OPERS ON THE PROJECTIVE LINE, WRONSKIAN RELATIONS, AND THE BETHE ANSATZ

TY J. BRINSON, DANIEL S. SAGE, AND ANTON M. ZEITLIN

ABSTRACT. It is well-known that the spectra of the Gaudin model may be described in terms of solutions of the Bethe Ansatz equations. A conceptual explanation for the appearance of the Bethe Ansatz equations is provided by appropriate G-opers: G-connections on the projective line with extra structure. In fact, solutions of the Bethe Ansatz equations are parameterized by an enhanced version of opers called Miura opers; here, the opers appearing have only regular singularities. Moreover, this geometric approach to the spectra of the Gaudin model provides a well-known example of the geometric Langlands correspondence. Feigin, Frenkel, Rybnikov, and Toledano Laredo have introduced an inhomogeneous version of the Gaudin model; this model incorporates an additional twist factor, which is an element of the Lie algebra of G. They exhibited the Bethe Ansatz equations for this model and gave an interpretation of the spectra in terms of opers with an irregular singularity. In this paper, we consider a new geometric approach to the study of the spectra of the inhomogeneous Gaudin model in terms of a further enhancement of opers called twisted Miura-Plücker opers. This approach involves a certain system of nonlinear differential equations called the *qq-system*, which were previously studied in [MV2] in the context of the Bethe Ansatz. We show that there is a close relationship between solutions of the inhomogeneous Bethe Ansatz equations and polynomial solutions of the qq-system and use this fact to construct a bijection between the set of solutions of the inhomogeneous Bethe Ansatz equations and the set of nondegenerate twisted Miura-Plücker opers. We further prove that as long as certain combinatorial conditions are satisfied, nondegenerate twisted Miura-Plücker opers are in fact Miura opers.

CONTENTS

1. Introduction

The Bethe Ansatz is a classical approach to computing the spectra of various quantum integrable systems, and in particular, spin chain models. This method is often very effective, but it is less easy to understand conceptually the reason for this effectiveness. The Gaudin model is one context in which such an explanation is known.

Let $\mathfrak{g}$ be a simple complex Lie algebra with universal enveloping algebra $U(\mathfrak{g})$ and Langlands dual algebra ${}^L\mathfrak{g}$. In the Gaudin model for $\mathfrak{g}$, one considers a family of mutually commuting elements in $U(\mathfrak{g})^{\otimes N}$ called Gaudin Hamiltonians, which depend on a collection of distinct complex numbers $z_1, \dots, z_N$. The Bethe Ansatz provides a method of constructing simultaneous eigenvectors of the Gaudin Hamiltonians on modules such as $V_{\boldsymbol{\lambda}} = \bigotimes_{i=1}^N V_{\lambda_i}$, where V_λ is the irreducible highest-weight module corresponding to the dominant integral weight λ. One starts with the unique (up to scalar) vector $|0\rangle \in V_{\boldsymbol{\lambda}}$ of highest weight $\sum \lambda_i$; it is a simultaneous eigenvector of Gaudin Hamiltonians. Given a set of distinct complex numbers $w_1, \dots, w_m$ labeled by simple roots α_{k_j} (defined in terms of fixed Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b}_+$), one then applies a certain order m lowering operator with poles at the w_j's to $|0\rangle$. If $\sum \lambda_i - \sum \alpha_{k_j}$ is dominant, then this vector is a highest weight vector in $V_{\boldsymbol{\lambda}}$ (and a simultaneous eigenvector of the Gaudin Hamiltonians) if and only if the Bethe Ansatz equations are satisfied:

$$\sum_{i=1}^N \frac{\langle \lambda_i, \check{\alpha}_{k_j} \rangle}{w_j - z_i} - \sum_{s \neq j} \frac{\langle \alpha_{k_s}, \check{\alpha}_{k_j} \rangle}{w_j - w_s} = 0, \quad j = 1, \dots, m. \tag{1.1}$$

In a series of papers [FFR1, F1, F2] , Frenkel and his collaborators introduced a geometric version of this result. They showed that the spectra of the Gaudin model is encoded by certain connections with extra structure associated to ${}^L\mathfrak{g}$ called *opers*. The opers appearing here have regular singularities at $z_1, \dots, z_N$ and ∞ and have trivial monodromy [FFR1, F2]. These opers also have apparent singularities at the w_j's, and the Bethe Ansatz equations are precisely the conditions for these singularities to be removable. Moreover, this approach allows one to give geometric meaning to solutions of the Bethe Ansatz equations without assuming that $\sum \lambda_i - \sum \alpha_{k_j}$ is dominant. In fact, they correspond bijectively to enhanced versions of opers called (nondegenerate) Miura opers. An important consequence of this geometric approach to the spectra of the Gaudin model is that it provides a well-known example of the geometric Langlands correspondence [F1].

More recently, Feigin, Frenkel, Rybnikov, and Toledano Laredo have worked on an "inhomogeneous" version of the Gaudin model [FFTL, FFR2] which involves an extra "twist parameter" $\chi \in \mathfrak{h}^*$. In these papers, the authors have given a similar geometric interpretation of the spectra in terms of opers, but here, the regular singularity at ∞ is replaced by a double pole with "2-residue"

$-\chi$. The Bethe Ansatz equations for this model are given by:

$$\sum_{i=1}^{N} \frac{\langle \lambda_i, \check{\alpha}_{k_j} \rangle}{w_j - z_i} - \sum_{s \neq j} \frac{\langle \alpha_{k_s}, \check{\alpha}_{k_j} \rangle}{w_j - w_s} = \langle \chi, \check{\alpha}_{k_j} \rangle, \quad j = 1, \ldots, m. \tag{1.2}$$

In this paper, we consider a new approach to the study of the spectra of the inhomogeneous Gaudin model in terms of *twisted Miura opers* and a certain system of nonlinear differential equations called the *qq-system*. The qq-system has also appeared in previous work of Mukhin and Varchenko on the Bethe Ansatz equations [MV1, MV2]. As we will see, there is a close relationship between solutions of the inhomogeneous Bethe Ansatz equations (1.2) and polynomial solutions of the qq-system. We will use this fact to construct a bijection between the set of solutions of the inhomogeneous Bethe Ansatz equations and the set of "nondegenerate" twisted Miura opers.

Since we will be primarily concerned with opers, it will be convenient to switch the roles of $\mathfrak{g}$ and ${}^L\mathfrak{g}$. From now on, we consider the Gaudin model for ${}^L\mathfrak{g}$, which will correspond to appropriate G-opers, where G is the simply connected group with Lie algebra $\mathfrak{g}$. The twist parameter may now be viewed as an element $Z \in \mathfrak{h}$.[1]

Let H be the maximal torus with Lie algebra $\mathfrak{h}$, and let B_+ and B_- be two opposite Borel subgroups containing H. Roughly speaking, an oper is a triple $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_-})$, where $\mathcal{F}_G$ is a principal G-bundle on $\mathbb{P}^1$ endowed with a meromorphic connection ∇ and $\mathcal{F}_{B_-}$ is a reduction of structure of the bundle to B_- such that ∇ satisfies a certain genericity condition with respect to $\mathcal{F}_{B_-}$. A Miura oper is an oper together with an additional reduction of structure $\mathcal{F}_{B_+}$ to the opposite Borel subgroup which is preserved by ∇. We now consider Miura opers whose underlying connection has regular singularities away from infinity, is monodromy-free, and is "Z-twisted". It turns out that the set of twisted Miura opers with the same underlying oper is a subvariety of the flag manifold called the Springer fiber over Z. Finally, given a Miura oper, we construct a family of Miura $\mathrm{GL}(2)$-opers parameterized by the fundamental weight. The underlying Miura oper is called a *Z-twisted Miura-Plücker G-oper* if the zero monodromy and Z-twistedness conditions hold on this family of Miura $\mathrm{GL}(2)$-opers and not necessarily on the G-oper itself.

In this paper, we show that solutions of the Z-twisted Bethe Ansatz equations for ${}^L\mathfrak{g}$ are parameterized by nondegenerate Z-twisted Miura G-opers. In order to accomplish this, we introduce a system of differential equations called the qq-system associated to G, the regular singularities z_j, and the twist parameter Z. This is a nonlinear system on a collection of rational functions $\{q^i_+(z), q^i_-(z)\}_{i\in\Delta}$, indexed by the set of simple roots Δ, which determine relations satisfied by the Wronskians $W(q^i_+(z), q^i_-(z))$. We first construct a surjection from nondegenerate polynomial solutions of the qq-system for Z to nondegenerate Z-twisted Miura-Plücker opers (Corollary 5.9).

[1] For much of the paper, we will in fact allow Z to be an element of a fixed Borel subalgebra $\mathfrak{b}_+$.

(In fact, we give a bijection between these solutions and "Z-twisted Miura-Plücker data" (Theorem 5.7).) Next, we prove that there is a surjective map from these polynomial solutions to solutions of the Bethe Ansatz equation (Theorem 5.11. We show that the fibers of these surjections coincide, thereby obtaining a one-to-one correspondence between nondegenerate Z-twisted Miura-Plücker opers and solutions of the Bethe Ansatz equations (Theorem 5.15).

We then introduce the crucial technical tool of *Bäcklund transformations*: transformations on twisted Miura-Plücker opers associated to Weyl group elements. These transformations change not only the Miura-Plücker oper, but also the twist factor. These transformations were first introduced in the context of qq-systems in [MV2]. We use Bäcklund transformations to show that, as long as certain combinatorial conditions are satisfied, nondegenerate twisted Miura-Plücker opers are in fact Miura opers (Theorem 6.18). As a corollary, we obtain the following important theorem (Theorem 6.19): under appropriate combinatorial hypotheses, there is a bijection between nondegenerate Z-twisted Miura opers and solutions to the Bethe Ansatz equations.

Our approach to this problem was inspired by recent work of Frenkel, Koroteev, and two of the authors on a q-deformation of the correspondence between opers and the spectra of the Gaudin model [KSZ, FKSZ]. These papers relate solutions of the Bethe Ansatz for the XXZ-model to certain q-difference equation versions of opers called twisted Miura-Plücker (G, q)-opers. The correspondence goes through the intermediary of the "QQ-system": a system of q-difference equations involving quantum Wronskians, which was introduced by Masoero, Raimondo, and Valeri [MRV1, MRV2] (see also [FH2]). However, we observe that our present results go beyond what is known about the XXZ model. In particular, the results of [KSZ, FKSZ] are limited to the case when the twist parameter is regular semisimple.

Acknowledgements. We are grateful to Edward Frenkel for his valuable comments. A.M.Z. is partially supported by Simons Collaboration Grant 578501 and NSF grant DMS-2203823. D.S.S is partially supported by Simons Collaboration Grant 637367.

2. G-OPERS WITH REGULAR SINGULARITIES

2.1. **Notation and group-theoretic background.** Let G be a connected, simply connected, simple algebraic group of rank r over $\mathbb{C}$. We fix a Borel subgroup B_- with unipotent radical $N_- = [B_-, B_-]$ and a maximal torus $H \subset B_-$. Let B_+ be the opposite Borel subgroup containing H and $N_+ = [B_+, B_+]$. Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of positive simple roots for the pair $H \subset B_+$. Let $\{\check{\alpha}_1, \dots, \check{\alpha}_r\}$ be the corresponding coroots; the elements of the Cartan matrix of the Lie algebra $\mathfrak{g}$ of G are given by $a_{ij} = \langle \alpha_j, \check{\alpha}_i \rangle$. The Lie algebra $\mathfrak{g}$ has Chevalley generators $\{e_i, f_i, \check{\alpha}_i\}_{i=1,\dots,r}$, so that $\mathfrak{b}_- = \mathrm{Lie}(B_-)$ is generated by the f_i's and the $\check{\alpha}_i$'s and $\mathfrak{b}_+ = \mathrm{Lie}(B_+)$ is generated by the e_i's and the $\check{\alpha}_i$'s. Similarly the Lie algebra $\mathfrak{n}_- = \mathrm{Lie}(N_-)$ is generated by the f_i's and $\mathfrak{n}_+ = \mathrm{Lie}(N_+)$ is generated by the e_i's. Let $\omega_1, \dots \omega_r$ be the fundamental weights, defined by $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$.

Let $W = N(H)/H$ be the Weyl group of G. For each i, we let $s_i \in W$ be the simple reflection corresponding to α_i. We also let w_0 be the longest element of W, so that $B_+ = w_0(B_-)$.

Recall that for any Borel subgroup B, the group G is partitioned into Bruhat cells BwB indexed by elements of W. Here, one chooses some maximal torus $T \subset B$ and sets $BwB = BnB$, where n is any lift of $w \in N(T)/T \cong W$. Since we defined W in terms of H, it is not immediately clear that this process makes sense. However, an argument involving the "abstract Cartan algebra" (see for example [CG, §3.1.22]) shows that the Bruhat cells are well-defined. We refer the reader to §2.1 of [FKSZ] for the details.

2.2. **Meromorphic G-opers.** We now define meromorphic G-opers. While the definitions below may be extended easily to an arbitrary smooth curve, we will restrict ourselves to the case of $\mathbb{P}^1$.

Let $\mathcal{F}_G$ be a principal G-bundle on $\mathbb{P}^1$ endowed with a connection ∇. This connection is automatically flat. Let $\mathcal{F}_{B_-}$ be a reduction of $\mathcal{F}_G$ to the Borel subgroup B_-. If ∇' is any connection which preserves $\mathcal{F}_{B_-}$, then $\nabla - \nabla'$ induces a well-defined one-form on $\mathbb{P}^1$ with values in the associated bundle $(\mathfrak{g}/\mathfrak{b}_-)_{\mathcal{F}_{B_-}}$. We denote this 1-form by $\nabla/\mathcal{F}_{B_-}$.

Following [BD], we will define a G-oper as a G-connection $(\mathcal{F}_G, \nabla)$ together with a reduction $\mathcal{F}_{B_-}$ of the G-bundle to the Borel subgroup B_-; this reduction is not preserved by the connection but instead satisfies a special "transversality condition" defined in terms of the 1-form $\nabla/\mathcal{F}_{B_-}$.

To define this transversality condition, let $\mathbf{O} \in [\mathfrak{n}_-, \mathfrak{n}_-]^\perp/\mathfrak{b}_- \in \mathfrak{g}/\mathfrak{b}_-$ be the open B_--orbit consisting of vectors stabilized by N_- and such that all of the simple root components with respect to the adjoint action of B_-/N_-, are non-zero. Here, the orthogonal complement is taken with respect to the Killing form.

Definition 2.1. A meromorphic G-*oper* on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_-})$, where $\mathcal{F}_G$ is a principal G-bundle on $\mathbb{P}^1$ equipped with a meromorphic connection ∇ and $\mathcal{F}_{B_-}$ is a reduction of $\mathcal{F}_G$ to B_- satisfying the following condition: there exists a Zariski open dense subset $U \subset \mathbb{P}^1$ together with a trivialization $\imath_{B_-}$ of $\mathcal{F}_{B_-}$ such that the restriction of the 1-form $\nabla/\mathcal{F}_{B_-}$ to U, written as an element of $\mathfrak{g}/\mathfrak{b}_-(z)$, belongs to $\mathbf{O}(z)$.

Note that this property does not depend on the choice of trivialization.

In terms of the particular trivialization $\imath_{B_-}$, the underlying connection of the G-oper can be written concretely as

$$\nabla = \partial_z + \sum_{i=1}^{r} \phi_i(z) e_i + b(z) \tag{2.1}$$

where $\phi_i(z) \in \mathbb{C}(z)$ and $b(z) \in \mathfrak{b}_-(z)$ are regular on U and moreover $\phi_i(z)$ has no zeros in U.

2.3. **Miura opers.** We will also need the notion of a Miura oper introduced in [F1, F2]. This is an oper together with a reduction of the underlying G-bundle to the opposite Borel subgroup that is preserved by the oper connection.

Definition 2.2. A *Miura G-oper* on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$, where $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_-})$ is a meromorphic G-oper on $\mathbb{P}^1$ and $\mathcal{F}_{B_+}$ is a reduction of the G-bundle $\mathcal{F}_G$ to B_+ that is preserved by the connection ∇.

Given a Miura G-oper, we refer to the G-oper obtained by forgetting $\mathcal{F}_{B_+}$ the underlying G-oper.

