|^2\pi(d\theta)dr\Big)^{\frac{q'}{2}}\Big]\\&\le C\Big[\mathbb{E}_{t'}|X(T)-X'(T)|^q+\mathbb{E}_{t'}\Big(\int_\tau^T|X(r)-X'(r)|dr\Big)^q\\&\quad+\Big(\sum_{t\le\theta_i<t'}|k(\theta_i,a_{i-1},a_i)-k(t',a_{i-1},a_i)|\Big)^q+\Big(\sum_{t\le\tau_j<t'}|\ell(t_j,\xi_j)-\ell(t',\xi_j)|\Big)^q\Big]^{\frac{q'}{q}}\\&\le C\Big[|X(t')-x'|^q+\mathbb{E}_{t'}[N^S(a(\cdot))(t'-t)]^q+\mathbb{E}_{t'}[N^I(\xi(\cdot))(t'-t)]^q\Big]^{\frac{q'}{q}}\\&\le C\mathbb{E}_{t'}\Big(1+|x|^{\delta q'}+\sup_{r\in[t,t']}|\xi(r)|^{\delta q'}+N^S(a(\cdot))^{q'}+N^I(\xi(\cdot))^{q'}\Big)(t'-t)^{\frac{q'}{2}}\end{aligned}$$

Thus, applying $\mathbb{E}_t$, one obtains (2.27). □

3. **Some properties of the value function.**

In this section, we present some basic properties of the value function. First, we have the following result which gives the boundedness of the value function.

Theorem 3.1. *Let* (H1)–(H4) *hold. Then for any* $p\ge1$,

$$0\le V^{m,a}(t,x)\le C_0,\qquad\forall(t,x,m,a)\in\mathcal{D}^p\times\mathbf{M}\times\mathbf{A},\tag{3.1}$$

for some absolute constant $C_0 > 0$ *and*

$$\begin{aligned} V^{m,a}(t,x) = &\inf_{\mathcal{U}[t,T]\times\mathcal{A}_0^{a,1+}[t,T]\times\mathcal{K}_0^{p,1+}[t.T]} J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot)), \\ &\forall(t,x,m,a)\in\mathcal{D}^p\times\mathbf{M}\times\mathbf{A}, \end{aligned} \tag{3.2}$$

with

$$\begin{aligned} \mathcal{A}_0^{a,1+}[t,T] &= \Big\{a(\cdot)=\sum_{i=1}^{N^S(a(\cdot))} a_{i-1}\mathbf{1}_{[\theta_{i-1}\theta_i)}(\cdot)\in\mathcal{A}^{a,1+}[t,T] \;\Big|\; \\ &\qquad k_0\mathbb{E}_t\big[N^S(a(\cdot))\big]\le 2C_0\Big\}, \\ \mathcal{K}_0^{p,1+}[t,T] &= \Big\{\xi(\cdot)=\sum_{j=1}^{N^I(\xi(\cdot))}\xi_j\mathbf{1}_{[\tau_j,T]}(\cdot)\in\mathcal{K}^{p,1+}[t,T] \;\Big|\; \\ &\qquad \ell_0\mathbb{E}_t\big[N^I(\xi(\cdot))+\sup_{s\in[t,T]}|\xi(s)|^\nu\big]\le 2C_0\Big\}. \end{aligned}$$

Proof. Let $(t,x,m,a)\in\mathcal{D}^p\times\mathbf{M}\times\mathbf{A}$. Fix any $u(\cdot)\in\mathcal{U}[t,T]$, the trivial switching control $a_0(\cdot)\in\mathcal{A}^{a,q}[t,T]$ and the trivial impulse control $\xi_0(\cdot)\in\mathcal{K}^{p,q}[t,T]$, for any $q>1$. Let the corresponding state process be $X_0(\cdot)$, and the corresponding adapted solution of BSDE (2.14) be $(\widehat{Y}_0(\cdot),Z_0(\cdot),\Gamma_0(\cdot\,,\cdot))$. By the boundedness of h and g,

$$\begin{aligned} 0 &\le J^{m,a}(t,x;u(\cdot),a_0(\cdot),\xi_0(\cdot)) = \mathbb{E}_t\Big[h(X_0(T),M(T),a) \\ &\quad+\int_t^T g\Big(r,X_0(r),M(r),u(r),a,Y_0(r),Z_0(r),\int_{\mathbb{R}}\Gamma_0(r,\theta)\pi(d\theta)\Big)dr\Big]\le C_0, \end{aligned} \tag{3.3}$$

for some absolute constant $C_0>0$. This proves (3.1). Now, for any $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}^{a,1+}[t,T]\times\mathcal{K}^{p,1+}[t,T]$, by the nonnegativity of h and g, one has

$$\begin{aligned} &J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot)) \\ &=\mathbb{E}_t\Big[h(X(T),M(T),a(T))+\sum_{i=1}^{N^S(a(\cdot))}k(\theta_i,a_{i-1},a_i)+\sum_{j=1}^{N^I(\xi(\cdot))}\ell(\tau_j,\xi_j) \\ &\qquad+\int_s^T g\Big(r,X(r),M(r),u(r),a(r),Y(r),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta)\pi(d\theta)\Big)dr\Big] \\ &\ge\mathbb{E}_t\Big[\sum_{i=1}^{N^S(a(\cdot))}k(\theta_i,a_{i-1},a_i)+\sum_{j=1}^{N^I(\xi(\cdot))}\ell(\tau_j,\xi_j)\Big] \\ &\ge\mathbb{E}_t\Big[k_0N^S(a(\cdot))+\ell_0N^I(\xi(\cdot))+\ell_0\sum_{j=1}^{N^I(\xi(\cdot))}|\xi_j|^\nu\Big] \\ &\ge\mathbb{E}_t\Big[k_0N^S(a(\cdot))+\ell_0N^I(\xi(\cdot))+\ell_0\sup_{s\in[t,T]}|\xi(s)|^\nu\Big]. \end{aligned} \tag{3.4}$$

Therefore, if $(a(\cdot),\xi(\cdot))\notin\mathcal{A}_0^{a,1+}[t,T]\times\mathcal{K}_0^{p,1+}[t,T]$, it must not be optimal. Hence, (3.2) follows. ∎

From Proposition 3.1, we see that for any $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}_0^{a,1+}[t,T]\times\mathcal{K}_0^{p,1+}[t,T]$, one has

$$0\le J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot))\le C,\qquad\forall(t,x)\in\mathcal{D}^p. \tag{3.5}$$

Further, in addition, if $a(\cdot)$ and $\xi(\cdot)$ are trivial on $[t,t')$, one has

$$\begin{aligned}&J^{m,a}\big(t,x;u(\cdot),a(\cdot),\xi(\cdot)\big)\\&=\mathbb{E}_t\Big[J^{M(t'),a}(t',X(t'-);u(\cdot),a(\cdot),\xi(\cdot)\big)\\&\qquad+\int_t^{t'}g\Big(r,X(r),M(r),u(r),a,Y(r),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta)\pi(d\theta)\Big)\Big],\end{aligned}\tag{3.6}$$

Next, we prove that the value function is continuous.

Theorem 3.2. *Let* $(\mathrm{H1})_\delta$, (H2)–(H4) *hold with* $0<\delta\le\nu$. *Then there exists a constant* $C_1>0$ *such that*

$$\begin{aligned}|V^{m,a}(t,x)-\mathbb{E}_t[V^{m,a}(t',x')]|\le C_1\mathbb{E}_t\Big(|x-x'|+(1+|x|^\delta)|t-t'|^{\frac12}\Big),\\ \forall(m,a)\in\mathbf{M}\times\mathbf{A},\ (t,x),(t',x')\in\mathcal{D}^p,\\ |t-t'|\ \textit{small}.\end{aligned}\tag{3.7}$$

Proof. We first let $t'=t$. Thus, for $(t,x),(t,x')\in\mathcal{D}^p$, $(m,a)\in\mathbf{M}\times\mathbf{A}$, and $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}_0^{a,1+}[t,T]\times\mathcal{K}_0^{p,1+}[t,T]$, let $X(\cdot)$ and $X'(\cdot)$ be the corresponding state processes (under the same controls). Also, let $(\widehat Y(\cdot),Z(\cdot),\Gamma(\cdot\,,\cdot))$ and $(\widehat Y'(\cdot),Z'(\cdot),\Gamma'(\cdot\,,\cdot))$ be the adapted solutions of the corresponding BSDE (2.14). Then by Proposition 2.2, (ii) (see (2.25))

$$|J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot))-J^{m,a}(t,x';u(\cdot),a(\cdot),\xi(\cdot)))\le|\widehat Y(t)-\widehat Y'(t)|\le C|x-x'|.$$

Hence, (3.7) follows.

