



# Morse inequalities for ordered eigenvalues of generic self-adjoint families

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## Abstract

In many applied problems one seeks to identify and count the critical points of a particular eigenvalue of a smooth parametric family of self-adjoint matrices, with the parameter space often being known and simple, such as a torus. Among particular settings where such a question arises are the Floquet–Bloch decomposition of periodic Schrödinger operators, topology of potential energy surfaces in quantum chemistry, spectral optimization problems such as minimal spectral partitions of manifolds, as well as nodal statistics of graph eigenfunctions. In contrast to the classical Morse theory dealing with smooth functions, the eigenvalues of families of self-adjoint matrices are not smooth at the points corresponding to repeated eigenvalues (called, depending on the application and on the dimension of the parameter space, the diabolical/Dirac/Weyl points or the conical intersections). This work develops a procedure for associating a Morse polynomial to a point of eigenvalue multiplicity; it utilizes the assumptions of smoothness and self-adjointness of the family to provide concrete answers. In particular, we define the notions of non-degenerate topologically critical point and generalized Morse family, establish that generalized Morse families are generic in an appropriate sense, establish a differential first-order conditions for criticality, as well as compute the local contribution of a topologically critical point to the Morse polynomial. Remarkably, the non-smooth contribution to the Morse polynomial turns out to depend only on the size of the eigenvalue multiplicity and the relative position of the eigenvalue of interest and not on the particulars of the operator family; it is expressed in terms of the homologies of Grassmannians.

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## 1 Introduction

Let  $\text{Sym}_n(\mathbb{R})$  and  $\text{Sym}_n(\mathbb{C})$  denote the spaces of  $n \times n$  real symmetric (correspondingly, complex Hermitian) matrices. When referring to both spaces at once, we will use the term “self-adjoint matrices” and use the notation  $\text{Sym}_n$ . The eigenvalues  $\{\widehat{\lambda}_i(A)\}_{i=1}^n$  of a matrix  $A \in \text{Sym}_n$  are real and will be numbered in the increasing order,

$$\widehat{\lambda}_1(A) \leq \widehat{\lambda}_2(A) \leq \cdots \leq \widehat{\lambda}_n(A). \quad (1.1)$$

Further, let  $M$  be a smooth (i.e.  $C^\infty$ ) compact  $d$ -dimensional manifold. A smooth  $d$ -parametric family of self-adjoint matrices (on  $M$ ) is a smooth map  $\mathcal{F} : M \rightarrow \text{Sym}_n$ .

The aim of this paper is to develop the Morse theory for the  $k$ -th ordered eigenvalue

$$\lambda_k := \widehat{\lambda}_k \circ \mathcal{F}$$

viewed as a function on  $M$ . This question is motivated by numerous problems in mathematical physics. The boundaries between isolating and conducting regimes in a periodic (crystalline) structure are determined by the extrema of eigenvalues of an operator<sup>1</sup> family defined on a  $d$ -dimensional torus  $M$  (for an introduction to the mathematics of this subject, see [49]). Other critical points of the eigenvalues give rise to special physically observable features of the density of states, the van Hove singularities [63]. Classifying all critical points of an eigenvalue (also on a torus) by their degree is used to study oscillation of eigenfunctions via the nodal–magnetic theorem [3, 4, 10, 20]. More broadly, the area of eigenvalue optimization encompasses questions from understanding the charge distribution in an atomic nucleus [28], configuration of atoms in a polyatomic molecule [24, 50], to shape optimization [40, 41] and optimal partition of domains and networks [8, 39]. The dimension of the manifold  $M$  in these applications can be very high or even infinite.

Morse theory is a natural tool for connecting statistics of the critical points with the topology of the underlying manifold. However, the classical Morse theory is formulated for functions that are sufficiently smooth, whereas the function  $\lambda_k$  is generically non-smooth at the points where  $\lambda_k(x)$  is a repeated eigenvalue of the matrix  $\mathcal{F}(x)$ . And it is these points of non-smoothness that play an outsized role in the applications [17, 24].

By Bronstein’s theorem [15], each  $\lambda_k$  is Lipschitz. Furthermore, by classical perturbation theory [48], the function  $\lambda_k$  is smooth along a submanifold  $N \subset M$  if the multiplicity of  $\lambda_k(x)$  is constant on  $N$ ; the latter property induces a stratification of  $M$ . There exist generalizations of Morse theory to Lipschitz functions [2], continuous functions [33, §45], as well as to stratified spaces [35]. These generalizations will provide the general foundation for our work, but the principal thrust of this paper is to leverage the properties of  $\text{Sym}_n$  and to get explicit — and beautiful — answers for the Morse data in terms of the local behavior of  $\mathcal{F}$  at a discrete set of points we will identify as “critical”. One of the surprising findings is that the Morse data attributable

<sup>1</sup>The particulars of the operator depend on what is being conducted: electrons, light, sound, etc.

to the non-smooth directions at a critical point does not depend on the particulars of the family  $\mathcal{F}$ .

To set the stage for our results we now review informally the main ideas of Morse theory, which links the topological invariants of the manifold  $M$  to the number and the indices of the critical points of a function  $\phi$  on  $M$ .

In more detail, if  $\phi$  is smooth, a point  $x \in M$  is called a *critical point* if the differential of  $\phi$  vanishes at  $x$ . The *Hessian* (second differential) of  $\phi$  at  $x$  is a quadratic form on the tangent space  $T_x M$ . In local coordinates it is represented by the matrix of second derivatives, the *Hessian matrix*. The *Morse index*  $\mu(x)$  is defined as the negative index of this quadratic form or, equivalently, the number of negative eigenvalues of the Hessian matrix. It is assumed that the second differential at every critical point of  $\phi$  is non-singular, i.e. the Hessian matrix has no zero eigenvalues; such critical points are called *non-degenerate*. Non-degenerate critical points are isolated and therefore there are only finitely many of them on  $M$ . A smooth function  $\phi$  is called a *Morse function* if all its critical points are non-degenerate.

The main result of the classical Morse theory quantifies the change in the topology of the level curves of  $\phi$  around a critical point. To be precise, for a point  $x \in M$  and its neighborhood  $U$  (which we will always assume to be homeomorphic to a ball) define the *local sublevel sets*,

$$U_x^{-\varepsilon}(\phi) := \{y \in U : \phi(y) \leq \phi(x) - \varepsilon\} \quad \text{and} \quad U_x^{+\varepsilon}(\phi) := \{y \in U : \phi(y) \leq \phi(x) + \varepsilon\}.$$

If  $x$  is a non-degenerate critical point of index  $\mu = \mu(x)$ , then, for a sufficiently small neighborhood  $U$  of  $x$  and sufficiently small  $\varepsilon > 0$ , the quotient space  $U_x^{+\varepsilon}(\phi)/U_x^{-\varepsilon}(\phi)$  is homotopy equivalent to the  $\mu$ -dimensional sphere  $\mathbb{S}^\mu$ . The global consequences of this are the *Morse inequalities*: given a Morse function  $\phi$ , denote by  $c_q$ ,  $q = 0, \dots, d$ , the number of its critical points of index  $q$ . Then there exist  $d$  integers  $r_q \geq 0$  such that

$$\begin{aligned} c_0 &= \beta_0 + r_1, \\ c_1 &= \beta_1 + r_1 + r_2, \\ c_2 &= \beta_2 + r_2 + r_3, \\ &\dots \\ c_{d-1} &= \beta_{d-1} + r_{d-1} + r_d, \\ c_d &= \beta_d + r_d, \end{aligned} \tag{1.2}$$

where  $\beta_q$  is the  $q$ -th Betti number of the manifold  $M$ , defined as the rank of the homology group<sup>2</sup>  $H_q(M) = H_q(M; \mathbb{Z})$ . To put it another way, the Betti numbers  $\beta_q$  give the lower bound for the number of critical points of index  $q$ ; extra critical points can only be created in pairs of adjacent index.

Equations (1.2) can be expressed concisely in terms of generating functions: one defines the *Morse polynomial*  $P_\phi(t)$  of a Morse function  $\phi$  as the sum of  $t^{\mu(x)}$  over all

<sup>2</sup>Throughout the paper, we use integer coefficient homology, unless specified otherwise.

critical points  $x \in M$  of  $\phi$ . On the topological side, the *Poincaré polynomial*  $P_M(t)$  of the manifold  $M$  is the sum of  $\beta_q t^q$ . Then the *Morse inequalities* are equivalent to the identity

$$(P_\phi(t) - P_M(t)) / (1 + t) = R(t), \quad (1.3)$$

where  $R(t)$  is a polynomial with nonnegative coefficients.

Now assume that  $\phi$  is just continuous; the local sublevel sets  $U_x^{\pm\epsilon}(\phi)$  are still well-defined. Mimicking the classical Morse theory of smooth functions we adopt the following definitions (cf. [33, §45, Def. 1, 2 and 3], the critical points are called bifurcation points there):

**Definition 1.1** A point  $x \in M$  is a *topologically regular point* of a continuous function  $\phi$  if there exists a small enough neighborhood  $U$  of  $x$  in  $M$  and  $\epsilon > 0$  such that  $U_x^{-\epsilon}(\phi)$  is a strong deformation retract of  $U_x^{+\epsilon}(\phi)$ . We say that a point is *topologically critical* if it is not topologically regular.

**Remark 1.2** If  $\phi$  is smooth, a topologically critical point  $x$  is also critical in the usual (differential) sense. The converse is, in general, not true: for example, if  $M = \mathbb{R}$  and  $\phi(x) = x^3$ , then  $x = 0$  is critical but not topologically critical. On the other hand, by the aforementioned main result of the classical Morse theory, if  $x$  is a *non-degenerate* critical point then it is also topologically critical.

**Definition 1.3** Given a continuous function  $\phi$  with a finite set of topologically critical points, the *Morse polynomial*  $P_\phi$  is the sum, over the topologically critical points  $x$ , of the Poincaré polynomials of the relative homology groups  $H_*(U_x^{+\epsilon}(\phi), U_x^{-\epsilon}(\phi))$ , where  $U$  is a small neighborhood of  $x$  and  $\epsilon > 0$  is sufficiently small.

**Remark 1.4** If  $\phi$  is a *smooth* Morse function, Definition 1.3 reduces to the classical one as the relative homology groups  $H_*(U_x^{+\epsilon}(\phi), U_x^{-\epsilon}(\phi))$  coincide with the reduced homology groups of the  $\mu(x)$ -dimensional sphere  $\mathbb{S}^{\mu(x)}$ , where  $\mu(x)$  is the Morse index of  $x$ , and so the contribution of  $x$  to the Morse polynomial  $P_\phi(t)$  is equal to  $t^{\mu(x)}$ .

With Definitions 1.1 and 1.3, the Morse inequalities (1.3) hold true for continuous functions  $\phi$  with finite number of topologically critical points (see, e.g., [33, §45, Theorem. 1]). The proof is essentially the same as the proof of the classical Morse inequality given in [51, §5] and is based on the exact sequence of pairs: the latter implies the subadditivity of relative Betti numbers and, more generally, of the partial alternating sums of relative Betti numbers, which implies the required Morse inequalities.

It is thus our goal to give a prescription for computing the Morse polynomial  $P_{\lambda_k}$  in terms of  $\mathcal{F}$  and its derivatives, under some natural assumptions on  $\mathcal{F}$ . To that end we will need to:

- (1) Provide an explicit characterization of non-smooth topologically critical and topologically regular points of  $\lambda_k$ ;

- (2) Give a natural definition of a *non-degenerate* non-smooth topologically critical point;
- (3) For a non-degenerate topologically critical point  $x$  of  $\lambda_k$ , find the relative homology

$$H_q(U_x^\varepsilon(\lambda_k), U_x^{-\varepsilon}(\lambda_k))$$

for a sufficiently small neighborhood  $U$  of  $x$  and sufficiently small  $\varepsilon > 0$ . As a by-product, this will determine the correct contribution from  $x$  to the Morse polynomial  $P_{\lambda_k}(t)$  of  $\lambda_k$ .

We remark that these questions are local in nature and we do not need to enforce compactness of  $M$  while answering them.

In this work, we completely implement the above objectives in the case of generic smooth families; additionally, our sufficient condition for a regular point is obtained for arbitrary families. The first objective is accomplished in the form of a “first derivative test”, with the derivative being applied to the smooth object: the family  $\mathcal{F}$  (see equation (1.5) and Theorems 1.5 and 1.12 for details).

The Morse contribution of a critical point (third objective) will consist of two parts: the classical index of the Hessian of  $\lambda_k$  in the directions of smoothness of  $\lambda_k$  and a contribution from the non-smooth directions which, remarkably, turns out to *depend only on the size of the eigenvalue multiplicity and the relative position of the eigenvalue of interest and not on the particulars of the operator family*. Theorem 1.14 expresses this contribution in terms of homologies of suitable Grassmannians; explicit formulas for the Poincaré polynomial are provided in Theorem 1.12. In Sect. 2.2 we mention some simple practical corollaries of our results as well as pose further problems.

### 1.1 A differential characterization of a topologically critical point

Our primary focus is on the points  $x \in M$  where the eigenvalue  $\lambda_k$  has multiplicity and is not differentiable. However, simple examples (for instance, Example 2.1 below) show that not every point of eigenvalue multiplicity is topologically critical.

Denote by  $\mathbf{E}_k$  the eigenspace of  $\lambda_k$  at a point  $x \in M$  of multiplicity  $\nu = \dim \mathbf{E}_k$ . The *compression* of a matrix  $X \in \text{Sym}_n$  to the space  $\mathbf{E}_k$  is the linear operator  $X_{\mathbf{E}_k} : \mathbf{E}_k \rightarrow \mathbf{E}_k$  acting as  $v \mapsto P_{\mathbf{E}_k} X v$ , where  $P_{\mathbf{E}_k}$  is the orthogonal projector onto  $\mathbf{E}_k$ . The matrix representation of  $X_{\mathbf{E}_k}$  can be computed as

$$X_{\mathbf{E}_k} := \mathcal{U}^* X \mathcal{U}, \quad (1.4)$$

where  $\mathcal{U} : \mathbb{F}^\nu \rightarrow \mathbb{F}^n$  is a linear isometry such that  $\text{Ran}(\mathcal{U}) = \mathbf{E}_k$  (explicitly, the columns of  $\mathcal{U}$  are an orthonormal basis of  $\mathbf{E}_k$ ). Introduce the linear operator  $\mathcal{H}_x : T_x M \rightarrow \text{Sym}_\nu$  acting as

$$\mathcal{H}_x : v \mapsto (d\mathcal{F}(x)v)_{\mathbf{E}_k}. \quad (1.5)$$

While the operator  $\mathcal{H}_x$  depends on the choice of the isometry  $\mathcal{U}$  in (1.4) (or, equivalently, the choice of basis in  $\mathbf{E}_k$ ), we will only use its properties that are invariant under unitary conjugation.

We recall that a matrix  $A \in \text{Sym}_v$  is *positive semidefinite* (notation:  $A \in \text{Sym}_v^+$ ) if all of its eigenvalues are non-negative, *positive definite* (notation:  $A \in \text{Sym}_v^{++}$ ) if all eigenvalues are strictly positive. We denote by  $S^\perp$  the orthogonal complement of a space  $S$  in  $\text{Sym}_v$  with respect to the *Frobenius inner product*  $\langle X, Y \rangle := \text{Tr}(XY)$ . For future reference we note that if  $X \in \text{Sym}_v^{++}$  and  $Y \in \text{Sym}_v^+$ ,  $Y \neq 0$ , then  $\langle X, Y \rangle > 0$  (see, e.g., [14, Example 2.24]).

Our first main result gives a sufficient condition for a point of eigenvalue multiplicity to be topologically regular.

**Theorem 1.5** *Let  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be a smooth family whose eigenvalue  $\lambda_k$  has multiplicity  $v \geq 1$  at the point  $x \in M$ . If  $\text{Ran } \mathcal{H}_x$  contains a positive definite matrix or, equivalently,<sup>3</sup>*

$$(\text{Ran } \mathcal{H}_x)^\perp \cap \text{Sym}_v^+ = 0, \quad (1.6)$$

*then  $x$  is topologically regular for  $\lambda_k$ .*

This theorem is proved in Sect. 3 by studying the Clarke subdifferential at the point  $x$ . We formulate the conditions in terms of both  $\text{Ran } \mathcal{H}_x$  and  $(\text{Ran } \mathcal{H}_x)^\perp$  because the former emerges naturally from the proof while the latter is simpler in practical computations: generically it is one- or zero-dimensional as we will see in Sect. 4.

**Remark 1.6** Condition (1.6) should be viewed as being analogous to the “non-vanishing gradient” in the smooth Morse theory. By what is sometimes called Hellmann–Feynman theorem (see Appendix A and references therein), the eigenvalues of  $\mathcal{H}_x v \in \text{Sym}_v$  give the slopes of the branches splitting off from the multiple eigenvalue  $\lambda_k(\mathcal{F}(x))$  when we leave  $x$  in the direction  $v$ . The regularity condition of Theorem 1.5 is equivalent to having a direction in which *all* eigenvalues are increasing.

To further illustrate this point, consider the special case  $v = 1$  when the eigenvalue  $\lambda_k$  is smooth. Let  $\psi$  be the eigenvector corresponding to  $\lambda_k$  at the point  $x$ . The operator  $\mathcal{H}_x : T_x M \rightarrow \mathbb{R}$  in this case maps  $v$  to  $\langle \psi, (d\mathcal{F}(x)v)\psi \rangle_{\mathbb{R}^n}$  which is equal to the directional derivative of  $\lambda_k(x)$  in the direction  $v$ . The condition of Theorem 1.5 is precisely that this derivative is non-zero in some direction, i.e. the gradient does not vanish.

Due to the topological nature of Definition 1.1, one cannot expect that a zero gradient-type condition alone would be sufficient for topological criticality (cf. Remark 1.2). To formulate a sufficient condition we need some notion of “non-degeneracy”, which will have a smooth (S) and non-smooth (N) parts.

**Definition 1.7** Let  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be a smooth family whose eigenvalue  $\lambda_k$  has multiplicity  $v \geq 1$  at the point  $x \in M$ . We say that  $\mathcal{F}$  satisfies the *non-degenerate criticality condition (N)* at the point  $x$  if

$$(\text{Ran } \mathcal{H}_x)^\perp = \text{span}\{B\}, \quad B \in \text{Sym}_n^{++}. \quad (1.7)$$

<sup>3</sup>Note that this equivalence is not immediate and is established in the beginning of the proof of Theorem 3.2 below.

**Remark 1.8** Condition (1.7) ensures non-degenerate criticality in the directions in which  $\lambda_k$  is non-smooth (hence “N”); a single condition plays two roles:

- it ensures that (1.6) is violated (intuitively, “the gradient is zero”), and
- it ensures that  $\text{Ran } \mathcal{H}_x$  has codimension 1, which will be interpreted in Sect. 4 as a type of transversality condition (intuitively, “non-degeneracy in the directions in which  $\lambda_k$  is non-smooth”).

Once condition (N) is satisfied at a point  $x$ , we need to pay special attention to a submanifold  $S$  where the multiplicity of  $\lambda_k$  remains the same.

**Proposition 1.9** *Let  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be a smooth family whose eigenvalue  $\lambda_k$  has multiplicity  $\nu \geq 1$  at the point  $x \in M$ . If  $\mathcal{F}$  satisfies the non-degenerate criticality condition (N) at the point  $x$ , then there exists a submanifold  $S \subset M$  such that for any  $y$  in a small neighborhood of  $x$  in  $M$ , the multiplicity of  $\lambda_k(y)$  is equal to  $\nu$  if and only if  $y \in S$ .*

*This submanifold, which we will call the (local) constant multiplicity stratum attached to  $x$ , has the following properties:*

- (1)  $S$  has codimension  $s(\nu) := \dim \text{Sym}_\nu(\mathbb{F}) - 1$  in  $M$ ,
- (2) the restriction  $\lambda_k|_S$  is a smooth function which has a critical point at  $x$ , i.e.  $d(\lambda_k|_S)(x) = 0$ .

The proof of the above Proposition is in Sect. 4.

**Definition 1.10** Assume  $\mathcal{F}$  satisfies the non-degenerate criticality condition (N) at the point  $x$  for the eigenvalue  $\lambda_k$  (in particular,  $x$  is a critical point of  $\lambda_k|_S$ ). We will say  $\mathcal{F}$  satisfies the *non-degenerate criticality condition (S)* if  $x$  is a *non-degenerate critical point* of  $\lambda_k|_S$ .