We next need to consider the relative position of the two reductions over any $x \in \mathbb{P}^1$. This relative position will be an element of the Weyl group. To define this, first note that the fiber $\mathcal{F}_{G,x}$ of $\mathcal{F}_G$ at x is a G-torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to B_- and B_+ respectively. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $gB_- \subset G$ and $\mathcal{F}_{B_+,x}$ with hB_+ for some $g, h \in G$. The quotient $g^{-1}h$ specifies an element of the double coset space $B_-\backslash G/B_+$. The Bruhat decomposition gives a bijection between this spaces and the Weyl group, so we obtain a well-defined element of G.

We say that $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ have *generic relative position* at $x \in \mathbb{P}^1$ if the relative position is the identity element of W. More concretely, this mean that the quotient $g^{-1}h$ belongs to the open dense Bruhat cell $B_-B_+ \subset G$.

The following result was proved in [F1, F2]. It will be convenient to give a different proof here.

Theorem 2.3. *For any Miura G-oper on $\mathbb{P}^1$, there exists an open dense subset $V \subset \mathbb{P}^1$ such that the reductions $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ are in generic relative position for all $x \in V$.*

Proof. Let U be a Zariski open dense subset on $\mathbb{P}^1$ as in Definition 2.1. Choosing a trivialization $\imath_{B_-}$ of $\mathcal{F}_G$ on U, we can write the connection ∇ in the form (2.1). On the other hand, using the B_+-reduction $\mathcal{F}_{B_+}$, we can choose another trivialization of $\mathcal{F}_G$ on U such that the connection in this gauge is preserved by ∇. In other words, there exists $g(z) \in G(z)$ such that

$$g(z)\partial_z g^{-1}(z) + g(z)(\sum_{i=1}^{r} \phi_i(z)e_i + b(z))g^{-1}(z) \in \mathfrak{b}_+(z) \tag{2.2}$$

This means that the relative position of the two reductions is determined by $g^{-1}(z)$. It thus suffices to show that $g^{-1}(z) \in B_-(z)B_+(z)$ or equivalently,

$$g(z) \in B_+(z)B_-(z) = B_+(z)N_-(z).$$

By the Bruhat decomposition, we know that $g(z) \in B_+(z)wN_-(z)$ for some $w \in W$, say $g(z) = b_+(z)wn_-(z)$ for some $b_+(z) \in B_+(z), n_-(z) \in N_-(z)$. Substituting this into (2.2) and simplifying gives

$$n_-(z)\partial_z n_-^{-1}(z) + n_-(z)(\sum_{i=1}^{r} \phi_i(z)e_i + b(z))n_-^{-1}(z) = \sum_{i=1}^{r} \phi_i(z)e_i + \tilde{b}(z) \in w^{-1}\mathfrak{b}_+(z)w, \tag{2.3}$$

where $\tilde{b}(z) \in \mathfrak{b}_-(z)$. It is well-known that $w^{-1}\mathfrak{b}_+w = \mathfrak{h} + (\mathfrak{n}_- \cap w^{-1}\mathfrak{b}_+w)) + (\mathfrak{n}_+ \cap w^{-1}\mathfrak{b}_+w)$. Since the strictly upper triangular component of the expression in (2.3) is $\sum_{i=1}^r \phi_i(z)e_i$, we conclude that $\phi_i(z)e_i \in w^{-1}\mathfrak{b}_+w$ for all i. This means that w preserves the set of simple roots, i.e., $w = 1$.

□

Corollary 2.4. *For any Miura G-oper on $\mathbb{P}^1$, there exists a trivialization of the underlying G-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper connection has the form*

$$\nabla = \partial_z + \sum_{i=1}^{r} g_i(z)\check{\alpha}_i + \sum_{i=1}^{r} \phi_i(z)e_i, \tag{2.4}$$

where $g_i(z), \phi_i(z) \in \mathbb{C}(z)$.

Proof. The previous theorem shows that $w = 1$ in (2.3), so there exists a gauge transformation $n_-(z)$ which takes the explicit form of the connection $\nabla = \partial_z + \sum_{i=1}^r \phi_i(z)e_i + b(z)$ into

$$n_-(z)\partial_z n_-^{-1}(z) + n_-(z)(\sum_{i=1}^{r} \phi_i(z)e_i + b(z))n_-^{-1}(z) = \sum_{i=1}^{r} \phi_i(z)e_i + \tilde{b}(z) \in \mathfrak{b}_+(z) \tag{2.5}$$

where $\tilde{b}(z) \in \mathfrak{b}_-(z)$. This implies that $\tilde{b}(z) \in \mathfrak{h}(z)$, and the statement follows by decomposing $\tilde{b}(z)$ with respect to the simple coroots. □

2.4. **Opers and Miura opers with regular singularities.** Let $\Lambda_1(z), \dots, \Lambda_r(z)$ be a collection of nonzero polynomials.

Definition 2.5. A *G-oper with regular singularities determined by* $\Lambda_1(z), \dots, \Lambda_r(z)$ is an oper on $\mathbb{P}^1$ whose connection (2.1) may be written in the form

$$\nabla = \partial_z + \sum_{i=1}^{r} \Lambda_i(z)e_i + b(z), \qquad b(z) \in \mathfrak{b}_-(z). \tag{2.6}$$

We will assume without loss of generality that the Λ_i's are monic, since this can always be arranged by a constant gauge change by an element of H. Let $\{z_1^i, \dots, z_{N_i}^i\}$ be the set of distinct roots of the Λ_i's. To each z_k^i, we associate the integral coweight $\check{\lambda}_k$ via

$$\Lambda_i(z) = \prod_{k=1}^{N_i} (z - z_k^i)^{\langle \alpha_i, \check{\lambda}_k \rangle}. \tag{2.7}$$

Definition 2.6. *A Miura G-oper with regular singularities determined by the polynomials $\Lambda_1(z), \dots, \Lambda_r(z)$* is a Miura G-oper whose underlying oper has regular singularities determined by the $\Lambda_i(z)$'s.

The following theorem is immediate from Corollary 2.4.

Theorem 2.7. *For every Miura G-oper with regular singularities determined by the polynomials* $\Lambda_1(z), \dots, \Lambda_r(z)$*, the underlying connection can be written in the form:*

$$\nabla = \partial_z + \sum_{i=1}^{r} \Lambda_i(z) e_i + \sum_{i=1}^{r} g_i(z) \check{\alpha}_i, \tag{2.8}$$

where $g_i(z) \in \mathbb{C}(z)$.

For the rest of the paper, all opers and Miura opers will have regular singularities with respect to the fixed collection of monic polynomials $\Lambda_1(z), \dots, \Lambda_r(z)$.

2.5. Z**-twisted opers.** We will primarily be interested in (Miura) opers whose underlying connection is gauge equivalent to a constant element of $\mathfrak{g}$.

Definition 2.8. A *Z-twisted G-oper* on $\mathbb{P}^1$ is a G-oper that is equivalent to the constant element $Z \in \mathfrak{g} \subset \mathfrak{g}(z)$ under the gauge action of $G(z)$.

Concretely, if the matrix form of the oper connection in a particular trivialization is given by $\nabla = \partial_z + A(z)$, then there exists $g(z) \in G(z)$ such that

$$A(z) = g(z)\partial_z g^{-1}(z) + g(z) Z g(z)^{-1}. \tag{2.9}$$

Remark 2.9. Note that for $Z \neq 0$, the constant connection $\partial_z + Z$ has a double pole at ∞ like the opers with a double pole at ∞ considered in [FFTL, FFR2]. We give a more detailed comparison of our work with the results of [FFTL] below in Remark 5.16.

To define Z-twisted Miura opers, we will assume that $Z \in \mathfrak{b}_+$. We introduce the notation

$$Z = Z^H + \sum_{i=1}^{r} c_i e_i + n, \qquad Z^H = \sum_{i=1}^{r} \zeta_i \check{\alpha}_i, \qquad \zeta_i, c_i \in \mathbb{C}, \qquad n \in [\mathfrak{n}_+, \mathfrak{n}_+]. \tag{2.10}$$

Definition 2.10. A *Z-twisted Miura G-oper* is a Miura G-oper on $\mathbb{P}^1$ that is equivalent to the constant element $Z \in \mathfrak{b}_+ \subset \mathfrak{b}_+(z)$ under the gauge action of $B_+(z)$, i.e., there exists $v(z) \in B_+(z)$ such that the matrix of the oper connection is given by

$$A(z) = v(z)\partial_z v^{-1}(z) + v(z) Z v(z)^{-1}. \tag{2.11}$$

For untwisted opers, there is a full flag variety G/B_+ of associated Miura opers. For twisted opers, we must introduce certain closed subvarieties of the flag manifold of the form $(G/B_+)_Z = \{gB_+ \mid g^{-1}Zg \in \mathfrak{b}_+\}$; these varieties are called *Springer fibers*. Springer fibers play an important role in representation theory. (See, for example, Chapter 3 of [CG].) For $\mathrm{SL}(n)$ (or $\mathrm{GL}(n)$), a Springer fiber may be viewed as the space of complete flags in $\mathbb{C}^n$ preserved by a fixed endomorphism.

Proposition 2.11. *The map from Miura Z-twisted opers to Z-twisted opers is a fiber bundle with fiber $(G/B_+)_Z$.*

Proof. Since the underlying connection of a Z-twisted oper is isomorphic to the connection $\partial_z + Z$, a Miura structure on such an oper is equivalent to a B_+-reduction that is preserved by $\partial_z + Z$. This is determined by a Borel subalgebra of $\mathfrak{g}$ that contains Z. The flag variety may be identified with the space of Borel subalgebras via $gB \mapsto g\mathfrak{b}_+ g^{-1}$, and the condition $Z \in g\mathfrak{b}_+ g^{-1}$ is equivalent to $gB \in (G/B_+)_Z$. ☐

2.6. **The associated Cartan connection.** Consider a Miura G-oper with regular singularities determined by polynomials $\Lambda_1(z), \dots, \Lambda_r(z)$. By Theorem 2.7, the underlying G-connection can be written in the form (2.8). Since it preserves the B_+-bundle $\mathcal{F}_{B_+}$ that is part of the data of the Miura G-oper, it may be viewed as a meromorphic B_+-connection on $\mathbb{P}^1$. Taking the quotient of $\mathcal{F}_{B_+}$ by $N_+ = [B_+, B_+]$ and using the fact that $B/N_+ \simeq H$, we obtain an H-bundle $\mathcal{F}_{B_+}/N_+$ endowed with an H-connection, which we denote by $\nabla^H = \partial_z + A^H(z)$. According to formula (2.8), it is given by the formula

$$A^H(z) = \sum_{i=1}^{r} g_i(z)\check{\alpha}_i. \tag{2.12}$$

We call $\nabla^H(z) = \partial_z + A^H(z)$ the *associated Cartan connection* of the Miura oper.

Now, if our Miura oper is Z-twisted, then we also have $A(z) = v(z)\partial_z v^{-1}(z) + v(z)Zv(z)^{-1}$, where $v(z) \in B_+(z)$. Since $v(z)$ can be written as

$$v(z) = \left(\prod_i y_i(z)^{\check{\alpha}_i}\right) n(z), \qquad n(z) \in N_+(z), \quad y_i(z) \in \mathbb{C}(z)^\times, \tag{2.13}$$

the Cartan connection $\nabla^H(z) = \partial_z + A^H(z)$ has the form:

$$A^H(z) = \sum_{i=1}^{r} (\zeta_i - y_i(z)^{-1}\partial_z y_i(z))\check{\alpha}_i, \tag{2.14}$$

with the ζ_i's defined in (2.10). We will refer to $\nabla^H(z)$ as a *Z-twisted Cartan connection*. This formula shows that $\nabla^H(z)$ is completely determined by Z^H, i.e., the diagonal part of Z, and the rational functions $y_i(z)$. Indeed, comparing this equation with (2.12) gives

$$g_i(z) = \zeta_i - y_i(z)^{-1}\partial_z y_i(z) \tag{2.15}$$

It is now easy to see that $\nabla^H(z)$ determines the $y_i(z)$'s uniquely up to scalar.

3. Nondegenerate Miura-Plücker opers

Our main goal is to link Miura opers to solutions of a certain system of equations which we will call the classical qq-system, which is in turn related to the system of Bethe Ansatz equations for the Gaudin model. We accomplish this in two steps. First, we introduce the notion of a Z-twisted Miura-Plücker G-oper. We associate to a Miura G-oper a collection of Miura $\mathrm{GL}(2)$-opers indexed by the fundamental weights of G. A Z-twisted Miura-Plücker oper is a Miura oper where the Z-twistedness condition is replaced by a slightly weaker condition imposed on these $\mathrm{GL}(2)$-opers. Second, we will restrict attention to opers satisfying certain nondegeneracy conditions defined in terms of the corresponding Cartan connection.

3.1. **The associated Miura $\mathrm{GL}(2)$-opers.** In this section, we associate to a Miura G-oper with regular singularities a collection of Miura $\mathrm{GL}(2)$-opers indexed by the fundamental weights.

Let V_i be the irreducible representation of G with highest weight given by the fundamental weight ω_i. Let $L_i \subset V_i$ be the B_+-stable line consisting of highest weight vectors. If we choose a nonzero element ν_{ω_i} in L_i, then the subspace of V_i of weight $\omega_i - \alpha_i$ is one-dimensional and is spanned by $f_i \cdot \nu_{\omega_i}$. Therefore, the two-dimensional subspace W_i of V_i spanned by the weight vectors ν_{ω_i} and $f_i \cdot \nu_{\omega_i}$ is a B_+-invariant subspace of V_i.

Now, let $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ be a Miura G-oper with regular singularities determined by polynomials $\Lambda_1(z), \dots, \Lambda_r(z)$ as in Definition 2.6. Recall that $\mathcal{F}_{B_+}$ is a B_+-reduction of a G-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$ preserved by the G-connection ∇. Therefore for each i, the vector bundle

$$\mathcal{V}_i = \mathcal{F}_{B_+} \times_{B_+} V_i = \mathcal{F}_G \times_G V_i$$

associated to V_i contains a rank two subbundle

$$\mathcal{W}_i = \mathcal{F}_{B_+} \times_{B_+} W_i$$

associated to $W_i \subset V_i$, and $\mathcal{W}_i$ in turn contains a line subbundle

$$\mathcal{L}_i = \mathcal{F}_{B_+} \times_{B_+} L_i$$

associated to $L_i \subset W_i$.

Denote by $\phi_i(\nabla)$ the connection on the vector bundle $\mathcal{V}_i$ (or equivalently, the $\mathrm{GL}(V_i)$-connection) corresponding to the above Miura oper connection ∇. Since ∇ preserves $\mathcal{F}_{B_+}$, we see that $\phi_i(\nabla)$ preserves the subbundles $\mathcal{L}_i$ and $\mathcal{W}_i$ of $\mathcal{V}_i$. Denote by ∇_i the corresponding connection on the rank 2 bundle $\mathcal{W}_i$.

Trivialize $\mathcal{F}_{B_+}$ on a Zariski open subset of $\mathbb{P}^1$ so that ∇ has the form (2.8) with respect to this trivialization. This trivializes the bundles $\mathcal{V}_i$, $\mathcal{W}_i$, and $\mathcal{L}_i$ as well, so that the connection $\nabla_i(z)$ can be expressed in terms of a 2×2 matrix whose entries are in $\mathbb{C}(z)$.

A direct computation using formula (2.8) yields the following result.

Lemma 3.1. *We have*

$$\nabla_i(z) = \partial_z + \begin{pmatrix} g_i(z) & \Lambda_i(z) \\ 0 & -g_i(z) - \sum_{k\neq i} a_{ki} g_k(z) \end{pmatrix}, \tag{3.1}$$

Using the trivialization of $\mathcal{W}_i$ in which $\nabla_i(z)$ has this form, we can decompose $\mathcal{W}_i$ as the direct sum of two line subbundles. The first is $\mathcal{L}_i$, generated by the basis vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The second, which we denote by $\widetilde{\mathcal{L}}_i$, is generated by the basis vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The subbundle $\mathcal{L}_i$ is ∇_i-invariant, whereas ∇_i satisfies the following $\mathrm{GL}(2)$-*oper condition* with respect to $\widetilde{\mathcal{L}}_i$.

Definition 3.2. A $\mathrm{GL}(2)$-*oper* on $\mathbb{P}^1$ is a triple $(\mathcal{W}, \nabla, \widetilde{\mathcal{L}})$, where $\mathcal{W}$ is a rank 2 bundle on $\mathbb{P}^1$, $\nabla : \mathcal{W} \longrightarrow \mathcal{W} \otimes K$ is a meromorphic connection on $\mathcal{W}$, K is the canonical bundle on $\mathbb{P}^1$, and $\widetilde{\mathcal{L}}$ is a line subbundle of $\mathcal{W}$ such that the induced map $\bar{\nabla} : \widetilde{\mathcal{L}} \longrightarrow (\mathcal{W}/\widetilde{\mathcal{L}}) \otimes K$ is an isomorphism on a Zariski open dense subset of $\mathbb{P}^1$.