Next, let $0\le t<t'\le T$, $(t,x),(t',x)\in\mathcal{D}^p$, and $(m,a)\in\mathbf{M}\times\mathbf{A}$. For any $\varepsilon>0$, let $(u'(\cdot),a'(\cdot),\xi'(\cdot))\in\mathcal{U}[t',T]\times\mathcal{A}_0^{a,1+}[t',T]\times\mathcal{K}_0^{p,1+}[t',T]$ such that

$$\begin{aligned}V^{M(t'),a}(t',X'(t'-))&\le J^{M(t'),a}(t',X'(t'-);u'(\cdot),a'(\cdot),\xi'(\cdot))\\&<V^{M(t'),a}(t',X'(t'-))+\varepsilon,\end{aligned}$$

where

$$a'(\cdot)=\sum_{i=1}^{N^S(a'(\cdot))}a_{i-1}\mathbf{1}_{[\theta_{i-1},\theta_i)}(\cdot),\qquad\xi'(\cdot)=\sum_{j=1}^{N^I(\xi'(\cdot))}\xi_j\mathbf{1}_{[\tau_j,T]}(\cdot),$$

and $X'(\cdot)$ is the state process satisfying the following:

$$\begin{aligned}X'(s)=x&+\int_t^s b(r,X'(r),M(r),u_0,a)dr\\&+\int_t^s\sigma(r,X'(r),M(r),u_0,a)dW(s),\qquad s\in[t,t'),\end{aligned}$$

for some fixed $u_0\in U$. Also, let $X'(\cdot)$ be the state with the initial tuple $(t'.X'(t'-),M(t'),a(t'-))$, under feasible controls $(u'(\cdot),a'(\cdot),\xi'(\cdot))$ on $[t',T]$, and $((\widehat Y'(\cdot).Z'(\cdot),\Gamma'(\cdot\,,\cdot))$ be the corresponding adapted solution of the BSDE on $[t',T]$, so that the cost functional can be defined. We now extend $(a'(\cdot),\xi'(\cdot))$ trivially on $[t,t')$ to get $(a(\cdot),\xi(\cdot))\in\mathcal{A}_0^{a,1+}[t,T]\times\mathcal{K}_0^{p,1+}[t,T]$, and let $u'(\cdot)=u_0\mathbf{1}_{[t,t')}+u(\cdot)\mathbf{1}_{[t',T]}(\cdot)$.

Correspondingly, we extend the adapted solution $((\widehat{Y}'(\cdot), Z'(\cdot), \Gamma'(\cdot\,,\cdot))$ from $[t', T]$ to $[t, T]$. Then we have (by (2.12))

$$\begin{aligned}
&J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot))\\
&=\mathbb{E}_t\Big[h(X'(T),M(T),a'(T))+\sum_{\theta_i\ge t}k(\theta_i,a_{i-1},a_i)+\sum_{\tau_j\ge t}\ell(\tau_j,\xi_j)\\
&\qquad+\int_t^T g\Big(r,X'(r),M(r),u'(r),a'(r),Y'(r),Z'(r),\int_{\mathbb{R}}\Gamma'(r,\theta)\pi(d\theta)\Big)dr\Big]\\
&=\mathbb{E}_t\Big[h(X'(T),M(T),a'(T))+\sum_{\theta_i\ge t'}k(\theta_i,a_{i-1},a_i)+\sum_{\tau_j\ge t'}\ell(\tau_j,\xi_j)\\
&\qquad+\int_{t'}^T g\Big(r,X'(r),M(r),u'(r),a'(r),Y'(r),Z'(r),\int_{\mathbb{R}}\Gamma'(r,\theta)\pi(d\theta)\Big)dr\\
&\qquad+\int_t^{t'} g\Big(r,X'(r),M(r),u_0,a,Y'(r),Z'(r),\int_{\mathbb{R}}\Gamma'(r,\theta)\pi(d\theta)\Big)dr\Big]\\
&\le\mathbb{E}_t\Big[J^{M(t'),a}(t',X(t'-);u'(\cdot),a'(\cdot),\xi(\cdot))+C(t'-t)\Big]\\
&\le\mathbb{E}_t\Big\{V^{M(t'),a}(t',X(t'-))+\varepsilon+C(t'-t)\Big\}\\
&=\mathbb{E}_t\Big[V^{m,a}(t',X(t'-))+\varepsilon\\
&\qquad+\sum_{m'=1}^{|\mathbf{M}|}[q_{mm'}(t'-t)+o(t'-t)]V^{m',a}(t',X(t'-))+C(t'-t)\Big]\\
&\le\mathbb{E}_t\Big[V^{m,a}(t',x)+C\mathbb{E}_t|X(t'-)-x|+\varepsilon+C(t'-t)\Big]\\
&\le\mathbb{E}_t\Big[V^{m,a}(t',x)+C(1+|x|^\delta)(t'-t)^{\frac{1}{2}}\Big]+\varepsilon.
\end{aligned}$$

This leads to (sending $\varepsilon\to 0$)

$$\mathbb{E}_t\Big[V^{m,a}(t,x)-V^{m,a}(t',x)\Big]\le C(1+|x|^\delta)(t'-t)^{\frac{1}{2}},\quad t'-t>0\text{ small.}\tag{3.8}$$

Next, for any $\varepsilon>0$, let $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}_0^{a,1+}[t,T]\times\mathcal{K}_0^{p,1+}[t,T]$ such that

$$V^{m,a}(t,x)\le J^{m,a}(t,x;u(\cdot),\xi(\cdot),a(\cdot))<V^{m,a}(t,x)+\varepsilon.$$

Now, let $u'(\cdot)=u(\cdot)\big|_{[t',T]}$ and

$$a'(\cdot)=\sum_{i=1}^{N^S(\xi(\cdot))}a_{i-1}\mathbf{1}_{[\theta_{i-1}\vee t',\theta_i\vee t')}(\cdot),\qquad \xi'(\cdot)=\sum_{j=1}^{N^I(\xi(\cdot))}\xi_j\mathbf{1}_{[\tau_j\vee t',T]}(\cdot).$$

Then $(u'(\cdot),a'(\cdot)),\xi'(\cdot))\in\mathcal{U}[t',T]\times\mathcal{A}_0^{a,1+}[t',T]\times\mathcal{K}_0^{p,1+}[t',T]$, and

$$(u'(r),a'(r),\xi'(r))=(u(r),a(r),\xi(r)),\qquad r\in[t',T].$$

Let $X'(\cdot)=X(\cdot\,;t',x,u'(\cdot),a'(\cdot),\xi'(\cdot))$ on $[t',T]$, and $(\widehat{Y}'(\cdot),Z'(\cdot),\Gamma'(\cdot\,,\cdot))$ be the adapted solution to the corresponding BSDE on $[t',T]$. Then, we have (noting the monotonicity of $t \mapsto k(t,a,a')$ and $t \mapsto \ell(t,\xi)$)

$$
\begin{aligned}
&V^{m,a}(t,x)+\varepsilon \geq J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot))\\
&=\mathbb{E}_t\Big[h(X(T),M(T),a(T))+\sum_{\theta_i\geq t}k(\theta_i,a_{i-1},a_i)+\sum_{\tau_j\geq t}\ell(\tau_j,\xi_j)\Big]\\
&\qquad+\int_t^T g\Big(r,X(r),M(r),u(r),a(r),\\
&\qquad\widehat{Y}(r)-\sum_{t\leq\theta_i<r}k(\theta_i,a_{i-1},a_i)-\sum_{\tau\leq\tau_j<r}\ell(\tau_j,\xi_j),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta(\pi(d\theta)\Big)dr\Big]\\
&\geq \mathbb{E}_t\Big[h(X'(T),M(T),a'(T))+\sum_{\theta_i\geq t}k(\theta_i\vee t',a_{i-1},a_i)+\sum_{\tau_j\geq t}\ell(\tau_j\vee t',\xi_j)\\
&\quad+\int_{t'}^T g\Big(r,X'(r),M(r),u'(r),a'(r),\\
&\qquad\widehat{Y}'(r)-\sum_{t\leq\theta_i<r}k(\theta_i\vee t',a_{i-1},a_i)-\sum_{\tau\leq\tau_j<r}\ell(\tau_j\vee t',\xi_j),Z'(r),\int_{\mathbb{R}}\Gamma'(r,\theta(\pi(d\theta)\Big)dr\Big]\\
&\quad-|h(X(T),M(T),a(T))-h(X'(T),M(T),a'(T))|\\
&\quad-\int_{t'}^T\Big|g\Big(r,X(r),M(r),u(r),a(r),\\
&\qquad\widehat{Y}(r)-\sum_{t\leq\theta_i<r}k(\theta_i,a_{i-1},a_i)-\sum_{\tau\leq\tau_j<r}\ell(\tau_j,\xi_j),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta(\pi(d\theta)\Big)\\
&\qquad-g\Big(r,X'(r),M(r),u'(r),a'(r),\\
&\qquad\widehat{Y}'(r)-\sum_{t\leq\theta_i<r}k(\theta_i\vee t',a_{i-1},a_i)-\sum_{\tau\leq\tau_j<r}\ell(\tau_j\vee t',\xi_j),Z'(r),\int_{\mathbb{R}}\Gamma'(r,\theta(\pi(d\theta)\Big)\Big|dr\Big]\\
&\qquad+\int_t^{t'}g\Big(r,X(r),M(r),u(r),a(r),Y(r),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta(\pi(d\theta)\Big)dr\Big]\\
&\geq \mathbb{E}_t\Big\{J^{M(t'),a}(t',x,u'(\cdot),a'(\cdot),\xi'(\cdot))-C|X(T)-X'(T)|\\
&\qquad-C\Big[\int_{t'}^T\Big(|X(r)-X'(r)|+|\widehat{Y}(r)-\widehat{Y}'(r)|+|Z(r)-Z'(r)|\\
&\qquad+\int_{\mathbb{R}}|\Gamma(r,\theta)-\Gamma'(r,\theta)|\pi(d\theta)\Big)dr+\sum_{i-1}^{N^S(a(\cdot))}\Big|k(\theta_i,a_{i-1},a_i)-k(\theta_i\vee t',a_{i-1},a_i)\Big|\\
&\qquad+\sum_{j=1}^{N^I(\xi(\cdot))}\Big|\ell(\tau_j,\xi_j)-\ell(\tau_j\vee t'.\xi_j)\Big|\Big]-C(t'-t)\Big\}\\
&\geq \mathbb{E}_t\Big\{V^{M(t'),a}(t',x)-C\Big[\sup_{r\in[t',T]}|X(r)-X'(r)|+\sup_{r\in[t',T]}|\widehat{Y}(r)-\widehat{Y}'(r)|\\
&\qquad+\Big(\int_{t'}^T|Z(r)-Z'(r)|^2dr\Big)^{\frac12}+\Big(\int_{t'}^T\int_{\mathbb{R}}|\Gamma(r-,\theta)-\Gamma'(r-,\theta)|^2\pi(d\theta)dr\Big)^{\frac12}\Big]\Big\}\\
&\qquad-C\Big(N^S(a(\cdot))+N^I(\xi(\cdot))\Big)(t'-t)-C(t'-t)\\
&\geq \mathbb{E}_t\Big[V^{m,a}(t',x)+\sum_{m'=1}^{|\mathbf{M}|}[q_{mm'}(t'-t)+o(t'-t)]V^{m',a}(t',x)-C|X(t')-X'(t')|
\end{aligned}
$$