Naturally, “S” stands for smooth criticality. It turns out that, together, conditions (N) and (S) are sufficient for topological criticality. To quantify the topological change in the sublevel sets we need additional terminology. The *relative index* of the  $k$ -th eigenvalue at point  $x$  is

$$i(x) = \#\{\lambda \in \text{spec}(\mathcal{F}(x)) : \lambda \leq \lambda_k(x)\} - k + 1. \quad (1.8)$$

In other words,  $i(x)$  is the sequential number of  $\lambda_k$  among the eigenvalues equal to it, but counting from the top. It is an integer between 1 and  $\nu(x)$ , the multiplicity of the eigenvalue  $\lambda_k(x)$  of the matrix  $\mathcal{F}(x)$ . We will need the quantity  $s(i) := \dim \text{Sym}_i(\mathbb{F}) - 1$ , which already appeared in a different role in Proposition 1.9. It is given explicitly by

$$s(i) := \dim \text{Sym}_i(\mathbb{F}) - 1 = \begin{cases} \frac{1}{2}i(i+1) - 1, & \mathbb{F} = \mathbb{R}, \\ i^2 - 1, & \mathbb{F} = \mathbb{C}. \end{cases} \quad (1.9)$$

Finally, we denote by  $\binom{n}{k}_q$  the  $q$ -binomial coefficient,

$$\binom{n}{k}_q := \frac{\prod_{i=1}^n (1 - q^i)}{\prod_{i=1}^k (1 - q^i) \prod_{i=1}^{n-k} (1 - q^i)},$$

which is well known ([47, Corollary 2.6]) to be a polynomial in  $q$ .

**Definition 1.11** A smooth family  $\mathcal{F} : M \rightarrow \text{Sym}_n(\mathbb{F})$  is called *generalized Morse* if, at every point  $x \in M$ ,  $\mathcal{F}$  either satisfies the regularity condition (1.6) or satisfies the non-degenerate criticality conditions (N) and (S).

**Theorem 1.12** Consider the eigenvalue  $\lambda_k$  of a smooth family  $\mathcal{F} : M \rightarrow \text{Sym}_n$ .

- (1) If  $\mathcal{F}$  satisfies non-degenerate criticality conditions (N) and (S) at  $x$ , then  $x$  is a topologically critical point of  $\lambda_k$ . The set of points  $x$  where conditions (N) and (S) are satisfied is discrete.
- (2) If  $M$  is a compact manifold and the family  $\mathcal{F}$  is generalized Morse, Morse inequalities (1.3) hold for the function  $\lambda_k : M \rightarrow \mathbb{R}$  with the Morse polynomial  $P_{\lambda_k}(t) := P_{\lambda_k}(t)$  given by

$$P_{\lambda_k}(t) := \sum_{x \in \text{CP}(\mathcal{F})} P_{\lambda_k}(t; x),$$

where the summation is over all topologically critical points  $x$  of  $\mathcal{F}$  and, denoting by  $v(x)$  the multiplicity of the eigenvalue  $\lambda_k$  of  $\mathcal{F}(x)$ , by  $i(x)$  its relative index, and by  $\mu(x)$  the Morse index of the restriction  $\lambda_k|_S$ ,

$$P_{\lambda_k}(t; x) := t^\mu \mathfrak{T}_v^i = t^{\mu+s(i)} \begin{cases} \binom{\lfloor (v-1)/2 \rfloor}{(i-1)/2}_{t^4}, & \mathbb{F} = \mathbb{R} \text{ and } i \text{ is odd,} \\ 0, & \mathbb{F} = \mathbb{R}, i \text{ is even, and } v \text{ is odd,} \\ t^{v-i} \binom{v/2-1}{i/2-1}_{t^4}, & \mathbb{F} = \mathbb{R}, i \text{ is even, and } v \text{ is even,} \\ \binom{v-1}{i-1}_{t^2}, & \mathbb{F} = \mathbb{C}. \end{cases} \quad (1.10)$$

The topological criticality claimed in part (1) follows immediately whenever the Poincaré polynomial of the relative  $\mathbb{Z}$ -homology — which is given in (1.10) — is non-zero. The case in the second line of (1.10) is more complicated, because that Poincaré polynomial is zero. To handle this case, in the final stages of the proof in Sect. 7 we will additionally calculate the Poincaré polynomial of the relative  $\mathbb{Z}_2$ -homology, see (7.3).

We also note that in equation (1.10) we introduced the notation  $\mathfrak{T}_v^i$  for the family-independent Morse contribution to  $P_{\lambda_k}(t; x)$ . This contribution arises from the “non-smooth” directions transverse to the “smooth” constant multiplicity stratum  $S$ . The index  $\mu = \mu(x)$  along the stratum  $S$  depends on the particulars of the family  $\mathcal{F}$ .

Now we state a result showing that for a “typical”  $\mathcal{F}$ , either Theorem 1.5 or Theorem 1.12 holds at every point  $x \in M$ .

**Theorem 1.13** *The set of generalized Morse families is open and dense in the Whitney topology in  $C^r(M, \text{Sym}_n)$  for  $2 \leq r \leq \infty$ .*

This result will be established in Sect. 4 as a part of Theorem 4.11. We will use transversality arguments similar to those in the proof of genericity of classical Morse functions (see, for example, [43, Chap. 4, Theorem 1.2]) via the strong (or jet) Thom transversality theorem for stratified spaces.

## 1.2 Geometrical description of the relative homology groups

In this subsection we provide some idea of what goes into the proof of Theorem 1.12, describing some geometric objects whose integer homology is quantified in (1.10).

First we introduce some notation. We denote by  $\text{Gr}_{\mathbb{F}}(k, n)$  the Grassmannian of (non-oriented)  $k$ -dimensional subspaces in  $\mathbb{F}^n$ . Theorem 1.14 below uses certain homologies of  $\text{Gr}_{\mathbb{R}}(k, n)$  with local coefficients, namely  $H_*(\text{Gr}_{\mathbb{R}}(k, n); \tilde{\mathbb{Z}})$ . The construction of this homology can be found, for instance, in [38, Sec. 3H] or [22, Chap. 5]; it will also be briefly summarized in Sect. 8.

Recall that for a given topological space  $Y$ , the *cone* of  $Y$  is  $\mathcal{C}Y := Y \times [0, 1]/(Y \times \{0\})$ , and the *suspension* of  $Y$  is  $\mathcal{S}Y := \mathcal{C}Y/(Y \times \{1\})$ . For example,  $\mathcal{S}\mathbb{S}^\mu = \mathbb{S}^{\mu+1}$ .

**Theorem 1.14** *In the context of Theorem 1.12, we have the following equivalent descriptions of the relative homology  $H_r(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k))$ ,*

(1)

$$H_r(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k)) = \begin{cases} \mathbb{Z}, & \text{if } i(x) = 1, r = \mu(x), \\ 0, & \text{if } i(x) = 1, r \neq \mu(x), \\ \tilde{H}_{r-\mu(x)}(\mathcal{S}\mathcal{R}_{v(x)}^{i(x)}), & \text{if } 1 < i(x) \leq v(x), \end{cases} \quad (1.11)$$

where

$$\mathcal{R}_v^i := \{R \in \text{Sym}_v^+ : \text{Tr } R = 1, \text{rank } R < i\}, \quad (1.12)$$

and  $\tilde{H}_q$  denotes the  $q$ -th reduced homology group.

(2)

$$\begin{aligned} & H_r(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k)) \\ &= \begin{cases} H_{r-\mu-s(i)}(\text{Gr}_{\mathbb{R}}(i-1, v-1)), & \mathbb{F} = \mathbb{R} \text{ and } i \text{ is odd,} \\ H_{r-\mu-s(i)}(\text{Gr}_{\mathbb{R}}(i-1, v-1); \tilde{\mathbb{Z}}), & \mathbb{F} = \mathbb{R} \text{ and } i \text{ is even,} \\ H_{r-\mu-s(i)}(\text{Gr}_{\mathbb{C}}(i-1, v-1)), & \mathbb{F} = \mathbb{C}. \end{cases} \end{aligned} \quad (1.13)$$

To prove Theorem 1.14, in Sect. 5 we will first separate out the contribution to the relative homology  $H_r(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k))$  of the local constant multiplicity stratum

$S$  and reduce the computation to the case when  $S$  is a single point. In the latter case, it will be shown that  $H_r(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k))$  reduces to the homology of the space  $\mathcal{SR}_v^i$ . In the next step, we will see that  $\mathcal{SR}_v^i$  is homotopy equivalent to the Thom space of a real bundle of rank  $s(i)$  over the Grassmannian  $\text{Gr}_{\mathbb{R}}(i-1, v-1)$ . The difference between the odd and the even  $i$  when  $\mathbb{F} = \mathbb{R}$  is that this real bundle is orientable in the former case and non-orientable in the latter. So, part (2) of the Theorem follows from the Thom isomorphism theorem in the oriented bundle case and more general tools such as the usual/twisted version of Poincaré–Lefschetz duality in the non-orientable bundle case [22, 32, 38].

The study of  $\mathbb{Z}_2$  and integer homology groups of the complex and real Grassmannians was at the heart of the development of algebraic topology and, in particular, the characteristic classes. Starting from the classical works of Ehresmann [26, 27], the answers appearing in (1.10) were explicitly calculated using the Schubert cell decomposition and combinatorics of the corresponding Young diagrams [46, Theorem IV, p. 108], [5]. The calculation of twisted homologies of real Grassmannians is less well-known but can be deduced from the classical work [18] and incorporated into a unified algorithm [16], or computed by means of the general theory of de Rham cohomologies of homogeneous spaces, see [36, chapter XI, pp. 494–496].

Examples of local contributions to the Morse polynomial for topologically critical points of multiplicities up to 8 are presented in Table 1 in the real case. The possible contribution from the smooth directions is ignored because those are specific to the family  $\mathcal{F}$ . In other words, we set  $\mu(x) = 0$  in equation (1.10). In the cases when the second line of (1.10) applies, the contribution of 0 *does not* mean that the point is regular; 0 appears because the polynomial ignores the torsion part of the corresponding homologies. We also observe that the contribution of the top eigenvalue ( $i = 1$ ) is always  $t^0$ ; the contribution of the bottom eigenvalue ( $i = v$ ) is always  $t^{s(v)}$ . By analogy with smooth Morse theory one can guess that the top eigenvalue always experiences a minimum, while the bottom eigenvalue always experiences a maximum ( $s(v)$  being the dimension of the space of non-smooth directions). This guess is rigorously established in Corollary 2.4 and its proof in Sect. 8.

**Table 1** Non-smooth contributions  $\mathfrak{T}_v^i(t)$  to the Morse polynomial from a topologically critical point of  $\lambda_k(x)$  in the real case ( $\mathbb{F} = \mathbb{R}$ , first three cases of equation (1.10))

| $v \setminus i$ | 1 | 2     | 3                    | 4                          | 5                          | 6                          | 7        | 8        |
|-----------------|---|-------|----------------------|----------------------------|----------------------------|----------------------------|----------|----------|
| 2               | 1 | $t^2$ |                      |                            |                            |                            |          |          |
| 3               | 1 | 0     | $t^5$                |                            |                            |                            |          |          |
| 4               | 1 | $t^4$ | $t^5$                | $t^9$                      |                            |                            |          |          |
| 5               | 1 | 0     | $t^5 + t^9$          | 0                          | $t^{14}$                   |                            |          |          |
| 6               | 1 | $t^6$ | $t^5 + t^9$          | $t^{11} + t^{15}$          | $t^{14}$                   | $t^{20}$                   |          |          |
| 7               | 1 | 0     | $t^5 + t^9 + t^{13}$ | 0                          | $t^{14} + t^{18} + t^{22}$ | 0                          | $t^{27}$ |          |
| 8               | 1 | $t^8$ | $t^5 + t^9 + t^{13}$ | $t^{13} + t^{17} + t^{21}$ | $t^{14} + t^{18} + t^{22}$ | $t^{22} + t^{26} + t^{30}$ | $t^{27}$ | $t^{35}$ |

## 2 Examples, applications and an open question

### 2.1 Examples

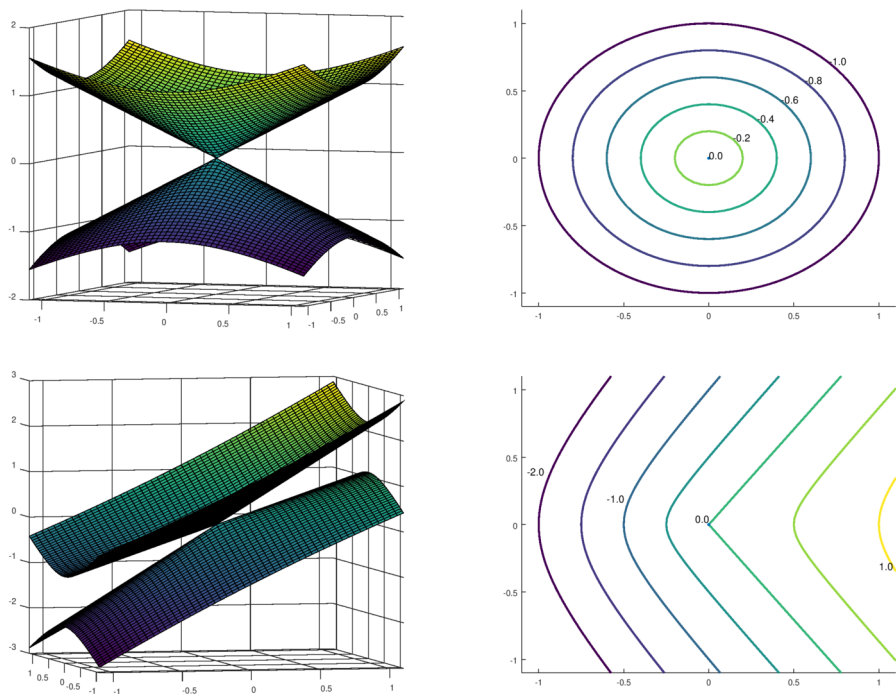
In this section we collect examples illustrating our criteria for regularity and criticality.

**Example 2.1** Consider the two families

$$\mathcal{F}_1(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}, \quad \text{and} \quad \mathcal{F}_2(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & 2x_1 \end{pmatrix}, \quad x = (x_1, x_2) \in \mathbb{R}^2. \quad (2.1)$$

Both families  $\mathcal{F}_1$  and  $\mathcal{F}_2$  have an isolated point of multiplicity 2 at  $(x_1, x_2) = (0, 0)$ . Focusing on the lower eigenvalue  $\lambda_1$ , its level curves in the case of  $\mathcal{F}_1$  undergo a significant change at the value 0 — they change from circles to empty, see Fig. 1(top). Therefore, the point  $(0, 0)$  is topologically critical and, visually,  $\lambda_1$  of  $\mathcal{F}_1$  has a maximum at  $(0, 0)$ . In contrast, the level curves and the sublevel sets of  $\mathcal{F}_2$  remain homotopically equivalent, see Fig. 1(bottom). The point  $(0, 0)$  is not topologically critical for  $\lambda_1$  of  $\mathcal{F}_2$ .

We now check the condition of Theorem 1.5 for the families  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . At the point  $x = (0, 0)$  the eigenspace  $\mathbb{E}_k$  is the whole of  $\mathbb{R}^2$  and no restriction is needed.



**Fig. 1** Eigenvalue surfaces (left) and contours of the first eigenvalue (right) for the families  $\mathcal{F}_1$  (top) and  $\mathcal{F}_2$  (bottom) from equation (2.1)

For the family  $\mathcal{F}_1$ , the mapping  $\mathcal{H}_x$  from (1.5) is

$$\mathcal{H}_x : v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto v_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + v_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and therefore

$$(\text{Ran } \mathcal{H}_x)^\perp = \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

satisfying non-degenerate criticality condition (N), (1.7). Since the point  $x = (0, 0)$  of eigenvalue multiplicity 2 is isolated, the criticality condition (S) is vacuously true. Theorem 1.12 applies at  $x = (0, 0)$  and the Morse data for the two sheets is given by the  $\nu = 2$  row of Table 1.

Proceeding to the family  $\mathcal{F}_2$ , a similar calculation yields

$$(\text{Ran } \mathcal{H}_x)^\perp = \text{span} \left\{ \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

contains no positive semidefinite matrices. Hence  $x$  is a topologically regular point for  $\mathcal{F}_2$  by Theorem 1.5.

**Example 2.2** The case of  $(\text{Ran } \mathcal{H}_x)^\perp$  being spanned by a semidefinite matrix which satisfies neither (1.6) nor condition (1.7), is borderline. As an example, consider the family

$$\mathcal{F}(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 x_2 + x_1^2 \end{pmatrix}. \quad (2.2)$$

For the point  $x = (0, 0)$  of multiplicity 2 we have

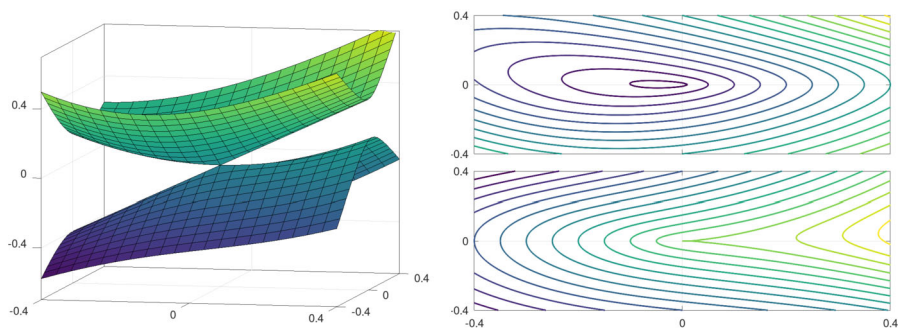
$$\text{Ran } \mathcal{H}_x = \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad (\text{Ran } \mathcal{H}_x)^\perp = \text{span} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Regularity condition (1.6) is violated and so is criticality condition (1.7). However, the constant multiplicity stratum  $S$  is well-defined: it is the isolated point  $\{x\}$ . As can be seen in Fig. 2, we have both behaviors (regular and critical) at once: the lower eigenvalue has a topologically regular point at  $x$  while the upper has a topologically critical point there.

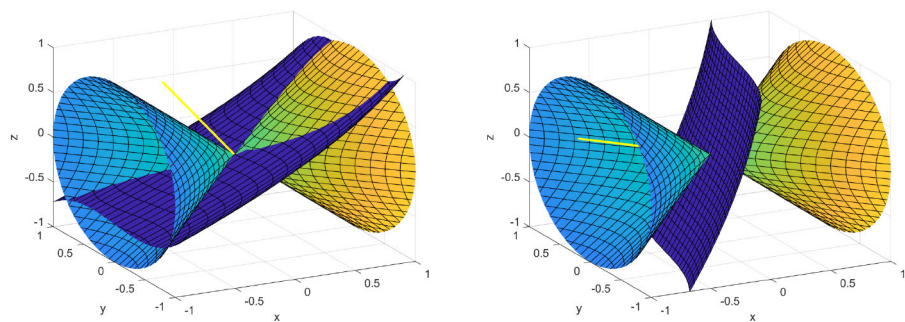
**Example 2.3** We now explore in detail the regularity and criticality conditions of Theorems 1.5 and 1.12 for families of  $2 \times 2$  matrices. We parametrize  $\text{Sym}_2(\mathbb{R})$  using  $\mathbb{R}^3$  via the mapping

$$(x, y, z) \mapsto \begin{pmatrix} x+y & z \\ z & x-y \end{pmatrix}. \quad (2.3)$$

In this parameterization, the Frobenius inner product (normalized by  $1/2$ ) becomes the Euclidean inner product, making orthogonality visual. In the  $(x, y, z)$  space, the sign definite matrices form the interior of the cone,  $x^2 < y^2 + z^2$ .



**Fig. 2** Eigenvalue surfaces (left) and contours for the family  $\mathcal{F}$  from equation (2.2). The point  $x = (0, 0)$  is topologically regular for the bottom eigenvalue and topologically critical (non-smooth minimum) for the top one



**Fig. 3** The cone whose interior consists of sign definite  $2 \times 2$  matrices is visualized in the 3-dimensional space parametrizing  $\text{Sym}_2(\mathbb{R})$  via (2.3). The monochrome surfaces represent the images of the families  $\mathcal{F}$  satisfying (on the left) the regularity condition and (on the right) criticality condition (N) at  $\mathcal{F}(0) = 0$ . The normal to the surface, representing a matrix in  $(\text{Ran } \mathcal{H}_0)^\perp$ , lies outside  $\text{Sym}_2^+$  in the left graph and inside  $\text{Sym}_2^{++}$  (up to an overall sign) in the right graph

Assume that  $\mathcal{F}$  depends on two parameters and satisfies  $\mathcal{F}(0) = 0$ , with  $x = 0$  being the only point of multiplicity 2. Figure 3 shows two such families. The regularity condition of Theorem 1.5 is equivalent to the tangent space at 0 to the image of  $\mathcal{F}$  intersecting the sign definite cone, Fig. 3 (left). Similarly, criticality condition (N), is equivalent to the tangent space to the image of  $\mathcal{F}$  having dimension 2 and lying outside the cone, which puts the normal to  $\mathcal{F}$  at 0 inside the cone, see Fig. 3 (right).

## 2.2 Some applications

Now we give some consequences of our main Theorems. We start with the observation that a maximum of an eigenvalue  $\lambda_k$  cannot occur at a point of multiplicity where  $\lambda_k$  coincides with an eigenvalue below it (the proof of Corollary 2.4 is given at the end of Sect. 8).