A Miura $\mathrm{GL}(2)$-*oper* on $\mathbb{P}^1$ is a quadruple $(\mathcal{W}, \nabla, \widetilde{\mathcal{L}}, \mathcal{L})$, where $(\mathcal{W}, \nabla, \widetilde{\mathcal{L}})$ is a $\mathrm{GL}(2)$-oper and $\mathcal{L}$ is an ∇-invariant line subbundle of $\mathcal{W}$.

Using this definition, one obtains an alternative definition of (Miura) $\mathrm{SL}(2)$-opers: they are the (Miura) $\mathrm{GL}(2)$-opers defined by the above triples (resp. quadruples) satisfying the additional property that in some trivialization on a Zariski-open dense subset of $\mathbb{P}^1$, the trace of the matrix of the connection is 0.

Our quadruple $(\mathcal{W}_i, \nabla, \widetilde{\mathcal{L}}_i, \mathcal{L}_i)$ is clearly a Miura $\mathrm{GL}(2)$-oper. It is not clear whether it is an $\mathrm{SL}(2)$-oper because the trace of the matrix in (3.1) is not necessarily 0.

We now make the further assumption that our Miura G-oper $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ with regular singularities is Z-twisted (see Definition 2.10). Recall that this implies that the associated Cartan connection $\nabla^H(z)$ has the form (2.14):

$$\nabla^H(z) = \prod_i y_i(z)^{\check{\alpha}_i} \left(\partial_z + Z^H\right) \prod_i y_i(z)^{-\check{\alpha}_i}, \qquad y_i(z) \in \mathbb{C}(z). \tag{3.2}$$

We claim that for Z-twisted Miura opers, there exists another trivialization of $\mathcal{W}_i$ in which the connection matrix of ∇_i has constant (though not necessarily zero) trace. This will be a particularly convenient gauge for ∇_i.

To prove the claim, let $A_i(z)$ denote the matrix in (3.1), and apply the gauge transformation by the diagonal matrix

$$u(z) = \begin{pmatrix} 1 & 0 \\ 0 & \prod_{j\neq i} y_j(z)^{a_{ji}} \end{pmatrix}.$$

This gives
(3.3)

$$\widetilde{\nabla}_i(z) = u(z)\nabla_i(z)u^{-1}(z) = \partial_z + \begin{pmatrix} \zeta_i - y_i(z)^{-1}\partial_z y_i(z) & \rho_i(z) \\ 0 & -\sum_{k\neq i} a_{ki}\zeta_k - \zeta_i + y_i(z)^{-1}\partial_z y_i(z)) \end{pmatrix},$$

where

$$\rho_i(z) = \Lambda_i(z)\prod_{k\neq i} y_k(z)^{-a_{ki}}. \tag{3.4}$$

Since $a_{ij} \leq 0$ for $i \neq j$, $\rho_i(z)$ is a polynomial if all $y_j(z)$'s are polynomials.

Let $G_i \cong \mathrm{SL}(2)$ be the subgroup of G corresponding to the $\mathfrak{sl}(2)$-triple spanned by $\{e_i, f_i, \check{\alpha}_i\}$. Note that the group G_i preserves W_i. Consider the Miura G_i-oper $(\mathcal{W}_i, \hat{\nabla}_i, \widetilde{\mathcal{L}}_i, \mathcal{L}_i)$ with $\widetilde{\mathcal{L}}_i = \operatorname{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$, $\mathcal{L}_i = \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$,

$$\hat{\nabla}_i = \partial_z + g_i\check{\alpha}_i + \rho_i(z)e_i = \begin{pmatrix} \zeta_i - y_i(z)^{-1}\partial_z y_i(z) & \rho_i(z) \\ 0 & -\zeta_i + y_i(z)^{-1}\partial_z y_i(z)) \end{pmatrix}, \tag{3.5}$$

We can now express the connection $\widetilde{\nabla}_i(z)$ as the sum of an $\mathrm{SL}(2)$-connection and a constant diagonal matrix:

$$\widetilde{\nabla}_i(z) = \begin{pmatrix} 0 & 0 \\ 0 & \sum_{j\neq i} -a_{ji}\zeta_j \end{pmatrix} + \hat{\nabla}_i(z) \tag{3.6}$$

$$= \partial_z + \begin{pmatrix} 0 & 0 \\ 0 & \sum_{j\neq i} -a_{ji}\zeta_j \end{pmatrix} + g_i(z)\check{\alpha}_i + \rho_i(z)e_i. \tag{3.7}$$

This shows that in this gauge, the trace of the matrix of the connection is constant with value $-\sum_{j\neq i} a_{ji}\zeta_j$.

Thus, a Z-twisted Miura G-oper gives rise to a collection of meromorphic Miura $\mathrm{SL}(2)$-opers $\hat{\nabla}_i(z)$ for $i = 1, \ldots, r$. It should be noted that $\hat{\nabla}_i(z)$ has regular singularities in the sense of Definition 2.5 if and only if $\rho_i(z)$ is a polynomial. For example, this holds for all i if all $y_j(z)$, $j = 1, \ldots, r$, are polynomials. We will use this observation below.

3.2. **Z-twisted Miura-Plücker opers.** Recall that a Z-twisted Miura G-opers is a Miura G-oper whose underlying connection can be written in the form (2.11):

$$\nabla(z) = v(z)(\partial_z + Z)v(z)^{-1}, \qquad v(z) \in B_+(z). \tag{3.8}$$

We will now relax this condition by imposing a twistedness condition only on the associated Miura $\mathrm{GL}(2)$-opers ∇_i (or equivalently, the Miura $\mathrm{SL}(2)$-opers $\hat{\nabla}_i$). More precisely, we will require the existence of an upper triangular gauge transformation $v(z) \in B_+(z)$ such that (3.8) holds upon restriction to W_i for all i.

Definition 3.3. A *Z-twisted Miura-Plücker*[2] *G-oper* is a meromorphic Miura G-oper on $\mathbb{P}^1$ with underlying connection ∇ satisfying the following condition: there exists $v(z) \in B_+(z)$ such that for all $i = 1, \dots, r$, the Miura $\mathrm{GL}(2)$-opers ∇_i associated to ∇ by formula (3.1) can be written in the form

$$\nabla_i(z) = v(z)(\partial_z + Z)v(z)^{-1}|_{W_i} = v_i(z)(\partial_z + Z_i)v_i(z)^{-1}, \tag{3.9}$$

where $v_i(z) = v(z)|_{W_i}$ and $Z_i = Z|_{W_i}$.

In other words, a Miura G-oper is a Z-twisted Miura-Plücker G-oper precisely when there is a trivialization of $\mathcal{F}_{B_+}$ in which all of the associated connections ∇_i have the constant matrix $Z_i \in \mathfrak{gl}(2)$. It is a Z-twisted Miura G-oper if ∇ has the constant matrix Z in this gauge. Thus, every Z-twisted Miura G-oper is automatically a Z-twisted Miura-Plücker G-oper, but the converse is not necessarily true if $G \neq \mathrm{SL}(2)$.

Note, however, that it follows from the above definition that the H-connection ∇^H associated to a Z-twisted Miura-Plücker G-oper can be written in the same form (3.2) as the H-connection associated to a Z-twisted Miura G-oper.

3.3. H**-nondegeneracy.** We now introduce the notion of H-nondegeneracy, the first of our two nondegeneracy conditions for Z-twisted Miura-Plücker opers. This condition actually applies to arbitrary Miura opers with regular singularities. Recall from Theorem 2.7 that the underlying connection can be represented in the form (2.8).

Definition 3.4. A Miura G-oper ∇ of the form (2.8) is called *H-nondegenerate* if the corresponding H-connection $\nabla^H(z)$ can be written in the form (2.14), with the rational functions $y_i(z)$ satisfying the following conditions:

(1) $y_i(z)$ has no multiple zeros or poles;
(2) for all i, the roots of $\Lambda_i(z)$ are distinct from the the zeros and poles of $y_i(z)$; and
(3) if $i \neq j$ and $a_{ij} \neq 0$, then the zeros and poles of $y_i(z)$ and $y_j(z)$ are distinct from each other.

3.4. **Nondegenerate Z-twisted Miura $\mathrm{SL}(2)$-opers.** We now turn to the second nondegeneracy condition. This condition applies to Z-twisted Miura-Plücker G-opers. In this subsection, we give the definition for $G = \mathrm{SL}(2)$. (Note that Z-twisted Miura-Plücker $\mathrm{SL}(2)$-opers are the same as

[2] The terminology arises from its relationshp to the Plücker description of B_+-bundles as explained in Section 4.1 of [FKSZ].

Z-twisted Miura $\mathrm{SL}(2)$-opers.) In the next subsection, we will give the definition for an arbitrary simple, simply connected complex Lie group G.

Consider a Miura $\mathrm{SL}(2)$-oper given by the formula (2.8), which for $\mathrm{SL}(2)$ becomes

$$\nabla = \partial_z + g(z)\check{\alpha} + \Lambda(z)e = \partial_z + \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & -g(z) \end{pmatrix}.$$

The corresponding Cartan connection is given by

$$\nabla^H(z) = \partial_z + g(z)\check{\alpha} = y(z)^{\check{\alpha}}(\partial_z + Z^H)y(z)^{-\check{\alpha}} = \partial_z + \begin{pmatrix} \zeta - y(z)^{-1}\partial_z y(z) & 0 \\ 0 & -\zeta + y(z)^{-1}\partial_z y(z) \end{pmatrix},$$

where $y(z)$ is a rational function. Let us assume that ∇ is H-nondegenerate, so that the zeros of $\Lambda(z)$ are distinct from the zeros and poles of $y(z)$.

If we apply a gauge transformation by an element $h(z)^{\check{\alpha}} \in H[z]$ to ∇, we obtain a new oper connection

$$\widetilde{\nabla}(z) = \partial_z + \widetilde{g}(z)\check{\alpha} + \widetilde{\Lambda}(z)e, \tag{3.10}$$

where

$$\widetilde{g}(z) = g(z) - h^{-1}(z)\partial_z h(z), \qquad \widetilde{\Lambda}(z) = \Lambda(z)h(z)^2. \tag{3.11}$$

It also has regular singularities, but for a different polynomial $\widetilde{\Lambda}(z)$, and $\widetilde{\nabla}(z)$ may no longer be H-nondegenerate. However, it turns out there is an essentially unique gauge transformation from $H[z]$ for which the resulting $\widetilde{\nabla}(z)$ is H-nondegenerate and $\widetilde{y}(z)$ is a polynomial. This choice allows us to fix the polynomial $\Lambda(z)$ determining the regular singularities of our $\mathrm{SL}(2)$-oper.

Lemma 3.5. (1) *There is an H-nondegenerate $\mathrm{SL}(2)$-oper $\widetilde{\nabla}(z)$ in the $H[z]$-gauge class of ∇, say with $\widetilde{\nabla}^H(z) = \partial_z + \widetilde{g}(z)\check{\alpha}$, for which the rational function $\widetilde{y}(z)$ is a polynomial. This oper is unique up to a scalar $a \in \mathbb{C}^\times$ that leaves $\widetilde{g}(z)$ unchanged, but multiplies $\widetilde{y}(z)$ and $\widetilde{\Lambda}(z)$ by a and a^2 respectively.*

(2) *This $\mathrm{SL}(2)$-oper $\widetilde{\nabla}$ may also be characterized by the property that $\widetilde{\Lambda}(z)$ has maximal degree subject to the constraint that it is H-nondegenerate.*

Proof. Write $y(z) = \frac{P_1(z)}{P_2(z)}$, where P_1, P_2 are relatively prime polynomials. For a nonzero polynomial $h(z) \in \mathbb{C}(z)^\times$, the gauge transformation of ∇ by $h(z)^{\check{\alpha}}$ is given by formulas (3.10) and (3.11). In order for $\widetilde{y}(z) = h(z)\frac{P_1(z)}{P_2(z)}$ to be a polynomial, we need $h(z)$ to be divisible by $P_2(z)$. If, however, $\deg(h/P_2) > 0$, then $\widetilde{y}(z)$ and $\widetilde{\Lambda}(z)$ would have a zero in common, so $\widetilde{A}(z)$ would not be H-nondegenerate. Hence, we must have $h(z) = aP_2(z)$ for some $a \in \mathbb{C}^\times$. Thus, $h(z)$ is uniquely defined by multiplication by a, which leaves $\widetilde{g}(z)$ unchanged, but multiplies $\widetilde{y}(z)$ and $\widetilde{\Lambda}(z)$ by a and a^2 respectively.

For the second statement, note that if $h(z)$ is a polynomial for which the zeros of $h(z)^2\Lambda(z)$ are distinct from the zeros and poles of $h(z)\frac{P_1(z)}{P_2(z)}$, we must have $h|P_2$. If $h(z)$ is not an associate of $P_2(z)$, we have $\deg(h) < \deg(P_2)$, so $\deg(h(z)^2\Lambda(z)) < \deg(\widetilde{\Lambda})$. □

This motivates the following definition.

Definition 3.6. A Z-twisted Miura $\mathrm{SL}(2)$-oper is called *nondegenerate* if it is H-nondegenerate and the rational function $y(z)$ appearing in formula (2.14) is a polynomial.

3.5. **Nondegenerate Z-twisted Miura-Plücker G-opers.** We now turn to the general case. Recall that to every Z^H-twisted Miura-Plücker G-oper ∇, we have associated a Miura $\mathrm{SL}(2)$-oper $\hat{\nabla}_i(z), i = 1, \dots, r$, given by formula (3.5). (It is obtained from the Miura $\mathrm{GL}(2)$-oper $\nabla_i = \nabla|_{W_i}$ using formulas (3.3) and (3.6)). It follows from the definition that if ∇ is Z-twisted with Z given by (2.10), then $\hat{\nabla}_i$ is $\zeta_i\check{\alpha}_i$-twisted.

Definition 3.7. Suppose that the rank of G is greater than 1. A Z-twisted Miura-Plücker G-oper ∇ is called *nondegenerate* if it is H-nondegenerate and each $\zeta_i\check{\alpha}_i$-twisted Miura $\mathrm{SL}(2)$-oper $\hat{\nabla}_i(z)$ is nondegenerate.

It turns out that this simply means that in addition to ∇ being H-nondegenerate, each $y_i(z)$ from formula (2.14) is a polynomial.

Proposition 3.8. *Let ∇ be a Z-twisted Miura-Plücker G-oper. The following statements are equivalent:*

(1) *∇ is nondegenerate.*
(2) *∇ is H-nondegenerate, and each $\hat{\nabla}_i(z)$ has regular singularities, i.e. $\rho_i(z)$ given by formula (3.4) is in $\mathbb{C}[z]$.*
(3) *Each $y_i(z)$ from formula (2.14) may be chosen to be a monic polynomial, and these polynomials satisfy the conditions in Definition 3.4.*

Proof. To prove that (2) implies (3), we need only show that if each $\rho_i(z)$ given by formula (3.4) is in $\mathbb{C}[z]$, then the $y_i(z)$'s are polynomials. Suppose $y_i(z)$ is not a polynomial, and choose $j \neq i$ such that $a_{ij} \neq 0$. Then $-a_{ij} > 0$, and so the denominator of $y_i(z)$ appears in the denominator of $\rho_j(z)$. Moreover, since the poles of $y_i(z)$ are distinct from the zeros of $\Lambda_j(z)$ and the other $y_k(z)$'s, the poles of $y_i(z)$ give rise to poles of $\rho_j(z)$. But then $\hat{\nabla}_j(z)$ would not have regular singularities.

Next, assume (3). By Definition 3.4, ∇ is H-nondegenerate. Since all the $y_i(z)$'s are polynomials, the same is true for the $\rho_i(z)$'s. (Here, we are using the fact that the off-diagonal elements of the Cartan matrix, a_{ij} with $i \neq j$, are less than or equal to 0.) Since $\rho_i(z)$ is a product of polynomials whose roots are distinct from the roots of $y_i(z)$, we see that the Cartan connection associated to $\hat{\nabla}_i(z)$ is nondegenerate.

Finally, (2) is a trivial consequence of (1). □

If we apply a gauge transformation by an element $h(z) \in H[z]$ to ∇, we get a new Z-twisted Miura-Plücker G-oper. However, the following proposition shows that it is only nondegenerate if $h(z) \in H$. As a consequence, the Λ_k's of a nondegenerate oper are determined up to scalar multiples. If we further impose the condition that each $y_i(z)$ is a *monic* polynomial, then $h(z) = 1$, and this fixes the Λ_k's.