$$
\begin{aligned}
&-\Big(1+|x|^\delta+\sup_{r\in[t,t']}|\xi(r)|^\delta+N^S(a(\cdot))+N^I(\xi(\cdot))\Big)(t'-t)^{\frac12}\Big]-C(t'-t)\\
&\geq \mathbb{E}_t\Big[V^{m,a}(t',x)-C(1+|x|^\delta)(t'-t)^{\frac12}\Big].
\end{aligned}
$$

In the above, we have used Theorem 3.1 (This is the place where we need the sublineariy of b and σ in x).

$$
\mathbb{E}_t\Big[\sup_{r\in[t,T]}|\xi(r)|^\delta+N^S(a(\cdot))+N^I(\xi(\cdot))\Big]\leq C.
$$

Sending $\varepsilon\to 0$, we obtain

$$
\mathbb{E}_t\Big[V^{m,a}(t',x)-V^{m,a}(t,x)\Big]\leq C(1+|x|^\delta)(t'-t)^{\frac12},\qquad t'-t>0 \text{ small.} \tag{3.9}
$$

Combining the above with (3.8), we complete the proof. □

Because of our framework, applying a similar arguments of Peng [9] (see also [5, 24]), we are able to show that $V^{m,a}(t,x)$ is a real-valued function on $[0,T]\times\mathbb{R}^n$. Then, from the above, we see that the value function $V^{m,a}(t,x)$ is continuous in $(t,x)\in[0,T]\times\mathbb{R}^n$.

4. **Dynamic programming principle.** In this section, we are going to present the dynamic programming principle for our Problem (C). We have the following form of the principle.

Theorem 4.1. *Let* $(\mathrm{H1})_\delta$, (H2)–(H4) *hold with* $0<\delta\leq\nu$. *Then*

$$
\begin{aligned}
V^{m,a}(t,x)\leq \min_{a'\neq a}\Big[V^{m,a'}(t,x)+k(t,a,a')\Big]\equiv \mathbf{N}^S[V^{m,a}](t,x),\\
(t,x,m,a)\in[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{A},
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
V^{m,a}(t,x)\leq \inf_{\xi\in K}\Big[V^{m,a}(t,x+\xi)+\ell(t,\xi)\Big]\equiv \mathbf{N}^I[V^{m,a}](t,x),\\
(t,x,m,a)\in[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{A},
\end{aligned} \tag{4.2}
$$

and for any $0\leq t<t'<T$,

$$
\begin{aligned}
&V^{m,a}(t,x)\\
&\leq \inf_{u(\cdot)\in\mathcal{U}[t,t']}\mathbb{E}_t\Big[\int_t^{t'}g\Big(r,X^0(r),M(r),u(r),a,Y^0(r),Z^0(r),\int_{\mathbb{R}}\Gamma^0(r,\theta)\pi(d\theta)\Big)dr\\
&\quad+V^{M(t'),a}(t',X^0(t'-))\Big],\quad (x,m,a)\in\mathbb{R}^n\times\mathbf{M}\times\mathbf{A},
\end{aligned} \tag{4.3}
$$

where

$$
\begin{aligned}
X^0(s)=x+\int_t^s b(r,X^0(r),M(r),u(r),a)dr\\
+\int_t^s\sigma(r,X^0(r),M(r),u(r),a)dW(r),\qquad s\in[t,t'),
\end{aligned} \tag{4.4}
$$

and

$$\begin{aligned}Y^0(s) = & V^{M(t'),a}(t',X^0(t'-))\\ & +\int_s^{t'} g(r,X^0(r),M(r),u(r),a,Y^0(r),Z^0(r),\int_{\mathbb{R}}\Gamma^0(r,\theta)\pi(d\theta))dr \\ & -\int_s^{t'} Z^0(r)dW(r)-\int_s^{t'}\Gamma^0(r-,\theta)\widetilde N(d\theta,dr),\quad s\in[t,t').\end{aligned} \tag{4.5}$$

Moreover. if in (4.1) *and* (4.2)*, the strict inequalities hold at* $(t,x,m,a)\in[0,T]\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{A}$*, then there exists a* $\tau\in(t,T]$ *such that*

$$\begin{aligned}& V^{m,a}(t,x)\\ & \geq \inf_{u(\cdot)\in\mathcal{U}[t,t']}\mathbb{E}_t\Big[\int_t^{t'} g\Big(r,X^0(r),M(r),u(r),a,Y^0(r),Z^0(r),\int_{\mathbb{R}}\Gamma^0(r,\theta)\pi(d\theta)\Big)dr \\ & \quad +V^{M(t'),a}(t',\widetilde X(t'))\Big],\qquad t'\in(t,\tau).\end{aligned} \tag{4.6}$$

where, $(X^0(\cdot),Y^0(\cdot),Z^0(\cdot),\Gamma^0(\cdot\,,\cdot))$ *is determined by* (4.4) *and* (4.5)*.*

Proof. Let $(t,x,m,a)\in[0,T)\times\mathbb{R}^n\times\mathbf{M}\times\mathbf{A}$ be given, and $a'\in\mathbf{A}\setminus\{a\}$. For any $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}^a[t,T]\times\mathcal{K}^p[t,T]$, we have

$$V^{m,a}(t,x)\leq J^{m,a}(t,x;u(\cdot),\xi(\cdot),a'(\cdot))=J^{m,a'}(t,x;u(\cdot),\xi(\cdot),a(\cdot))+k(t,a,a'),$$

where $a'(\cdot)=\{(a_i',\theta_i')\}_{i\geq1}$ with

$$a_0'=a',\ \theta_0'=t,\quad a_{i+1}'=a_i,\ \theta_{i+1}'=\theta_i,\ i\geq0,$$

or formally, $a'(\cdot)=a'\mathbf{1}_{[t,t)}(\cdot)+a(\cdot)$. Thus,

$$V^{m,a}(t,x)\leq\min_{a'\neq a}\Big[V^{m,a'}(t,x)+k(t,a,a')\Big]\equiv\mathbf{N}^S[V^{m,a}](t,x),$$

proving (4.1). Next, let $\xi'\in K$, and

$$\xi'(\cdot)=\xi'\mathbf{1}_{[t,T]}(\cdot)+\xi(\cdot).$$

Then

$$V^{m,a}(t,x)\leq J^{m,a}(t,x;u(\cdot),\xi'(\cdot),a(\cdot))=J^{m,a}(t,x+\xi;u(\cdot),\xi(\cdot),a(\cdot))+\ell(t,\xi).$$

Thus,

$$V^{m,a}(t,x)\leq\inf_{\xi\in K}\Big[V^{m,a}(t,x+\xi)+\ell(t,\xi)\Big]\equiv\mathbf{N}^I[V^{m,a}](t,x),$$

proving (4.2). Next, for given $0\leq t<t'<T$, take any $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathcal{A}^a_{t'}[t,T]\times\mathcal{K}^p_{t'}[t,T]$, we have

$$\begin{aligned}& V^{m,a}(t,x)\leq J^{m,a}\big(t,x;u(\cdot),a(\cdot),\xi(\cdot)\big)\\ & =\mathbb{E}_t\Big[J^{M(t'),a}\big(t',X^0(t'-);u(\cdot)\big|_{[t',T]},a(\cdot)|_{[t',T]},\xi(\cdot)\big|_{[t',T]}\big)\\ & \qquad+\int_t^{t'} g\Big(r,X^0(r),M(r),u,a,Y^0(r),Z^0(r),\int_{\mathbb{R}}\Gamma^0(r,\theta)\pi(d\theta)\Big)dr\Big].\end{aligned} \tag{4.7}$$

Then, taking infimum with respect to $(u(\cdot),a(\cdot),\xi(\cdot))\in\mathcal{U}[t',T]\times\mathcal{A}^a[t',T]\times\mathcal{K}^p[t',T]$, we obtain (4.3).