**Corollary 2.4** *Let  $x$  be a non-degenerate topologically critical point of the eigenvalue  $\lambda_k$  of a generalized Morse family (generic by Theorem 1.13)  $\mathcal{F} : M \rightarrow \text{Sym}_n(\mathbb{R})$ ,*

where  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . Then  $x$  is a local maximum (resp. minimum) of  $\lambda_k$  if and only if the following two conditions hold simultaneously:

- (1) the eigenvalue  $\lambda_k$  is the bottom (resp. top) eigenvalue among those coinciding with  $\lambda_k(x)$  at  $x$ ; equivalently, the relative index  $i(x, k) = v(x)$  (resp.  $i(x) = 1$ ).
- (2) the restriction of  $\lambda_k$  to the local constant multiplicity stratum attached to  $x$  has a local maximum (resp. minimum) at  $x$ .

Consequently, for a generalized Morse family  $\mathcal{F}$  over a compact manifold  $M$ , we have the strict inequalities

$$\begin{aligned} \max_{x \in M} \lambda_{k-1}(x) &< \max_{x \in M} \lambda_k(x), & k = 2, \dots, n; \\ \min_{x \in M} \lambda_{k-1}(x) &< \min_{x \in M} \lambda_k(x), & k = 2, \dots, n. \end{aligned}$$

Similarly<sup>4</sup> to the classical Morse theory, Theorem 1.12 can be used to obtain lower bounds on the number of critical points of a particular type, smooth or non-smooth. Our particular example is motivated by condensed matter physics, where the density of states (either quantum or vibrational) of a periodic structure has singularities caused by critical points [54, 60] in the “dispersion relation” — the eigenvalue spectrum as a function of the wave vector ranging over the reciprocal space. Van Hove [63] classified the singularities (which are now known as “Van Hove singularities”) and pointed out that they are unavoidably present due to Morse theory applied to the reciprocal space, which is a torus due to periodicity of the structure.

Of primary interest is to estimate the number of *smooth* critical points which produce stronger singularities. Below we make the results of [63] rigorous, sharpening the estimates in  $d = 3$  dimensions. We also mention that higher dimensions, now open to analysis using Theorem 1.12 are not a mere mathematical curiosity: they are accessible to physics experiments through techniques such as periodic forcing or synthetic dimensions [57].

**Corollary 2.5** *Assume that  $M$  is the torus  $\mathbb{T}^d$ ,  $d = 2$  or  $3$ . Let  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be a generalized Morse family (generic by Theorem 1.13). Then the number  $c_\mu(k)$  of smooth critical points of  $\lambda_k$  of Morse index  $\mu = 0, \dots, d$  satisfied the following lower bounds.*

- (1) *In  $d = 2$  any ordered eigenvalue has at least two smooth saddle points, i.e.*

$$c_1(k) \geq 2, \quad k = 1, \dots, n.$$

- (2) *In  $d = 3$ ,*

$$c_1(1) \geq 3, \tag{2.4}$$

$$c_1(k) + c_2(k-1) \geq 4, \quad k = 2, \dots, n, \tag{2.5}$$

$$c_2(n) \geq 3. \tag{2.6}$$

<sup>4</sup>The similarity is natural since our Theorem 1.12 reproduces the classical Morse inequalities if one sets  $n = 1$ .

**Remark 2.6** Only the simpler estimates (2.4) and (2.6) for the bottom and top eigenvalue appear in [63] for  $d = 3$ ; the guaranteed existence of smooth critical points in the intermediate eigenvalues (2.5) is a new result. The intuition behind this result is as follows: when a point of eigenvalue multiplicity affects the count of smooth critical points of  $\lambda_k$ , it also affects the count of smooth critical point for neighboring ordered eigenvalues, such as  $\lambda_{k-1}$ , and it does so in a strictly controllable fashion since the Morse data depends little on the particulars of the family  $\mathcal{F}$ . Carefully tracing these contributions across different ordered eigenvalues leads to sharper estimates.

**Proof of Corollary 2.5** It follows from Proposition 1.9(1) that the maximal multiplicity of the eigenvalue is 2 (otherwise the codimension of  $S$  is larger than the dimension  $d = 2$  or 3 of the manifold).

In the case  $d = 2$ , the non-smooth critical points are isolated. According to the first row of Table 1, such points do not contribute any  $t^1$  terms. Therefore, the coefficient of  $t$  in  $P_{\lambda_k}$  is  $c_1(k)$  and, by Morse inequalities (1.3), it is greater or equal than the first Betti number of  $\mathbb{T}^2$ , which is 2.

In the case  $d = 3$  we need a more detailed analysis of the Morse inequalities (1.3) for  $\lambda_k$ . We write them as

$$\sum_{p=0}^3 (c_p(k) + d_p(k))t^p = (1+t)^3 + (1+t)(\alpha_0(k) + \alpha_1(k)t + \alpha_2(k)t^2),$$

where  $d_p(k)$  is the contribution to the polynomial  $P_{\lambda_k}$  coming from the points of multiplicity 2,  $(1+t)^3$  is the Poincaré polynomial of  $\mathbb{T}^3$ , and where  $\alpha_p(k)$  are the nonnegative coefficients of the remainder term  $R(t)$  in (1.3). Then similarly to (1.2), we have

$$c_0(k) + d_0(k) = 1 + \alpha_0(k) \geq 1, \quad (2.7)$$

$$c_1(k) + d_1(k) = 3 + \alpha_0(k) + \alpha_1(k) \geq 2 + c_0(k) + d_0(k), \quad (2.8)$$

$$c_2(k) + d_2(k) = 3 + \alpha_1(k) + \alpha_2(k) \geq 2 + c_3(k) + d_3(k), \quad (2.9)$$

$$c_3(k) + d_3(k) = 1 + \alpha_2(k) \geq 1. \quad (2.10)$$

We also observe that if  $\lambda_k$  has a non-smooth critical point  $x$  counted in  $d_0(k)$ , then  $\nu(x) = 2$  with  $\mu(x) = 0$  and  $i(x) = 1$  (since this is the only way to obtain  $t^0$  in (1.10) for  $\nu = 2$ ). This implies that  $\lambda_{k-1}(x) = \lambda_k(x)$  with the same constant multiplicity curve  $S$  and the same point  $x$  is a critical point of  $\lambda_{k-1}$  with  $\nu = 2$ ,  $\mu = 0$  and  $i = 2$ . From Table 1 we have  $P_{\lambda_{k-1}}(t; x) = t^2$ , namely  $x$  contributes to  $d_2(k-1)$ . This argument can be done in reverse and also extended to points contributing to  $d_1(k)$  (with  $\nu = 2$ ,  $\mu = 1$  and  $i = 1$ ), resulting in

$$d_0(k) = d_2(k-1), \quad d_1(k) = d_3(k-1), \quad k = 2, \dots, n, \quad (2.11)$$

$$d_0(1) = d_1(1) = 0, \quad d_2(n) = d_3(n) = 0. \quad (2.12)$$

The boundary values in (2.12) are obtained by noting that we cannot have  $\lambda_1(x) = \lambda_0(x)$  or  $\lambda_n(x) = \lambda_{n+1}(x)$  since eigenvalues  $\lambda_0$  and  $\lambda_{n+1}$  do not exist.

For  $k = 1$ , (2.12) substituted into (2.7) and (2.8) gives  $c_0(1) \geq 1$  and  $c_1(1) \geq 2 + c_0(1) \geq 3$ , establishing (2.4). Estimate (2.6) is similarly established from (2.12), (2.10) and (2.9).

Replacing  $k$  with  $k - 1$  in estimate (2.9) and using (2.11) gives

$$c_2(k - 1) + d_0(k) \geq 2 + c_3(k - 1) + d_1(k) \geq 2 + d_1(k).$$

Adding this last inequality to line (2.8) results in (2.5) after cancellations and the trivial estimate  $c_0(k) \geq 0$ .  $\square$

**Remark 2.7** It is straightforward to extend (2.4)–(2.6) to an arbitrary compact 3-dimensional manifold  $M$  with Betti numbers  $\beta_r$ , obtaining

$$\begin{aligned} c_1(1) &\geq \beta_1, \\ c_1(k) + c_2(k - 1) &\geq \beta_1 + \beta_2 - \beta_0 - \beta_3, \quad k = 2, \dots, n, \\ c_2(n) &\geq \beta_2. \end{aligned}$$

These inequalities extend to  $d = 3$  the results of Valero [62] who studied critical points of principal curvature functions (eigenvalues of the second fundamental form) of a smooth closed orientable surface.

Independence of the transverse Morse contributions from the particulars of the family  $\mathcal{F}$  also allows one to sort the terms in the Morse polynomial. This is illustrated by the next simple result.

Let  $\text{Conseq}_{k,n}$  be the set of all subsets of  $\{1, \dots, n\}$  containing  $k$  and consisting of consecutive numbers, i.e. subsets of the form  $\{j_1, j_1 + 1, \dots, j_2\} \ni k$ . Given  $J \in \text{Conseq}_{k,n}$ , let  $i(k; J)$  be the sequential number of  $k$  in the set  $J$  but counting from the top (cf. (1.8)). As usual,  $|J|$  will denote the cardinality of  $J$ .

Let  $\mathcal{F} : M \rightarrow \text{Sym}_n(\mathbb{R})$  be a generalized Morse family. For any set  $J \in \text{Conseq}_{k,n}$ , let

$$S(k, J) = \{x \in M : \lambda_j(x) = \lambda_k(x) \text{ if and only if } j \in J\}.$$

By our assumptions,  $S(k, J)$  are smooth embedded submanifolds of  $M$  and the restrictions  $\lambda_k|_{S(k, J)}$  of the eigenvalue  $\lambda_k$  to  $S(k, J)$  are smooth.

**Corollary 2.8** *Given a generalized Morse family  $\mathcal{F} : M \rightarrow \text{Sym}_n(\mathbb{R})$  the following inequality holds*

$$\sum_{J \in \text{Conseq}_{k,n}} \mathfrak{T}_{|J|}^{i(k; J)}(t) P_{\lambda_k|_{S(k, J)}}(t) \geq P_{\lambda_k}(t) \geq P_M(t), \quad (2.13)$$

where  $P(t) \geq Q(t)$  if and only if the all coefficients of the polynomials  $P(t) - Q(t)$  are nonnegative, the polynomials  $\mathfrak{T}_{|J|}^{i(k; J)}$  are defined in (1.10), and  $P_{\lambda_k|_{S(k, J)}}(t)$  are the Morse polynomials of the smooth functions  $\lambda_k|_{S(k, J)}$  on  $S(k, J)$ . In particular  $P_{\lambda_k|_{S(k, \{k\})}}$  is the total contribution of all smooth critical points of  $\lambda_k$ .

**Proof** We only need to prove the left inequality in (2.13). By (1.10), the contribution of a topologically critical point  $x \in S(k, J)$  to  $P_{\lambda_k}(t)$  is  $t^{\mu(x)} \mathfrak{T}_{|J|}^{i(k; J)}(t)$ . Therefore, the left-hand side of (2.13) is different from  $P_{\lambda_k}(t)$  in that the former also includes contributions from smooth critical points of  $\lambda_k|_{S(k, J)}$  that do not give rise to a topologically critical point of  $\lambda_k$ . However, those contributions are polynomials with non-negative coefficients, producing the inequality.  $\square$

We demonstrate Corollary 2.8 in a simple example involving an intermediate eigenvalue. Letting  $n = 3$ ,  $k = 2$ , and using the first two rows of Table 1, inequality (2.13) reads:

$$P_{\lambda_2|_{S(2, \{2\})}}(t) + t^2 P_{\lambda_2|_{S(2, \{2, 3\})}}(t) + P_{\lambda_2|_{S(2, \{1, 2\})}}(t) \geq P_{\lambda_2}(t) \geq P_M(t). \quad (2.14)$$

Note that the term with  $P_{\lambda_2|_{S(2, \{1, 2, 3\})}}(t)$  does not appear in (2.14) because  $\mathfrak{T}_3^2(t) = 0$  according to the second row of Table 1. Further simplifications of inequalities (2.14) are possible if it is known a priori that  $\lambda$  is a perfect Morse function when restricted to the connected components of the constant multiplicity strata  $S_{k, \{k-1, k\}}(\mathcal{F})$  and  $S_{k, \{k, k+1\}}(\mathcal{F})$ .

### 2.3 An open question

Finally, we mention an *open question* which naturally follows from our work: to classify Morse contributions from points where the multiplicity  $\nu$  is higher than what is suggested by the codimension calculation in the von Neumann–Wigner theorem [64]. Such points often arise in physical problems due to presence of a discrete symmetry; for an example, see [11, 29]. At a point of “excessive multiplicity”, the transversality condition (4.1) is not satisfied because  $d < s(\nu)$ , but one can still define an analogue of the “non-degeneracy in the non-smooth direction” (cf. Remark 1.8). It appears that the Morse indices are independent of the particulars of the family  $\mathcal{F}$  when the “excess”  $s(\nu) - d$  is equal to 1, but whether this persists for higher values of  $s(\nu) - d$  is still unclear.

## 3 Regularity condition: proof of Theorem 1.5

In this section we establish Theorem 1.5, namely the sufficient condition for a point to be regular (see Definition 1.1).

Recall the definition of the Clarke directional derivative<sup>5</sup> of a locally Lipschitz function  $f : M \rightarrow \mathbb{R}$  (for details, see, for example, [19, 55]). Given  $v \in T_x M$ , let  $\widehat{V}$  be a vector field in a neighborhood of  $x$  such that  $\widehat{V}(x) = v$  and let  $e^{t\widehat{V}}$  denote the local flow generated by the vector field  $\widehat{V}$ . Then the *Clarke generalized directional derivative* of  $f$  at  $x$  in the direction  $v$  is

$$f^\circ(x, v) = \limsup_{\substack{y \rightarrow x \\ t \rightarrow 0+}} \frac{f(e^{t\widehat{V}} y) - f(y)}{t}.$$

<sup>5</sup>This is usually a stepping stone to defining the Clarke subdifferential, but we will limit ourselves to Clarke directional derivative which is both simpler and sufficient for our needs.

Independence of this definition of the choice of  $\widehat{V}$  follows from the flow-box theorem and the chain rule for the Clarke subdifferential, see [55, Thm 1.2(i) and Prop 1.4(ii)].

**Definition 3.1** The point  $x$  is called a *critical point* of  $f$  in the Clarke sense, if

$$0 \leq f^\circ(x, v) \quad \text{for all } v \in T_x M.$$

Otherwise, the point  $x$  is said to be *regular in the Clarke sense*.

The assumptions of Theorem 1.5 will be shown to imply that the point  $x$  is regular in the Clarke sense, whereupon we will use the following result.

**Theorem 3.2** [2, Proposition 1.2] *A point regular in the Clarke sense is topologically regular in the sense of Definition 1.1.*

**Proof of Theorem 1.5** We first establish that condition (1.6), namely

$$(\text{Ran } \mathcal{H}_x)^\perp \cap \text{Sym}_v^+ = 0,$$

is equivalent to existence of a matrix  $C \in \text{Ran } \mathcal{H}_x$  which is (strictly) positive definite. Despite being intuitively clear, the proof of this fact is not immediate and we provide it for completeness; a similar result is known as Fundamental Theorem of Asset Pricing in mathematical finance [25]. Assume the contrary,

$$\text{Ran } \mathcal{H}_x \cap \text{Sym}_v^{++} = \emptyset.$$

The set  $\text{Sym}_v^{++}$  is open and convex (the latter can be seen by Weyl's inequality for eigenvalues). A suitable version of the Helly–Hahn–Banach separation theorem (for example, [56, Thm 7.7.4]) implies existence of a functional vanishing on  $\text{Ran } \mathcal{H}_x$  and positive on  $\text{Sym}_v^{++}$ . By Riesz Representation Theorem, this functional is  $\langle D, \cdot \rangle$  for some  $D \in \text{Sym}_v$ , for which we now have  $D \in (\text{Ran } \mathcal{H}_x)^\perp$  and  $\langle D, P \rangle > 0$  for all  $P \in \text{Sym}_v^{++}$ . In particular,  $D$  is non-zero and belongs to the dual cone of  $\text{Sym}_v^{++}$ , namely to  $\text{Sym}_v^+$  [14], contradicting condition (1.6).

Secondly, results of [21, Theorem 4.2] (see also<sup>6</sup> [42, Sect. 6]) show that

$$\begin{aligned} \lambda_k^\circ(x, v) &\leq \max \left\{ \langle u, (d\mathcal{F}(x)v)u \rangle : u \in \mathbf{E}_k, \|u\| = 1 \right\} \\ &= \lambda^{\max} \left( (d\mathcal{F}(x)v)_{\mathbf{E}_k} \right) = \lambda^{\max}(\mathcal{H}_x(v)), \end{aligned}$$

where  $\mathbf{E}_k$  is the eigenspace of the eigenvalue  $\lambda_k(x)$  of  $\mathcal{F}(x)$ ; the middle equality is by the variational characterization of the eigenvalues and the last by the definition (1.5) of  $\mathcal{H}_x$ .

We already established that there exists  $v$  such that  $\mathcal{H}_x(v) \in \text{Sym}_v^{++}$ . Then  $\mathcal{H}_x(-v)$  is negative definite and we have

$$\lambda_k^\circ(x, v) \leq \lambda^{\max}(\mathcal{H}_x(-v)) < 0.$$

<sup>6</sup>Note that there is a misprint in the direction of the inequality in [42, Sect. 6].

The point  $x$  is regular in the Clarke sense and, therefore, regular in the sense of Definition 1.1.  $\square$

## 4 Transversality and its consequences

**Definition 4.1** We say that a family  $\mathcal{F}$  is *transverse (with respect to eigenvalue  $\lambda_k$ )* at a point  $x$  if

$$\mathcal{I}_v + \text{Ran } \mathcal{H}_x = \text{Sym}_v, \quad (4.1)$$

where  $v$  is the multiplicity of  $\lambda_k$  at the point  $x$  and  $\mathcal{I}_v := \text{span}(I_v) \subset \text{Sym}_v$  is the space of multiples of the identity matrix.

In this section we explore the consequences of the transversality condition, equation (4.1). In particular, in Lemma 4.2 we interpret condition (4.1) as transversality of the family  $\mathcal{F}$  and the subvariety of  $\text{Sym}_n(\mathbb{F})$  of matrices with multiplicity. We then show that transversality at a non-degenerate topologically critical point allows us to work separately in the smooth and non-smooth directions. In particular, we establish that a non-degenerate topologically critical point satisfies the sufficient conditions of Goresky–MacPherson’s stratified Morse theory. The latter allows us to separate the Morse data at a topologically critical point into a smooth part and a transverse part; the latter will be shown in Sect. 5 to be independent of the particulars of the family  $\mathcal{F}$ .

Let  $\mathcal{Q}_{k,v}^n$  be the subset of  $\text{Sym}_n(\mathbb{F})$ , where  $\mathbb{F}$  is  $\mathbb{R}$  or  $\mathbb{C}$ , consisting of the matrices whose eigenvalue  $\lambda_k$  has multiplicity  $v$ . It is well-known [6] that the set  $\mathcal{Q}_{k,v}^n$  is a semialgebraic submanifold of  $\text{Sym}_n$  of codimension<sup>7</sup>

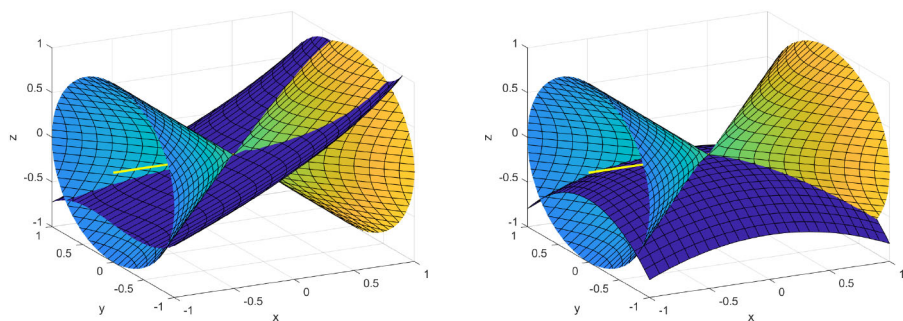
$$s(v) := \dim \text{Sym}_v(\mathbb{F}) - 1 = \begin{cases} \frac{1}{2}v(v+1) - 1, & \mathbb{F} = \mathbb{R}, \\ v^2 - 1, & \mathbb{F} = \mathbb{C}. \end{cases} \quad (4.2)$$

In particular, if  $v > 1$  (the eigenvalue  $\lambda_k$  is not simple), then  $\text{codim } \mathcal{Q}_{k,v}^n \geq 2$ , if  $\mathbb{F} = \mathbb{R}$  and  $\text{codim } \mathcal{Q}_{k,v}^n \geq 3$ , if  $\mathbb{F} = \mathbb{C}$ . We remark that we use real dimension in all (co)dimension calculations.