Proposition 3.9. *If ∇ is a nondegenerate Z-twisted Miura-Plücker G-oper and $h(z) \in H[z]$, then $h(z)\nabla h(z)^{-1}$ is nondegenerate if and only if $h(z)$ is a constant element of H.*

Proof. Write $h(z) = \prod h_i(z)^{\check{\alpha}_i}$. Gauge transformation of ∇ by $h(z)$ induces a gauge transformation of ∇_i by $h_i(z)$. Since ∇_i is nondegenerate, Lemma 3.5 implies that the new Miura $\mathrm{SL}(2)$-oper is nondegenerate if and only $h_i \in \mathbb{C}^\times$. □

4. SL(2)-OPERS AND THE BETHE ANSATZ EQUATIONS

Before exploring the relationship between Miura G-opers and the Bethe Ansatz equations in general, we briefly describe what happens for $G = \mathrm{SL}(2)$. These results are immediate corollaries of the results in the following sections. However, in this case, one can give simpler proofs; see [KSZ] for the details.

Let $Z^H = \mathrm{diag}(\zeta, -\zeta)$. A nondegenerate Z^H-twisted Miura $\mathrm{SL}(2)$-oper can be represented in matrix form as

$$\nabla(z) = \partial_z + (\zeta - y(z)^{-1}\partial_z y(z))\check{\alpha} + \Lambda(z)e = \begin{pmatrix} \zeta - y(z)^{-1}\partial_z y(z) & \Lambda(z) \\ 0 & -\zeta + y(z)^{-1}\partial_z y(z) \end{pmatrix},$$

where the polynomials $y(z)$ and $\Lambda(z)$ have no roots in common and $y(z)$ is monic with no multiple roots. This connection is gauge equivalent to $\partial_z + \zeta\check{\alpha} + \Lambda(z)e$ via a gauge transformation by a matrix of the form

$$v(z) = y(z)^{\check{\alpha}} e^{\frac{q_-(z)}{q_+(z)}e},$$

where $q_-(z), q_+(z)$ are relatively prime polynomials with $q_+(z)$ monic.

One can now show that $y(z) = q_+(z)$ and the polynomials $q_+(z)$ and $q_-(z)$ satisfy the following differential equation involving their Wronskian:

$$q_+(z)\partial_z q_-(z) - q_-(z)\partial_z q_+(z) + 2\zeta q_+(z)q_-(z) = \Lambda(z)$$

This is the $\mathrm{SL}(2)$-version of a system of equations called the qq-system. In fact, there is a bijection between nondegenerate Z^H-twisted Miura opers together with a choice of the matrix $v(z)$ and nondegenerate polynomial solutions of the qq-system; here, a polynomial solution of the qq-system is called nondegenerate if $q_+(z)$ is monic with no multiple roots and has no roots in common with $\Lambda(z)$.

Nondegenerate solutions lead to solutions of the Bethe Ansatz equation for the inhomogeneous Gaudin model. Indeed, let $\Lambda(z)=\prod_{k=1}^{N}(z-z_k)^{\ell_i}$ and $q_+(z)=\prod_{i=1}^{n}(z-w_i)$ with $w_i\neq w_j$ if $i\neq j$ and $w_i\neq z_k$. One can then show that

$$2\zeta+\sum_{k=1}^{N}\frac{\ell_k}{w_i-z_k}-\sum_{k=1}^{n}\frac{2}{w_i-w_k}=0,\quad k=1,\dots,r. \tag{4.1}$$

In fact, there is a one-to-one correspondence between Z^H-twisted Miura opers and solutions of the Bethe Ansatz equation.

5. Miura-Plücker opers, Wronskian relations, and the Bethe Ansatz equations for the Gaudin model

We now return to the general situation, with G an arbitrary simple, simply connected complex Lie group. We show that a Z-twisted Miura-Plücker G-oper is also Z^H-twisted. We then establish a one-to-one correspondence between the set of nondegenerate Z^H-twisted Miura-Plücker G-opers and the set of solutions of a system of Bethe Ansatz equations associated to G. A key element of the construction is an intermediate object between these two sets: solutions to a system of nonlinear differential equations called the *qq-system*, which imposes relations on certain Wronskians indexed by the simple roots.

5.1. **Reduction to the semisimple case.** Let ∇ be a Z-twisted Miura-Plücker oper for $Z\in\mathfrak{b}_+$. As in (2.10), we write $Z=Z^H+\sum_{i=1}^{r}c_ie_i+n_+$ with $Z^H=\sum_{i=1}^{r}\zeta_i\check{\alpha}_i\in\mathfrak{h}$ and $n_+\in[\mathfrak{n}_+,\mathfrak{n}_+]$.

We now show that a Z-twisted Miura-Plücker oper is also Z^H-twisted.

Proposition 5.1. *i) There exist an element $u(z)\in N_+(z)$ so that $u(z)(\partial_z+Z)u(z)^{-1}=\partial_z+Z^H+\tilde{n}_+(z)$, where $\tilde{n}_+(z)\in[\mathfrak{n}_+,\mathfrak{n}_+](z)$.*
ii) Any Z-twisted Miura-Plücker oper is Z^H-twisted.

Proof. To prove the first statement, we will construct $u(z)$ as a product of r elements corresponding to the simple roots. Assume that $\langle\alpha_i,Z^H\rangle\neq 0$, and set $u_i(z)=\exp\Big(-\frac{c_i}{\langle\alpha_i,Z^H\rangle}e_i\Big)$. We obtain

$$u_i(z)(\partial_z+Z)u_i(z)^{-1}=\partial_z+Z^H+\sum_{j=1,i\neq j}^{r}c_je_j+\dots, \tag{5.1}$$

where the dots stand for terms in $[\mathfrak{n}_+,\mathfrak{n}_+](z)$. Similarly, if $\langle\alpha_i,Z^H\rangle=0$, set $u_i(z)=\exp(zc_ie_i)$, which again leads to (5.1). Then $u(z)=\prod_{i=1}^{r}u_i(z)$, where the order of the $u_i(z)$'s does not matter, satisfies the desired conditions.

Recall that we have $v(z) \in B_+(z)$ such that $\nabla_i(z) = v(z)(\partial_z + Z)v(z)^{-1}|_{W_i}$ for all i. Set $v^u(z) = v(z)u(z)^{-1} \in B_+(z)$, with $u(z)$ as in the first part. It follows that

$$\begin{aligned}\nabla_i(z) &= v(z)(\partial_z + Z)v(z)^{-1}|_{W_i} \\ &= v(z)u(z)^{-1}(\partial_z + Z^H)u(z)v(z)^{-1}|_{W_i} = v_i^u(z)(\partial_z + Z_i^H)v_i^u(z)^{-1}.\end{aligned} \tag{5.2}$$

where $v_i^u(z) = v(z)u^{-1}(z)|_{W_i}$ and $Z_i^H = Z^H|_{W_i}$. Thus any Z-twisted Miura-Plücker oper is Z^H-twisted.

□

For the rest of the paper, we will restrict attention to opers with a semisimple twist. However, we will retain the notation Z^H for clarity.

5.2. **Twisted Miura-Plücker data and qq-systems.** We now introduce a nonlinear system of differential equations depending on the polynomials $\Lambda_1(z), \dots, \Lambda_r(z)$ and the semisimple element Z^H. As we will see, it may be viewed as a functional realization of the Bethe Ansatz equations.

Recall that the Wronskian of two rational functions $q_+(z)$ and $q_-(z)$ is given by

$$W(q_+, q_-)(z) = q_+(z)\partial_z q_-(z) - q_-(z)\partial_z q_+(z).$$

Definition 5.2. The *qq-system* associated to $\mathfrak{g}$, the semisimple element $Z^H \in \mathfrak{h}$, and the collection of monic polynomials $\Lambda_1(z), \dots, \Lambda_r(z)$ is the system of equations

$$W(q_+^i, q_-^i)(z) + \langle \alpha_i, Z^H \rangle q_+^i(z) q_-^i(z) = \Lambda_i(z) \prod_{j \neq i} \left[q_+^j(z) \right]^{-a_{ji}} \tag{5.3}$$

for $i = 1, \dots, r$.

These qq-systems were previously studied in [MV2].

A polynomial solution $\{q_+^i(z), q_-^i(z)\}_{i=1,\dots,r}$ of (5.3) is called *nondegenerate* if each $q_+^i(z)$ is monicand the $q_+^i(z)$'s satisfy the conditions in Definition 3.4. Note that nondegeneracy only depends on the $q_+^i(z)$'s.

It is an immediate consequence of the definition that for nondegenerate polynomial solutions, $q_+^i(z)$ and $q_-^i(z)$ are relatively prime. Indeed, if w is a common root of $q_+^i(z)$ and $q_-^i(z)$, then it is a root of the left-hand side of the ith qq-equation. It follows that w is also a root of some factor on the right-hand side, which contradicts nondegeneracy.

Remark 5.3. This system of equations (5.3) has also been considered in [MV1] in the context of differential operators corresponding to Miura opers for $Z = 0$.

Remark 5.4. If $\mathfrak{g}$ is not simply-laced, let $\tilde{\mathfrak{g}}$ be the associated simply-laced Lie algebra, i.e., the Lie algebra whose Dynkin diagram has the multiple bond replaced by a simple bond. (We will systematically use tilde superscripts to denote objects associated to this new Lie algebra.) We suppose further that $\mathfrak{g}$ has a unique short simple root, hence is of type B_n or G_2. In this case, we show that a solution to the qq-system for $\mathfrak{g}$ gives rise to a solution to the qq-system for $\tilde{\mathfrak{g}}$.

Let $\{q^i_+(z), q^i_-(z)\}$ be a solution to the qq-system for $\mathfrak{g}$ for fixed Z^H and Λ_i's. We let k and ℓ be the indices of the simple roots connected by the multiple bond, with k corresponding to the short simple root. Note that the Cartan matrices of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ only differs in the k,ℓ entry.

Fix a semisimple element $\tilde{Z}^{\tilde{H}} \in \tilde{\mathfrak{h}}$ by the equations $\langle\tilde{\alpha}_i, \tilde{Z}^{\tilde{H}}\rangle = (1+\delta_{ik}(-a_{k\ell}-1))\langle\alpha_i, Z^H\rangle$. Define polynomials $\tilde{q}^i_\pm(z)$ and $\tilde{\Lambda}_i(z)$ by

$$\tilde{q}^i_\pm(z) = \begin{cases} (q^k_\pm(z))^{-a_{k\ell}} & i = k, \\ q^i_\pm(z) & \text{otherwise}, \end{cases} \qquad \tilde{\Lambda}_i(z) = \begin{cases} -a_{k\ell}(q^k_+(z)q^k_-(z))^{-a_{k\ell}-1}\Lambda_k(z) & i = k, \\ (q^k_+(z))^{-a_{k\ell}-1}\Lambda_\ell(z) & i = \ell, \\ \Lambda_i(z) & \text{otherwise}. \end{cases}$$

(Note that $\tilde{\Lambda}_k$ is no longer monic.)

It is now easy to check that the $\tilde{q}^i_\pm(z)$'s satisfy the qq-system for $\tilde{\mathfrak{g}}$ given by

$$W(\tilde{q}^i_+, \tilde{q}^i_-)(z) + \langle\tilde{\alpha}_i, \tilde{Z}^{\tilde{H}}\rangle\tilde{q}^i_+(z)\tilde{q}^i_-(z) = \tilde{\Lambda}_i(z)\prod_{j\neq i}\left[\tilde{q}^j_+(z)\right]^{-\tilde{a}_{ji}}.$$

The kth equation is just the original kth equation multiplied by $-a_{k\ell}(q^k_+(z)q^k_-(z))^{-a_{k\ell}-1}$. The left-hand sides of the new and old ℓth equations coincide, and the additional factor in $\tilde{\Lambda}_\ell(z)$ ensures that the same holds for the right-hand sides. Finally, suppose $i \neq k, \ell$. Since i is not connected to k, $\tilde{a}_{ki} = a_{ki} = 0$, q^i_+ and $\tilde{q}^i_+$ do not appear on the right-hand side of the ith equation, so the new and old equations are identical. Note that this is where the construction fails if types C_n and F_4.

We remark that this construction always leads to degenerate solutions of the qq-system for $\tilde{\mathfrak{g}}$.

Remark 5.5. The q-deformed version of the system (5.3) is known as a QQ-system [FH2]. It plays a similar role in the study of the Bethe Ansatz equations for the XXZ model. It also arises in the ODE/IM correspondence [MRV1, MRV2], in the representation theory of quantum groups [FH2], and in enumerative geometry [KPSZ, KSZ, KZ2, KZ1].

In order to describe the relationship between solutions of the qq-system and Miura-Plücker opers, we need the notion of a Z^H-*twisted Miura-Plücker datum*. Recall that if ∇ is a Z^H-twisted Miura-Plücker oper, then by Theorem 2.7, it can be written in the form (2.8):

$$\nabla = \partial_z + \sum_{i=1}^r g_i(z)\check{\alpha}_i + \sum_{i=1}^r \Lambda_i(z)e_i, \qquad g_i(z) \in \mathbb{C}(z)^\times. \tag{5.4}$$

Moreover, there exists $v(z) \in B_+(z)$ such that for all $i = 1,\dots,r$, the Miura GL(2)-opers ∇_i associated to ∇ can be written in the form (3.9):

$$\nabla_i = v_i(z)(\partial_z + Z^H_i)v_i(z)^{-1}, \qquad i = 1,\dots,r, \tag{5.5}$$

where $v_i(z) = v(z)|_{W_i}$ and $Z_i^H = Z^H|_{W_i}$.

The element $v(z)$ is not uniquely determined by the Miura-Plücker oper. First, note that the subgroup $[N_+(z), N_+(z)]$ acts trivially on the representations W_i. Next, it is obvious from (5.5) that the constant maximal torus H fixes $\partial_z + Z_i^H$. It follows that any element of the coset $v(z)H[N_+(z), N_+(z)]$ also satisfies (5.5). We call such a coset a *framing* of the Miura-Plücker oper.

Definition 5.6. A Z^H*-twisted Miura-Plücker datum* is a pair $(\nabla, v(z)H[N_+(z), N_+(z)])$ consisting of a Z^H-twisted Miura-Plücker oper together with a framing. The datum is called nondegenerate if the underlying Miura-Plücker oper is nondegenerate.

Theorem 5.7. *There is a one-to-one correspondence between the set of nondegenerate Z^H-twisted Miura-Plücker data and the set of nondegenerate polynomial solutions of the qq-system* (5.3).

Proof. Let $(\nabla, v(z)H[N_+(z), N_+(z)])$ be a nondegenerate Z^H-twisted Miura-Plücker datum. We will fix the representative of the framing coset by setting

$$v(z) = \prod_{i=1}^r y_i(z)^{\check{\alpha}_i} \prod_{i=1}^r e^{-\frac{q_-^i(z)}{q_+^i(z)} e_i}, \tag{5.6}$$

where $q_+^i(z), q_-^i(z)$ are relatively prime polynomials with $q_+^i(z)$ monic for each $i = 1, \dots, r$ and each $y_i(z)$ is a monic polynomial.

We now show that the $q_+^i(z), q_-^i(z)$'s give a nondegenerate solution to the qq-system and in fact,

$$y_i(z) = q_+^i(z), \qquad i = 1, \dots, r. \tag{5.7}$$

We first compute the matrix of $v(z)$ and Z^H acting on the two-dimensional subspace W_i introduced in Section 3.1. A short calculation shows that

$$v(z)|_{W^i} = \begin{pmatrix} y_i(z) & 0 \\ 0 & y_i^{-1}(z) \prod_{j\neq i} y_j^{-a_{ji}}(z) \end{pmatrix} \begin{pmatrix} 1 & -\frac{q_-^i(z)}{q_+^i(z)} \\ 0 & 1 \end{pmatrix} \tag{5.8}$$

and

$$Z^H|_{W_i} = \begin{pmatrix} \zeta_i & 0 \\ 0 & -\zeta_i - \sum_{j\neq i} a_{ji}\zeta_j \end{pmatrix}. \tag{5.9}$$

We now apply (3.1) and (5.5) to relate the $y_i(z)$'s and $q_\pm^i(z)$'s. First, comparing the diagonal entries on both sides of (5.5) gives formula (2.15):

$$g_i(z) = \zeta_i - y_i^{-1}(z)\partial_z y_i(z). \tag{5.10}$$

Next, by comparing the upper triangular entries on both sides of (5.5), we obtain

$$\left[\partial_z\left(\frac{q_-^i(z)}{q_+^i(z)}\right) + \Big(\sum_j a_{ji}\zeta_j\Big)\frac{q_-^i(z)}{q_+^i(z)}\right]\Big[y_i(z)\Big]^2 = \Lambda_i(z)\prod_{j\neq i} y_j(z)^{-a_{ji}}. \tag{5.11}$$

Multiplying through by $q_+^i(z)^2$ gives

$$\Big[W(q_+^i(z),q_-^i(z))+\Big(\sum_j a_{ji}\zeta_j\Big)q_-^i(z)q_+^i(z)\Big]\Big[y_i(z)\Big]^2=\Big[q_+^i(z)\Big]^2\Lambda_i(z)\prod_{j\neq i}y_j(z)^{-a_{ji}}. \tag{5.12}$$

The nondegeneracy conditions for our oper imply that $y_i(z)|q_+^i(z)$. Write $q_+^i(z)=y_i(z)p(z)$. We will show that $p(z)$ has degree 0. Suppose that $p(z)$ has a root c with multiplicity $m\geq 1$. Note that c is a root of q_+^i of multiplicity either m or $m+1$, depending on whether c is a (necessarily simple) root of $y_i(z)$.