Finally, let the strict inequalities hold in the (4.1)–(4.2). Then, by the continuity of the value function, there exists a $\rho > 0$ such that

$$V^{m,a}(t',x') < \min\{\mathbf{N}^S[V^{m,a}](t',x'), \mathbf{N}^I[V^{m,a}](t',x')\}, \qquad |t-t'|+|x-x'|<\rho.$$

Hence, one can find a $\tau \in (t,T)$ such that starting from (t,x,m,a), within $[t,\tau]$, if $(a(\cdot),\xi(\cdot))$ is not trivial, then for any $u(\cdot) \in \mathcal{U}[t,T]$, the triple $(u(\cdot),a(\cdot),\xi(\cdot))$ cannot be optimal. Therefore, for $t' \in (t,\tau]$, we have

$$V^{m,a}(t,x) = \inf_{\mathcal{U}[t,T]\times\mathcal{A}^a_{t'}[t,T]\times\mathcal{K}^p_{t'}[t,T]} J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot)),$$

where

$$\begin{aligned}
\mathcal{A}^a_{t'}[t,T] &= \Big\{a(\cdot) \in \mathcal{A}^a[t,T] \;\Big|\; a(\cdot) \text{ is trivial on } [t,t')\Big\},\\
\mathcal{K}^p_{t'}[t,T] &= \Big\{\xi(\cdot) \in \mathcal{K}^p[t,T] \;\Big|\; \xi(\cdot) \text{ is trivial on } [t,t')\Big\}.
\end{aligned} \tag{4.8}$$

Note that for any $(u(\cdot),a(\cdot),\xi(\cdot)) \in \mathcal{U}[t,T]\times\mathcal{A}^a_{t'}[t,T]\times\mathcal{K}^p_{t'}[t,T]$, one has

$$\begin{aligned}
&J^{m,a}(t,x;u(\cdot),a(\cdot),\xi(\cdot))\\
&= \mathbb{E}_t\Big[h(X(T),M(T),a(T)) + \sum_{i=1}^{N^S(a(\cdot))} k(\theta_i,a_{i-1},a_i) + \sum_{j=1}^{N^I(\xi(\cdot))} \ell(\tau_j,\xi_j)\\
&\quad + \int_t^T g\Big(r,X(r),M(r),u(r),a(r),Y(r),Z(r),\int_{\mathbb{R}}\Gamma(r,\theta)\pi(d\theta)\Big)dr\Big]\\
&= \mathbb{E}_t\Big[J^{m,a}(t',X^0(t'-);u(\cdot),a(\cdot),\xi(\cdot))\\
&\quad + \int_t^{t'} g\Big(r,X^0(r),M(r),u(r),a(r),Y^0(r),Z^0(r),\int_{\mathbb{R}}\Gamma^0(r,\theta)\pi(d\theta)\Big)dr\Big]\\
&\geq \mathbb{E}_t\Big[V^{M(t'),a}(t',X^0(t'-))\\
&\quad + \int_t^{t'} g\Big(r,X^0(r),M(r),u(r),a,Y^0(r),Z^0(r),\int_{\mathbb{R}}\Gamma^0(r,\theta)\pi(d\theta)\Big)dr\Big].
\end{aligned}$$

Then by taking infimum over $\mathcal{U}[t,T]\times\mathcal{A}^a_{t'}[t,T]\times\mathcal{K}^p_{t'}[t,T]$, we get (4.6). ☐

Note that (4.3) can be written as

$$\begin{aligned}
&0 \leq \inf_{u(\cdot)\in\mathcal{U}[t,t']}\Big[V^{M(t'),a}(t',X^0(t'-)) - V^{m,a}(t,x)\\
&\qquad + \int_t^{t'} g\Big(r,X^0(r),M(r),u(r),a,Y^0(r),Z^0(r),\int_{\mathbb{R}}\Gamma^0(r,\theta)\pi(d\theta)\Big)dr\\
&\qquad - \int_t^{t'} Z^0(r)dW(r) - \int_t^{t'}\int_{\mathbb{R}}\Gamma^0(r-,\theta)\widetilde{N}(d\theta,dr)\Big],\\
&\quad (x,m,a) \in \mathbb{R}^n\times\mathbf{M}\times\mathbf{A},
\end{aligned} \tag{4.9}$$

and (4.6) can be written as

$$
\begin{aligned}
0 \geq \inf_{u(\cdot)\in\mathcal{U}[t,t']} & \Big[V^{M(t'),a}(t', X^0(t'-)) - V^{m,a}(t,x) \\
& + \int_t^{t'} g\Big(r, X^0(r), M(r), u(r), a, Y^0(r), Z^0(r), \int_{\mathbb{R}} \Gamma^0(r,\theta)\pi(d\theta)\Big) dr \\
& - \int_t^{t'} Z^0(r) dW(r) - \int_t^{t'} \int_{\mathbb{R}} \Gamma^0(r-,\theta)\widetilde{N}(d\theta, dr)\Big], \\
& (x, m, a) \in \mathbb{R}^n \times \mathbf{M} \times \mathbf{A},
\end{aligned}
\tag{4.10}
$$

Hence,

$$
\begin{aligned}
0 = \inf_{u(\cdot)\in\mathcal{U}[t,t']} & \Big[V^{M(t'),a}(t', X^0(t'-)) - V^{m,a}(t,x) \\
& + \int_t^{t'} g\Big(r, X^0(r), M(r), u(r), a, Y^0(r), Z^0(r), \int_{\mathbb{R}} \Gamma^0(r,\theta)\pi(d\theta)\Big) dr \\
& - \int_t^{t'} Z^0(r) dW(r) - \int_t^{t'} \int_{\mathbb{R}} \Gamma(r-,\theta)\widetilde{N}(d\theta, dr)\Big], \\
& (x, m, a) \in \mathbb{R}^n \times \mathbf{M} \times \mathbf{A},
\end{aligned}
\tag{4.11}
$$

This is the dynamic programming principle without conditional expectation.

5. **HJB quasi-variational inequality and viscosity solution.** In this section, we will derive the Hamilton-Jacobi-Bellman quasi-variational inequality (HJB-QVI, for short) for our value function, and show that the value function is a viscosity solution of such a HJB-QVI. To begin with, we first present the following lemma, see [16].

Lemma 5.1. *Let $(X(\cdot), M(\cdot))$ satisfy (1.1) and (1.4). Let $\varphi : [0,T]\times\mathbb{R}^n\times\mathbf{M}\to\mathbb{R}$ be smooth, Then*

$$
\begin{aligned}
& \varphi(s, X(s), M(s)) - \varphi(t, X(t), M(t)) = \int_t^s \Big\{\varphi_r(r, X(r), M(r)) \\
& \quad + \frac{1}{2}\mathrm{tr}\Big[\varphi_{xx}(r, X(r), M(r))\sigma(r, X(r), M(r), u(r), a)\sigma(r, X(r), M(r), u(r), a)^\top\Big] \\
& \quad + \varphi_x(r, X(r), M(r)) b(r, X(r), M(r), u(r), a) + \sum_{m'=1}^{|\mathbf{M}|} q_{M(r)m'}\varphi(r, X(r), m')\Big\} dr \\
& \quad + \int_t^s \varphi_x(r, X(r), M(r))\sigma(r, X(r), M(r), u(r), a) dW(r) \\
& \quad + \int_t^s \int_{\mathbb{R}} \Big[\varphi\big(r, X(r), M(r-) + \mu(M(r-),\theta)\big) - \varphi(r, X(r), M(r-))\Big]\widetilde{N}(d\theta, dr) \\
& \equiv \int_t^s \Big(\varphi_r + \mathbb{A}^{u,a}\varphi\Big) dr + \int_t^r \varphi_x \sigma dW(r) + \int_t^s \int_{\mathbb{R}} \triangle_\theta \varphi \widetilde{N}(d\theta, dr),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbb{A}^{u,a}\varphi(t,x,m)\\
&=\frac{1}{2}\mathrm{tr}\,[\varphi_{xx}(t,x,m)\sigma(t,x,m,a)\sigma(t,x,m,a)^\top]\\
&\quad+\varphi_x(t,x,m)b(t,x,m,a)+\sum_{m'=1}^{|\mathbf{M}|}q_{mm'}\varphi(t,x,m'),\\
&\triangle_\theta\varphi(r,X(r),M(r-))\\
&=\varphi\big(r,X(r),M(r-)+\mu(M(r-),\theta)\big)-\varphi(r,X(r),M(r-))\\
&=\sum_{m'=1}^{|\mathbf{M}|}\Big[\varphi(r,X(r),m')-\varphi(r,X(r),M(r-))\Big]\mathbf{1}_{\Delta_{M(r-)m'}}(\theta).
\end{aligned}
\tag{5.1}
$$

Clearly, one has

$$
\begin{aligned}
\int_{\mathbb{R}}\triangle_\theta\varphi(t,x,m)\pi(d\theta)&=\sum_{m'=1}^{|\mathbf{M}|}\Big[\varphi(t,x,m')-\varphi(t,x,m)\Big]q_{mm'}\\
&=\sum_{m'=1}^{|\mathbf{M}|}q_{mm'}\varphi(t,x,m').
\end{aligned}
\tag{5.2}
$$

This will be used below.