**Lemma 4.2** *Let  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be a smooth family whose eigenvalue  $\lambda_k$  has multiplicity  $v$  at the point  $x \in M$  (i.e.  $\mathcal{F}(x) \in \mathcal{Q}_{k,v}^n$ ). Then  $\mathcal{F}$  is transverse at  $x$  in the sense of (4.1) if and only if*

$$\text{Ran } d\mathcal{F}(x) + T_{\mathcal{F}(x)}\mathcal{Q}_{k,v}^n = T_{\mathcal{F}(x)}\text{Sym}_n \quad (\cong \text{Sym}_n). \quad (4.3)$$

<sup>7</sup>The reason for this codimension to be equal to  $\dim \text{Sym}_v(\mathbb{F}) - 1$  is as follows: Symmetric matrices with non-repeated eigenvalues can be encoded by their eigenvalues and unit eigenvectors. When an eigenvalue is repeated  $v$  times, there is a loss of  $v - 1$  parameters from the eigenvalues plus an extra freedom of choice of an orthonormal basis in the corresponding eigenspace. Thus the desired codimension is equal to  $v - 1$  plus the dimension of the space of orthonormal bases of  $\mathbb{F}^v$ , adding up to  $\dim \text{Sym}_v(\mathbb{F}) - 1$ .



**Fig. 4** Examples of two families visualized in the 3-dimensional space parametrizing  $\text{Sym}_2(\mathbb{R})$  via (2.3). The subspace  $\mathcal{I}_2$  is drawn as a line. The monochrome surfaces represent the images of the families  $\mathcal{F}$ , satisfying  $\mathcal{F}(0) = 0$ . The family on the left is transverse at  $x = 0$  and the family on the right is not. The cones are drawn for comparison with Fig. 3

**Remark 4.3** It is easy to see that when  $v = 1$ , both conditions (4.1) and (4.3) are satisfied independently of  $\mathcal{F}$ . When  $v = n$ , conditions (4.1) and (4.3) become identical. The transversality condition in the case  $v = n = 2$  is illustrated in Fig. 4.

**Proof of Lemma 4.2** Consider the linear mapping  $h : \text{Sym}_n \rightarrow \text{Sym}_v$  acting as a compression to  $\mathbf{E}_k$ . Namely,  $A \mapsto A\mathbf{E}_k = \mathcal{U}^* A \mathcal{U}$ , where  $\mathcal{U}$  is a linear isometry  $\mathbb{F}^v \rightarrow \mathbf{E}_k$ , see (1.4). The mapping  $h$  is onto: for any  $B \in \text{Sym}_v$ , choosing  $\tilde{B} = \mathcal{U} B \mathcal{U}^* \in \text{Sym}_n$  yields  $h(\tilde{B}) = \mathcal{U}^* \mathcal{U} B \mathcal{U}^* \mathcal{U} = I_v B I_v = B$ . Furthermore, by Hellmann–Feynman theorem (Theorem A.1),  $A \in T_{\mathcal{F}(x)} Q_{k,v}^n$  if and only if  $h(A) \in \mathcal{I}_v$  (informally, a direction is tangent to  $Q_{k,v}^n$  if and only if the eigenvalues remain equal to first order). Finally, by definition of  $\mathcal{H}_x$  we have  $h(\text{Ran } d\mathcal{F}(x)) = \text{Ran } \mathcal{H}_x$ .

Assuming condition (4.3) and applying to it the mapping  $h$ , we get

$$\text{Sym}_v = h(\text{Sym}_n) = h(\text{Ran } d\mathcal{F}(x)) + h(T_{\mathcal{F}(x)} Q_{k,v}^n) = \text{Ran } \mathcal{H}_x + \mathcal{I}_v,$$

establishing (4.1). Conversely, assume  $\mathcal{F}$  violates condition (4.3), meaning that

$$\text{Ran } d\mathcal{F}(x) + T_{\mathcal{F}(x)} Q_{k,v}^n = G + T_{\mathcal{F}(x)} Q_{k,v}^n$$

for some linear subspace  $G$  of  $\dim G < \text{codim } T_{\mathcal{F}(x)} Q_{k,v}^n = s(v)$ . Applying  $h$  to both sides we get

$$\text{Ran } \mathcal{H}_x + \mathcal{I}_v = h(\text{Ran } d\mathcal{F}(x) + T_{\mathcal{F}(x)} Q_{k,v}^n) = h(G + T_{\mathcal{F}(x)} Q_{k,v}^n) = h(G) + \mathcal{I}_v.$$

Counting dimensions, we arrive to  $\dim(\text{Ran } \mathcal{H}_x + \mathcal{I}_v) < s(v) + 1 = \dim(\text{Sym}_v)$ , and therefore (4.1) cannot hold.  $\square$

**Corollary 4.4** (1) If  $x$  satisfies non-degenerate criticality condition (N), see Definition 1.7, then  $\mathcal{F}$  is transverse at  $x$ .

- (2) If  $\mathcal{F}$  is transverse at a point  $x$ , the constant multiplicity stratum  $S$  of  $x$  is a submanifold of  $M$  of codimension  $s(v)$  and the function  $\lambda_k$  restricted to  $S$  is smooth.

**Proof** Recall that non-degenerate criticality condition (N) states that  $(\text{Ran } \mathcal{H}_x)^\perp$  is spanned by a positive definite matrix. In particular, the codimension of  $\text{Ran } \mathcal{H}_x$  is 1. Furthermore, the identity matrix  $I_v$  is not in  $\text{Ran } \mathcal{H}_x$  because the identity cannot be orthogonal to a positive definite matrix. Therefore condition (4.1) holds.

Now let  $\mathcal{F}$  be transverse at  $x$  and let  $v = v(x)$  be the multiplicity of the eigenvalue  $\lambda_k$  at  $x$ . Then  $S$  is the connected component of  $\mathcal{F}^{-1}\left(Q_{k,v}^n\right)$  containing  $x$ . Transversality implies  $S$  is a submanifold of codimension  $\text{codim } Q_{k,v}^n = s(v)$ . The smoothness of  $\lambda_k$  restricted to  $S$  is a standard result of perturbation theory for linear operators (see, for example, [48, Section II.1.4 or Theorem II.5.4]). To see it, one uses the Cauchy integral formula for the *total eigenprojector* (or *Riesz projector*), i.e. the projector onto the span of eigenspaces of the eigenvalues lying in a small neighborhood around  $\lambda_k(x)$ . It follows that the total eigenprojector is smooth in a sufficiently small neighborhood of  $x \in M$  (the neighborhood on  $M$  needs to be small enough so that no eigenvalues cross the contour of integration). Once restricted to  $y \in S$ , the eigenprojector is simply  $\lambda_k(y)I_v$ , therefore  $\lambda_k(y)$  is also smooth.  $\square$

**Lemma 4.5** *If  $x$  satisfies non-degenerate criticality condition (N), then  $\text{Ran } \mathcal{H}_x|_{T_x S} = 0$  and  $x$  is a critical point of the locally smooth function  $\lambda_k|_S$ .*

**Proof** By Hellmann–Feynman theorem, see Appendix A, the eigenvalues of  $\mathcal{H}_x v \in \text{Sym}_v$  give the slopes of the eigenvalues splitting off from the multiple eigenvalue  $\lambda_k(\mathcal{F}(x))$  when we leave  $x$  in the direction  $v$ . Leaving in the direction  $v \in T_x S$ , where  $S$  is the constant multiplicity stratum attached to  $x$ , must produce equal slopes, i.e.  $\mathcal{H}_x(v)$  is a multiple of the identity matrix  $I_v$  for every  $v \in T_x S$ . But a non-zero multiple of the identity cannot be orthogonal to  $B \in \text{Sym}_v^{++}$ , therefore  $\text{Ran } \mathcal{H}_x|_{T_x S} = 0$ . In other words, the slopes of the branches splitting off from the multiple eigenvalue  $\lambda_k(x)$  are all zero.  $\square$

**Proof of Proposition 1.9** Corollary 4.4 and Lemma 4.5, combined, give the conclusions of Proposition 1.9.  $\square$

The next step is to enable ourselves to focus on the directions transverse to  $S$ .

**Corollary 4.6** *Let  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be a smooth family whose eigenvalue  $\lambda_k$  satisfies conditions (N) and (S) (Definitions 1.7 and 1.10) at the point  $x \in M$ . Let  $S$  be the constant multiplicity stratum at  $x$  and let  $N$  be a submanifold of  $M$  of dimension  $\dim N = \text{codim}_M S = s(v)$  which intersects  $S$  transversely at  $x$ .*

*Then the eigenvalue  $\lambda_k$  of the restriction  $\mathcal{F}|_N$  also satisfies conditions (N) and (S).*

**Proof** Transversality and dimension count imply that the constant multiplicity stratum of  $\mathcal{F}|_N$  is the isolated point  $x$ . Therefore condition (S) for  $\mathcal{F}|_N$  is vacuously true at  $x$ .

Condition (N) for  $\mathcal{F}$ , combined with Lemma 4.5, yields  $\text{Ran } \mathcal{H}_x|_{T_x S} = 0$ . We thus obtain

$$\text{Ran } \mathcal{H}_x = \text{Ran } \mathcal{H}_x|_{T_x N} + \text{Ran } \mathcal{H}_x|_{T_x S} = \text{Ran } \mathcal{H}_x|_{T_x N}.$$

In other words, the space  $\text{Ran } \mathcal{H}_x$  remains unchanged after restricting  $\mathcal{F}$  to  $N$ , therefore condition (N) holds for  $\mathcal{F}|_N$ .  $\square$

The next step is to separate the Morse data at a critical point  $x$  into a smooth part (along  $S$ ) and a transverse part (along  $N$ ). For this purpose we will use the stratified Morse theory of Goresky and MacPherson [35]. We now show that a point satisfying non-degenerate criticality conditions (N) and (S) is nondepraved in the sense of [35, definition in Sec. I.2.3]. The setting of [35] calls for a smooth function on a certain manifold which is then restricted to a stratified subspace of that manifold. To that end we consider the graph of the function  $\lambda_k$  on  $M$ , i.e. the set  $Z_k := \{(x, \lambda_k(x)) : x \in M\}$  as a stratified subspace of  $\tilde{M} := M \times \mathbb{R}$  and the (smooth) function  $\pi : \tilde{M} \rightarrow \mathbb{R}$  which is the projection to the second component of  $\tilde{M}$ . As before, the stratification (both on  $M$  and on  $Z_k$ ) is induced by the multiplicity of the eigenvalue  $\lambda_k(x)$ .

Recall that a subspace  $Q$  of  $T_z \tilde{M}$  is called a *generalized tangent space* to a stratified subspace  $Z \subset \tilde{M}$  at the point  $z \in Z$ , if there exists a stratum  $\mathcal{R}$  of  $Z$  with  $z \in \mathcal{R}$ , and a sequence of points  $\{z_i\} \subset \mathcal{R}$  converging to  $z$  such that

$$Q = \lim_{i \rightarrow \infty} T_{z_i} \mathcal{R}. \quad (4.4)$$

**Proposition 4.7** *Let the family  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be transverse and the point  $x \in M$  satisfy conditions of Theorem 1.12. Let  $z := (x, \lambda_k(x))$  be the corresponding point on the stratified subspace  $Z_k \subset \tilde{M}$  defined above and  $\tilde{S}$  be the stratum of  $Z_k$  containing  $z$ . Then the following two statements hold:*

- (1) *For each generalized tangent space  $Q$  at  $z$  we have  $d\pi(z)|_Q \neq 0$  except when  $Q = T_z \tilde{S}$ .*
- (2)  *$x$  is isolated in the set of all points  $y$  that are critical for  $\lambda_k|_{S_y}$ , where  $S_y$  is the constant multiplicity stratum attached to  $y$ .*

**Remark 4.8** A point  $z := (x, \lambda_k(x))$  satisfying conditions (1)–(2) of Proposition 4.7 and such that  $x$  is non-degenerate as a smooth critical point of  $\lambda_k|_S$  is a nondepraved point of the map  $\pi|_{Z_k}$  in the sense of Goresky–MacPherson [35, Sec. I.2.3].<sup>8,9</sup>

<sup>8</sup>The definition of a *nondepraved point* in [35, Sec. I.2.3] contains three conditions. Conditions (c) and (a) of [35, Sec. I.2.3] correspond to parts (1) and (2) of Proposition 4.7, respectively. The third condition — condition (b) of [35, Sec. I.2.3] — holds automatically in our case because  $x$  is non-degenerate as a smooth critical point of  $\lambda_k|_S$ , by condition (S) assumed in Theorem 1.12). Thus we omit here the general description of condition (b), which is rather technical.

<sup>9</sup>We also mention that [35] uses the term “critical” for the points  $y$  that are critical when the function in question is restricted to their respective stratum of constant multiplicity.

**Corollary 4.9** *Let the family  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be transverse and the point  $x \in M$  satisfy conditions of item (1) of Theorem 1.12. Let  $z := (x, \lambda_k(x))$  be the corresponding point on the stratified subspace  $Z_k \subset \tilde{M}$  defined above and  $\pi : M \times \mathbb{R} \rightarrow \mathbb{R}$  be the projection to the second component of  $\tilde{M}$ . Then  $z$  is a nondepraved point of the map  $\pi|_{Z_k}$  in the sense of Goresky–MacPherson [35, Sec. I.2.3].*

**Remark 4.10** Let us discuss informally the idea behind part (1) of Proposition 4.7, the proof of which is fairly technical. When we leave  $x$  in the direction not tangent to  $S$ , the multiplicity of eigenvalue  $\lambda_k$  is reduced as other eigenvalues split off. Part (1) stipulates that among the directions in which the multiplicity splits in a prescribed manner, there is at least one direction in which the slope of  $\lambda_k$  is not equal to zero. This is again a consequence of transversality: the space of directions is too rich to produce only zero slopes.

**Proof of Proposition 4.7, part (1)** First, if  $p : M \times \mathbb{R} \rightarrow M$  denotes the projection to the first component of  $\tilde{M} = M \times \mathbb{R}$ , then  $dp(\tilde{x}) : T_{\tilde{x}}\tilde{M} \rightarrow T_x M$  is the corresponding projection to the first component of  $T_{\tilde{x}}\tilde{M} \cong T_x M \times \mathbb{R}$ ; here  $\tilde{x} = (x, \lambda)$  for some  $\lambda \in \mathbb{R}$ .

Since  $d\pi|_{T_z\tilde{S}} = d(\lambda_k|_S) \circ dp|_{T_z\tilde{S}}$ , where  $S = p(\tilde{S})$  is the constant multiplicity stratum of  $x$ , we conclude from Lemma 4.5 that  $d\pi(z)|_Q = 0$  when  $Q = T_z\tilde{S}$ .

Let now  $Q \neq T_z\tilde{S}$  and assume that

$$d\pi(z)|_Q = 0. \quad (4.5)$$

Let  $\nu$  be the multiplicity of the eigenvalue  $\lambda_k(x)$  of  $\mathcal{F}(x)$  and  $\mathbf{E}_k$ ,  $\dim \mathbf{E}_k = \nu$ , be the corresponding eigenspace. Let  $\mathcal{R}$  be the stratum used for the definition of  $Q$  in (4.4) and  $\nu_{\mathcal{R}} < \nu$  be the multiplicity of  $\lambda_k$  on  $p(\mathcal{R})$ .

Let  $(z_i) \subset \mathcal{R}$  be the sequence defining  $Q$  and let  $x_i = p(z_i)$ . Let  $\mathbf{E}_k(x_i) \subset \mathbb{F}^n$  denote the  $\nu_{\mathcal{R}}$ -dimensional eigenspace of the eigenvalue  $\lambda_k$  of  $\mathcal{F}(x_i)$  and let  $\mathcal{U}_i$  be a choice of linear isometry from  $\mathbb{F}^{\nu_{\mathcal{R}}}$  to  $\mathbf{E}_k(x_i)$ . Finally, let  $W_i \subset T_{x_i}M$  denote the first component of the tangent space at  $z_i$  to  $\mathcal{R}$ , namely  $W_i = dp(z_i)(T_{z_i}\mathcal{R})$ .

We would like to use Hellmann–Feynman theorem at  $x_i$ . In the directions from  $W_i$ , the eigenvalue  $\lambda_k$  retains multiplicity  $\nu_{\mathcal{R}}$  in the linear approximation. In other words, directional derivatives of the eigenvalue group of  $\lambda_k$  are all equal. Formally,

$$\mathcal{U}_i^*(d\mathcal{F}(x_i)w)\mathcal{U}_i = D_w\lambda_k(x_i)I_{\nu_{\mathcal{R}}}, \quad \text{for all } w \in W_i; \quad (4.6)$$

here  $D_w\lambda_k$  is the directional derivative of  $\lambda_k$ . This expression is invariant with respect to the choice of isometry  $\mathcal{U}_i$ .

Using compactness of the Grassmannians and, if necessary, passing to a subsequence, the spaces  $\mathbf{E}_k(x_i)$  converge to a subspace  $\mathbf{E}_k^{\mathcal{R}}$  of the  $\nu$ -dimensional eigenspace  $\mathbf{E}_k$  of the matrix  $\mathcal{F}(x)$ . The isometries  $\mathcal{U}_i$  (adjusted if necessary) converge to a linear isometry  $\mathcal{U}_{\mathcal{R}}$  from  $\mathbb{F}^{\nu_{\mathcal{R}}}$  to  $\mathbf{E}_k^{\mathcal{R}}$ . Tangent subspaces  $W_i$  also converge to the subspace  $W_0 := dp(z)Q$ . Passing to the limit in (4.6), the derivative on the right-hand side of (4.6) must tend to 0 due to (4.5). Recalling the definition of  $\mathcal{H}$  in (1.5), we get

$$\mathcal{U}_{\mathcal{R}}^*(d\mathcal{F}(x)w)\mathcal{U}_{\mathcal{R}} = \mathcal{U}_{\mathcal{R}}^*\mathcal{U}(\mathcal{H}_x w)\mathcal{U}^*\mathcal{U}_{\mathcal{R}} = 0, \quad \text{for all } w \in W_0.$$

In other words, the matrix  $\mathcal{H}_x w$  with  $w$  restricted to  $W_0$  maps vectors from  $V = \text{Ran}(\mathcal{U}^* \mathcal{U}_{\mathcal{R}}) \subset \mathbb{F}^v$  to vectors orthogonal to  $V$ . We can express this as

$$\text{Ran } \mathcal{H}_x|_{W_0} \subset \text{Sym}_v(V, V^\perp), \quad (4.7)$$

where  $\text{Sym}_v(X, Y)$  denotes the set of all  $v \times v$  self-adjoint matrices that map  $X$  to  $Y$ . The space  $V$  is  $v_{\mathcal{R}}$ -dimensional<sup>10</sup> and, in a suitable choice of basis, a  $v_{\mathcal{R}} \times v_{\mathcal{R}}$  subblock of  $\mathcal{H}_x w$  is identically zero. Therefore, the dimension of  $\text{Sym}_v(V, V^\perp)$  is

$$\dim \text{Sym}_v(V, V^\perp) = \dim \text{Sym}_v - \dim \text{Sym}_{v_{\mathcal{R}}} = s(v) - s(v_{\mathcal{R}}). \quad (4.8)$$

On the other hand, we have the following equalities,

$$\text{codim } \text{Ker } \mathcal{H}_x = \dim \text{Ran } \mathcal{H}_x = \dim \text{Sym}_v - 1 = s(v) = \text{codim } T_x S.$$

The first is the rank-nullity theorem, the second is because  $\text{Ran } \mathcal{H}_x$  has codimension 1 (by condition (N)), the third is the definition of  $s(v)$  and the last is from the properties of  $S$ . Using  $T_x S \subset \text{Ker } \mathcal{H}_x$  (Lemma 4.5) and counting dimensions, we conclude

$$\text{Ker } \mathcal{H}_x = T_x S.$$

Now we want to show that the stratification on  $M$  induced by the multiplicity of the eigenvalue  $\lambda_k(x)$  satisfies *Whitney condition A*: any generalized tangent space at  $z$  contains the tangent space of the stratum containing  $z$ . For this note that the *discriminant variety* of  $\text{Sym}_n$ ,

$$\text{Discr}_n := \bigcup_{1 \leq k \leq n, v > 1} \mathcal{Q}_{k,v}^n,$$

is an algebraic variety. Therefore, by classical results of Whitney [65],  $\text{Discr}_n$  admits a stratification satisfying Whitney condition A. Consequently, if  $\mathcal{F}: M \rightarrow \text{Sym}_n$  is a transverse family then the fact that  $\text{Discr}_n$  satisfies Whitney condition A implies that the stratifications on  $M$  induced by the multiplicity of the eigenvalue  $\lambda_k(x)$  satisfies Whitney condition A as well.

Whitney condition A gives the inclusion  $T_x S \subset W_0$  and therefore  $\text{Ker } \mathcal{H}_x \subset W_0$ . Using the rank-nullity theorem again, we get

$$\begin{aligned} \dim \text{Ran } \mathcal{H}_x|_{W_0} &= \text{codim}_{W_0} \text{Ker } \mathcal{H}_x|_{W_0} = \text{codim}_{W_0} \text{Ker } \mathcal{H}_x = \dim W_0 - \dim \text{Ker } \mathcal{H}_x \\ &= \text{codim}_{T_x M} \text{Ker } \mathcal{H}_x - \text{codim}_{T_x M} W_0 = s(v) - s(v_{\mathcal{R}}). \end{aligned} \quad (4.9)$$

Comparing (4.7), (4.8) and (4.9) we conclude that

$$\text{Sym}_v(V, V^\perp) = \text{Ran } \mathcal{H}_x|_{W_0}.$$

<sup>10</sup>From properties of isometries and the inclusion  $\mathbf{E}_k^{\mathcal{R}} \subset \mathbf{E}_k$  it can be seen that  $(\mathcal{U}^* \mathcal{U}_{\mathcal{R}})^* \mathcal{U}^* \mathcal{U}_{\mathcal{R}} = I_{v_{\mathcal{R}}}$ .