Now, rewrite the previous equation as

$$q_-^i(z)\partial_z q_+^i(z)=q_+^i(z)\partial_z q_-^i(z)+\Big(\sum_j a_{ji}\zeta_j\Big)q_-^i(z)q_+^i(z)-p(z)^2\Lambda_i(z)\prod_{j\neq i}y_j(z)^{-a_{ji}}. \tag{5.13}$$

Suppose that c is not a root of $y_i(z)$. Then c is a root of the left-hand side of (5.13) with multiplicity $m-1$. Since c is a zero of the three terms on the right-hand side have multiplicities $\geq m$, m, and $2m$ respectively, we have a contradiction. On the other hand, if c is a root of $y_i(z)$, then c is a root of the left-hand side with multiplicity m while it is a root of the three terms on the right-hand side with multiplicities $\geq m+1$, $m+1$, and $2m$. Again, we have a contradiction, so $p(z)$ is a constant. Since $q_+^i(z)$ and $y_i(z)$ are monic, $p(z)=1$ and $q_+^i(z)=y_i(z)$.

Dividing out by $y_i(z)^2$ in (5.12), we see that the polynomials $q_+^i(z), q_-^i(z)$, $i=1,\dots,r$, satisfy the system of equations (5.3) and are nondegenerate. Thus, we obtain a map from the set of nondegenerate Z-twisted Miura G-opers to the set of nondegenerate solutions of (5.3).

To show that this map is a bijection, we construct its inverse. Suppose that we are given a nondegenerate solution $\{q_+^i(z),q_-^i(z)\}_{i=1,\dots,r}$ of the system (5.3). We then define ∇ by formula (5.4), where we set

$$g_i(z)=\zeta_i-q_+^i(z)^{-1}\partial_z q_+^i(z),$$

i.e.

$$\nabla=\partial_z+\sum_{i=1}^r\Big[\zeta_i-q_+^i(z)^{-1}\partial_z q_+^i(z)\Big]\check{\alpha}_i+\sum_{i=1}^r\Lambda_i(z)e_i. \tag{5.14}$$

We also set

$$v(z)=\prod_{i=1}^r q_+^i(z)^{\check{\alpha}_i}\prod_{j=1}^r e^{-\frac{q_-^j(z)}{q_+^j(z)}e_i}. \tag{5.15}$$

Note that this means that we are setting $y_i(z)=q_+^i(z)$ for all i. Equations (5.5) are now satisfied for all i. Indeed, the Wronskian equations imply that the off-diagonal part of (5.5) holds while the diagonal part is automatic. Moreover, the nondegeneracy conditions on ∇ are satisfied by

Proposition 3.8. Therefore, $(\nabla, v(z)H[N_+(z), N_+(z)])$ defines a nondegenerate Z^H-twisted Miura-Plücker G-oper. This completes the proof. □

Remark 5.8. The inverse map is defined even for degenerate solutions of the qq-system. Thus, a polynomial solution of the qq-system gives rise to a Z^H-twisted Miura-Plücker datum without the assumption of nondegeneracy.

Corollary 5.9. *There is a surjective map from the set of nondegenerate polynomial solutions of the qq-system (5.3) to the set of nondegenerate Z^H-twisted Miura-Plücker opers whose fibers consist of all solutions with fixed $q^i_+(z)$'s for each $i = 1, \dots, r$.*

Proof. In the correspondence of the theorem, the Miura-Plücker oper is defined entirely in terms of the $q^i_+(z)$'s. The desired map is the composition of the inverse map with the map that forgets the framing. □

In the next section, we will describe the fibers of this map explicitly.

5.3. **The qq-system and the Bethe Ansatz equations.** We now derive the equations determining the zeros of a nondegenerate polynomial solution $\{q^i_+(z), q^i_-(z)\}_{i=1,\dots,r}$ of the qq-system. These equations are precisely the Bethe Ansatz equations for the inhomogeneous Gaudin model that were introduced in [FFTL, FFR2].

We begin by reformulating the qq-system. Multiplying both sides of (5.3) by $q^i_+(z)^{-2}e^{\langle\alpha_i, Z^H\rangle z}$ and recalling that $a_{ii} = 2$, we see that the qq-system is equivalent to

$$\partial_z\left[e^{\langle\alpha_i,Z^H\rangle z}\left(\frac{q^i_-(z)}{q^i_+(z)}\right)\right] = \Lambda_i(z)\left(\prod_j q^j_+(z)^{-a_{ji}}\right)e^{\langle\alpha_i,Z^H\rangle z}, \quad i = 1,\dots,r. \tag{5.16}$$

Let $\{w^i_\ell\}$ be the roots of $q^i_+(z)$. To derive the Bethe Ansatz equations, recall that a meromorphic function $f(z)$ with a double pole at w has residue 0 if and only if $\partial_z \log(f(z)(z-w)^2)|_{z=w} = 0$. By nondegeneracy, we can apply this remark to the right-hand side of (5.16) at w^i_ℓ, thereby obtaining the system of equations

$$\begin{gathered}\langle\alpha_i, Z^H\rangle + \partial_z \log\left[\Lambda_i(z)\prod_j q^j_+(z)^{-a_{ji}}(z-w^i_\ell)^2\right]\Bigg|_{z=w^i_\ell} = 0,\\ i = 1,\dots,r; \quad \ell = 1,\dots,\deg(q^i_+(z)).\end{gathered} \tag{5.17}$$

These equations can be recast in a more familiar form by computing the logarithmic derivatives explicitly. Recall from (2.7) that the roots of the $\Lambda_j(z)$'s are denoted by $z^i_1, \dots, z^i_{N_i}$ and the multiplicity of the root z_k in the $\Lambda_j(z)$'s is determined by the dominant integral coweight $\check\lambda_k$. A simple computation now gives the Bethe Ansatz equations

$$\langle\alpha_i, Z^H\rangle + \sum_{j=1}^{N_i} \frac{\langle\alpha_i, \check{\lambda}_j\rangle}{w^i_\ell - z^i_j} - \sum_{(j,s)\neq(i,\ell)} \frac{a_{ji}}{w^i_\ell - w^j_s} = 0, \qquad i = 1,\dots,r, \quad \ell = 1,\dots,\deg(q^i_+(z)). \tag{5.18}$$

Remark 5.10. These are the Bethe Ansatz equations corresponding to the representation $\otimes_{j=1}^N V_{\check{\lambda}_j}$ and the coweight $\check{\mu} = \sum \check{\lambda}_j - \sum \deg q^i_+(z)\check{\alpha}_i$. The $\check{\lambda}_i$'s are dominant, but we are not assuming that $\check{\mu}$ is dominant.

Next, we show that the map from nondegenerate polynomial solutions of the qq-equations to solutions of the Bethe Ansatz equations is surjective; moreover, the fibers are affine spaces of dimension equal to the number of simple roots which kill Z^H.

We start by considering some properties of the rational functions $\phi_i(z) = q^i_-(z)/q^i_+(z)$. First, we get an equivalent form of the ith qq-equations by dividing (5.3) by $q^i_+(z)^2$:

$$\partial_z\phi_i(z) + \langle\alpha_i, Z^H\rangle\phi_i(z) = \Lambda_i(z)\left(\prod_j q^j_+(z)^{-a_{ji}}\right). \tag{5.19}$$

For convenience, we set $\xi_i = \langle\alpha_i, Z^H\rangle$.

Since $e^{\xi_i z}\Lambda_i(z)\left(\prod_j q^j_+(z)^{-a_{ji}}\right)$ has a double pole at w^i_k and residue 0, we obtain the partial fraction decomposition

$$\Lambda_i(z)\left(\prod_j q^j_+(z)^{-a_{ji}}\right) = p_i(z) + \sum b^i_k\left(\frac{1}{(z-w^i_k)^2} - \frac{\xi_i}{z-w^i_k}\right), \tag{5.20}$$

where $p_i(z)$ is a polynomial. If we write

$$\phi_i(z) = h_i(z) + \sum \frac{c^i_k}{z-w^i_k}, \tag{5.21}$$

with $h_i(z)$ a polynomial, then (5.19) can be expressed in terms of partial fraction decompositions as

$$\partial_z h_i(z) + \xi_i h_i(z) - \sum \frac{c^i_k}{(z-w^i_k)^2} + \sum \frac{\xi_i c^i_k}{z-w^i_k} = p_i(z) + \sum b^i_k\left(\frac{1}{(z-w^i_k)^2} - \frac{\xi_i}{z-w^i_k}\right). \tag{5.22}$$

In other words,

$$c^i_k = -b^i_k \qquad \text{and} \qquad \partial_z h_i(z) + \xi_i h_i(z) = p_i(z). \tag{5.23}$$

We will use these conditions to define a polynomial solution of the qq-systems associated to a solution of the Bethe Ansatz equations. Fix such a solution, i.e., a collection of w^i_ℓ's satisfying 5.18. Notice that for this solution to make sense, w^i_ℓ is not a root of Λ_i and if $a_{ji} \neq 0$ and $(i,\ell) \neq (j,s)$, then $w^i_\ell \neq w^j_s$. Set $q^i_+(z) = \prod_\ell (z - w^i_\ell)$. We must show that there exist polynomials $q^i_-(z)$ which extend the $q^i_+(z)$'s to a solution of (5.3); this solution will automatically be nondegenerate.

In order to define $q^i_-(z)$, we will construct a rational function $\phi_i(z)$ whose poles are precisely the roots of $q^i_+(z)$ and set $q^i_-(z) = \phi_i(z) q^i_+(z)$. We define $\phi_i(z)$ via the partial fraction decomposition (5.21), so that the qq-equations are satisfied if and only if 5.23 holds. Thus, after setting $c^i_k = -b^i_k$, we just need $h_i(z)$ to be a polynomial solution of the differential equation $\partial_z h_i(z) + \xi_i h_i(z) = p_i(z)$. If $\xi_i = 0$, then $h_i(z)$ can be any indefinite integral of $p_i(z)$. If $\xi_i \neq 0$, then there is a unique indefinite integral of $e^{\xi_i z} p_i(z)$ such that $h_i(z) = e^{-\xi_i z} \int^z e^{\xi_i x} p_i(x)\, dx$ is a polynomial.

We thus obtain the following theorem, which was first proven in [MV2].

Theorem 5.11. (1) *If $\langle \alpha_i, Z^H \rangle \neq 0$ for all i (for example, if Z^H is regular semisimple), then there is a bijection between the solutions of the Bethe Ansatz equations (5.17) and the nondegenerate polynomial solutions of the qq-system (5.3).*

(2) *If $\langle \alpha_l, Z^H \rangle = 0$, for $l = i_1, \dots, i_k$ and is nonzero otherwise, then $\{q^i_+(z)\}_{i=1,\dots,r}$ and $\{q^i_-(z)\}_{i \neq i_1,\dots,i_k}$, are uniquely determined by the Bethe Ansatz equations, but each $\{q^{i_j}_-(z)\}$ for $j = 1, \dots k$ is only determined up to an arbitrary transformation $q^{i_j}_-(z) \longrightarrow q^{i_j}_-(z) + c_j q^{i_j}_+(z)$, where $c_j \in \mathbb{C}$.*

Remark 5.12. The map $\{q^i_+(z), q^i_-(z)\} \mapsto \{q^i_+(z)\}$ taking polynomial solutions of the qq-system to the "positive part" has fibers which are affine spaces of the dimension given in the theorem, even when the solutions are degenerate. Indeed, choose $q^1_+(z), \dots, q^r_+(z)$ for which there exists a (not necessarily nondegenerate) polynomial solution of the qq-system. The possible $q^i_-(z)$'s are determined by integrating (5.16):

$$q^i_-(z) = q^i_+(z) e^{-\langle \alpha_i, Z^H \rangle z} \int^z e^{\langle \alpha_i, Z^H \rangle x} \Lambda_i(x) \prod_j q^j_+(x)^{-a_{ji}}\, dx, \tag{5.24}$$

Here, we must choose the integration constant so that $q^i_-(z)$ is a polynomial. By hypothesis, there exists at least one such constant.

If $\langle \alpha_i, Z^H \rangle \neq 0$ for all i (for example, if Z^H is regular semisimple), then it is clear that only one integration constant is possible, so the $q^i_-(z)$'s are uniquely determined. However, if $\langle \alpha_i, Z^H \rangle = 0$, then $q^i_-(z)$ is only determined up to adding a constant multiple of $q^i_+(z)$:

$$q^i_-(z) = q^i_+(z) \left[c_i + \int^z \Lambda_i(x) \prod_j q^j_+(x)^{-a_{ji}}\, dx \right], \tag{5.25}$$

where $c_i \in \mathbb{C}$ is arbitrary.

Remark 5.13. The previous remark shows that the degrees of the q^i_-'s are essentially determined by the degrees of the q^i_+'s and the Λ_i's. If $\langle \alpha_i, Z^H \rangle \neq 0$, then it is obvious from the ith qq-equation that $\deg q^i_- = \deg \Lambda_i - \deg q^i_+ - \sum_{j \neq i} a_{ji} \deg q^j_+$. On the other hand, if $\langle \alpha_i, Z^H \rangle = 0$, then it follows from the theorem that there is a solution with $\deg q_-^i \neq \deg q_+^i$. In this case, $\deg W(q_+^i, q_-^i) = \deg q_+^i + \deg q_-^i - 1$, so $\deg q_-^i = 1 + \deg \Lambda_i - \deg q_+^i - \sum_{j \neq i} a_{ji} \deg q_+^j$. If this degree is greater than $\deg q_+^i$, then every possible q_-^i has this degree. If it is less than $\deg q_+^i$, then every other possible q_-^i has degree equal to $\deg q_+^i$.

An immediate consequence of this theorem is the algebraicity of the set of q_+^i's giving rise to nondegenerate solutions of the qq-system. More precisely, fix nonnegative integers $d_1, \dots, d_r$. Let $\mathcal{Q}_{d_1,\dots,d_r}$ be the set of monic polynomials $p_1, \dots p_r$ such that there exists a nondegenerate polynomial solution of the qq-equations (for the given Z^H and Λ_i's) satisfying $q_+^i = p_i$ and $\deg p_i = d_i$ for all i.

Corollary 5.14. *The set $\mathcal{Q}_{d_1,\dots,d_r}$ is an affine variety.*

This theorem states that there is a surjection from nondegenerate polynomial solutions of the qq-system and solutions of the Bethe Ansatz equation whose fibers consist of all solutions with fixed $q_+^i(z)$'s. Combining this with Corollary 5.9 gives the following result:

Theorem 5.15. *There is a one-to-one correspondence between nondegenerate Z^H-twisted Miura-Plücker opers and solutions of the Bethe Ansatz equations* (5.18).

Remark 5.16. Let ${}^L G$ be the adjoint group with Lie algebra $^L\mathfrak{g}$ that is Langlands dual to $\mathfrak{g}$. Theorem 6.7 of [FFTL] states the equivalence between the Miura ${}^L G$-opers with Cartan connection of the form (see equation 6.7 of [FFTL])

$$\partial_z + Z - \sum_{i=1}^{N} \frac{\lambda_i}{z - z_i} + \sum_{k=1}^{r} \sum_{j=1}^{m} \frac{\alpha_k}{z - w_j^k},$$

where $\{w_j^k\}$ satisfy the Bethe Ansatz equations, and the joint eigenvalues on Bethe vectors of the Gaudin Hamiltonians corresponding to the Lie algebra $\mathfrak{g}$. Using a gauge transformation by $\prod_{i=1}^{N} (z - z_i)^{\lambda_i} \in H^L(z)$, one can transform the connection of those Miura opers to ours.