Next, we present a useful lemma below. In the case that the Poisson process is absent, such a result was proved in [11].

Lemma 5.2. *Suppose η, $\zeta(\cdot)$, and $\gamma(\cdot\,,\cdot)$ are proper random variable or processes. Then the following holds, as long as each term makes sense:*

$$
\begin{aligned}
&\mathbb{E}\Big[|\eta|^p+\Big(\int_t^{t'}|\zeta(r)|^2dr\Big)^{\frac{p}{2}}+\Big(\int_t^{t'}\int_{\mathbb{R}}\gamma(r-,\theta)|^2\pi(d\theta)dr\Big)^{\frac{p}{2}}\Big]\\
&\le K\mathbb{E}\Big|\eta+\int_t^{t'}\zeta(r)dW(r)+\int_t^{t'}\int_{\mathbb{R}}\gamma(r-,\theta)\widetilde{N}(d\theta,dr)\Big|^p.
\end{aligned}
\tag{5.3}
$$

Proof. For fixed $0\le t<t'\le T$, let

$$
\xi=\eta+\int_t^{t'}\zeta(r)dW(r)+\int_t^{t'}\int_{\mathbb{R}}\gamma(r-\theta)\widetilde{N}(d\theta,dr),
$$

which is $\mathcal{F}_t$-measurable. Let $(Y(\cdot),Z(\cdot),\Gamma(\cdot\,,\cdot))$ be the adapted solution to the following BSDE:

$$
Y(s)=\xi-\int_s^{t'}Z(r)dW(r)-\int_s^{t'}\int_{\mathbb{R}}\Gamma(r-,\theta)\widetilde{N}(d\theta,dr),\qquad r\in[t,t'].
$$

Then we have

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[t,t']}|Y(s)|^p+\Big(\int_t^{t'}|Z(r)|^2dr\Big)^{\frac{p}{2}}+\Big(\int_t^{t'}\int_{\mathbb{R}}|\Gamma(r-,\theta)|^2\pi(d\theta)dr\Big)^{\frac{p}{2}}\Big]\\
&\le K\mathbb{E}|\xi|^p.
\end{aligned}
\tag{5.4}
$$

Now,

$$\begin{aligned} &Y(t)+\int_t^{t'}Z(r)dW(r)+\int_t^{t'}\int_{\mathbb{R}}\Gamma(r-.\theta)\widetilde N(d\theta,dr)=\xi\\ &=\eta+\int_t^{t'}\zeta(r)dW(r)+\int_t^{t'}\int_{\mathbb{R}}\gamma(r-,\theta)\widetilde N(d\theta,dr).\end{aligned}$$

By taking conditional expectation $\mathbb{E}_t[\,\cdot\,]$, we see that

$$Y(t)=\eta.$$

Consequently,

$$\int_t^{t'}\Big(Z(r)-\zeta(r)\Big)dW(r)+\int_t^{t'}\int_{\mathbb{R}}\Big(\Gamma(r-,\theta)-\gamma(r-,\theta)\Big)\widetilde N(d\theta,dr)=0,$$

which leads to

$$\begin{aligned} Z(r)&=\zeta(r), && r\in[t,t'],\ \text{a.s.}\\ \Gamma(r-,\theta)&=\gamma(r-,\theta), && r\in[t,t'],\ \theta\in\mathbb{R},\ \text{a.s.}\end{aligned}$$

Then (5.3) follows from (5.4). ☐

For the terminal cost, we need the following compatibility condition.

(H5) The following hold:

$$h(x,m,a)\le\min\big\{\mathbf{N}^S[\cdot\,,m,a](x),\mathbf{N}^I[h(\cdot\,,m,a)](x)\big\},\tag{5.5}$$

where

$$\begin{aligned} \mathbf{N}^S\big[h(\cdot\,,m,a)\big](x)&=\min_{a;\neq a}\Big[h(x,m,a')+k(T,a,a')\Big],\\ \mathbf{N}^I\big[h(\cdot\,,m,a)\big](x)&=\inf_{\xi\in K}\Big[h(x+\xi,m,a)+\ell(T,\xi)\Big].\end{aligned}$$

The above condition means that a switching or an impulse at T is not necessary.

We now formally derive the HJB-QVI.

Proposition 5.3. *Let* (H1)–(H5) *hold. Suppose $V^{m,a}(t,x)$ is smooth in $(t,x)\in[0,T]\times\mathbb{R}^n$, for all $(m,a)\in\mathbf{M}\times\mathbf{A}$. Then it satisfies the following HJB-QVI:*

$$\left\{\begin{aligned} &\max\Big\{V_t^{m,a}(t,x)+\mathbb{H}^{m,a}\Big(t,x,V^{m,a}(t,x),V_x^{m,a}(t,x),V_{xx}^{m,a}(t,x),\\ &\qquad\sum_{m'=1}^{|\mathbf{M}|}q_{mm'}V^{m',a}(t,x)\Big)+\sum_{m'=1}^{|\mathbf{M}|}q_{mm'}V^{m',a}(t,x),\\ &\qquad V^{m,a}(t,x)-\mathbf{N}^S[V^{m,a}](t,x),V^{m,a}(t,x)-\mathbf{N}^I[V^{m,a}](t,x)\Big\}=0,\\ &\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\quad (m,a)\in\mathbf{M}\times\mathbf{A},\\ &V^{m,a}(T,x)=h^{m,a}(x),\qquad x\in\mathbb{R}^n,\quad (m,a)\in\mathbf{M}\times\mathbf{A}.\end{aligned}\right.\tag{5.6}$$

where

$$\begin{aligned} &\mathbb{H}^{m,a}(t,x,V,p,P,\gamma)\\ &=\inf_{u\in U}\Big\{\frac12\mathrm{tr}\Big[P\sigma(t,x,m,u,a)\sigma(t,x,m,u,a)^\top\Big]+pb(t,x,m,u,a)\\ &\quad+g(t,x,m,a,V,p\sigma(t,x,m,u,a),\gamma)\Big\},\end{aligned}$$

and

$$h^{m,a}(x) = h(x, m, a).$$

Proof. First, from Theorem 4.1, we see that

$$V^{m,a}(t,x) \le \min\{\mathbf{N}^S[V^{m,a}](t,x), \mathbf{N}^I[V^{m,a}](t,x)\}. \tag{5.7}$$

Thus, it suffices to consider the case that the strict inequality holds in (5.7), i.e., the strict inequalities hold in (4.1) and (4.2). Then, (4.11) holds, for some $\tau \in (t,T]$. Hence, for $t' \in (t,\tau)$,

$$\begin{aligned}
0 = \inf_{u(\cdot)\in\mathcal{U}[t,t']} \Big\{ \int_t^{t'} & \Big[g\Big(r, X^0(r), M(r), u(r), a, Y^0(r), Z^0(r), \int_{\mathbb{R}} \Gamma^0(r,\theta)\pi(d\theta)\Big) \\
& + V_r^{M(r),a}(r, X^0(r)) + \mathbb{A}^{u(\cdot)} V^{M(r),a}(r, X^0(r))\Big] dr \\
& + \int_t^{t'} \Big(V_x^{M(t'),a}(r, X^0(r))\sigma(r, X^0(r), u(r), r) - Z^0(r)\Big) dW(r) \\
& + \int_t^{t'}\int_{\mathbb{R}} \Big(\triangle V^{M(t'),a}(r, X^0(r)) - \Gamma^0(r-,\theta)\Big)\widetilde{N}(d\theta, dr)\Big\}.
\end{aligned}$$