Consequently,<sup>11</sup>

$$(\text{Ran } \mathcal{H}_x)^\perp \subset \left( \text{Ran } \mathcal{H}_x|_{W_0} \right)^\perp = \text{Sym}_v \left( V, V^\perp \right)^\perp = \text{Sym}_v(V^\perp, 0), \quad (4.10)$$

i.e.  $V^\perp$  is in the kernel of the matrices from  $(\text{Ran } \mathcal{H}_x)^\perp$  which contradicts condition (N), see (1.7).  $\square$

**Proof of Proposition 4.7, part (2)** Assume, by contradiction, that  $x$  is an accumulation point of a sequence  $(x_i)$  of points which are critical on their respective strata. Passing to a subsequence if necessary, we can assume that all  $z_i := (x_i, \lambda_k(x_i))$  belong to the same stratum  $\mathcal{R}$  and that the sequence of spaces  $T_{z_i}\mathcal{R}$  converges to a space  $Q$ .

Note that  $Q$  is a nontrivial generalized tangent space to  $Z_k$  at  $x$ . Since  $x_i$  are critical for  $\lambda_k$  restricted to the stratum  $p(\mathcal{R})$ , we have  $d\pi(z_i)|_{T_{z_i}\mathcal{R}} = d\lambda_k|_{p(\mathcal{R})}(x_i) = 0$  and finally  $d\pi(z)|_Q = 0$ , which is a contradiction to part (1) of the Proposition.  $\square$

We finish the section with establishing the comforting<sup>12</sup> result of Theorem 1.13: the set of generalized Morse families is open and dense. We restate Theorem 1.13 in an expanded form.

**Theorem 4.11** (Theorem 1.13) *The set of families  $\mathcal{F}$  having the below properties for every  $\lambda_k$  is open and dense in the Whitney topology of  $C^r(M, \text{Sym}_n)$ ,  $2 \leq r \leq \infty$ :*

- (1) *at every point  $x$ ,  $\mathcal{F}$  is transverse in the sense of Definition 4.1,*
- (2) *at every point  $x$ , either  $\text{Ran } \mathcal{H}_x$  or  $(\text{Ran } \mathcal{H}_x)^\perp$  contains a positive definite matrix,*
- (3) *in the latter case,  $\lambda_k$  restricted to the constant multiplicity stratum of  $x$  has a non-degenerate critical point at  $x$ .*

*In particular, a family  $\mathcal{F}$  satisfying the above properties is generalized Morse (Definition 1.11).*

**Remark 4.12** Observe that property (1) of Theorem 4.11 does not imply property (2): a counter-example is provided by Example 2.2. Furthermore, when (1) and the second case of (2) hold — and thus non-degenerate criticality condition (N) is fulfilled — Lemma 4.5 shows that  $\lambda_k|_S$  has a critical point at  $x$ . Property (3) posits non-degeneracy of this point, strengthening the conclusion to non-degenerate criticality condition (S).

**Proof of Theorem 4.11** Lemma 4.2 showed that the transversality in the sense of Definition 4.1 is equivalent to the transversality between  $\mathcal{F}$  and the submanifold  $Q_{k,v}^n$  at  $x$ .

<sup>11</sup>Note that in equation (4.10) the same notation  $\perp$  is used for two different operations: on one hand, for the operation of taking orthogonal complement for subspaces  $\mathbb{F}^v$  and, on the other hand, for the operation of taking orthogonal complement for subspaces of  $\text{Sym}_v$ .

<sup>12</sup>In every particular case of  $\mathcal{F}$ , one still needs to establish non-degeneracy of the critical point “by hand”. In some well-studied cases, such as discrete magnetic Schrödinger operators [3, 31], degenerate critical points are endemic.

As was mentioned before the discriminant variety  $\text{Discr}_n$  admits a stratification satisfying Whitney condition A. For such stratifications,<sup>13</sup> we have the stratified version of the weak<sup>14</sup> Thom transversality theorem (see [30, Proposition 3.6] or informal discussions in [7, Sect. 2.3]). Namely, for any  $1 \leq r \leq \infty$ , the set of maps in  $C^r(M, \text{Sym}_n)$  that are transverse to  $\text{Discr}_n$  is open and dense in the Whitney topology in  $C^r(M, \text{Sym}_n)$ .

This establishes that property 1 holds for families  $\mathcal{F}$  from an open and dense set in the Whitney topology of  $C^r(M, \text{Sym}_n)$ .

Properties 2–3 are more challenging because they involve properties of the derivatives of  $\mathcal{F}$ . Let  $J^1(M, \text{Sym}_n)$  denote the space of the 1-jets of smooth families of self-adjoint matrices and let  $\Gamma^1(\mathcal{F}) \subset J^1(M, \text{Sym}_n)$  denote the graph of the 1-jet extension of a smooth family  $\mathcal{F}: M \rightarrow \text{Sym}_n$ ,

$$\Gamma^1(\mathcal{F}) := \{(x, \mathcal{F}(x), d\mathcal{F}(x)): x \in M\}.$$

We will show that our conclusions follows from the transversality (in the differential topological sense) of  $\Gamma^1(\mathcal{F})$  to certain stratified subspaces of  $J^1(M, \text{Sym}_n)$ . Then the proposition will follow from a stratified version of the strong (or jet) Thom transversality theorem (see [7, p. 38 and p. 42] as well as [30, Proposition 3.6]): the set of families whose 1-jet extension graph is transverse to a closed stratified subspace is open and dense in the Whitney topology of  $C^r(M, \text{Sym}_n)$  with  $2 \leq r \leq \infty$ . The theorem holds if the stratified subspace satisfies Whitney condition A.

The jet space  $J^1(M, \text{Sym}_n)$  is the space of triples  $(x, A, L)$  such that

$$x \in M, \quad A \in \text{Sym}_n, \quad L \in \text{Hom}(T_x M, T_A \text{Sym}_n).$$

Given an integer  $k$ ,  $1 \leq k \leq n$ , a matrix  $A \in \mathcal{Q}_{k,v}^n$  and a “differential”  $L \in \text{Hom}(T_x M, T_A \text{Sym}_n)$ , introduce the linear subspace

$$\text{Ran } L_{k,A} := \left\{ \mathcal{U}_{k,A}^*(Lv) \mathcal{U}_{k,A} : v \in T_x M \right\} \subset \text{Sym}_v(\mathbb{F}), \quad (4.11)$$

where  $\mathcal{U}_{k,A}$  is a linear isometry from  $\mathbb{F}^v$  to the  $v$ -dimensional eigenspace  $\mathbf{E}_k(A)$  of the eigenvalue  $\lambda_k$  of  $A$ .

We define the following subsets of  $J^1(M, \text{Sym}_n)$ .

$$T^c := \bigcup_{1 \leq k \leq n, v \geq 1} \left\{ (x, A, L) : A \in \mathcal{Q}_{k,v}^n, (\text{Ran } L_{k,A})^\perp \neq 0 \right\}.$$

$$T_0^c := \bigcup_{1 \leq k \leq n, v \geq 1} \left\{ (x, A, L) : A \in \mathcal{Q}_{k,v}^n, \exists B \in (\text{Ran } L_{k,A})^\perp \setminus \{0\}, \det B = 0 \right\}.$$

We note that the subspace  $\text{Ran } L_{k,A}$  does not depend on the base point  $x$  but it depends the particular choice of the isometry  $\mathcal{U}_{k,A}$ . Nevertheless, the properties of

<sup>13</sup>And in fact only for them [61].

<sup>14</sup>The word “weak” here is used to distinguish it from the jet version of the Thom transversality theorem which is usually called strong [7].

$\text{Ran } L_{k,A}$  used in the definitions of  $T^c$  and  $T_0^c$  above are independent of the choice of the isometry  $\mathcal{U}_{k,A}$ .

**Lemma 4.13**  *$T^c$  and  $T_0^c$  are stratified spaces satisfying Whitney condition A. Every stratum of  $T^c$  has codimension at least  $d$  in  $J^1(M, \text{Sym}_n)$ , where  $d = \dim M$ ; every stratum of  $T_0^c$  has codimension at least  $d + 1$ .*

**Proof** Obviously the sets  $T^c$  and  $T_0^c$  are closed, with stratification induced by  $\nu$  and the dimension of  $(\text{Ran } L_{k,A})^\perp$ . Besides, they are smooth<sup>15</sup> fiber bundles over  $M$  with semialgebraic fibers and therefore satisfy Whitney condition A.

Semialgebraicity of the fibers of  $T^c$  and  $T_0^c$  follows from the Tarski–Seidenberg theorem stating that semialgebraicity is preserved under projections ([13, Theorem 2.2.1], [53, Theorem 8.6.6]). Indeed, let  $\Pi : J^1(M, \text{Sym}_n) \rightarrow M$  be the canonical projection. For each  $x \in M$ , we view  $\Pi^{-1}(x) \cong \text{Sym}_n \times \text{Hom}(T_x M, \text{Sym}_n)$  as a vector space by canonically identifying  $T_A \text{Sym}_n$  with  $\text{Sym}_n$ . Focusing on  $T^c$ , the set

$$\left\{ (A, L, \lambda) : \det(A - \lambda I) = 0, (\text{Ran } L_{k,A})^\perp \neq 0 \text{ for some } k \right\}$$

is semialgebraic in the vector space  $\Pi^{-1}(x) \times \mathbb{R}$ . Its projection on  $\Pi^{-1}(x)$  is exactly the fiber  $T^c \cap \Pi^{-1}(x)$  and it is semialgebraic by the Tarski–Seidenberg theorem. The argument for  $T_0^c$  is identical.

Now we prove that every stratum of  $T^c$  has codimension at least  $d$ . Let  $\Pi_1 : J^1(M, \text{Sym}_n) \rightarrow M \times \text{Sym}_n$  be the canonical projection. Recall that the codimension of  $Q_{k,\nu}$  in  $\text{Sym}_n$  is  $s(\nu) := \dim \text{Sym}_\nu - 1$ .

We consider two cases. If  $\nu$  is such that  $d \leq s(\nu)$ , then  $\dim \text{Ran } L_{k,A} \leq d < \dim \text{Sym}_\nu$  and therefore  $(\text{Ran } L_{k,A})^\perp \neq 0$  for every  $L$ . We get  $\Pi_1^{-1}(M \times Q_{k,\nu}) = T^c$  and has codimension  $s(\nu) \geq d$ .

Assume now that  $\nu$  is such that  $d > s(\nu)$ . Then for an  $A \in Q_{k,\nu}$  the codimension of the top stratum of  $\Pi_1^{-1}(x, A) \cap T^c$  in  $\Pi_1^{-1}(x, A)$  is equal to the codimension of the subset of matrices of the rank  $\dim \text{Sym}_\nu - 1 = s(\nu)$  in the space of all  $(\dim \text{Sym}_\nu) \times d$  matrices, i.e. it is equal to<sup>16</sup>  $d - s(\nu)$ . Hence, the codimension in  $J^1(M, \text{Sym}_n)$  of the top stratum of  $T^c$  is at least  $d - s(\nu)$  plus  $s(\nu)$ , the codimension of  $\Pi_1^{-1}(M \times Q_{k,\nu})$  in  $J^1(M, \text{Sym}_n)$ .

To estimate the codimension of the strata of  $T_0^c$ , we note that on the top strata of  $T^c$ ,  $(\text{Ran } L_{k,A})^\perp$  must be one-dimensional. This implies that the codimension of the intersection of  $T_0^c$  with such strata is at least  $d + 1$ , while the codimension of intersections of  $T_0^c$  with the lower strata of  $T^c$  is automatically not less than  $d + 1$ .  $\square$

We continue the proof of Theorem 4.11. Let  $\mathcal{F} : M \rightarrow \text{Sym}_n$  be a transverse family in the sense of Definition 4.1 so that the graph  $\Gamma^1(\mathcal{F})$  of its 1-jet extension is

<sup>15</sup>For bundles whose fibers are stratified spaces, smoothness is defined in the usual way — as smoothness of trivializing maps. Smooth maps between stratified submanifolds are maps which are restrictions of smooth maps on the corresponding ambient manifolds, see [35, p. 13].

<sup>16</sup>Here we use that the codimension of the set of  $n_1 \times n_2$  matrices of rank  $r$  is equal to  $(n_1 - r)(n_2 - r)$ .

transverse to  $T^c$  and  $T_0^c$ . As we mentioned above the set of such maps is open and dense in the required topology. From the transversality of  $T_0^c$  to the  $d$ -dimensional  $\Gamma^1(\mathcal{F})$  we immediately get

$$\Gamma^1(\mathcal{F}) \cap T_0^c = \emptyset. \quad (4.12)$$

Choose an arbitrary point  $z$  and eigenvalue  $\lambda_k$  (of multiplicity  $\nu$ ). If the corresponding  $\text{Ran } \mathcal{H}_z$  contains a positive definite matrix, properties 2–3 hold trivially. We therefore focus on the opposite case:  $\text{Ran } \mathcal{H}_z \cap \text{Sym}_\nu^{++} = \emptyset$ . In the proof of Theorem 1.5 in Sect. 3 we saw that this means  $(\text{Ran } \mathcal{H}_z)^\perp$  contains a positive semidefinite matrix  $B$ . We want to show that  $B$  is actually positive definite.

Assume the contrary, namely  $\det B = 0$ ; we will work locally in  $J^1(M, \text{Sym}_n)$  around the point in the graph  $\Gamma^1(\mathcal{F})$ ,

$$Z = (z, A, L) := (z, \mathcal{F}(z), d\mathcal{F}(z)).$$

We first observe that  $\text{Ran } L_{k,A}$  defined in (4.11) coincides with  $\text{Ran } \mathcal{H}_z$  defined via (1.5). Since  $B \in (\text{Ran } \mathcal{H}_z)^\perp \setminus \{0\}$  and  $\det B = 0$ , we conclude that  $Z \in \Gamma^1(\mathcal{F}) \cap T_0^c$ , contradicting (4.12). Property 2 is now verified.

We now verify property 3. We have a positive definite  $B \in (\text{Ran } \mathcal{H}_z)^\perp$ , therefore, by Lemma 4.5,  $z$  is a smooth critical point along its constant multiplicity stratum  $S = S_z$ . Also from the existence of  $B$ , we have

$$Z \in \Gamma^1(\mathcal{F}) \cap T^c.$$

Denote by  $T_{k,v}^c$  the stratum of  $T^c$  containing the point  $Z$ . By definition of transversality to a stratified space,  $\Gamma^1(\mathcal{F})$  is transverse to  $T_{k,v}^c$  in  $J^1(M, \text{Sym}_n)$ . By dimension counting and transversality,  $(\text{Ran } L_{k,A})^\perp$  is 1-dimensional along  $T_{k,v}^c$ .

Define two submanifolds of  $J^1(M, \text{Sym}_n)$ ,

$$\begin{aligned} J_{k,v} &:= \left\{ (x, A, L) \in J^1(M, \text{Sym}_n) : A \in Q_{k,v}^n \right\}, \\ J_S &:= \left\{ (x, A, L) \in J^1(M, \text{Sym}_n) : x \in S, A \in Q_{k,v}^n, L(T_x S) \subset T_A Q_{k,v}^n \right\} \subset J_{k,v}. \end{aligned}$$

To see that  $J_S$  is a manifold, we note that for each fixed  $(x, A) \in S \times Q_{k,v}^n$ , the set of admissible  $L$  in  $J_S$  is a vector space smoothly depending on  $(x, A)$ . In other words,  $J_S$  is a smooth vector bundle over  $S \times Q_{k,v}^n$ .

We now use the following simple fact (twice): If  $U$ ,  $V$ , and  $W$  are submanifolds of  $M$  such that  $W$  is transverse to  $U$  in  $M$  and  $U \subset V$ , then  $W \cap V$  is transverse to  $U$  in  $V$ . Since  $T_{k,v}^c \subset J_{k,v}$ , we conclude that  $\Gamma^1(\mathcal{F}) \cap J_{k,v}$  is transverse to  $T_{k,v}^c$  in  $J_{k,v}$ . And now, since

$$\Gamma^1(\mathcal{F}) \cap J_{k,v} = \{ (x, \mathcal{F}(x), d\mathcal{F}(x)) : x \in S \} \subset J_S, \quad (4.13)$$

we conclude that  $T_{k,v}^c \cap J_S$  is transverse to  $\Gamma^1(\mathcal{F}) \cap J_{k,v}$  in  $J_S$ .

We have successfully localized our  $x$  to  $S$ . The space (4.13) looks similar to the graph of the 1-jet extension of  $\mathcal{F}|_S$ , except that the differential  $d\mathcal{F}(x)$  is defined on  $T_x M$  and not on  $T_x S$ . Consider the map  $\Psi : J_S \rightarrow J^1(S, \mathbb{R})$ ,

$$\Psi(x, A, L) := \left( x, \widehat{\lambda}_k(A), d(\widehat{\lambda}_k|_{Q_{k,v}^n})(A) \circ L|_{T_x S} \right),$$

which is well-defined and smooth because  $\widehat{\lambda}_k$  (defined in (1.1)) is smooth when restricted to  $Q_{k,v}^n$ ,  $d(\widehat{\lambda}_k|_{Q_{k,v}^n})(A) : T_A Q_{k,v}^n \rightarrow \mathbb{R}$  and  $L|_{T_x S} : T_x S \rightarrow T_A Q_{k,v}^n$  by definition of  $J_S$ .

We want to show that  $\Psi$  is a submersion and therefore preserves transversality. To prove submersivity of a map it is enough to prove that, for any point  $q$  in the domain, any smooth curve in the codomain of the map passing through the image of  $q$  is the image of a smooth curve in the domain passing through  $q$ .

Let  $(x_0, A_0, L_0)$  be an arbitrary point on  $J_S$ . We will work in a local chart around  $x_0 \in M$  in which  $S$  is a subspace. Let  $P$  denote the projection in  $T_x M$  onto  $T_x S$ , which now does not depend on the point  $x \in S$ . Consider a smooth curve  $(x_t, f_t, g_t)$  in  $J^1(S, \mathbb{R})$  such that  $\Psi(x_0, A_0, L_0) = (x_0, f_0, g_0)$ . Then the smooth curve

$$(x_t, A_0 + (f_t - f_0)I, L_0 + I(g_t - g_0)P),$$

is in  $J_S$  and is mapped to  $(x_t, f_t, g_t)$  by  $\Psi$ . To see this, observe that all sets  $Q_{k,v}^n$  are invariant under the addition of a multiple of the identity matrix and also that  $\widehat{\lambda}_k(A + \mu I) = \widehat{\lambda}_k(A) + \mu$  and therefore  $d(\widehat{\lambda}_k|_{Q_{k,v}^n})(A)I = 1$ .

We now have that  $\Psi(T_{k,v}^c \cap J_S)$  is transverse to  $\Psi(\Gamma^1(\mathcal{F}) \cap J_{k,v})$  in  $J^1(S, \mathbb{R})$ . It is immediate that

$$\Psi(\Gamma^1(\mathcal{F}) \cap J_{k,v}) = \Gamma^1(\widehat{\lambda}_k \circ \mathcal{F}|_S).$$

We now argue that

$$\Psi(T_{k,v}^c \cap J_S) = \{(x, \widehat{\lambda}_k(A), 0) : x \in S, A \in Q_{k,v}^n\}. \quad (4.14)$$

Indeed, at  $Z$ , the space  $(\text{Ran } L_{k,A})^\perp$  is spanned by a positive definite matrix and this property holds in a small neighborhood of  $Z$  in  $T_{k,v}^c$ . By Lemma 4.5,  $L_{k,A}|_{T_x S} = 0$ , while by Hellmann-Feynman theorem,

$$d(\widehat{\lambda}_k|_{Q_{k,v}^n})(A) \circ L|_{T_x S} = \frac{1}{v} \text{Tr}(\mathcal{U}_{k,A}^* L|_{T_x S} \mathcal{U}_{k,A}) = \frac{1}{v} \text{Tr}(L_{k,A}|_{T_x S}) = 0.$$

Finally, it is well known that the transversality of the graph  $\Gamma^1(\widehat{\lambda}_k \circ \mathcal{F}|_S)$  to the 0-section space (4.14) is equivalent to the non-degeneracy of the critical point  $z$  of  $\widehat{\lambda}_k \circ \mathcal{F}|_S$ , see [43, Sect. 6.1] or [9, Lem 5.23]. Property 3 is now established.  $\square$

## 5 Topological change in the sublevel sets: part (1) of Theorem 1.14

In this section we describe the change in the sublevel sets of the eigenvalue  $\lambda_k$  when passing through a non-degenerate topologically critical point  $x$ . It will be expressed

in terms of the data introduced in Theorem 1.12, namely the Morse index  $\mu(x)$  of  $\lambda_k$  restricted to the local constant multiplicity stratum  $S$  attached to the point  $x$ , the multiplicity  $\nu$ , and the relative index  $i = i(x)$  introduced in equation (1.8). As a result of the section, we will establish part (1) of Theorem 1.14.