5.4. **Regularity of the connection at the $\{w_\ell^i\}$'s.** The expression (5.14) for a nondegenerate Miura-Plücker oper appears to have singularities at the roots of the q_+^i's. However, there exists a gauge in wich the connection is in fact regular; in other words, the connection (5.14) has trivial monodromy at $\{w_\ell^i\}$'s. To show this, it will be convenient to describe the Bethe Ansatz equations in terms of the Cartan connection $\nabla^H = \partial_z + A^H(z)$, with $A^H(z)$ defined in (2.12):

$$\left(\frac{2}{z - w_\ell^i} + \langle \alpha_i, A^H(z) \rangle + \partial_z \log \Lambda_i(z) \right) \Big|_{z = w_\ell^i} = 0, \qquad i = 1, \dots, r, \quad \ell = 1, \dots, \deg(q_+^i(z)). \tag{5.26}$$

We now apply gauge change by $g_{i,\ell}(z) = \exp\left[\frac{-f_i}{\Lambda_i(z)(z-w^i_\ell)}\right]$ to

$$\nabla = \partial_z + \sum_{i=1}^{r} \Lambda_i(z) e_i + A^H(z).$$

The only terms in which $z - w^i_\ell$ appears in the denominator are those which involve $\check{\alpha}_i$ and f_i. The former gives $\frac{1}{z-w^i_\ell}\check{\alpha}_i + \langle \alpha_i, A^H(z)\rangle \check{\alpha}_i$, and since $\langle \alpha_i, A^H(z)\rangle$ has a simple pole at w^i_ℓ with residue -1, this expression is regular at w^i_ℓ. The terms involving f_i are

$$-\partial_z\left(\frac{1}{\Lambda_i(z)(z-w^i_\ell)}\right) f_i - \frac{f_i}{\Lambda_i(z-w^i_\ell)^2} - \frac{\langle \alpha_i, A^H(z)\rangle f_i}{\Lambda_i(z)(z-w^i_\ell)} = -\frac{\partial_z \log \Lambda_i + 2(z-w^i_\ell)^{-1} + \langle \alpha_i, A^H(z)\rangle}{\Lambda_i(z-w^i_\ell)} f_i.$$

The residue term of this vanishes by the Bethe Ansatz equations (5.26), and we conclude that the matrix of ∇ in this gauge is manifestly regular at w^i_ℓ.

Thus, we have proved the following theorem:

Theorem 5.17. *The nondegenerate Z-twisted Miura-Plücker oper connections have trivial monodromy at the $\{w^i_\ell\}$'s.*

6. BÄCKLUND TRANSFORMATIONS

In this section, we show that nondegenerate Z^H-twisted Miura-Plücker opers are in fact Z^H-twisted Miura opers. Thus, solutions of the Bethe Ansatz equations are in fact parameterized by Z^H-twisted Miura opers. The proof relies on the important technical tool of *Bäcklund transformations*: transformations on twisted Miura-Plücker opers associated to elements of the Weyl group. These transformations were first introduced in the context of qq-systems in [MV2], where they were referred to as reproduction procedures. When $Z^H = 0$, it was shown in [MV1] that these reproduction procedures act on the differential operators underlying opers as in Proposition 6.1 below. If $Z \neq 0$, the Backlünd transformations coincide with the exponential reproduction procedure of [MV2]. Moreover, it was proved in [MV2, Theorem 6.7] that the population obtained from the exponential reproduction procedure for regular semisimple Z can be identified with an orbit of the Weyl group of $\mathfrak{g}$. For completeness, we will reprove some of the results of [MV2]. We will then establish the full correspondence between qq-systems and Z-twisted Miura G-opers.

6.1. Simple Bäcklund transformations.

Our goal is to define transformations which take a Z^H-twisted Miura-Plücker oper to a $w(Z^H)$-twisted Miura-Plücker oper, where w is an element of the Weyl group. As a first step, we consider the case of a simple reflection s_i.

Recall that a polynomial solution of the qq-system gives rise to a connection (5.14) defined in terms of the q^j_+'s and Z^H. We now exhibit a gauge transformation which takes this connection to another connection in the form (5.14), but with q^i_+ and Z^H replaced by q^i_- and $s_i(Z^H)$. This gauge transformation is by an element of $N_-(z)$, so it does not preserve the Miura-Plücker oper structure.

Proposition 6.1. *Let $\{q_+^j, q_-^j\}_{j=1,\dots,r}$ be a polynomial solution of the qq-system (5.3), and let ∇ be the connection of the corresponding Z-twisted Miura-Plücker oper in the form (5.14). Let $\nabla^{(i)}$ be the connection obtained from ∇ via the gauge transformation by $e^{\mu_i(z) f_i}$, where*

$$\mu_i(z) = \Lambda_i(z)^{-1}\left[\partial_z \log\left(\frac{q_-^i(z)}{q_+^i(z)}\right) + \langle \alpha_i, Z^H\rangle\right] \tag{6.1}$$

Then $\nabla^{(i)}$ is obtained by making the following substitutions in (5.14):

$$\begin{aligned} q_+^j(z) &\mapsto q_+^j(z), \qquad && j \neq i, \\ q_+^i(z) &\mapsto q_-^i(z), \qquad && Z \mapsto s_i(Z^H) = Z^H - \langle \alpha_i, Z^H\rangle\, \check{\alpha}_i. \end{aligned} \tag{6.2}$$

Proof. A short computation shows that

$$\nabla^{(i)} = e^{\mu_i(z) f_i}\, \nabla\, e^{-\mu_i(z) f_i} = \tag{6.3}$$

$$\partial_z + A^H(z) - \Lambda_i(z)\mu_i(z)\check{\alpha}_i + \sum_{k=1}^r \Lambda_k(z) e_k + f_i\Big(\mu_i(z)\langle \alpha_i, A^H(z)\rangle - \mu_i'(z) - \mu_i(z)^2\Lambda_i(z)\Big),$$

where we remind that $A^H(z) = \sum_{i=1}^r (\zeta_i - \partial_z \log q_+^i(z))\check{\alpha}_i$.

In this expression, the diagonal term is

$$Z^H - \langle \alpha_i, Z^H\rangle\, \check{\alpha}_i - \sum_j \frac{\partial_z q_+^j(z)}{q_+^j(z)}\check{\alpha}_j - \partial_z \log\left(\frac{q_-^i(z)}{q_+^i(z)}\right)\check{\alpha}_i = s_i(Z^H) - \frac{\partial_z q_-^i(z)}{q_-^i(z)}\check{\alpha}_i - \sum_{j\neq i}\frac{\partial_z q_+^j(z)}{q_+^j(z)}\check{\alpha}_j$$

as desired.

Thus the statement of the theorem is true if μ_i satisfies the *Riccati equation*:

$$\frac{\mu_i'(z)}{\mu_i(z)} + \mu_i(z)\Lambda_i(z) = \langle \alpha_i, A^H(z)\rangle. \tag{6.4}$$

Setting $h_i(z) = \Lambda_i(z)\mu_i(z)$, this equation is equivalent to

$$\frac{h_i'(z)}{h_i(z)} + h_i(z) = \langle \alpha_i, A^H(z)\rangle + \partial_z \log(\Lambda_i(z)). \tag{6.5}$$

This identity now follows by taking the logarithmic derivative of (5.16):

$$\frac{h_i'(z)}{h_i(z)} + h_i(z) = \partial_z \log h_i(z) + \partial_z \log\left(\frac{q^i_-(z)}{q^i_+(z)}\right) + \langle \alpha_i, Z^H\rangle z = \partial_z \log\left(\partial_z\left[\left(\frac{q^i_-(z)}{q^i_+(z)}\right)e^{\langle\alpha_i, Z^H\rangle z}\right]\right)$$
$$= \partial_z \log\left(\Lambda_i(z)\left[\prod_j q^j_+(z)^{-a_{ji}}\right]e^{\langle\alpha_i, Z^H\rangle z}\right) = \partial_z \log(\Lambda_i(z)) + \langle\alpha_i, A^H(z)\rangle.$$

□

Remark 6.2. Note that $\mu_i(z)$ can be rewritten using the qq-system equations as:

$$\mu_i(z) = \frac{\prod_{j\neq i} q^j_+(z)^{-a_{ji}}}{q^i_+(z) q^i_-(z)}. \tag{6.6}$$

6.2. **General Bäcklund transformations.** We would like to construct Bäcklund transformations associated to an arbitrary element w of the Weyl group by taking a reduced expression for w and composing simple Bäcklund transformations associated to the given simple reflections. However, in general, it is not possible to compose Bäcklund transformations. The problem is that, even if one starts with a nondegenerate solution of the qq-system, the connection $\nabla^{(i)}$ defined in Proposition 6.1 is not necessarily the underlying connection of a nondegenerate $s_i(Z^H)$-twisted Miura-Plücker oper. It thus does not give rise to the necessary initial data for another Bäcklund transformation, namely a solution of the qq-system for $s_i(Z^H)$.

Definition 6.3. Let $\{q^j_+(z), q^j_-(z)\}$ be a polynomial solution of the qq-system for Z^H.

(1) The solution is called *i-composable* if the polynomials $q^1_+(z), \dots, q^{i-1}_+(z), q^i_-(z), q^{i+1}_+(z), \dots, q^r_+(z)$ are the positive polynomials of a solution to the qq-system for $s_i(Z^H)$.
(2) The solution is called *i-generic* if it is nondegenerate and if the collection of polynomials $q^1_+(z), \dots, q^{i-1}_+(z), q^i_-(z), q^{i+1}_+(z), \dots, q^r_+(z)$ satisfy the conditions in Definition 3.4.

We will also refer to a twisted Miura-Plücker datum as i-composable or i-generic if it comes from such a solution of the qq-system.

It is immediate from Proposition 6.1 and Theorem 5.7 that if $\{q^j_+(z), q^j_-(z)\}$ is i-composable, then $\nabla^{(i)}$ is the underlying connection of a $s_i(Z^H)$-twisted Miura-Plücker oper.

Remark 6.4. Assume that $\mathcal{Q}_{d_1,\dots,d_r}$ is nonempty. While it is easy to see that i-genericity is a Zariski-open condition on the variety $\mathcal{Q}_{d_1,\dots,d_r}$, it is not clear that this open subset is nonempty. In other words, $q^i_-(z)$ may have multiple roots or it may share a root with $\Lambda_i(z)$ or with $q^j_+(z)$ for $j \neq i$ such that $a_{ji} \neq 0$. However, if $\langle\alpha_i, Z^H\rangle = 0$, the set of i-generic polynomial solutions is nonempty. Indeed, if $q^i_-(z)$ does not satisfy the conditions in Definition 3.4, one can replace it by $q^i_-(z) + c q^i_+(z)$ for an appropriate nonzero scalar c. In particular, when $Z = 0$, the set of nondegenerate polynomial solutions of the qq-system that are i-generic for all i is nonempty.

Lemma 6.5. *If $\{q^j_+, q^j_-\}_{j=1,\dots,r}$ is an i-generic polynomial solution of the qq-system, then it is i-composable. In particular,*

(1) *The connection $\nabla^{(i)}$ constructed in Proposition 6.1 is the underlying connection of a nondegenerate $s_i(Z^H)$-twisted Miura-Plücker oper.*
(2) *Any corresponding (necessarily nondegenerate) polynomial solution $\{\widetilde{q}^j_+, \widetilde{q}^j_-\}_{j=1,\dots,r}$ of the qq-system for $s_i(Z^H)$ has $\widetilde{q}^i_+ = q^i_-$ and $\widetilde{q}^j_+ = q^j_+$ for $j \neq i$. Moreover, one may take $\widetilde{q}^i_- = -q^i_+$.*

Proof. We will show that the polynomials $\{\widetilde{q}^j_+\}$ defined above give rise to a solution of the Bethe Ansatz equations for $s_i(Z^H)$. It will then follow from Theorem 5.11 that there exist polynomials $\widetilde{q}^j_-$ such that $\{\widetilde{q}^j_+, \widetilde{q}^j_-\}_{j=1,\dots,r}$ is a nondegenerate polynomial solution of the qq-system; moreover, this solution will correspond to a $s_i(Z^H)$-twisted Miura-Plücker datum with underlying connection $\nabla^{(i)}$. We will show explicitly that one can take $\widetilde{q}^i_- = -q^i_+$.

First, note that $W(q^i_-, -q^i_+) = W(q^i_+, q^i_-)$ and $\langle \alpha_i, s_i(Z^H)\rangle q^i_-(z)(-q^i_+(z)) = \langle \alpha_i, Z^H\rangle q^i_+(z) q^i_-(z)$. It is now immediate that the ith equation of the qq-system for $s_i(Z^H)$ is satisfied by the $\widetilde{q}^j_+$'s and $\widetilde{q}^i_- = -q^i_+$. As in the proof of Theorem 5.11, this implies that the Bethe Ansatz equations (5.17) involving the roots of $\widetilde{q}^i_+ = q^i_-$ are satisfied.

Next, rewrite the ith equation of the original qq-system as

$$\partial_z \log(q^i_-(z)) - \partial_z \log(q^i_+(z)) + \langle Z^H, \alpha_i\rangle = \frac{\Lambda_i(z)}{q^i_+(z) q^i_-(z)} \prod_{j\neq i} \left[q^j_+(z)\right]^{-a_{ji}}. \tag{6.7}$$

Evaluating this expression at a root w^j_ℓ of $q^j_+(z)$ for $j \neq i$ and using nondegeneracy, one obtains

$$\partial_z \log(q^i_-(z))\Big|_{w^j_\ell} + \langle \alpha_i, Z^H\rangle = \partial_z \log(q^i_+(z))\Big|_{w^j_\ell}. \tag{6.8}$$

One gets the remaining Bethe Ansatz equations by substituting this into (5.17):

$$\begin{aligned} 0 &= \langle \alpha_j, Z^H\rangle + \partial_z \log\left[\Lambda_j(z) \prod_k q^k_+(z)^{-a_{kj}} (z - w^j_\ell)^2\right]\Bigg|_{z=w^j_\ell} \\ &= \langle \alpha_j, Z^H\rangle - a_{ij}\langle \alpha_i, Z^H\rangle + \partial_z \log\left[\Lambda_j(z) q^i_-(z)^{-a_{ij}} \prod_{k\neq i} q^k_+(z)^{-a_{kj}} (z - w^j_\ell)^2\right]\Bigg|_{z=w^j_\ell} \\ &= \langle \alpha_j, s_i(Z^H)\rangle + \partial_z \log\left[\Lambda_j(z) \prod_k \widetilde{q}^k_+(z)^{-a_{kj}} (z - w^j_\ell)^2\right]\Bigg|_{z=w^j_\ell}. \end{aligned} \tag{6.9}$$

□

Thus, the ith simple Bäcklund transformation may be viewed as taking an i-generic Miura-Plücker datum to a nondegenerate $s_i(Z^H)$-twisted Miura-Plücker oper.

Definition 6.6. Let $w = s_{i_1} \dots s_{i_k}$ be a reduced decomposition of an element w of the Weyl group.

(1) A polynomial solution of the qq-system (5.3) for Z^H is called $(i_1, \dots, i_k)$-*composable* if for each ℓ, $1 \le \ell \le k$, the connection $\nabla^{(i_k)\dots(i_{k-\ell+1})}$ comes from a polynomial solution of the qq-system for $s_{i_{k-\ell+1}} \dots s_{i_k}(Z^H)$.

(2) The solution is called $(i_1, \dots, i_k)$-*generic*) if it is nondegenerate and for each ℓ, $1 \le \ell \le k$, the connection $\nabla^{(i_k)\dots(i_{k-\ell+1})}$ comes from a nondegenerate polynomial solution of the qq-system for $s_{i_{k-\ell+1}} \dots s_{i_k}(Z^H)$.

(3) A Z^H-twisted Miura-Plücker oper is called $(i_1, \dots, i_k)$-*composable* (resp. $(i_1, \dots, i_k)$-*generic*) if it arises from such a solution of the qq-system.

It is immediate that $(i_1, \dots, i_k)$-genericity implies $(i_1, \dots, i_k)$-composability.

Remark 6.7. Note that in this definition, we only assume the existence of a sequence of transformations as described in Lemma 6.5 for a particular reduced decomposition of w. We do not assume that such a sequence exists for other reduced decompositions of w.