Thus, for any $\varepsilon > 0$, there exists a $u_\varepsilon(\cdot) \in \mathcal{U}[t,t']$, which is continuous such that

$$\begin{aligned}
0 \le \widetilde{Y}_\varepsilon(t) \equiv \int_t^{t'} & \Big[g\Big(r, X_\varepsilon^0(r), M(r), u_\varepsilon(r), a, Y_\varepsilon^0(r), Z_\varepsilon^0(r), \int_{\mathbb{R}} \Gamma_\varepsilon^0(r,\theta)\pi(d\theta)\Big) \\
& + V_s^{M(r),a}(r, X_\varepsilon^0(r)) + \mathbb{A}^{u_\varepsilon(\cdot)} V^{M(r),a}(r, X_\varepsilon^0(r))\Big] dr \\
& + \int_t^{t'} \Big(V_x^{M(t'),a}(r, X_\varepsilon^0(r))\sigma(r, X_\varepsilon^0(r), u_\varepsilon(r), a) - Z_\varepsilon^0(r)\Big) dW(r) \\
& + \int_t^{t'}\int_{\mathbb{R}} \Big(\triangle_\theta V^{M(t'),a}(r, X_\varepsilon^0(r)) - \Gamma_\varepsilon^0(r-,\theta)\Big)\widetilde{N}(d\theta, dr) < \varepsilon.
\end{aligned}$$

Then, applying Lemma 5.2, we obtain

$$\begin{aligned}
& \mathbb{E}\Big\{\Big|\int_t^{t'} \Big[g\Big(r, X_\varepsilon^0(r), M(r), u_\varepsilon(r), a, Y_\varepsilon^0(r), Z_\varepsilon^0(r), \int_{\mathbb{R}} \Gamma_\varepsilon^0(r,\theta)\pi(d\theta)\Big) \\
& \quad + V_s^{M(r),a}(r, X_\varepsilon^0(r)) + \mathbb{A}^{u_\varepsilon(\cdot)} V^{M(r),a}(r, X_\varepsilon^0(r))\Big] ds\Big|^p \\
& \quad + \Big(\int_t^{t'} |V_x^{M(r),a}(r, X_\varepsilon^0(r))\sigma(r, X_\varepsilon^0(r), u_\varepsilon(r), a) - Z_\varepsilon^0(r)|^2 dr\Big)^{\frac{p}{2}} \\
& \quad + \int_t^{t'}\int_{\mathbb{R}} |\triangle_\theta V^{M(r),a}(r, X_\varepsilon^0(r)) - \Gamma_\varepsilon^0(r-,\theta)|^2 \pi(d\theta) dr\Big)^{\frac{p}{2}}\Big\} \\
& \le K\mathbb{E}|\widetilde{Y}_\varepsilon(t)|^p < \varepsilon^p.
\end{aligned}$$

Hence, for small $\varepsilon > 0$, we formally have (for some $u \in U$)

$$\begin{aligned}
Z_\varepsilon^0(t) &\approx V_x^{m,a}(t,x)\sigma(t,x,u,a), \\
\int_{\mathbb{R}} \Gamma_1^0(r-,\theta)\pi(d\theta) &\approx \int_{\mathbb{R}} [V^{M(r-)+\mu(M(r-),\theta),a}(t,x) - V^{m,a}(t,x)]\pi(d\theta) \\
&= \sum_{m.=1}^{|\mathbf{M}|} q_{mm'} V^{m',a}(t,x).
\end{aligned}$$

Consequently, at the limit (as $\varepsilon \to 0$)

$$\begin{aligned} & V_t^{m,a}(t,x) + \mathbb{A}^u V^{m,a}(t,x) \\ & \quad + g\Big(t,x,m,u,a,V^{m,a}(t,x),V_x^{m,a}(t,x)\sigma(t,x,u,a),\sum_{m'=1}^{|\mathbf{M}|} q_{mm'}V^{m',a}(t,x)\Big) = 0. \end{aligned}$$

Then, we formally have derived our HJB-QVI (5.6). □

The above is very formal, the purpose of which is to obtain the form of the HJB-QVI (5.6). We now want to make the above rigorous.

In what follows, we will denote

$$\mathbf{V}^a(t,x) = \begin{pmatrix} V^{1,a}(t,x) \\ V^{2,a}(t,x) \\ \vdots \\ V^{|\mathbf{M}|,a}(t,x) \end{pmatrix}, \quad \mathbf{h}^a(x) = \begin{pmatrix} h^{1,a}(t,x) \\ h^{2,a}(t,x) \\ \vdots \\ h^{|\mathbf{M}|,a}(t,x) \end{pmatrix}, \quad \boldsymbol{\varphi}(t,x) = \begin{pmatrix} \varphi^{1}(t,x) \\ \varphi^{2}(t,x) \\ \vdots \\ \varphi^{|\mathbf{M}|}(t,x) \end{pmatrix}.$$

Next, by saying $\mathbf{V}^a(t,x) \geq \boldsymbol{\varphi}(t,x)$, we mean the componentwise inequalities:

$$V^{m,a}(t,x) \geq \varphi^m(t,x), \qquad 1 \leq m \leq |\mathbf{M}|.$$

Also, $\mathbf{V}^a(t,x) \leq \boldsymbol{\varphi}(t,x)$ has the similar meaning.

The following definition is similar to that given in [4].

Definition 5.4. (i) Function $\mathbf{V}^a(\cdot\,,\cdot)$, $a \in \mathbf{A}$, is called a *viscosity sub-solution* of (5.6) if

$$\mathbf{V}^a(T,x) \leq \mathbf{h}^a(T,x), \qquad \forall x \in \mathbb{R}^n, \tag{5.8}$$

and for any smooth function $\boldsymbol{\varphi}(\cdot\,,\cdot)$ with

$$\begin{aligned} V^{m,a}(t,x) - \varphi^m(t,x) = 0, \\ \mathbf{V}^a(t',x') - \boldsymbol{\varphi}(t',x') \leq 0, \end{aligned} \qquad \forall (t',x') \in [0,T] \times \mathbb{R}^n. \tag{5.9}$$

it holds

$$\begin{aligned} \max\Big\{ & \varphi_t^m(t,x) + \mathbb{H}^{m,a}\Big(t,x,\varphi^m(t,x),\varphi_x^m(t,x),\varphi_{xx}^m(t,x), \\ & \sum_{m'=1}^{|\mathbf{M}|} q_{mm'}V^{m',a}(t,x)\Big) + \sum_{m'=1}^{|\mathbf{M}|} q_{mm'}[V^{m',a}(t,x)], \\ & V^{m,a}(t,x) - \mathbf{N}^S[V^{m,a}](t,x), V^{m,a}(t,x) - \mathbf{N}^I[V^{m,a}](t,x)\Big\} \geq 0. \end{aligned}$$

(ii) Function $\mathbf{V}^a(\cdot\,,\cdot)$, $a \in \mathbf{A}$, is called a *viscosity super-solution* of (5.6) if

$$\mathbf{V}^a(T,x) \geq \mathbf{h}^a(T,x), \qquad \forall x \in \mathbb{R}^n, \tag{5.10}$$

and for any smooth function $\boldsymbol{\varphi}(\cdot\,,\cdot)$ with

$$\begin{aligned} V^{m,a}(t,x) - \varphi^m(t,x) = 0, \\ \mathbf{V}^a(t',x') - \boldsymbol{\varphi}(t',x') \geq 0, \end{aligned} \qquad \forall (t',x') \in [0,T] \times \mathbb{R}^n, \tag{5.11}$$

it holds

$$\begin{aligned}&\max\Big\{\varphi_t^m(t,x)+\mathbb{H}^{m,a}\Big(t,x,\varphi^m(t,x),\varphi_x^m(t,x),\varphi_{xx}^m(t,x),\\&\sum_{m'=1}^{|\mathbf{M}|}q_{mm'}V^{m',a}(t,x)\Big)+\sum_{m'=1}^{|\mathbf{M}|}q_{mm'}[V^{m',a}(t,x)],\\&V^{m,a}(t,x)-\mathbf{N}^S[V^{m,a}](t,x),V^{m,a}(t,x)-\mathbf{N}^I[V^{m,a}](t,x)\Big\}\le 0.\end{aligned}$$

(iii) Function $\mathbf{V}^a(\cdot\,,\cdot)$, $a\in\mathbf{A}$, is a *viscosity solution* of (5.6), if it is both viscosity sub-solution and viscosity super-solution of (5.6).

We now present the following result.

Theorem 5.5. *Let* (H1)$_\delta$, (H2)–(H5) *hold with* $0<\delta\le\nu$. *Then* $V^{m,a}(\cdot\,,\cdot)$ *is a viscosity solution of HJB-QVI* (5.6).

Proof. We prove that $V^{m,a}(t,x)$ is a viscosity subsolution to (5.6). Clearly, we need only to show the case that the strict inequality holds in (5.7). Then, there exists a $\tau\in(t,T]$ such that for $t'\in(t,\tau)$, on $[t,t']$, the switching and impulse controls must be trivial (do not play any role).