First we reduce our considerations to the directions transverse to the constant multiplicity stratum  $S$  at a critical point  $x$ .

**Lemma 5.1** *Let  $N$  be a submanifold of  $M$  of dimension  $\dim N = \operatorname{codim}_M S = s(\nu)$  which intersects  $S$  transversely at  $x$ . Then, for small enough  $U$  and  $\varepsilon > 0$ ,*

$$H_r\left(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k)\right) \cong H_{r-\mu(x)}\left(U_x^{+\varepsilon}(\lambda_k|_N), U_x^{-\varepsilon}(\lambda_k|_N)\right).$$

**Proof** By definition, see [35, Sec I.3.5], the *local Morse data* is

$$\left(U_x^{-\varepsilon, +\varepsilon}(\lambda_k), \partial U_x^{-\varepsilon}(\lambda_k)\right),$$

where

$$U_x^{-\varepsilon, +\varepsilon} := U_x^{+\varepsilon} \setminus U_x^{-\varepsilon}. \quad (5.1)$$

The *normal data* and the *tangential data* is simply the data of  $\lambda_k$  restricted to the submanifolds  $N$  and  $S$ , respectively, see [35, Sec I.3.6]. The normal data is

$$(J, K) \cong \left(U_x^{-\varepsilon, +\varepsilon}(\lambda_k|_N), \partial U_x^{-\varepsilon}(\lambda_k|_N)\right), \quad (5.2)$$

and, by the local version of the main theorem of the classical Morse theory [51, Theorem 3.2], the tangential data is

$$(P, Q) \cong \left(\mathbb{B}^{\mu(x)}, \partial \mathbb{B}^{\mu(x)}\right),$$

where  $\mathbb{B}^{\mu(x)}$  denotes the  $\mu(x)$ -dimensional ball.

We already established in Corollary 4.9 that  $(x, \lambda_k(x))$  is a nondepraved point of the corresponding map. We can now use [35, Thm I.3.7] to decompose the local Morse data into a product of tangential and normal data. More precisely, if the tangential data is  $(P, Q)$  and the normal data is  $(J, K)$ , the local Morse data is homotopy equivalent to  $(P \times J, (P \times K) \cup (Q \times J))$ .

We want to compute

$$\begin{aligned} H_r\left(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k)\right) &\cong H_r\left(U_x^{-\varepsilon, +\varepsilon}(\lambda_k), \partial U_x^{-\varepsilon}(\lambda_k)\right) \\ &\cong H_r\left(P \times J, (P \times K) \cup (Q \times J)\right), \end{aligned} \quad (5.3)$$

the first equality being by Excision Theorem and the second by [35, Thm I.3.7] (and homotopy invariance). By the relative version of the Künneth theorem, see [23,

Proposition 12.10], we have the following short exact sequence

$$\begin{aligned} 0 \rightarrow \bigoplus_{j+k=r} H_j(P, Q) \otimes H_k(J, K) &\rightarrow H_r(P \times J, (P \times K) \cup (Q \times J)) \\ &\rightarrow \bigoplus_{j+k=r-1} \text{Tor}_1(H_j(P, Q), H_k(J, K)) \rightarrow 0. \end{aligned} \quad (5.4)$$

Since

$$H_j(P, Q) = H_j(\mathbb{B}^{\mu(x)}, \partial \mathbb{B}^{\mu(x)}) = \tilde{H}_j(\mathbb{S}^{\mu(x)}) = \begin{cases} 0, & j \neq \mu(x), \\ \mathbb{Z}, & j = \mu(x), \end{cases} \quad (5.5)$$

where  $\tilde{H}_*$  stands for the reduced homology, are free, the torsion product terms in (5.4) are all 0. We therefore get

$$\begin{aligned} H_r(P \times J, (P \times K) \cup (Q \times J)) &\cong \bigoplus_{j+k=r} H_j(P, Q) \otimes H_k(J, K) \\ &= \mathbb{Z} \otimes H_{r-\mu(x)}(J, K) = H_{r-\mu(x)}(J, K), \end{aligned} \quad (5.6)$$

where we used (5.5) again. Combining (5.3) with (5.6) and (5.2), we obtain the result.  $\square$

The preceding lemma tells us that we can restrict our attention to the case  $M = N$ . In this case the constant multiplicity stratum attached to  $x$  is the isolated point itself and  $\dim M = \dim N = s(v)$  (see equation (4.2) for the formula defining  $s(v)$  and some explanations). Since the considerations are purely local, we can assume that  $M = \mathbb{R}^{s(v)}$ ,  $x = 0$ , and  $\mathcal{F}(x) = 0$ .

**Lemma 5.2** *Let  $\mathcal{F} : \mathbb{R}^{s(v)} \rightarrow \text{Sym}_v(\mathbb{F})$ ,  $\mathcal{F}(0) = 0$ , be a smooth family satisfying at  $x = 0$  non-degenerate criticality conditions (N) and (S). Then there exists a neighborhood  $U$  of 0 in  $\mathbb{R}^{s(v)}$ , such that for sufficiently small  $\varepsilon > 0$  the sublevel set  $U_0^{+\varepsilon}(\lambda_k)$  deformation retracts to the set  $D_{k,\varepsilon} \cup U_0^{-\varepsilon}(\lambda_k)$ , where*

$$D_{k,\varepsilon} := \left\{ x \in U : -\varepsilon \leq \lambda_1(\mathcal{F}(x)) = \dots = \lambda_k(\mathcal{F}(x)) \leq 0 \right\}. \quad (5.7)$$

**Remark 5.3** It is instructive to consider what happens in the boundary cases  $k = 1$  and  $k = v$ . We will see that condition (N) implies that  $\mathcal{F}$  is injective and  $\mathcal{F}(U)$  does not contain any semidefinite matrices except 0 (for a suitably small  $U$ ). Therefore, when  $k = 1$ ,

$$U_0^\varepsilon(\lambda_1) = U = D_{1,\varepsilon} \cup U_0^{-\varepsilon}(\lambda_1)$$

and no retraction is needed.

Similarly, when  $k = v$ , we have  $U_0^{-\varepsilon}(\lambda_v) = \emptyset$  and

$$D_{v,\varepsilon} = \{x \in U : \mathcal{F}(x) = 0\} = 0,$$

and the Lemma reduces to the claim that  $U_0^\varepsilon(\lambda_\nu)$  deformation retracts to a point. Furthermore, the set defined in (5.1) is  $U_0^{-\varepsilon, +\varepsilon}(\lambda_\nu) = U_0^\varepsilon(\lambda_\nu)$ . In the proof of Lemma 5.2, we will see that  $U_0^\varepsilon(\lambda_\nu)$  is diffeomorphic to the intersection of a smooth  $s(\nu)$ -dimensional manifold through 0 with a small ball around 0. Therefore, we obtain that the normal Morse data in (5.2) is homeomorphic to the pair

$$(J, K) = (\mathbb{B}^{s(\nu)}, \emptyset).$$

**Proof of Lemma 5.2** Since  $\mathcal{F}(0) = 0 \in \text{Sym}_\nu$ , the eigenspace  $\mathbf{E}_k$  in (1.5) is the whole space  $\mathbb{R}^\nu$  and  $\mathcal{H}_0 = d\mathcal{F}(0)$ . From condition (N) we get that  $\text{Ran } d\mathcal{F}(0) = \text{span}\{B\}^\perp$  with  $B \in \text{Sym}_\nu^{++}$ . By the definition of  $s(\nu)$  and dimension counting we conclude that  $d\mathcal{F}(0): \mathbb{R}^{s(\nu)} \rightarrow \text{Sym}_\nu$  is injective (since  $\mathcal{F}$  is a map between vector spaces, we can consider its differential to be a map between the same vector spaces).

Choose a neighborhood  $W$  of 0 such that  $d\mathcal{F}(x)$  remains close to  $d\mathcal{F}(0)$  for all  $x \in W$  (and, in particular, injective) and the suitably scaled normal to  $\text{Ran } d\mathcal{F}(x)$  remains close to  $B$  (and, in particular, positive definite). For future reference we note that, under these smallness conditions,  $\mathcal{F}$  is a diffeomorphism from  $W$  to  $\mathcal{F}(W)$  and the latter set contains no positive or negative semidefinite matrices except 0.

Denote by  $\mathfrak{B}_\delta$  the open ball in  $\text{Sym}_\nu$  of radius  $\delta$  around the origin in the *operator norm*. Choose  $\delta$  sufficiently small so that  $\partial\mathfrak{B}_\delta \cap \mathcal{F}(\partial W) = \emptyset$ . This is possible because  $d\mathcal{F}(0)$  is injective and the operator norm on  $\mathcal{F}(\partial W)$  is bounded from below. Now we take  $U = \mathcal{F}^{-1}(\mathfrak{B}_\delta \cap \mathcal{F}(W))$ . This set is non-empty because it contains 0; it has the useful property that the operator norm (equivalently, spectral radius) of  $\mathcal{F}(x)$  is equal to  $\delta$  for  $x \in \partial U$  and is strictly smaller than  $\delta$  on  $U$ .

Given a matrix  $F_0 \in \mathcal{F}(U_0^\varepsilon(\lambda_k))$  we will describe the retraction trajectory  $\Gamma_{F_0}(t)$ ,  $t \in [0, 1]$ , starting at  $F_0$ . The trajectory will be piecewise smooth, with the pieces described recursively. Define, for  $m \leq k$ ,

$$G_m^k := \{F \in \text{Sym}_\nu : \dots < \lambda_m(F) = \dots = \lambda_k(F) \leq \dots\},$$

which is the set of matrices with a gap below the eigenvalue  $\lambda_m(F)$  but no gap between  $\lambda_m(F)$  and  $\lambda_k(F)$ . It is easy to see that

$$\overline{G_m^k} \setminus G_m^k = \bigcup_{1 \leq m' < m} G_{m'}^k, \quad (5.8)$$

which we will call the *egress set* of  $G_m^k$ .

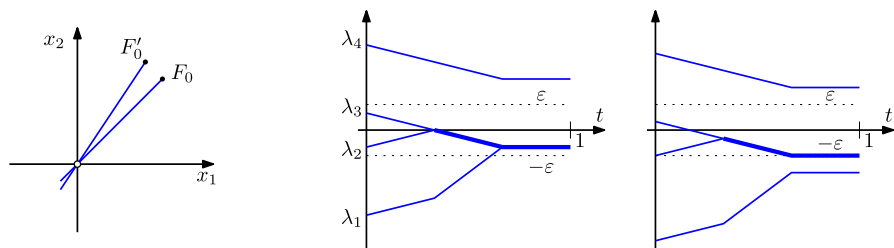
Assume that  $\lambda_k(F_0) > -\varepsilon$  and that  $F_0 \in G_{m_0}^k$  for some  $m_0 > 1$ ; let  $t_0 = 0$ . Define two complementary spectral (Riesz) projectors corresponding to  $F_0$ ,

$$P_- := P(\{\lambda < \lambda_k(F_0)\}), \quad P_+ := P(\{\lambda \geq \lambda_k(F_0)\}),$$

and consider the affine plane in  $\text{Sym}_\nu$  defined by

$$\{F_0 - 2\varepsilon s P_+ + r P_- : s, r \in \mathbb{R}\}. \quad (5.9)$$

Since the projector  $P_+ \in \text{Sym}_\nu^+$  is non-zero and the normal to  $d\mathcal{F}(x)$  is positive definite locally around  $x = 0$ , this affine plane is transverse to  $\mathcal{F}(U)$  in  $\text{Sym}_\nu$ . Their



**Fig. 5** Left: the curves  $\gamma_{F_0}^2(t)$  for  $k = 2$  and  $\mathcal{F}_1$  from equation (2.1), for a pair of initial points  $F_0$ . The curves are shown in the 2-dimensional plane  $\mathcal{F}_1(U)$ . The egress set for  $G_2^2$  is the point  $(0, 0)$ . Note that the curves intersect on the egress set, which is the reason we chose to specify the flow rather than the vector field. Middle and right: evolution of the eigenvalues of  $\Gamma_{F_0}(t)$  for a pair of  $F_0$  with  $k = 3$  and the family  $\mathcal{F}(U) = \{F \in \text{Sym}_4 : \text{Tr}(F) = 0\}$ . Egress points correspond to points where  $\lambda_k$  increases its multiplicity (the latter is shown with thicker lines)

intersection is nonempty because it contains  $F_0$  and thus, by the Implicit Function Theorem, it is a 1-dimensional embedded submanifold of  $\mathcal{F}(U)$ . Denote by  $\gamma_{F_0}^{m_0}$  the connected component of the intersection that contains  $F_0$ .

Furthermore, implicit differentiation of the equation  $F_0 - 2\varepsilon s P_+ + r P_- = \mathcal{F}(x)$  at a point  $\mathcal{F}(x) \in \gamma_{F_0}^{m_0}$  shows that

$$\frac{dr}{ds} = 2\varepsilon \frac{\langle B_x, P_+ \rangle}{\langle B_x, P_- \rangle}, \quad r(0) = 0, \quad (5.10)$$

where  $B_x$  is either positive or negative definite symmetric matrix that spans the orthogonal complement to the differential  $d\mathcal{F}(x)$ . Since  $\langle B_x, P_- \rangle > 0$  for any  $x \in U$ , the set of points of  $\gamma_{F_0}^{m_0}$  where  $\gamma_{F_0}^{m_0}$  can be locally represented as a function of  $s$  is both open and closed in the subspace topology of  $\gamma_{F_0}^{m_0}$ . We conclude that  $\gamma_{F_0}^{m_0}$  can be represented by a function of  $s$  globally, i.e. as long as the closure of  $\gamma_{F_0}^{m_0}$  in  $\text{Sym}_v$  does not hit the boundary of  $\mathcal{F}(U)$ . In a slight abuse of notation, we will refer to this function as  $\gamma_{F_0}^{m_0}$ . Figure 5(left) shows examples of the curves  $\gamma_{F_0}^1(s)$  for the family  $\mathcal{F}_1$  from equation (2.1) and two different initial points  $F_0$ .

The matrices on the curve  $\gamma_{F_0}^{m_0}(s)$  have fixed eigenspaces but their eigenvalues change with  $s$ . For small positive  $s$  the eigenvalues  $\lambda_{m_0} = \lambda_k$  and above decrease with the constant speed  $2\varepsilon$  while the eigenvalues below  $\lambda_{m_0}$  increase because the derivative in (5.10) is positive. This closes the gap below the eigenvalue  $\lambda_{m_0}$  and decreases the spectral radius (operator norm) of  $\gamma_{F_0}^m(s)$ . Therefore, the curve will intersect the egress set (5.8) at some time  $\hat{s} > 0$  before it reaches the boundary  $\mathcal{F}(\partial U)$ .

Setting  $F_1 := \gamma_{F_0}^{m_0}(\hat{s})$  and  $t_1 = t_0 + \hat{s}$ , we determine  $m_1 < m_0$  such that  $F_1 \in G_{m_1}^k$  and repeat the process starting at  $(t_1, F_1)$ . We then join the pieces together,

$$\Gamma_{F_0}(t) = \gamma_{F_j}^{m_j}(t - t_j), \quad t_j \leq t \leq t_{j+1}.$$

There are two ways in which we will terminate this recursive process. If an egress point  $F_n \in G_1^k$  is reached (which has no eigenvalues strictly smaller than  $\lambda_k(F_n)$  and equation (5.10) becomes undefined due to  $P_- = 0$ ), we continue  $\Gamma_{F_0}$  as a constant,

$\Gamma_{F_0}(t) = F_n$  for  $t \geq t_n$ . An example of this is shown in Fig. 5(middle). The case  $m_0 = 1$  which we previously excluded can now be absorbed into this rule.

Alternatively, since  $\lambda_k$  decreases from an initial value below  $\varepsilon$  at the constant rate  $2\varepsilon$ , we will reach a point in  $U_0^{-\varepsilon}(\lambda_k)$  at some  $\tilde{t} \leq 1$ . In this case we also continue  $F(t)$  as a constant for  $t > \tilde{t}$ , see Fig. 5(right) for an example (with  $\tilde{t} = 2/3$  in this particular case). The case  $\lambda_k(F_0) \leq -\varepsilon$  can now be absorbed into the above description by setting  $\tilde{t} = 0$ .

The preceding paragraphs show that the final values  $\mathcal{F}^{-1}(\Gamma_{F_0}(1))$  belong to the set  $D_{k,\varepsilon} \cup U_0^{-\varepsilon}(\lambda_k)$ , see equation (5.7), and that  $x \mapsto \Gamma_{\mathcal{F}(x)}(t)$  acts as identity on  $D_{k,\varepsilon} \cup U_0^{-\varepsilon}(\lambda_k)$  for all  $t$ . This suggests that we have a deformation retraction

$$(x, t) \mapsto \mathcal{F}^{-1}(\Gamma_{\mathcal{F}(x)}(t)),$$

if we establish that the trajectories  $\Gamma_F(t)$  define a continuous mapping  $\mathcal{F}(U) \times [0, 1] \rightarrow \mathcal{F}(U)$ .

We first note that each trajectory is continuous in  $t$  by construction. Therefore, we need to show that starting at a point  $F'$  which is near  $F$  will result in  $\Gamma_{F'}(t)$  being near  $\Gamma_F(t)$ . A perturbation of arbitrarily small norm may split multiple eigenvalues, therefore if  $F \in G_m^k$  with  $m < k$ , then, in general,  $F' \in G_{m'}^k$  with  $m \leq m'$  (in fact, generically,  $m' = k$ ). However,

$$\begin{aligned} |\lambda_m(F') - \lambda_k(F')| &= |\lambda_m(F') - \lambda_m(F)| + |\lambda_k(F) - \lambda_k(F')| \\ &\leq C|F - F'|, \end{aligned}$$

with some  $F$ -independent<sup>17</sup> constant  $C$ , and therefore after a time of order  $C|F - F'|/2\varepsilon$ , the  $k$ -th eigenvalue  $\Gamma_{F'}$  will collide with  $m$ -th eigenvalue. To put it more precisely, there is  $\tau$ ,  $0 < \tau \leq C|F - F'|/2\varepsilon$ , such that  $\Gamma_{F'}(\tau) \in G_m^k$ . By choosing  $|F - F'|$  to be sufficiently small (while  $\varepsilon$  is small but fixed), we ensure that  $\Gamma_F(\tau)$  is still in  $G_m^k$ . By noting that the trajectories  $\Gamma_{F'}(t)$  are continuous in  $t$  uniformly with respect to  $F'$ , we conclude that  $\Gamma_{F'}(\tau)$  is close to  $\Gamma_F(\tau)$ .

For two initial points  $F$  and  $F'$  in the same set  $G_m^k$ , the curves  $\gamma_F^m(s)$  and  $\gamma_{F'}^m(s)$  will remain nearby for any bounded time  $s < 1$ . This can be seen, for example, as stability of the transverse intersection of the manifold  $\mathcal{F}(U)$  and the manifold (5.9). The stability is with respect to the parameters  $F$ ,  $P_+$  and  $P_-$  and the spectral projections are continuous in  $F$  precisely because  $F'$  belongs the same set  $G_m^k$ .

We now chain the two argument in the alternating fashion: short time to bring two points to the same set  $G_m^k$ , long time along smooth trajectories until one of the trajectories reaches an egress point, then short time to bring them to the same set  $G_{m_1}^k$  and so on. Since we iterate a bounded number of times, the composition is a continuous mapping.  $\square$

As before (see the paragraph before the formulation of Theorem 1.14), let  $\mathcal{C}Y$  and  $\mathcal{S}Y$  be the cone and the suspension of a topological space  $Y$ . Also let  $\Sigma Y =$

<sup>17</sup>The constant is independent of  $F$  but may depend on the norm used for  $F$ ; in case of the operator norm, Weyl inequality yields  $C = 2$ .

$\Sigma Y / (\{y_0\} \times I)$  be the *reduced suspension* of  $Y$ , where  $y_0 \in Y$ . Note that if  $Y$  is a CW-complex, then  $\Sigma Y$  is homotopy equivalent to  $\mathcal{S}Y$ . In Lemma 5.2 we saw that  $U_0^\varepsilon(\lambda_k)$  is homotopy equivalent to the union of  $U_0^{-\varepsilon}(\lambda_k)$  and the space  $D_{k,\varepsilon}$  which we aim to understand further. We will now show that  $D_{k,\varepsilon}$  is a cone of the space  $\mathcal{R}_v^i$  introduced in equation (1.12).