We will need a technical result for $(i_1 \dots i_k)$-composable solutions of the qq-system, showing the existence of an element of $B_-(z)$ which intertwines the action of ∇ and $s_{i_1} \dots s_{i_k}(Z^H)$ on highest weight vectors.

Proposition 6.8. *Let $w = s_{i_1} \dots s_{i_k}$ be a reduced decomposition. Then, for each $(i_1 \dots i_k)$-composable solution of the qq-system (5.3), there exists an element $b_-(z) \in B_-(z)$ of the form*

$$b_-(z) = e^{c_{i_k}(z) f_{i_k}} \dots e^{c_{i_2}(z) f_{i_2}} e^{c_{i_1}(z) f_{i_1}} h(z),$$

where $c_{i_j}(z)$ are non-zero rational functions and $h(z) \in H(z)$, such that

$$b_-(z) w(Z^H) v = \partial_z b_-(z) v + A(z) b_-(z) v. \tag{6.10}$$

Here, $A(z)$ is given by equation (5.14) and v is a highest weight vector in any irreducible finite-dimensional representation of G.

Proof. Let ∇^w be the $w(Z^H)$-twisted Miura-Plücker oper obtained by iterating the Bäcklund transformations defined in Proposition 6.1:

$$\nabla^w = e^{\mu_{i_1}(z) f_{i_1}} \dots e^{\mu_{i_k}(z) f_{i_k}} \nabla e^{-\mu_{i_k}(z) f_{i_k}} \dots e^{-\mu_{i_1}(z) f_{i_1}}. \tag{6.11}$$

Let $\{\overline{q}^i_+\}_{i=1,\dots,r}$ be the "plus" part of the corresponding solution to the qq-system. We claim that

$$b_-(z) = e^{-\mu_{i_k}(z) f_{i_k}} \dots e^{-\mu_{i_1}(z) f_{i_1}} \prod_j \left[\overline{q}^j_+(z)\right]^{\check{\alpha}_j} \tag{6.12}$$

satisfies (6.10).

Let V be an irreducible representation with highest weight λ, and let $v \in V$ be a highest weight vector. First observe that

$$\nabla^w v = w(Z^H)v - \left(\sum_{j=1}^{r} \frac{\partial_z \bar{q}_+^j(z)}{\bar{q}_+^j(z)} \check{\alpha}_j \right) v. \tag{6.13}$$

For brevity, write $E(z) = e^{-\mu_{i_k}(z) f_{i_k}} \dots e^{-\mu_{i_1}(z) f_{i_1}}$. We now compute:

$$\begin{aligned}
(\partial_z + A(z)) b_-(z) v &= \prod_j \left[\bar{q}_+^j(z) \right]^{\langle \check{\alpha}_j, \lambda \rangle} (\partial_z + A(z)) E(z) v + b_-(z) \left(\sum_j \frac{\partial_z \bar{q}_+^j(z)}{\bar{q}_+^j(z)} \check{\alpha}_j \right) v \\
&= \prod_j \left[\bar{q}_+^j(z) \right]^{\langle \check{\alpha}_j, \lambda \rangle} E(z) \nabla^w v + b_-(z) \left(\sum_j \frac{\partial_z \bar{q}_+^j(z)}{\bar{q}_+^j(z)} \check{\alpha}_j \right) v \\
&= b_-(z) \left[w(Z^H) v - \left(\sum_{j=1}^{r} \frac{\partial_z \bar{q}_+^j(z)}{\bar{q}_+^j(z)} \check{\alpha}_j \right) v \right] + b_-(z) \left(\sum_j \frac{\partial_z \bar{q}_+^j(z)}{\bar{q}_+^j(z)} \check{\alpha}_j \right) v \\
&= b_-(z) w(Z^H) v,
\end{aligned}$$

as desired.

□

6.3. Z^H**-twisted Miura-Plücker opers with admissible combinatorics are** Z^H**-twisted Miura opers.** We now prove one of the main results of the paper, namely, that Z^H-twisted Miura-Plücker opers satisfying certain combinatorial conditions are in fact nondegenerate Z^H-twisted Miura opers. We begin by outlining the argument.

The first step is to define a class of Z^H-twisted Miura-Plücker opers for which one can give an explicit construction of an upper triangular matrix which diagonalizes the oper, thereby showing that it is a Z^H-twisted Miura oper. The desired condition will be called w_0-*genericity* (or more generally, w_0-*composability*); it will be a special case of the genericity considered in Definition 6.6.

Next, we observe that the behavior of the qq-system and its iterates under Bäcklund transformations depend on certain underlying combinatorics: the set of roots killing Z^H, the degrees of the Λ_i's, and the degrees of the q_+^i's. This combinatorial data essentially determine the degrees of the q_-^i's and inductively, the degrees of the polynomials appearing as solutions of the new qq-systems obtained after applying Bäcklund transformations. We will call this combinatorial data *admissible* if there exists a w_0-generic solution of the qq-system with the given combinatorics.

Finally, we show that twisted Miura-Plücker opers with admissible combinatorics are in fact Miura opers. To do this, we introduce formal variables associated to the given admissible combinatorics: for the coordinates of a certain affine variety determined by the set of roots, for the zeros of the q_+^i's and other $\tilde{q}_\pm^i$'s that appear upon an appropriate iteration of Bäcklund transformations, and for the zeros of the Λ_i's. We construct a ring R by adjoining these formal variables to $\mathbb{C}(z)$ and taking a suitable localization. One can now define a qq-system $\{Q^i_+, Q^i_-\}$ over R which has the property that upon specializing the formal variables appropriately, one obtains an ordinary qq-system with the given combinatorics. Moreover, $\{Q^i_+, Q^i_-\}$ is w_0-generic because it specializes to an ordinary w_0-generic qq-system. We can use this fact to deduce that Miura-Plücker opers with the given combinatorics are in fact Miura opers.

6.3.1. *w_0-composability and w_0-genericity.* We begin by describing a sufficient condition for a Z^H-twisted Miura-Plücker oper to be a Z^H-twisted Miura oper. Let w_0 be the longest element of the Weyl group. We call a solution of the qq-system (or the corresponding Miura-Plücker oper) w_0-generic (resp. *w_0-decomposable*) if there exists a reduced decomposition $w_0 = s_{i_1} \dots s_{i_\ell}$ such that the solution (or oper) is $(i_1, \dots, i_\ell)$-generic (resp. composable). (For any $w \in W$, one defines w-genericity and w-composability similarly.)

We will need the following well-known fact about the product of Bruhat cells (see e.g. [H, Lemma 29.3.A]):

Lemma 6.9. *i) If $u, v \in W$ satisfy $\ell(u) + \ell(v) = \ell(uv)$, then $B_- u B_- v B_- = B_- uv B_-$.*
ii) If $w \in W$ has a reduced decomposition $w = s_{i_1} s_{i_2} \dots s_{i_k}$, then

$$e^{a_{i_1} e_{i_1}} e^{a_{i_2} e_{i_2}} \dots e^{a_{i_k} e_{i_k}} \in B_- w N_-, \quad e^{a_{i_1} f_{i_1}} e^{a_{i_2} f_{i_2}} \dots e^{a_{i_k} f_{i_k}} \in B_+ w N_+$$

if $a_{i_j} \neq 0$ for all j.

Theorem 6.10. *Every w_0-composable (resp. w_0-generic) Z^H-twisted Miura-Plücker G-oper is a Z^H-twisted Miura G-oper (resp. a nondegenerate Z^H-twisted Miura G-oper).*

Proof. Let

$$\nabla = \partial_z + A(z) = \partial_z + \sum_{i=1}^r \left[\zeta_i - q^i_+(z)^{-1} \partial_z q^i_+(z)\right] \check{\alpha}_i + \sum_{i=1}^r \Lambda_i(z) e_i$$

be the w_0-composable Z^H-twisted Miura-Plücker oper coming from a w_0-composable solution $\{q^j_+\}$ of the qq-system. By Proposition 6.8, there exists an element $b_-(z) \in B_-(z)$ such that

$$b_-(z) w_0(Z^H) v = (\partial_z + A(z)) b_-(z) v,$$

where v is any highest weight vector in a finite-dimensional irreducible representation of G. Moreover, if $w_0 = s_{i_1} \dots s_{i_\ell}$ is a reduced expression for which the solution is $(i_1, \dots, i_\ell)$-composable, then

$$b_-(z) = e^{c_{i_\ell} f_{i_\ell}} \dots e^{c_{i_2} f_{i_2}} e^{c_{i_1} f_{i_1}} h(z)$$

with $c_{i_j}(z) \in \mathbb{C}(z)^\times$ and $h(z) \in H(z)$.

By Lemma 6.9 and the fact that w_0 is an involution,

$$b_-(z) = b_+(z) w_0 n_+(z),$$

for some $b_+(z) \in B_+(z)$ and $n_+(z) \in N_+(z)$, so if v is a highest weight vector in an irreducible representation,

$$b_+(z)Z^H w_0 v = (\partial_z + A(z))b_+(z)w_0 v.$$

Therefore, if we set

$$u(z) = Z^H - b_+^{-1}(z)\partial_z b_+(z) + b_+^{-1}(z)A(z)b_+(z) \in \mathfrak{b}_+(z), \tag{6.14}$$

then

$$u(z)w_0 v = 0.$$

for any irreducible finite-dimensional representation of G with highest weight vector v. Thus, $u(z)$ is an element of $\mathfrak{b}_+(z)$ which fixes the lowest weight vector $w_0 v$ of any irreducible finite-dimensional representation of G. This means that $u(z) = 0$. Equation (6.14) then implies that $A(z)$ satisfies

$$A(z) = b_+(z)(\partial_z + Z^H)b_+(z)^{-1} \tag{6.15}$$

for some $b_+(z) \in B_+(z)$. Thus, we have proved that every w_0-composable Z^H-twisted Miura-Plücker oper is a Z^H-twisted Miura oper. Equivalently, every w_0-composable solution of the qq-system gives rise to a Z^H-twisted Miura oper. By definition, if the original solution is in fact w_0-generic, then the corresponding Z^H-twisted Miura oper is nondegenerate. □

6.3.2. *Admissible combinatorial data.* Let $d_1, \dots, d_r$ and $N_1, \dots, N_r$ be nonnegative integers, and let Ψ be a collection of roots. Set $\mathfrak{h}_\Psi = \{Y \in \mathfrak{h} \mid \beta(Y) = 0 \iff \beta \in \Psi\}$; it is an affine cone.

Definition 6.11. The combinatorial datum $(\mathbf{d} = (d_1, \dots, d_r), \mathbf{N} = (N_1, \dots, N_r), \Psi)$ is called w_0-*admissible* (or simply *admissible*) if there exists a w_0-generic solution of the qq-system with $Z_H \in \mathfrak{h}_\Psi$ and for all i, $\deg \Lambda_i = N_i$ and $\deg q_+^i = d_i$.

Remark 6.12. One may similarly define w-admissibility. In this language, e-admissibility combinatorics simply means that there exists a nondegenerate polynomial solution with the given combinatorics.

We now give a more explicit formulation of admissibility in the two opposite extremes $Z^H = 0$ and Z^H regular semisimple, i.e. Ψ equals Φ (the set of all roots) or $\varnothing$.

Proposition 6.13. *The combinatorial datum* $(\mathbf{d}, \mathbf{N}, \Phi)$ *is admissible if and only if there exists a nondegenerate polynomial solution of the qq-system with* $Z^H = 0$ *and for all* i, $\deg q_+^i(z) = d_i$ *and* $\deg \Lambda_i = N_i$.

Proof. By induction, it suffices to show that for any nondegenerate solution and for any i, one can modify $q_-^i(z)$ so that the solution is i-generic. This was shown in Remark 6.4. □

We now assume that Z^H is regular semisimple. In this case, one can characterize admissibility explicitly in terms of certain inequalities that must be satisfied by the d_j's and N_j's.

We first observe that a Bäcklund transformation induces a transformation on the set of d_j's. Indeed, as we have seen in Remark 5.13, if Z^H is regular semisimple, then $\deg q^i_- = \deg \Lambda_i - \deg q^i_+ - \sum_{j\neq i} a_{ji} \deg q^j_+$. Accordingly, the ith Bäcklund transformation takes $d^j \mapsto d^{(i)}_j$, where

$$d^{(i)}_j = \begin{cases} N_i - d_i - \sum_{k\neq i} a_{ki} d_k & \text{if } j = i \\ d_j & \text{otherwise.} \end{cases} \tag{6.16}$$

The following necessary condition for the existence of an $(i_1, \dots, i_k)$-composable solution of the qq-system with fixed regular semisimple combinatorics is now immediate.

Lemma 6.14. *If there exists an $(i_1, \dots, i_k)$-composable polynomial solution with combinatorial datum $(\mathbf{d}, \mathbf{N}, \varnothing)$, then for $0 \leq s \leq k$*[3] *and $1 \leq j \leq r$,*

$$d^{(i_k)\dots(i_{k-s+1})}_j \leq N_j - \sum_{\ell\neq j} a_{pj} d^{(i_k)\dots(i_{k-s+1})}_p. \tag{6.17}$$

It turns out that if $\mathfrak{g}$ is simply-laced, then this necessary condition is in fact sufficient. Moreover, one can find a generic solution with the given combinatorics. In order to prove this, we will consider a limit of the qq-system, the *infinite qq-system*.

Let $\xi_i = \langle \alpha_i, Z^H\rangle$. To take the limit of the ith qq-equation as ξ_i goes to infinity, we need to rewrite the equation. Since the right-hand side of the equation is monic, we have $q^i_-(z) = \xi_i^{-1} \bar{q}^i_-(z)$, where $\bar{q}^i_-(z)$ is monic. The ith qq-equation is thus equivalent to

$$\xi_i^{-1} W(q^i_+, \bar{q}^i_-)(z) + q^i_+(z)\bar{q}^i_-(z) = \Lambda_i(z) \prod_{j\neq i} \left[q^j_+(z)\right]^{-a_{ji}} \tag{6.18}$$

Upon taking the limit, the Wronskian term disappears.

Definition 6.15. The *infinite qq-system* associated to $\mathfrak{g}$ and the collection of monic polynomials $\Lambda_1(z), \dots, \Lambda_r(z)$ is the system of equations

$$q^i_+(z) q^i_-(z) = \Lambda_i(z) \prod_{j\neq i} \left[q^j_+(z)\right]^{-a_{ji}} \qquad \text{for } i = 1, \dots, r, \tag{6.19}$$

where the $q^j_+(z)$'s (and hence the $q^i_-(z)$'s) are assumed to be monic.

It is easier to understand the significance of the infinite qq-system in the q-deformed case [FKSZ]. The q-difference analog of the qq-system, known as the QQ-system, expresses the relations between the so-called Baxter Q-operators in the corresponding XXZ integrable model [FH1], [FH2], acting on a tensor product $\mathcal{H}$ of finite-dimensional representations of the quantum group $U_q(\widehat{\mathfrak{g}})$.

[3] By convention, the case $s = 0$ corresponds to the original d_j's.

(This tensor product is the underlying Hilbert space of the XXZ model). The Baxter Q-operators can be expressed as weighted half-traces $Q^i_\pm(z) = \mathrm{Tr}_{V^i_\pm}\left[(\mathcal{Z}\otimes I)R\right]$ in the so-called prefundamental representations $\{V^i_\pm\}_{i=1,\dots,r}$ of $U_q(\widehat{b}_+)$ (see [HJ]) of the normalized universal R-matrix $R \in U_q(\widehat{b}_+)\widehat{\otimes}U_q(\widehat{b}_-)$; here, the weight $\mathcal{Z} = \prod_i \hat{\zeta}_i^{\check{\alpha}_i}$ is a deformation of the classical Z^H. The Q-operators act on $\mathcal{H}$ through the second factor of the R-matrix, i.e., through $U_q(\widehat{b}_-) \subset U_q(\widehat{\mathfrak{g}})$.

One can define the infinite version of such Baxter Q-operators by considering the limit as the corresponding multiplicative weight parameters $\hat{\xi}_i = \prod_j \hat{\zeta}_j^{\,-a_{ji}}$ goes to zero. One can even write an explicit formula expressing the expansion coefficients of the Q-operator in terms of their infinite analogues and the generators of the quantum group. This was done explicitly in [FH1] and [PSZ] in the case of $\mathfrak{g} = \mathfrak{sl}(2)$. The latter reference, together with the subsequent works [KPSZ], [KZ2], [KZ1], identified the infinite version of the QQ-system relations with the relations in the classical equivariant K-theory ring on a certain quiver variety while the finite version gives the relations in the quantum K-theory ring. The parameters $\hat{\xi}_i$ are known as Kähler parameters.