Now, let $\boldsymbol{\varphi}(\cdot)$ be smooth so that (5.8) and (5.9) hold. For any $u(\cdot)\in\mathcal{U}]t,t']$, let $(X(\cdot),Y(\cdot),Z(\cdot),\Gamma(\cdot\,,\cdot))$ be the state process and the adapted solution of the relevant BSDE on $[t,t']$, corresponding to (t,x) and $u(\cdot)$. Then,

$$V^{M(t'),a}(t',X(t'))-\varphi^{M(t')}(t',X(t'))\le V^{m,a}(t,x)-\varphi^m(t,x)=0,\tag{5.12}$$

which implies

$$V^{M(t'),a}(t,X(t'))\le\varphi^{M(t')}(t',X(t')).\tag{5.13}$$

Next, we introduce two BSDEs:

$$\begin{aligned}Y^V(s)=&\,V^{M(t'),a}(t',X(t'))\\&+\int_s^{t'}g\Big(r,X(r),M(r),u(r),a,Y^V(r),Z^V(r),\int_{\mathbb{R}}\Gamma^V(r,\theta)\pi(d\theta)\Big)dr\\&-\int_s^{t'}Z^V(r)dW(r)-\int_s^{t'}\!\int_{\mathbb{R}}\Gamma^V(r-,\theta)\widetilde{N}(d\theta,dr),\quad s\in[t,t'],\end{aligned}\tag{5.14}$$

and

$$\begin{aligned}Y^\varphi(s)=&\,\varphi^{M(t')}(t',X(t'))\\&+\int_s^{t'}g\Big(r,X(r),M(r),u(r),a,Y^\varphi(r),Z^\varphi(r),\int_{\mathbb{R}}\Gamma^\varphi(r,\theta)\pi(d\theta)\Big)dr\\&-\int_s^{t'}Z^\varphi(r)dW(r)-\int_s^{t'}\!\int_{\mathbb{R}}\Gamma^\varphi(r-,\theta)\widetilde{N}(d\theta,dr),\quad s\in[t,t'].\end{aligned}\tag{5.15}$$

The above two BSDEs have the same generators, but the terminal conditions satisfy (5.13). By comparison of BSDEs, we have

$$Y^V(r)\le Y\varphi(r),\qquad r\in[t,t'].$$

Consequently, noting (5.12) and the DPP (the right-hand side of (5.14) is $\geq V^{m,a}(t,x)$),

$$
\begin{aligned}
\varphi^m(t,x) = V^{m,a}(t,x) &\leq Y^V(t) \leq Y^\varphi(t) \\
&= \varphi^{M(t')}(t', X(t')) \\
&\quad + \textstyle\int_r^{t'} g\Big(r, X(r), M(r), u(r), a, Y^\varphi(r), Z^\varphi(r), \int_{\mathbb{R}} \Gamma^\varphi(r,\theta)\pi(d\theta)\Big)dr \\
&\quad - \int_r^{t'} Z^\varphi(r)dW(r) - \int_r^{t'}\int_{\mathbb{R}} \widetilde{\Gamma}^\varphi(r-,\theta)\widetilde{N}(d\theta,dr),
\end{aligned}
$$

Then, by Itô's formula, one has

$$
\begin{aligned}
0 \leq \int_t^{t'} &\Big[g\Big(r, X(r), M(r), u(r), a, Y^\varphi(r), Z^\varphi(r), \int_{\mathbb{R}} \Gamma\varphi(r,\theta)\pi(d\theta)\Big) \\
&+\varphi_r^{M(r)}(r, X(r)) + \mathbb{A}^u\varphi^{M(r)}(r, X(r))\Big]dr \\
&+ \int_t^{t'} [\varphi_x^{M(r)}(r, X(r))\sigma(r, X(r), u, a) - Z^\varphi(r)]dW(r) \\
&+ \int_t^{t'} \int_{\mathbb{R}} [\Gamma^\varphi(r-,\theta) - \triangle_\theta\varphi^{M(r)}(r, X(r))]\widetilde{N}(d\theta, dr).
\end{aligned}
\tag{5.16}
$$

Now, let

$$
\begin{aligned}
F(r,x,m,u,a,\widetilde{y},\widetilde{z},\widetilde{\gamma}) &= \varphi_r^m(r,x) + \mathbb{A}^u\varphi^m(r,x) \\
&+g\Big(r,x,m,u,a,\widetilde{y}+\varphi^m(r,x),\widetilde{z}+\varphi_x^m(r,x)\sigma(r,x,m,u,a),\widetilde{\gamma}+\triangle_\theta\varphi^m(r,x)\Big).
\end{aligned}
\tag{5.17}
$$

Define

$$
\begin{aligned}
\widetilde{Y}(r) &= Y^\varphi(r) - \varphi^{M(r)}(r,x), \\
\widetilde{Z}(r) &= Z^\varphi(r) - \varphi_x^{M(r)}(r, X(r))\sigma(r, X(r), M(r), u(r), a), \\
\widetilde{\Gamma}(s,\theta) &= \Gamma^\varphi(r,\theta) - \triangle_\theta\varphi^{M(r)}(r, X(r)).
\end{aligned}
\tag{5.18}
$$

Clearly,

$$
\begin{aligned}
&F\Big(r, X(r), M(r), u(r), a, \widetilde{Y}(r), \widetilde{Z}(r), \int_{\mathbb{R}} \widetilde{\Gamma}(r,\theta)\pi(d\theta)\Big) \\
&= \varphi_r^{M(r)} + \mathbb{A}^u\varphi^{M(r)} \\
&\quad +g\Big(r, X(r), M(r), u(r), a, Y^\varphi(r), Z^\varphi(r), \int_{\mathbb{R}} \Gamma^\varphi(r,\theta)\pi(d\theta)\Big),
\end{aligned}
\tag{5.19}
$$

and

$$
\begin{aligned}
&F\Big(r, X(r), M(r), u(r), a, 0, 0, 0\Big) \\
&= \varphi_r^{M(r)} + \mathbb{A}^u\varphi^{M(r)} + g\Big(r, X(r), M(r), u(r), a, \varphi^{M(r)}(r, X(r)), \\
&\quad \varphi_x^{M(r)}(r, X(r))\sigma(r, X(r), M(r), u(r), a), \int_{\mathbb{R}} \triangle_\theta\varphi^{M(r)}(r, X(r))\pi(d\theta)\Big),
\end{aligned}
\tag{5.20}
$$

Hence, in terms of (5.19), one can rewrite (5.16) as

$$
\begin{aligned}
&\mathbb{E}\int_t^{t'} F\Big(r, X(r), M(r), u(r), a, \widetilde{Y}(r), \widetilde{Z}(r), \int_{\mathbb{R}} \widetilde{\Gamma}(r,\theta)\pi(d\theta)\Big)dr \geq 0, \\
&\quad \forall u(\cdot) \in \mathcal{U}[t,t'].
\end{aligned}
\tag{5.21}
$$

By definition (see (5.14) and (5.15)),

$$\begin{aligned}\widetilde{Y}(s) = \int_s^{t'} & F(r, X(r), M(r), u(r), a, \widetilde{Y}(r), \widetilde{Z}(r), \int_{\mathbb{R}} \widetilde{\Gamma}(r,\theta)\pi(d\theta))dr \\ & - \int_s^{t'} \widetilde{Z}(r)dW(r) - \int_s^{t'}\int_{\mathbb{R}} \widetilde{\Gamma}(r-,\theta)\widetilde{N}(d\theta, dr), \qquad s \in [t, t'].\end{aligned}$$

We now introduce the following BSDE (replacing $(X(r), M(r))$ by (x, m)):

$$\begin{aligned}\widetilde{\eta}(s) = \int_s^{t'} & F\Big(r, x, m, u(r), a, \widetilde{\eta}(r), \widetilde{\zeta}(r), \int_{\mathbb{R}} \widetilde{\gamma}(r,\theta)\pi(d\theta)\Big)dr \\ & - \int_s^{t'} \widetilde{\zeta}(r)dW(r) - \int_s^{t'}\int_{\mathbb{R}} \widetilde{\gamma}(r-,\theta)\widetilde{N}(d\theta, dr), \qquad s \in [t, t'].\end{aligned} \tag{5.22}$$

Then, we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{r\in[t,t']} |\widetilde{Y}(r) - \widetilde{\eta}(r)|^2 + \int_t^{t'} |\widetilde{Z}(r) - \widetilde{\zeta}(r)|^2 dr \\ &\quad + \int_t^{t'}\int_{\mathbb{R}} |\widetilde{\Gamma}(r,\theta) - \widetilde{\gamma}(r,\theta)|^2 \pi(d\theta)dr\Big] \\ &\le C\mathbb{E}\Big[\int_t^{t'} \Big(|X(r) - x| + |M(r) - m|\Big)dr\Big]^2 \\ &\le C\Big[\Big(\int_t^{t'} (r-t)^{\frac{1}{2}} dr\Big)^2 + (t'-t)\int_t^{t'} \sum_{m'=1}^{|\mathbf{M}|} (m'-m)^2 [q_{mm'}(r-t) + o(r-t)]dr\Big] \\ &\le C(t'-t)^3.\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}\Big|\int_t^{t'} \Big[F\Big(r, X(r), M(r), u(r), a, \widetilde{Y}(r), \widetilde{Z}(r), \int_{\mathbb{R}} \widetilde{\Gamma}(r,\theta)\pi(d\theta)\Big) \\ &\quad - F\Big(r, x, m, u(r), a, \widetilde{\eta}(r), \widetilde{\zeta}(r), \int_{\mathbb{R}} \widetilde{\gamma}(r,\theta)\pi(d\theta)\Big)\Big]dr\Big| \\ &\le C\mathbb{E}\Big[\sup_{r\in[t,t']} |\widetilde{Y}(r) - \widetilde{\eta}(r)|^2 + \int_t^{t'} |\widetilde{Z}(r) - \widetilde{\zeta}(r)|^2 dr \\ &\quad + \int_t^{t'}\int_{\mathbb{R}} |\widetilde{\Gamma}(r,\theta) - \widetilde{\gamma}(r,\theta)|^2 \pi(d\theta)dr\Big]^{\frac{1}{2}} \le C(t'-t)^{\frac{3}{2}}.\end{aligned} \tag{5.23}$$