In the present setting (namely,  $\mathcal{F}(0) = 0 \in \text{Sym}_v(\mathbb{F})$ ), the relative index  $i$  is related to  $k$  via  $i = v - k + 1$ , cf. (1.8). Notationally, it will be more convenient to use  $k$  instead of  $i$ , so we introduce a slight change in the notation,

$$\mathcal{R}_v^i = \mathcal{R}_{k,v} := \{R \in \text{Sym}_v^+ : \text{Tr } R = 1, \dim \text{Ker } R \geq k\}. \quad (5.11)$$

**Lemma 5.4** *Let  $\mathcal{F}$ ,  $U$  and  $D_{k,\varepsilon}$  be as in Lemma 5.2. Then, for sufficiently small  $\varepsilon > 0$ , the topological space  $D_{k,\varepsilon}$  is homeomorphic to  $\mathcal{C}\mathcal{R}_{k,v}$  and the topological space*

$$(U_0^{-\varepsilon}(\lambda_k) \cup D_{k,\varepsilon}) / U_0^{-\varepsilon}(\lambda_k), \quad 1 \leq k < v, \quad (5.12)$$

*is homeomorphic to  $\mathcal{S}\mathcal{R}_{k,v}$ .*

**Proof** The choice of  $U$  ensured that  $\mathcal{F}$  is a homeomorphism from  $D_{k,\varepsilon} \subset U$  to

$$\mathcal{F}(D_{k,\varepsilon}) := \{F \in \mathcal{F}(U) : -\varepsilon \leq \lambda_1(F) = \dots = \lambda_k(F) \leq 0\}.$$

We will now describe the homeomorphism from  $\mathcal{C}\mathcal{R}_{k,v}$  to  $\mathcal{F}(D_{k,\varepsilon})$ .

Given a point  $R \in \mathcal{R}_{k,v}$ , consider the intersection of  $\mathcal{F}(U)$  with the plane

$$\{-\varepsilon t I + r R : t, r \in \mathbb{R}\}. \quad (5.13)$$

Mimicking the proof of Lemma 5.2, we conclude that the intersection is a 1-dimensional submanifold which has a connected component  $\phi_R$  containing the matrix 0. Moreover, implicit differentiation at  $\mathcal{F}(x) \in \phi_R$  yields

$$\frac{dr}{dt} = \varepsilon \frac{\langle B_x, I \rangle}{\langle B_x, R \rangle}, \quad (5.14)$$

therefore the submanifold can be represented by a function of  $t$ ,

$$\Phi(R, t) = -\varepsilon t I + r(t) R, \quad r(0) = 0.$$

When  $t \in [0, 1]$ , we also have  $\Phi(R, t) \in \mathcal{F}(D_{k,\varepsilon})$  because equation (5.14) implies  $r(t) \geq 0$ . We remark that  $\langle B_x, R \rangle$  is bounded away from zero uniformly in  $x \in U$  and  $R \in \mathcal{R}_{k,v}$ , therefore, when  $\varepsilon$  is sufficiently small,  $\Phi(R, t)$  will remain in  $\mathcal{F}(U)$  until  $t > 1$ . Thus the function  $\Phi$  is a well-defined<sup>18</sup> mapping from  $\mathcal{C}\mathcal{R}_{k,v}$  to  $\mathcal{F}(D_{k,\varepsilon})$ . It is evidently continuous.

The properties of  $\mathcal{F}$  imply that  $\mathcal{F}(U)$  contains no multiples of identity and no positive semidefinite matrices except for the zero matrix. Therefore, for every  $F \in \mathcal{F}(D_{k,\varepsilon})$ ,  $F \neq 0$ ,

$$R_F := \frac{F - \lambda_1(F)I}{\text{Tr}(F - \lambda_1(F)I)} \in \mathcal{R}_{k,v}, \quad (5.15)$$

<sup>18</sup>Namely,  $\Phi(R, 0) = 0$  for all  $R$ .

is well-defined, and we also have  $-\varepsilon \leq \lambda_1(F) < 0$ . Thus

$$\Phi': F \mapsto \begin{cases} \left(R_F, -\frac{\lambda_1(F)}{\varepsilon}\right), & \text{if } F \neq 0, \\ (*, 0), & \text{if } F = 0, \end{cases}$$

is a well-defined continuous mapping from  $\mathcal{F}(D_{k,\varepsilon})$  to  $\mathcal{CR}_{k,v}$ . It remains to verify that  $\Phi'$  is the inverse of  $\Phi$ . It is immediate that  $\Phi' \circ \Phi = \text{id}$ . To prove that  $\Phi \circ \Phi' = \text{id}$  we observe that the intersection  $\phi_{R_F}$ , corresponding to  $R_F$  of equation (5.15), contains  $F$ ; we only need to show that  $F$  and 0 belong to the same connected component of  $\phi_R$ .

The point  $F$  on the plane (5.13) corresponds to  $t = -\lambda_1(F)/\varepsilon > 0$  and some  $r = r'$ . Decreasing  $t$  from this point decreases  $r(t)$  and therefore decreases the operator norm of  $\Phi$ . Thus we will not hit the boundary of  $\mathcal{F}(D_{k,\varepsilon})$  as long as  $t \geq 0$ . Therefore, we will arrive at the matrix 0 while staying on the same connected component.

We have established the first part of the lemma. To understand the quotient in (5.12), we note that

$$\begin{aligned} & (U_0^{-\varepsilon}(\lambda_k) \cup D_{k,\varepsilon}) / U_0^{-\varepsilon}(\lambda_k) \\ &= D_{k,\varepsilon} / (U_0^{-\varepsilon}(\lambda_k) \cap D_{k,\varepsilon}) \\ &\cong \mathcal{F}(D_{k,\varepsilon}) / \{F \in \mathcal{F}(D_{k,\varepsilon}) : -\varepsilon = \lambda_1(F) = \dots = \lambda_k(F)\} \\ &= \mathcal{F}(D_{k,\varepsilon}) / \{\Phi(R, 1) : R \in \mathcal{R}_{k,v}\} \cong \mathcal{SR}_{k,v}, \end{aligned}$$

completing the proof.  $\square$

**Proof of Theorem 1.14, part (1)** We review how the preceding lemmas link together to give the proof of the theorem. Lemma 5.1 shows that the smooth part  $\mathcal{F}|_S$  gives the classical contribution to the sublevel set quotient and we can focus on understanding the transverse part  $\mathcal{F}|_N$ . We remark that by Corollary 4.6 the point  $x$  remains non-degenerate topologically critical when we replace  $\mathcal{F}$  with  $\mathcal{F}|_N$ .

Combining Lemmas 5.1, 5.2, and 5.4, we compute the  $r$ -th homology group

$$\begin{aligned} H_r(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k)) &\cong H_{r-\mu(x)}(U_x^{+\varepsilon}(\lambda_k|_N), U_x^{-\varepsilon}(\lambda_k|_N)) \\ &\cong H_{r-\mu(x)}(U_x^{-\varepsilon}(\lambda_k|_N) \cup D_{k,\varepsilon}, U_x^{-\varepsilon}(\lambda_k|_N)) \\ &\cong \tilde{H}_{r-\mu(x)}((U_x^{-\varepsilon}(\lambda_k|_N) \cup D_{k,\varepsilon}) / U_x^{-\varepsilon}(\lambda_k|_N)) \\ &\cong \tilde{H}_{r-\mu(x)}(\mathcal{SR}_{k,v}). \end{aligned}$$

Taking into account (5.11), we obtain the claim for  $k < v$ . The answer for  $k = v$  (equivalently,  $i = 1$ ) was already established in Remark 5.3.  $\square$

## 6 Topological change in the sublevel sets: part (2) of Theorem 1.14

We will go from part (1) to part (2) of Theorem 1.14 by relating the space  $\mathcal{R}_{k,v}$  to the Thom space of a particular vector bundle. Recall [52] that the *Thom space*  $\mathcal{T}(E)$  of a real vector bundle  $E$  over a manifold is the quotient of the unit ball bundle  $\mathcal{B}(E)$  of  $E$  by the unit sphere bundle of  $E$  with respect to some Euclidean metric on  $E$ . If the base manifold of the bundle  $E$  is compact, then the Thom space of  $E$  is the Alexandroff (one point) compactification of the total space of  $E$ . As before, we denote by  $\text{Gr}_{\mathbb{F}}(k, n)$  the Grassmannian of (non-oriented)  $k$ -dimensional subspaces in  $\mathbb{F}^n$ .

Given a vector bundle  $E$  denote by  $S^2E$  the *symmetric tensor product* of  $E$ . Namely,  $S^2E$  is the vector bundle over the same base as  $E$ ; the fiber of  $S^2E$  over a point is equal to the symmetric tensor product with itself of the fiber of  $E$  over the same point. Choosing a Euclidean metric on  $E$  we can identify  $S^2E$  with the bundle whose fiber over a point is the space of all self-adjoint isomorphisms of the fiber of  $E$  over the same point. Then by  $S_0^2E$  we denote the bundle of traceless elements of  $S^2E$ . Obviously

$$S^2E \cong S_0^2E \oplus \theta^1, \quad (6.1)$$

where  $\theta^1$  is the trivial rank 1 bundle over the base of  $E$ . Finally, let  $\text{Taut}_{\mathbb{F}}(k, n)$  be the tautological bundle over the Grassmannian  $\text{Gr}_{\mathbb{F}}(k, n)$ : the fiber of this bundle over  $\Lambda \in \text{Gr}_{\mathbb{F}}(k, n)$  is the vector space  $\Lambda$  itself.

**Lemma 6.1** *Recall the relative index  $i$  of the eigenvalue, equation (1.8), which in the present situation is equal to  $i = v - k + 1$ . Then the space*

$$\mathcal{R}_{k,v} := \{R \in \text{Sym}_v^+ : \text{Tr } R = 1, \dim \text{Ker } R \geq k\}$$

*with  $1 \leq k < v$  is homotopy equivalent to the Thom space of the real vector bundle over the Grassmannian  $\text{Gr}_{\mathbb{F}}(i - 1, v - 1)$ ,*

$$E_{i,v} := S_0^2 \text{Taut}_{\mathbb{F}}(i - 1, v - 1) \oplus \text{Taut}_{\mathbb{F}}(i - 1, v - 1). \quad (6.2)$$

*The rank of the bundle is  $s(i) - 1$ , where  $s(i)$  is given by (1.9). The bundle is non-orientable if  $\mathbb{F} = \mathbb{R}$  and  $i$  is even, and orientable otherwise.*

**Remark 6.2** Let us consider the boundary case  $k = 1$  or, equivalently,  $i = v$ . The Grassmannian  $\text{Gr}_{\mathbb{F}}(i - 1, v - 1)$  is a single point, so the vector bundle  $E_{v,v}$  is simply a real vector space of dimension  $s(v) - 1$ . Its Thom space is the one-point compactification of  $\mathbb{R}^{s(v)-1}$ , namely the sphere  $\mathbb{S}^{s(v)-1}$ . Correspondingly, the cone  $\mathcal{CR}_{1,v}$  is homotopy equivalent to the ball  $\mathbb{B}^{s(v)}$ . Therefore, we get that the normal data at the bottom eigenvalue is homotopy equivalent to the pair

$$(J, K) = (\mathbb{B}^{s(v)}, \partial \mathbb{B}^{s(v)}).$$

**Proof of Lemma 6.1** The homotopy equivalence has been established in [1, Theorem 1] and the proof thereof. For completeness we review the main steps here.

Fixing an arbitrary unit vector  $e \in \mathbb{F}^v$  we define

$$\mathcal{P}_{k,v} := \left\{ \frac{1}{v-k} P \in \text{Sym}_v : P^2 = P, \dim \text{Ker } P = k, e \in \text{Ker } P \right\} \\ \cong \text{Gr}_{\mathbb{F}}(v-k, v-1).$$

One can show that  $\mathcal{R}_{k,v} \setminus \mathcal{P}_{k,v}$  is contractible: if  $P_e = ee^*$  is the projection onto  $e$ , consider

$$(A, t) \mapsto \phi_k((1-t)A + tP_e), \quad A \in \mathcal{R}_{k,v} \setminus \mathcal{P}_{k,v}, \quad t \in [0, 1] \quad (6.3)$$

where  $\phi_k(M)$  acts on the eigenvalues of  $M$  as

$$\lambda_j(M) \mapsto \max[0, \lambda_j(M) - \lambda_k(M)], \quad (6.4)$$

followed by a normalization to get unit trace. Using interlacing inequalities<sup>19</sup> for the rank one perturbation (up to rescaling) of  $A$  by  $P$ , one can show that (6.3) is a well-defined retraction. In particular, (6.4) does not produce a zero matrix (which cannot be trace-normalized) and the result of (6.3) is not in  $\mathcal{P}_{k,v}$  for any  $t$ .

We now obtain that  $\mathcal{R}_{k,v}$  is homotopy equivalent to the Thom space of the normal bundle of  $\mathcal{P}_{k,v}$  in  $\mathcal{R}_{k,v}$ . Indeed, a tubular neighborhood  $T$  of  $\mathcal{P}_{k,v}$  in  $\mathcal{R}_{k,v}$  is diffeomorphic to the normal bundle of  $\mathcal{P}_{k,v}$ , while the above retraction allows one to show  $\mathcal{R}_{k,v}$  is homotopy equivalent to  $\mathcal{R}_{k,v} / (\mathcal{R}_{k,v} \setminus T)$ .

The normal bundle of  $\mathcal{P}_{k,v}$  in  $\mathcal{R}_{k,v}$  is a Whitney sum of the normal bundle of  $\mathcal{P}_{k,v}$  in

$$\widehat{\mathcal{P}}_{k,v} := \left\{ \frac{1}{v-k} P \in \text{Sym}_v : P^2 = P, \dim \text{Ker } P = k \right\},$$

and the normal bundle of  $\widehat{\mathcal{P}}_{k,v}$  in  $\mathcal{R}_{k,v}$ . The fiber in the former bundle is  $(\text{Ker } P)^\perp$ : it consists of the directions in which  $e$  can rotate out of  $\text{Ker } P$ . Therefore the former bundle is  $\text{Taut}_{\mathbb{F}}(v-k, v-1)$ . The fiber in the normal bundle of  $\widehat{\mathcal{P}}_{k,v}$  in  $\mathcal{R}_{k,v}$  consists of all self-adjoint perturbations to the operator  $\frac{1}{v-k} P$  that preserve its kernel and unit trace. Identifying these with the space of traceless self-adjoint operators on  $(\text{Ker } P)^\perp$ , we get  $S_0^2 \text{Taut}_{\mathbb{F}}(v-k, v-1)$ . We get (6.2) after recalling that  $v-k = i-1$ .

To calculate the rank we use

$$\text{rank}(\text{Taut}_{\mathbb{F}}(i-1, v-1)) = \begin{cases} i-1, & \mathbb{F} = \mathbb{R}, \\ 2(i-1) & \mathbb{F} = \mathbb{C}, \end{cases}$$

and

$$\text{rank}(S_0^2 \text{Taut}_{\mathbb{F}}(i-1, v-1)) = \begin{cases} \frac{1}{2}(i-1)i-1, & \mathbb{F} = \mathbb{R}, \\ (i-1)^2-1 & \mathbb{F} = \mathbb{C}, \end{cases}$$

<sup>19</sup>A particularly convenient form for this task can be found in [12, Thm 4.3].

giving  $\frac{1}{2}(i-1)(i+2) - 1$  in total in the real case and  $i^2 - 2$  in the complex case.

Recall that a real vector bundle  $E$  is orientable if and only if its first Stiefel–Whitney class  $w_1(E) \in H^1(B, \mathbb{Z}_2)$  vanishes (here  $B$  is the base of the bundle). The first Stiefel–Whitney class is additive with respect to the Whitney sum, therefore

$$w_1(E_{i,v}) = w_1(S_0^2 \mathcal{E}) + w_1(\mathcal{E}), \quad \mathcal{E} = \text{Taut}_{\mathbb{R}}(i-1, v-1).$$

Using additivity on equation (6.1) gives  $w_1(S_0^2 \mathcal{E}) = w_1(S^2 \mathcal{E})$  because  $w_1$  is zero for the trivial bundle. The classical formulas for the Stiefel–Whitney classes of symmetric tensor power (see, for example, [32, Sect. 19.5.C, Theorem 3]) yield  $w_1(S^2 \mathcal{E}) = (\text{rank } \mathcal{E} + 1)w_1(\mathcal{E})$  and, finally,

$$w_1(E_{i,v}) = (\text{rank } \mathcal{E} + 2)w_1(\mathcal{E}), \quad \mathcal{E} = \text{Taut}_{\mathbb{R}}(i-1, v-1).$$

Since the real tautological bundle  $\mathcal{E}$  is not orientable and has rank  $i-1$ ,  $w_1(E_{i,v})$  vanishes if and only if  $i+1$  is zero modulo 2, completing the proof of the lemma.  $\square$

Recall that the oriented Grassmannian  $\widetilde{\text{Gr}}_{\mathbb{R}}(k, n)$  consisting of the *oriented*  $k$ -dimensional subspaces in  $\mathbb{R}^n$  is a double cover of  $\text{Gr}_{\mathbb{R}}(k, n)$ . Let  $\tau$  denote the orientation-reversing involution on  $\widetilde{\text{Gr}}_{\mathbb{R}}(k, n)$ . In the space of  $q$ -chains of  $\widetilde{\text{Gr}}_{\mathbb{R}}(k, n)$  over the ring  $\mathbb{Z}$  we distinguish the subspace of chains which are skew-symmetric with respect to  $\tau$ :  $\tau(\alpha) = -\alpha$ , where  $\alpha$  is a chain. The subspaces of skew-symmetric  $q$ -chains are invariant under the boundary operator and therefore define a complex. The homology groups of this complex will be denoted  $H_q(\text{Gr}_{\mathbb{R}}(k, n); \widetilde{\mathbb{Z}})$ . In the sequel we refer to them as *twisted homologies*, as they are homologies with local coefficients in the module of twisted integers  $\widetilde{\mathbb{Z}}$ , i.e.  $\mathbb{Z}$  considered as the module corresponding to the nontrivial action of  $\mathbb{Z}_2$  on  $\mathbb{Z}$ .

**Proof of Theorem 1.14, part (2)** In the case  $1 < i(x) \leq v(x)$ , we start from equation (1.11),

$$\begin{aligned} H_r \left( U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k) \right) &\cong \widetilde{H}_{r-\mu(x)} \left( \mathcal{SR}_{k,v} \right) \cong \widetilde{H}_{r-\mu(x)} \left( \Sigma \mathcal{R}_{k,v} \right) \\ &\cong \widetilde{H}_{r-\mu(x)} \left( \Sigma \left( \mathcal{T}(E_{i,v}) \right) \right), \end{aligned}$$

where  $E_{i,v}$  is given by (6.2).

Recall [44, Cor. 16.1.6] that the reduced suspension of a Thom space of a vector bundle is homeomorphic to the Thom space of the Whitney sum of this bundle with the trivial rank 1 bundle  $\theta^1$ , i.e.

$$\Sigma \left( \mathcal{T}(E_{i,v}) \right) \cong \mathcal{T}(\widehat{E}_{i,v}), \quad \widehat{E}_{i,v} := E_{i,v} \oplus \theta^1 \quad (6.5)$$

The bundle  $\widehat{E}_{i,v}$  is orientable if and only if  $E_{i,v}$  is orientable; its rank is one plus the rank of  $E_{i,v}$ . Lemma 6.1 supplies both pieces of information.

The bundle  $\widehat{E}_{i,v}$  is orientable if  $\mathbb{F} = \mathbb{C}$  or if  $\mathbb{F} = \mathbb{R}$  and  $i$  is odd, and we can use the homological version of the Thom isomorphism theorem [52, Lemma 18.2], which gives

$$\widetilde{H}_{r-\mu(x)}(\mathcal{T}(\widehat{E}_{i,v})) = H_{r-\mu(x)-s(i)}(\mathrm{Gr}_{\mathbb{F}}(i-1, v-1)),$$

which is the right-hand side of (1.13) in the corresponding cases.

When  $\mathbb{F} = \mathbb{R}$  and  $i$  is even, the bundle  $\widehat{E}_{i,v}$  is nonorientable (1.13) results from the Thom isomorphism for nonorientable bundles [59, Theorem 3.10],<sup>20</sup>

$$\widetilde{H}_{r-\mu(x)}(\mathcal{T}(\widehat{E}_{i,v})) = H_{r-\mu(x)-s(i)}(\mathrm{Gr}_{\mathbb{F}}(i-1, v-1); \widetilde{\mathbb{Z}}). \quad (6.6)$$

In the special case  $k = v$ , not covered by Lemma 6.1, we compute directly using Lemma 5.1 and Remark 5.3,

$$\begin{aligned} H_r(U^{\lambda^c+\varepsilon}(\lambda_v), U^{\lambda^c-\varepsilon}(\lambda_v)) &\cong H_{r-\mu(x)}(U^{\lambda^c+\varepsilon}(\lambda_v|_N), U^{\lambda^c-\varepsilon}(\lambda_v|_N)) \\ &\cong H_{r-\mu(x)}(\mathbb{B}^{s(v)}, \emptyset) \cong H_{r-\mu(x)}(\{x\}) \\ &\cong H_{r-\mu(x)-s(i)}(\mathrm{Gr}_{\mathbb{F}}(i-1, v-1)), \end{aligned}$$

since  $i = 1$ ,  $s(i) = 0$  and  $\mathrm{Gr}_{\mathbb{F}}(0, v-1)$  is a single point.  $\square$

**Remark 6.3** One can also derive (6.6), using Poincaré and Poincaré–Lefschetz dualities in their usual and skew form, mimicking the proof of [52, Lemma 18.2]. This alternative derivation is included as Appendix B.