In particular, these results for Baxter Q-operators imply that one can find solutions of the QQ-system which are the deformations of solutions of its infinite analogue. Upon taking the limit which reduces the XXZ model to Gaudin model (see e.g. Section 6 of [KSZ]), we see that this is true for the qq-system as well.

For example, in the $\mathfrak{sl}(2)$ case, the infinite qq-system is simply the single equation $q_+(z)q_-(z) = \Lambda(z)$. If we set $\Lambda(z) = \prod_{j=1}^N (z - z_j)$, then a solution is obtained by dividing the z_j's into $w_1, \dots, w_d$ and $v_1, \dots, v_{N-d}$ and setting $q_+^\infty(z) = \prod_{k=1}^d (z - w_k)$ and $q_-^\infty(z) = \prod_{\ell=1}^{N-d}(z - v_\ell)$. Then following the discussed above q-deformed case, if $\Lambda(z)$ has no repeated roots, then for large enough ξ, there are deformations w_k^ξ and v_ℓ^ξ such that $q_+^\xi(z) = \prod_{k=1}^d (z - w_k^\xi)$, $q_-^\xi(z) = \prod_{\ell=1}^{N-d}(z - v_\ell^\xi)$ are a solution of the finite qq-system (for the same $\Lambda(z)$) with parameter ξ. Moreover, given the initial choice of $q_+^\infty(z)$, the solution is unique and is indeed given by a formula that allows one to view the z_j's as free parameters.

Lemma 6.16. *If $\Lambda(z)$ has no repeated roots, then the finite solution $q_+^\xi(z), q_-^\xi(z)$ is nondegenerate for large ξ.*

Proof. Since $q_+^\infty(z)$ has no multiple roots and is relatively prime to $q_-^\infty(z)$, the same holds for the finite solutions for large ξ. For such ξ, suppose that $q_+^\xi(z)$ has a root w in common with $\Lambda(z)$. We see that $\partial_z q_+^\xi(z) q_-^\xi(z)$ vanishes at w, since every other term in the qq-equation vanishes. This implies that w is either a root of $q_-^\xi(z)$ or a multiple root of $q_+^\xi(z)$, a contradiction. □

We can generalize this procedure to define generic solutions to the qq-system for simply-laced $\mathfrak{g}$. Assume that $d_j \leq N_j - \sum_{\ell\neq j} a_{kj} d_k$ for all j. We can then choose $Z_j, W_j \subset \mathbb{C}$ such that $|Z_j| = N_j$, $|W_j| = d_j$, the Z_j's are pairwise disjoint, $Z_j \cap W_k = \varnothing$ unless $a_{jk} \neq 0$, and $W_j \subset Z_j \cup \bigcup_{a_{kj}<0} W_k$. Let $V_j = Z_j \cup \bigcup_{a_{kj}<0} W_k \setminus W_j$. Setting $\Lambda_j(z) = \prod_{a\in Z_j}(z - a)$, $q_+^{j,\infty}(z) =$

$\prod_{w\in W_j}(z-w)$, and $q_-^{j,\infty}(z)=\prod_{v\in V_j}(z-v)$ gives a solution of (6.19). Since $\mathfrak{g}$ is simply-laced, $\Lambda_j(z)\prod_{k\neq j} q_+^{k,\infty}(z)^{-a_{kj}}$ is multiplicity-free. One can now apply the results above to obtain unique deformations $q_+^{i,Z^H}(z), q_-^{i,Z^H}(z)$ satisfying the qq-equations. By the lemma, these are nondegenerate solutions for large Z^H.

Suppose further that the system of inequalities $d_j^{(i)}\leq N_j-\sum_{\ell\neq j} a_{kj}d_k^{(i)}$ is also satisfied. This guarantees that we have a solution $q_+^{j,\infty,(i)}(z), q_-^{j,\infty,(i)}(z)$ to the infinite qq-system with the same $\Lambda_i(z)$, with $q_+^{i,\infty,(i)}(z)=q_-^{j,\infty}(z)$ and $q_-^{i,\infty,(i)}(z)=q_+^{j,\infty}(z)$, and with $q_+^{j,\infty,(i)}(z)=q_+^{j,\infty}(z)$ for $j\neq i$. We again can deform this infinite solution to obtain a unique solution of the qq-equations for $s_i(Z^H)$. By uniqueness, these finite solutions are the ith Bäcklund transformation of the previous solutions, i.e., they are just $q_\pm^{j,Z^H,(i)}(z), q_+^{j,Z^H,(i)}(z)$. Since these solutions are nondegenerate for large Z^H, we see that $\{q_+^{j,Z^H}(z), q_-^{j,Z^H}(z)\}$ is i-generic for large Z^H.

It is clear that we can iterate this process, so we obtain the following theorem:

Theorem 6.17. *Suppose that $\mathfrak{g}$ is simply-laced. Then, there exists an $(i_1,\ldots,i_k)$-generic solution of the qq-equations with combinatorial datum $(\mathbf{d},\mathbf{N},\varnothing)$ if and only if the system of inequalities (6.17) are satisfied. In particular, $(\mathbf{d},\mathbf{N},\varnothing)$ is admissible if and only if the system of inequalities is satisfied for some reduced decomposition of w_0.*

6.3.3. *Removing the hypothesis of w_0-genericity.* We now show that the w_0-genericity hypothesis in Theorem 6.10 is unnecessary as long as the combinatorial datum is admissible.

Theorem 6.18. *Every nondegenerate Z^H-twisted Miura-Plücker oper with admissible combinatorics is a (nondegenerate) Z^H-twisted Miura oper. In particular, this is the case when*

1. *$Z^H=0$ and there exists a nondegenerate polynomial solution of the qq-system with degrees $(\mathbf{d},\mathbf{N})$, and*
2. *$\mathfrak{g}$ is simply-laced, Z^H is regular semisimple, and the system of inequalities (6.17) is satisfied for some reduced decomposition of w_0.*

Proof. Let $\nabla=\partial_z+A(z)$ be a nondegenerate Z^H-twisted Miura-Plücker oper with admissible combinatorial datum $(\mathbf{d},\mathbf{N},\Psi)$ and with corresponding polynomials $q_+^i(z)$'s. We must show the existence of $v(z)\in B_+(z)$ such that $A(z)=v(z)(\partial_z+Z^H)v(z)^{-1}$. We will accomplish this by considering a solution to the qq-system over a ring R defined in terms of certain formal variables.

Let $w_0=s_{i_1}\ldots s_{i_\ell}$ be a reduced decomposition for which there exists an $(i_1,\ldots,i_\ell)$-generic solution of the qq-equations. We now introduce formal variables for the roots of various polynomials: the $\Lambda_i(z)$'s, the positive polynomials $\tilde{q}_+^j(z)$'s one obtains by iterating Bäcklund transformations along this reduced word, and the negative polynomials $\tilde{q}_-^{i_s}(z)$ corresponding to the simple reflection at each step. (All of these degrees are uniquely determined except for possibly the degrees of the $\tilde{q}_-^{i_s}(z)$'s. However, we can always choose the degree to be the generic one specified in Remark 5.13 while maintaining nondegeneracy.) Thus, we have the formal variables

- $\{\mathbf{z}_k^i\}$ for $1 \leq i \leq r$ and $1 \leq k \leq N_i$;
- $\{\mathbf{w}_k^{j,s}\}$ for $0 \leq s \leq \ell - 1$, $1 \leq j \leq r$, and $1 \leq k \leq d_j^{(i_\ell)\ldots(i_{\ell-s+1})}$; and
- $\{\mathbf{v}_k^{i_{\ell-s},s}\}$ with $1 \leq j \leq r$ and k less than the generic degree of $q_-^{(i_\ell)\ldots(i_{\ell-s+1}),i_{\ell-s}}(z)$.

Let R be the ring $\mathbb{C}[\mathfrak{h}_\Psi] \otimes \mathbb{C}(z)[\{\mathbf{w}_k^{j,s}\}, \{\mathbf{v}_k^{i_{\ell-s},s}\}, \{\mathbf{z}_k^i\}]$, localized at the $(z - \mathbf{w}_k^{i_{\ell-s},s})$'s, the $(z - \mathbf{v}_k^{i_{\ell-s},s})$'s, the $(\mathbf{w}_k^{i,0} - \mathbf{z}_j^i)$'s, and the $(\mathbf{w}_k^{i,0} - \mathbf{w}_s^{j,0})$'s, and satisfying the Bethe equations (5.18). Set $Q_+^i(z, \{\mathbf{w}_k^{i,0}\}) = \prod_k (z - \mathbf{w}_k^{i,0})$. We view the Q_+^i's as the "plus" polynomials of a qq-system defined over R (with the twist parameter given by a generic $\boldsymbol{Z^H} = \sum_{i=1}^r \zeta_i \check{\alpha}_i$ and the singularities given by $\boldsymbol{\Lambda}_i = \prod (z - \mathbf{z}_j^i)$'s). Note that this data specializes to the data for our original ∇.

By Theorem 5.11, we can complete the Q_+^i's to a solution $\{Q_+^i, Q_-^i\}$ of this qq-equation over R. This solution corresponds to the connection $\partial_z + \mathcal{A}(z, \{\mathbf{w}_k^i\}, \{\boldsymbol{\zeta}_i\}, \{\boldsymbol{\Lambda}_i\})$, where

$$\mathcal{A}(z, \{\mathbf{w}_k^i\}, \{\boldsymbol{\zeta}_i\}, \{\boldsymbol{\Lambda}_i\}) = \sum_{i=1}^r \Big[\boldsymbol{\zeta}_i - Q_+^i(z)^{-1}\partial_z Q_+^i(z)\Big]\check{\alpha}_i + \sum_{i=1}^r \boldsymbol{\Lambda}_i(z) e_i.$$

Again, our original connection ∇ is a specialization of this connection.

We claim that the qq-system $\{Q_+^i, Q_-^i\}$ over R is w_0-generic. To see this, it suffices to show that some specialization of this qq-system is w_0-generic. This exists by the definition of admissibility.

Note that in the definition of the ith Bäcklund transformation, μ_i's (see (6.6)) is a rational function with $q_+^i(z) q_-^i(z)$ in the denominator. It follows that all the μ_i's needed in iterating Bäcklund transformations for $\{Q_+^i, Q_-^i\}$ lie in R. One can thus use Bäcklund transformations and the procedure of Theorem 6.10 to construct a matrix $U(z, \{\mathbf{w}_k^i\}, \{\boldsymbol{\zeta}_i\}, \{\boldsymbol{\Lambda}_i\}) \in B_+(R)$ satisfying the equation

$$\mathcal{A}(z) = U(z, \{\mathbf{w}_k^i\}, \{\boldsymbol{\zeta}_i\}, \{\boldsymbol{\Lambda}_i\}) \Big(\partial_z + \sum_{i=1}^r \boldsymbol{\zeta}_i \check{\alpha}_i\Big) U(z, \{\mathbf{w}_k^i\}, \{\boldsymbol{\zeta}_i\}, \{\boldsymbol{\Lambda}_i\})^{-1}. \tag{6.20}$$

Let $v(z)$ be the specialization of $U(z, \{\mathbf{w}_k^i\}, \{\boldsymbol{\zeta}_i\}, \{\boldsymbol{\Lambda}_i\})$ at the data for our original ∇. We then obtain $A(z) = v(z)(\partial_z + Z^H) v(z)^{-1}$ as desired.

□

Theorem 6.19. *There is a one-to-one correspondence between the set of nondegenerate Z^H-twisted Miura G-opers with admissible combinatorial data and the set of solutions to the Z^H-twisted Bethe Ansatz equations for ${}^L\mathfrak{g}$ with the same combinatorial data.*

Proof. This follows immediately from Theorems 5.15 and 6.18. □

Remark 6.20. In [FKSZ], the authors studied a difference equation version of the qq-system involving quantum Wronskians called the *QQ-system*. In that paper, it is shown that there is a bijection between twisted Miura-Plücker (G,q)-opers (with regular semisimple twist parameter) and solutions to the Bethe Ansatz equations for the XXZ model, and this correspondence goes through the intermediary of polynomials solutions of the QQ-system. There is an analogue of w_0-genericity in this context, and as for ordinary opers, a w_0-generic Miura-Plücker q-oper is in fact a Miura q-oper. The methods of this paper can be used to prove the q-oper analogue of Theorem 6.18: a Miura-Plücker q-opers with admissible combinatorics is a Miura q-oper.

References

[BD] A. Beilinson and V. Drinfeld, *Opers*, math/0501398.

[CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, Inc., Boston, MA, 1997.

[F1] E. Frenkel, *Opers on the projective line, flag manifolds and Bethe Ansatz*, Mosc. Math. J. **4** (2003), 655–705, math/0308269.

[F2] E. Frenkel, *Gaudin Model and Opers*, Infinite Dimensional Algebras and Quantum Integrable Systems, eds. P. Kulish, e.a., Progress in Math. **237** (2004), 1–60, math/0407524.

[FFR1] B. Feigin, E. Frenkel, and N. Reshetikhin, *Gaudin Model, Bethe Ansatz and Critical Level*, Commun. Math. Phys. **166** (1994), 27–62, hep-th/9402022.

[FFR2] B. Feigin, E. Frenkel, and L. Rybnikov, *Opers with irregular singularity and spectra of the shift of argument subalgebra*, Duke Math. J. **155** (2010), 337–363.

[FFTL] B. Feigin, E. Frenkel, and V. Toledano Laredo, *Gaudin models with irregular singularities*, Adv. Math. **223** (2010), 873–948, math/0612798.

[FH1] E. Frenkel and D. Hernandez, *Baxter's relations and spectra of quantum integrable models*, Duke Math. J. **164** (2015), no. 12, 2407–2460, 1308.3444.

[FH2] E. Frenkel and D. Hernandez, *Spectra of quantum KdV Hamiltonians, Langlands duality, and affine opers*, Commun. Math. Phys. **362** (2018), 361–414, 1606.05301.

[FKSZ] E. Frenkel, P. Koroteev, D. S. Sage, and A. M. Zeitlin, *q-Opers, QQ-Systems, and Bethe Ansatz*, J. Eur. Math. Soc. (JEMS) **26** (2024), no. 1, 355–405, 2002.07344.

[H] J. Humphreys, *Linear algebraic groups*, Springer, 1975.

[HJ] D. Hernandez and M. Jimbo, *Asymptotic Representations and Drinfeld Rational Fractions*, Compositio Mathematica **148** (2012), no. 5, 1593–1623, 1104.1891.

[KPSZ] P. Koroteev, P. Pushkar, A. Smirnov, and A. M. Zeitlin, *Quantum K-theory of quiver varieties and many-body systems*, Selecta Math. New. Ser. **27 (5)** (2021), 87, 1705.10419.

[KSZ] P. Koroteev, D. S. Sage, and A. M. Zeitlin, *(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality*, Commun. Math. Phys. **381 (2)** (2021), 435–470, 1811.09937.

[KZ1] P. Koroteev and A. M. Zeitlin, *3d Mirror Symmetry for Instanton Moduli Spaces*, Commun. Math. Phys. **403** (2023), 1005–1068, 2105.00588.

[KZ2] P. Koroteev and A. M. Zeitlin, *Toroidal q-Opers*, J. Inst. Math. Jussieu **22** (2023), 581–642, 2007.11786.

[MRV1] D. Masoero, A. Raimondo, and D. Valeri, *Bethe Ansatz and the Spectral Theory of Affine Lie Algebra-Valued Connections I. The simply-laced Case*, Commun. Math. Phys. **344** (2016), no. 3, 719–750, 1501.07421.

[MRV2] D. Masoero, A. Raimondo, and D. Valeri, *Bethe Ansatz and the Spectral Theory of Affine Lie algebra–Valued Connections II: The Non Simply–Laced Case*, Communications in Mathematical Physics **349** (2017), no. 3, 1063–1105, 1511.00895.

[MV1] E. Mukhin and A. Varchenko, *Miura Opers and Critical Points of Master Functions*, Cent. Eur. J. Math. **3**, math/0312406.

[MV2] E. Mukhin and A. Varchenko, *Quasi-polynomials and the Bethe Ansatz*, Geom. Top. Mon. **13** (2008), math/0604048.

[PSZ] P. P. Pushkar, A. Smirnov, and A. M. Zeitlin, *Baxter Q-operator from quantum K-theory*, Adv. Math. **360** (2020), 106919, 1612.08723.

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803, USA
Email address: tbrins2@lsu.edu

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803, USA
Email address: sage@math.lsu.edu

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803, USA
Email address: zeitlin@math.lsu.edu