This implies

$$\begin{aligned}\mathbb{E}\widetilde{\eta}(t) = \mathbb{E}\int_t^{t'} & F\Big(r, x, m, u(r), a, \widetilde{\eta}(r), \widetilde{\zeta}(r), \int_{\mathbb{R}} \widetilde{\gamma}(r,\theta)\pi(d\theta)\Big)dr \\ & \ge -C(t'-t)^{\frac{3}{2}}, \qquad \forall u(\cdot) \in \mathcal{U}[t, t'].\end{aligned} \tag{5.24}$$

Now, for any $u_0 \in U$, let $\widetilde{\eta}_0(\cdot)$ be the solution of the following:

$$\widetilde{\eta}_0(s) = \int_s^{t'} F(r, x, m, u_0, a, \widetilde{\eta}_0(r), 0, 0)dr, \qquad s \in [t, t'], \tag{5.25}$$

which is a deterministic function. The above means $(\widetilde{\eta}_0(\cdot), 0, 0)$ is the adapted solution to BSDE (5.22) with $u(\cdot)$ replaced by u_0 (which is deterministic). Thus, we have (see (5.24))

$$\int_t^{t'} F\Big(r, x, m, u_0, a, \widetilde{\eta}_0(r), 0, 0\Big)dr \geq -C(t'-t)^{\frac{3}{2}}, \qquad \forall u_0 \in U. \tag{5.26}$$

On the other hand, from (5.25), we have

$$|\widetilde{\eta}_0(s)| \leq C\int_r^{t'}(1 + |\widetilde{\eta}_0(r)|)dr, \qquad s \in [t, t'].$$

Then, by Gronwall's inequality, one has

$$|\widetilde{\eta}_0(s)| \leq C(t'-t). \qquad s \in [t, t']$$

This yields

$$\begin{aligned}&\int_t^{t'} F(r, x, m, u_0, a, 0, 0, 0)dr \geq \int_t^{t'}\Big[F(r, x, m, u_0, a, \widetilde{\eta}_0(r), 0, 0) - C|\widetilde{\eta}_0(r)|\Big]dr\\ &\geq -C(t'-t)^{\frac{3}{2}}, \qquad \forall u_0 \in U.\end{aligned}$$

Now, dividing $t'-t$ and sending $t' \to t$, we obtain

$$\inf_{u\in U} F(t, x, m, u, a, 0, 0, 0) \geq 0.$$

This gives

$$\begin{aligned}&\varphi_t^m(t,x) + \inf_{u\in U}\Big[\mathbb{A}^u\varphi^m(t,x) + g\Big((t, x, m, u, a, V^{m,a}(t,x), \varphi_x^m(t,x)\sigma(t,x,u,a),\\ &\sum_{m'=1}^{|\mathbf{M}|} q_{mm'}V^{m',a}(t,x)\Big)\Big] \geq 0.\end{aligned}$$

Thus, $V^{m,a}(t,x)$ is a viscosity sub-solution of (5.6).

Similarly, with a little modification, we can show that $V^{m,a}(t,x)$ is a viscosity super-solution of HJB-QVI (5.6). ☐

At this moment, by following lines of [13], with very little modification, we are able to show that HJB-QVI (5.6) admits a unique viscosity solution. Therefore, we could state the following result.

Theorem 5.6. *Let* $(\mathrm{H1})_\delta$, (H2)–(H5) *hold with* $0 < \delta \leq \nu$. *Then the value function* $V^{m,a}(t,x)$ *of Problem (C) is the unique viscosity solution of HJB-QVI* (5.6).

6. **Concluding remarks.**

We have studied an optimal control problem for an SDE with regime switching, having three types of controls appear, and with the cost functional being of recursive type governed by a BSVIE. Due to several complicated situations happen at the same time, some compatibility conditions are needed to ensure our problem to be meaningful and treatable. The main efforts of the paper are to study the properties of the value function, mainly, the continuity of the value function in the time variable. Also, in proving that the value function is a viscosity solution to the HJB-QVI (5.6), we have adopted the method of [8]. Some necessary details were added so that it is easier to read.

Acknowledgement. The first author would like to thank Professor Zhiyong Yu of Shandong University, and Professor Hongwei Mei of Texas Tech University, for fruitful discussions. Also, the authors would to like thank the anonymous referees for their careful reading and suggestive comments.

REFERENCES

[1] D. Duffie and L. G. Epstein, Stochastic differential utility, *Econometrica*, **60** (1992), 353-394.

[2] D. Duffie and L. G. Epstein, Asset pricing with stochastic differential utility, *Review Financial Studies*, **5** (1992), 411-436.

[3] D. Duffie and P. L. Lions, PDE solutions of stochastic differential utility, *J. Math. Econom*, **21** (1992), 577-606.

[4] P. Gassiat, F. Gozzi and H. Pham, Dynamic programming for an investment/consumptin problem in illiquid markets with regime-switching, *Banach Center Publ.*, **105** (2015), 103-118.

[5] M. Hu and S. Ji, Dynamic programming principle for stchastic recursive optimal control probelm drived by G-Brwonian motion, *Stoch.Process Appl.*, **127** (2017), 107-134.

[6] H. Mei and J. Yong, Equilibrium strategies for time-inconsistent stochastic switching systems, *ESAIM COCV*, **25** (2019), 60 pp.

[7] B. Oksendsl and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, 2^{nd} edition, Springer, Berlin, 2007.

[8] S. Peng, A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation, *Stoch, Stoch. Rep.*, **38** (1992), 119-134.

[9] S. Peng, Backeard stochastic differential equations— stochastic optimizaton theory and viscoosity of HJB equations, *Science Press, Beijing*, (1997), 85-138 (in Chinese).

[10] Y. Ren, On solutions of backward stochastc Volterra integral equatons with jumps in Hilbert spaces, *J. Optim. Theory Appl.*, **144** (2010), 319-333.

[11] Y. Shi, T. Wang and J. Yong, Mean-field backward stochastic Volterre integral equations, *Discrete Conin. Dyn. Syst., Ser. B*, **18** (2013), 1929-1967.

[12] R. Situ, On solutions of bckward stochastic differential equations with jumps and applications, *Stoch. Proc. Appl.*, **66** (1997), 209-236.

[13] S. Tang and J. Yong, Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach, *Stochastics Stochastics Reports*, **45** (1993), 145-176.

[14] H. Wang, J. Sun and J. Yong, Recursive utility processes, dynamic risk measures and quadratic backward stochastic Volterra integral equations, *Appl. Math. Optim.*, **84** (2021), 145-190.

[15] S. Yao, $\mathbb{L}^p$ solutions of backward stochatic differential equations with jumps, *Stoch. Proc. Appl.*, **127** (2017), 3465-3511.

[16] G. Yin and C. Zhu, *Hybrid Switching Diffusions — Properties and Applications*, Stoch. Model. Appl. Probab., 63, Springer, New York, 2010.

[17] J. Yong, Systems governed by ordinary differential equations with continuous, switching and impulse controls, *Appl. Math. Optim.*, **20** (1989), 223-235.

[18] J. Yong, Differential games with switching strategies, *J. Math. Anal. Appl.*, **145** (1990), 455-469.

[19] J. Yong, A zero-sum differential game in a finite duration with switching strategies, *SIAM J. Control Optim.*, **28** (1990), 1234-1250.

[20] J. Yong, Zero-sum differential games involving impulse controls, *Appl. Math. Optim.*, **29** (1994), 243-261.

[21] J. Yong, Backward stochastic Volterra integral equations and some related problems, *Stoch. Proc. Appl.*, **116** (2006), 779-795.

[22] J. Yong, Well-posedness and regularity of backward stochastic Volterra integral equations, *Prob. Theory Rel. Fields*, **142** (2008), 21-77.

[23] J. Yong, Backward stochastic Volterra integral equations — A brief survey, *Appl. Math. J. Chinese Univ.*, **28** (2013), 383-394.

[24] J. Yong, Stoahcsitc optimal control —concise introduction, *Math. Contro Rel. Fields*, **12** (2022), 1039-1136.

[25] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Appl. Math. (N. Y.), 43, Springer-Verlag, New York, 1999.

[26] L. Zhang, A BSDE approach to stochastic differential games involving impulse controls and HJB equation, *J. Syst. Sci. Complex.*, **35** (2022), 766-801.

Received September 2023; revised December 2023; early access January 2024.