## 7 Proof of part (1) of Theorem 1.12: criticality

$\mathbb{F} = \mathbb{C}$ . In the setting of Theorem 1.14, the Poincaré polynomials of the relative homology groups  $H_*(U^{\lambda_k(x)+\varepsilon}(\lambda_k), U^{\lambda_k(x)-\varepsilon}(\lambda_k))$  is equal to

$$t^{\mu(x)+s(i)} P_{\mathrm{Gr}_{\mathbb{C}}(i-1, v-1)}(t). \quad (7.1)$$

Betti numbers for complex Grassmannians were established by Ehresmann, see [26, Theorem on p. 409, section II.7]. The  $r$ -th Betti number is zero if  $r$  is odd and is equal to the number of Young diagrams with  $r/2$  cells that fit inside the  $k \times (n-k)$  rectangle, if  $r$  is even. The Poincaré polynomial  $P_{\mathrm{Gr}_{\mathbb{C}}(k,n)}$  is nothing but the generating function for this *restricted partition problem*. The latter is well known to be of the form

$$P_{\mathrm{Gr}_{\mathbb{C}}(k,n)}(t) = \binom{n}{k}_{t^2},$$

<sup>20</sup>An analogous result for cohomologies can be found in [58].

see [5, Theorem 3.1, p. 33]. By (7.1) and (7.1), the Poincaré polynomials of the relative homology groups  $H_*(U^{\lambda_k(x)+\varepsilon}(\lambda_k), U^{\lambda_k(x)-\varepsilon}(\lambda_k))$  does not vanish, so  $x$  is a critical points.

$\mathbb{F} = \mathbb{R}$ . The calculation of the Poincaré polynomials of integer homologies will be done in the next section, but it is equal to zero in some cases and so does not lead to the proof of criticality. Instead, we show that  $\mathbb{Z}_2$ -homology groups  $H_*(U_x^{+\varepsilon}(\lambda_k), U_x^{-\varepsilon}(\lambda_k); \mathbb{Z}_2)$  are nontrivial.

Let  $P_{Y, \mathbb{Z}_2}(t)$  be the Poincaré polynomial of  $\mathbb{Z}_2$ -homologies of a topological space  $Y$ , i.e. the coefficient of  $t^i$  in  $P_{Y, \mathbb{Z}_2}(t)$  is equal to the number of copies of  $\mathbb{Z}_2$ s in  $H_i(Y, \mathbb{Z}_2)$ . The Poincaré polynomial of  $\mathbb{Z}_2$ -homologies of the Grassmannian  $\text{Gr}_{\mathbb{R}}(k, n)$  is well known ([5, Theorem 3.1, p. 33], [52, §7]) to be

$$P_{\text{Gr}_{\mathbb{R}}(k, n), \mathbb{Z}_2}(t) = \binom{n}{k}_t. \quad (7.2)$$

Moreover, when the coefficients are  $\mathbb{Z}_2$ , there is no difference between symmetric and skew-symmetric chains, therefore if  $P_{Y, \widetilde{\mathbb{Z}_2}}(t)$  is the Poincaré polynomial of the twisted  $\mathbb{Z}_2$ -homologies of  $Y$ , then  $P_{Y, \widetilde{\mathbb{Z}_2}}(t) = P_{Y, \mathbb{Z}_2}(t)$ . Based on this and (7.2), in the setting of Theorem 1.14, the Poincaré polynomial of the relative  $\mathbb{Z}_2$ -homology groups  $H_*(U^{\lambda_k(x)+\varepsilon}(\lambda_k), U^{\lambda_k(x)-\varepsilon}(\lambda_k); \mathbb{Z}_2)$  is equal to

$$t^{\mu(x)+s(i)} \binom{v-1}{i-1}_t, \quad (7.3)$$

which is not zero. The proof of critically in the case of  $\mathbb{F} = \mathbb{R}$  is complete.

## 8 Proof of Theorem 1.12, part (2): computing the Poincaré polynomials

Theorem 1.12 will be obtained as a combination of the next two lemmas. Lemma 8.1 provides an expression for the Poincaré polynomial of twisted homologies  $H_*(\text{Gr}_{\mathbb{R}}(i-1, v-1); \widetilde{\mathbb{Z}})$  by relating it to the Poincaré polynomial of the oriented Grassmannian  $\widetilde{\text{Gr}}_{\mathbb{R}}$ . Lemma 8.2 below collates known expressions for the Poincaré polynomials of Grassmannians and oriented Grassmannians.

**Lemma 8.1** *In the setting of Theorem 1.14, the Poincaré polynomials of the relative homology groups  $H_*(U^{\lambda_k(x)+\varepsilon}(\lambda_k), U^{\lambda_k(x)-\varepsilon}(\lambda_k))$  is equal to*

$$t^{\mu(x)+s(i)} \begin{cases} P_{\text{Gr}_{\mathbb{R}}(i-1, v-1)}(t), & \mathbb{F} = \mathbb{R} \text{ and } i \text{ is odd,} \\ P_{\widetilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1)}(t) - P_{\text{Gr}_{\mathbb{R}}(i-1, v-1)}(t), & \mathbb{F} = \mathbb{R} \text{ and } i \text{ is even,} \\ P_{\text{Gr}_{\mathbb{C}}(i-1, v-1)}(t), & \mathbb{F} = \mathbb{C}. \end{cases} \quad (8.1)$$

where  $P_Y(t)$  denotes the Poincaré polynomial of the manifold  $Y$ .

**Proof** Since we already established part (2) of Theorem 1.14, we only need to show that the Poincaré polynomial of the homology groups  $H_*(\text{Gr}_{\mathbb{R}}(i-1, v-1); \widetilde{\mathbb{Z}})$  is equal to  $P_{\widetilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1)}(t) - P_{\text{Gr}_{\mathbb{R}}(i-1, v-1)}(t)$ .

We will use homologies with coefficients in  $\mathbb{Q}$  (or  $\mathbb{R}$ ). Indeed, since the Betti numbers ignore the torsion part of  $H_r(\cdot; \mathbb{Z})$ , the Universal Coefficients Theorem (see, e.g. [38, Sect. 3.A]) implies they can be calculated as the rank of  $H_r(\cdot; G)$  with any torsion-free abelian group  $G$ . The benefit of using  $\mathbb{Q}$  is that now any chain  $c$  in  $\tilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1)$  can be uniquely represented as a sum of a symmetric and a skew-symmetric chains with coefficients in  $\mathbb{Q}$ ,

$$c = \frac{1}{2}(c + \tau(c)) + \frac{1}{2}(c - \tau(c)),$$

where  $\tau$  is the orientation reversing involution of  $\tilde{\text{Gr}}(i-1, v-1)$  (viewed as a double cover of  $\text{Gr}(i-1, v-1)$ ). The analogous statement is of course wrong in integer coefficients, as  $\frac{1}{2} \notin \mathbb{Z}$ .

Since the boundary operator preserves the parity of a chain, the homology  $H_r(\tilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1); \mathbb{Q})$  decomposes into the direct sum of homologies of  $\tau$ -symmetric and  $\tau$ -skew-symmetric chains on  $\tilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1)$ . The former homology coincides with the usual homology of  $\text{Gr}_{\mathbb{R}}(i-1, v-1)$ . The latter yields, by definition, the twisted  $\mathbb{Q}$ -homology of  $\text{Gr}_{\mathbb{R}}(i-1, v-1)$ . To summarize, we obtain

$$\begin{aligned} H_r(\tilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1); \mathbb{Q}) \\ = H_r(\text{Gr}_{\mathbb{R}}(i-1, v-1); \mathbb{Q}) \oplus H_r(\text{Gr}_{\mathbb{R}}(i-1, v-1); \tilde{\mathbb{Q}}). \end{aligned} \quad (8.2)$$

The sum in (8.2) translates into the sum of Poincaré polynomials, yielding the middle line in (8.1).  $\square$

**Lemma 8.2** *Let  $k, n \in \mathbb{N}$ ,  $k \leq n$ . The Poincaré polynomials of the Grassmannians  $\text{Gr}_{\mathbb{C}}(k, n)$ ,  $\text{Gr}_{\mathbb{R}}(k, n)$  and  $\tilde{\text{Gr}}_{\mathbb{R}}(k, n)$  are given by*

$$P_{\text{Gr}_{\mathbb{C}}(k, n)}(t) = \binom{n}{k}_{t^2}, \quad (8.3)$$

$$P_{\text{Gr}_{\mathbb{R}}(k, n)}(t) = \begin{cases} \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}_{t^4}, & \text{if } k(n-k) \text{ is even,} \\ (1+t^{n-1}) \binom{n/2-1}{(k-1)/2}_{t^4}, & \text{if } k(n-k) \text{ is odd,} \end{cases} \quad (8.4)$$

$$P_{\tilde{\text{Gr}}_{\mathbb{R}}(k, n)}(t) = \begin{cases} (1+t^{n-k}) \binom{(n-1)/2}{(k-1)/2}_{t^4} & \text{if } k \text{ is odd, } n \text{ is odd,} \\ (1+t^{n-1}) \binom{n/2-1}{(k-1)/2}_{t^4}, & \text{if } k \text{ is odd, } n \text{ is even,} \\ \frac{(1+t^k)(1+t^{n-k})}{1+t^n} \binom{n/2}{k/2}_{t^4}, & \text{if } k \text{ is even, } n \text{ is even.} \end{cases} \quad (8.5)$$

**Remark 8.3** We will not need the Poincaré polynomial in the last line of (8.5) and we include it for completeness only. The case of even  $k$  and odd  $n$  is covered by the first line of (8.5) since  $\tilde{\text{Gr}}_{\mathbb{R}}(k, n) = \tilde{\text{Gr}}_{\mathbb{R}}(n-k, n)$ .

**Proof** The complex case was already discussed in the beginning of Sect. 7.

The  $r$ -th Betti number of the real Grassmannian has a similar combinatorial description [46, Theorem IV, p. 108]: it is equal to the number of Young diagrams of  $r$  cells that fit inside the  $k \times (n - k)$  rectangle and have even length differences for each pair of columns and for each pair of rows. From this it can be shown that the Poincaré polynomial  $P_{\text{Gr}_{\mathbb{R}}(k,n)}$  satisfies (8.4) (see also [16, Theorem 5.1]). We remark that for  $k$  and  $n$  both even, (8.4) is a consequence of (8.3) because the corresponding Young diagrams must be made up from  $2 \times 2$  squares.

Finally, the oriented Grassmannian is a homogeneous space, namely

$$\tilde{\text{Gr}}(k, n) \cong SO(n)/(SO(k) \times SO(n - k)).$$

The corresponding Poincaré polynomial has been computed within the general theory of de Rham cohomologies of homogeneous spaces, see, for example, [36, Chap. XI]. Up to notation, the first line of (8.5) corresponds to [36, Lines 2–3, col. 3 of Table II on p. 494], the second line of (8.5) corresponds to [36, Lines 2–3, col. 1 of Table III on p. 496] and the third line of (8.5) corresponds to [36, Lines 2–3, col. 2 of Table III on p. 495].  $\square$

**Proof of Theorem 1.12, part (2)** We first establish equation (1.10). Using Lemma 8.1 as well as Lemma 8.2 with

$$k := i - 1 \quad \text{and} \quad n := v - 1,$$

- (1) The first line of (1.10) is obtained directly from the first line of (8.1) and the first line of (8.4).
- (2) The second line of (1.10) is obtained from the second line of (8.1) by combining the second lines of (8.4) and (8.5).
- (3) The third line of (1.10) is obtained from the second line of (8.1) by combining the first lines of (8.4) and (8.5).
- (4) Finally, the last line of (1.10) is obtained directly from the last line of (8.1) and (8.3).

Finally, Morse inequalities (1.3) when  $M$  is compact are established by [33, § 45] using the tools identical to [51, § 5].  $\square$

**Proof of Corollary 2.4** A critical point is a point of local maximum if and only if the local Morse data is homotopy equivalent to  $(\mathbb{B}^d, \partial \mathbb{B}^d)$ .

If  $x$  is a maximum, its contribution to the Poincaré polynomial is equal to  $t^d$ , which occurs only in the cases described by Corollary 2.4.

To establish sufficiency, we compute the local Morse data at  $x$ . If condition (1) is satisfied, the normal data at the point  $x$  has been computed in Remark 6.2,  $(J, K) = (\mathbb{B}^{s(v)}, \partial \mathbb{B}^{s(v)})$ . From condition (2) we get the tangential data

$$(P, Q) = (\mathbb{B}^{d-s(v)}, \partial \mathbb{B}^{d-s(v)}).$$

By [35, Thm I.3.7], the local Morse data is then

$$(P \times J, (P \times K) \cup (Q \times J)) \cong (\mathbb{B}^d, \partial \mathbb{B}^d),$$

implying the point is a maximum.

Similarly, a critical point is a point of local minimum if and only if the local Morse data is homotopy equivalent to  $(\mathbb{B}^d, \emptyset)$ . If  $x$  is a minimum, its contribution to the Poincaré polynomial is equal to 1, which occurs only in the cases described by Corollary 2.4.

Conversely, condition (1) implies the normal data is

$$(J, K) = (\mathbb{B}^{s(v)}, \emptyset),$$

see Remark 5.3. From condition (2), the tangential data is

$$(P, Q) = (\mathbb{B}^{d-s(v)}, \emptyset).$$

Combining these using [35, Thm I.3.7] gives the required result.  $\square$

## Appendix A: Hellmann–Feynman Theorem

In this section we review the mathematical formulation of the formula that is known in physics as Hellmann–Feynman Theorem or first-order perturbation theory. We base our formulation on [48, Thm II.5.4] (see also [37]).

**Theorem A.1** *Let  $T : \mathbb{R} \rightarrow \text{Sym}_n(\mathcal{F})$  be differentiable at  $x = 0$ . Let  $\lambda$  be an eigenvalue of  $T(0)$  of multiplicity  $v$ ,  $\mathbf{E} \subset \mathcal{F}^n$  be its eigenspace. Then, for small enough  $x$ , there are exactly  $v$  eigenvalues of  $T(x)$  close to  $\lambda$  and they are given by*

$$\lambda_j(x) = \lambda + x\mu_j + o(x), \quad j = 1, \dots, v,$$

where  $\{\mu_j\}$  are the eigenvalues of the  $v \times v$  matrix  $(T'(0))_{\mathbf{E}}$ , see (1.4).

## Appendix B: Twisted Thom space homologies from Poincaré–Lefschetz duality

The Poincaré–Lefschetz duality (see, e.g. [38, Theorem 3.43]) states that if  $Y$  is compact orientable  $n$ -dimensional manifold with boundary  $\partial Y$ , then

$$H_r(Y, \partial Y) \cong H^{n-r}(Y), \quad H^r(Y, \partial Y) \cong H_{n-r}(Y). \quad (\text{B.1})$$

There is also a twisted analogue<sup>21</sup> of Poincaré–Lefschetz duality for non-oriented manifolds: if  $Y$  is compact non-orientable  $n$ -dimensional manifold with boundary

<sup>21</sup>See [34, Prop 15.2.10] or [22, Chap. 5]. It is also sometimes known as Poincaré–Verdier duality, see [45, VI.3].

$\partial Y$ , then

$$H_r(Y, \partial Y) \cong H^{n-r}(Y; \widetilde{\mathbb{Z}}), \quad H^r(Y, \partial Y) \cong H_{n-r}(Y; \widetilde{\mathbb{Z}}). \quad (\text{B.2})$$

Here, the twisted homology  $H_*(Y; \widetilde{\mathbb{Z}})$  was already introduced in Sect. 6. To define twisted cohomology groups, denote by  $\widetilde{Y}$  the orientation cover of  $Y$  and by  $\tau$  the corresponding orientation-reversing involution.  $H^*(Y; \widetilde{\mathbb{Z}})$  are the cohomologies of the cochain complex defined on the spaces of cochains  $c$  satisfying  $c(\tau(\alpha)) = -c(\alpha)$  for every chain  $\alpha$  in  $\widetilde{Y}$  (see [38, Se. 3H] for a more general point of view). Such cochains will be called *skew-symmetric cochains*. Note that the space of skew-symmetric cochains can be identified with the dual space to the space of skew-symmetric chains, as expected.

(a) Assume now that  $v$  is even. Then the base  $\text{Gr}(i-1, v-1)$  of the vector bundle  $\widehat{E}_{i,v}$  is non-orientable and, since the vector bundle is also non-orientable, the total space  $\mathcal{B}(\widehat{E}_{i,v})$  is orientable. By the usual Poincaré–Lefschetz duality (B.1),

$$H_r(\mathcal{B}(\widehat{E}_{i,v}), \partial \mathcal{B}(\widehat{E}_{i,v})) \cong H^{\dim \widehat{E}_{i,v}-r}(\mathcal{B}(\widehat{E}_{i,v})) \cong H^{\dim \widehat{E}_{i,v}-r}(\text{Gr}_{\mathbb{R}}(i-1, v-1)),$$

where  $\dim \widehat{E}_{i,v}$  is the dimension of the total space  $\mathcal{B}(\widehat{E}_{i,v})$ . In the last identification we used that the base  $\text{Gr}(i-1, v-1)$  is the deformation retract of the total space of the bundle.

Further, since  $\text{Gr}(i-1, v-1)$  is non-orientable when  $v$  is even, we use the twisted analog of Poincaré duality for nonorientable manifolds (see [38, Theorem 3H.6] as well as (B.2) with  $\partial Y = \emptyset$ ) to get

$$H^{\dim \widehat{E}_{i,v}-r}(\text{Gr}_{\mathbb{R}}(i-1, v-1)) \cong H_{r-s(i)}(\text{Gr}_{\mathbb{R}}(i-1, v-1); \widetilde{\mathbb{Z}}),$$

where we used

$$\dim \widehat{E}_{i,v} - \dim \text{Gr}_{\mathbb{R}}(i-1, v-1) = \text{rank } \widehat{E}_{i,v} = s(i).$$

(b) Consider the case of odd  $v$ . Then the base  $\text{Gr}(i-1, v-1)$  is orientable, the bundle is non-orientable and therefore the total space  $\mathcal{B}(\widehat{E}_{i,v})$  is non-orientable. By the twisted Poincaré–Lefschetz duality (B.2),

$$H_r(\mathcal{B}(\widehat{E}_{i,v}), \partial \mathcal{B}(\widehat{E}_{i,v})) \cong H^{\dim \widehat{E}_{i,v}-r}(\mathcal{B}(\widehat{E}_{i,v}), \widetilde{\mathbb{Z}})$$

The orientation double cover  $\widetilde{E}_{i,v}$  of  $\widehat{E}_{i,v}$  can be constructed from the tautological bundle of the oriented Grassmannian  $\widetilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1)$  in the same way as  $\widehat{E}_{i,v}$  was constructed from the tautological bundle of the Grassmannian  $\text{Gr}_{\mathbb{R}}(i-1, v-1)$  by relations (6.2) and (6.5). In particular,  $\widetilde{E}_{i,v}$  is a bundle of rank  $s(i)$  over the oriented Grassmannian  $\widetilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1)$ . Therefore, retracting the unit ball bundle  $\mathcal{B}(\widetilde{E}_{i,v})$  of  $\widetilde{E}_{i,v}$  to its base, we get that the integer cohomology groups of  $\mathcal{B}(\widetilde{E}_{i,v})$  are isomorphic to the integer cohomology groups of the oriented Grassmannian  $\widetilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1)$ , i.e.

$$H^{\dim \widehat{E}_{i,v}-r}(\mathcal{B}(\widetilde{E}_{i,v})) \cong H^{\dim \widehat{E}_{i,v}-r}(\widetilde{\text{Gr}}_{\mathbb{R}}(i-1, v-1)).$$

Moreover, the retraction can be made to preserve the spaces of skew-symmetric chains, which implies that

$$H^{\dim \widehat{E}_{i,v}-r}(\mathcal{B}(\widehat{E}_{i,v}); \widetilde{\mathbb{Z}}) \cong H^{\dim \widehat{E}_{i,v}-r}(\mathrm{Gr}_{\mathbb{R}}(i-1, v-1); \widetilde{\mathbb{Z}}).$$

When  $v$  is odd,  $\mathrm{Gr}_{\mathbb{R}}(i-1, v-1)$  is orientable and so is  $\widetilde{\mathrm{Gr}}_{\mathbb{R}}(i-1, v-1)$ . Moreover, the map from the usual Poincaré duality (see [38, Thm. 3.30] as well as (B.1) with  $\partial T = \emptyset$ ) applied to  $\widetilde{\mathrm{Gr}}_{\mathbb{R}}(i-1, v-1)$  sends the equivalence classes of skew-symmetric cochains to the corresponding skew-symmetric chains. Thus, we arrive to

$$H^{\dim \widehat{E}_{i,nu}-r}(\mathrm{Gr}_{\mathbb{R}}(i-1, v-1); \widetilde{\mathbb{Z}}) \cong H_{r-s(i)}(\mathrm{Gr}_{\mathbb{R}}(i-1, v-1); \widetilde{\mathbb{Z}}).$$

To summarize, we get the corresponding line in (1.13) whether  $v$  is even or odd.

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