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On Eisenhart's Type Theorem for Sub-Riemannian Metrics on Step 2 Distributions with ad-Surjective Tanaka Symbols

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Abstract—The classical result of Eisenhart states that, if a Riemannian metric g admits a Riemannian metric that is not constantly proportional to g and has the same (parameterized) geodesics as g in a neighborhood of a given point, then g is a direct product of two Riemannian metrics in this neighborhood. We introduce a new generic class of step 2 graded nilpotent Lie algebras, called ad-*surjective*, and extend the Eisenhart theorem to sub-Riemannian metrics on step 2 distributions with ad-surjective Tanaka symbols. The class of ad-surjective step 2 nilpotent Lie algebras contains a well-known class of algebras of H-type as a very particular case.

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1. INTRODUCTION

1.1. Affine Equivalence in Riemannian Geometry: Nonrigidity and Product Structure

The paper is devoted to a problem in sub-Riemannian geometry, but we start with a historical overview of the same problem in Riemannian geometry. Recall that two Riemannian metrics g_1 and g_2 on a manifold M are called projectively equivalent if they have the same geodesics, as unparameterized curves, namely, for every geodesic $\gamma(t)$ of g_1 there exists a reparameterization $t = \varphi(\tau)$ such that $\gamma\big(\varphi(\tau)\big)$ is a geodesic of g_2. They are called affinely equivalent if they are projective equivalent and the reparameterizations $\varphi(\tau)$ above are affine functions, i. e., they are of the form $\varphi(\tau) = a\tau + b$. We will write $g_1 \overset{p}{\sim} g_2$ and $g_1 \overset{a}{\sim} g_2$ in the case of projective and affine equivalence, respectively. In the sequel, we will mainly be interested in the local version of the same definitions for germs of Riemannian metrics at a point when conditions on the coincidence of geodesics hold in a neighborhood of this point.

From the form of the equation for Riemannian geodesics, it follows immediately that two Riemannian metrics are affinely equivalent if and only if they have the same geodesics as parameterized curves, which in turn is equivalent to the condition that they have the same Levi-Civita connection, i. e., one metric is parallel with respect to the Levi-Civita connection of the other.

Obviously, given any Riemannian metric g and a positive constant c, the metrics cg and g are affinely equivalent. The metric cg will be said a *constantly proportional metric* to the metric g.

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The Riemannian metric g is called *affinely rigid* if the metrics constantly proportional to it are the only affinely equivalent metric to it.

A class of Riemannian metrics g that are not affinely rigid are the metrics admitting a product structure, i. e., when the ambient manifold M can be represented as $M = M_1 \times M_2$, where each M_1 and M_2 are of positive dimension, and there exist Riemannian metrics g_1 and g_2 on M_1 and M_2, respectively, such that, if $\pi_i : M \to M_i$, $i = 1, 2$, they are canonical projections, then

$$g = \pi_1^* g_1 + \pi_2^* g_2. \tag{1.1}$$

Then, obviously, for every two positive constants C_1 and C_2

$$g \overset{a}{\sim} (C_1 \pi_1^* g_1 + C_2 \pi_2^* g_2) \tag{1.2}$$

and the metric $(C_1 \pi_1^* g_1 + C_2 \pi_2^* g_2)$ is not constantly proportional to g if $C_1 \neq C_2$, i. e., the metric g is not affinely rigid. In 1923 L. P. Eisenhart proved that locally the converse is true, i. e., the following theorem holds.

Theorem 1 ([5]). *If a Riemannian metric g is not affinely rigid near a point q_0, i.e., admits a locally affinely equivalent nonconstantly proportional Riemannian metric in a neighborhood of a point q_0, then the metric g is the direct product of two Riemannian metrics in a neighborhood of q_0.*

This theorem is closely related to (and actually is a local version of) the De Rham decomposition theorem [4] on the direct product structure of a simply connected complete Riemannian manifolds in terms of the decomposition of the tangent bundle into invariant subbundles with respect to the action of the holonomy group. Indeed, if $g_1 \overset{a}{\sim} g_2$ and these metrics are not constantly proportional, then the eigenspaces of the transition operators between these metrics (see (3.4) below for the definition of the transition operator) form such a decomposition of the tangent bundle with respect to the action of the holonomy group of both g_1 and g_2 (recall that they have the same Levi-Civita connection).

1.2. Affine Equivalence of Sub-Riemannian Metrics: the Main Conjecture

First, recall that a distribution D on a manifold M is a subbundle of the tangent bundle TM. A sub-Riemannian manifold/structure is a triple (M, D, g), where M is a smooth manifold, D is a bracket-generating distribution, and for any q, $g(q)$ is an inner product on $D(q)$ which depends smoothly on q. We say that g is a sub-Riemannian metric on (M, D). A Riemannian manifold/structure/metric appears as the particular case where $D = TM$.

In the sequel, we will assume that the distribution D is bracket-generating, i. e., for every point $q \in M$ the iterative Lie brackets of vector fields tangent to a distribution D (i. e., of sections of D) span the tangent space T_qM. In more detail, one can define a filtration

$$D = D^1 \subset D^2 \subset \dots D^j(q) \subset \dots \tag{1.3}$$

of the tangent bundle, called a *weak derived flag*, as follows: set $D = D^1$ and define recursively

$$D^j = D^{j-1} + [D, D^{j-1}], \quad j > 1. \tag{1.4}$$

If $X_1, \dots, X_m$ are m vector fields constituting a local basis of a distribution D, then $D^j(q)$ is the linear span of all iterated Lie brackets of these vector fields, of length not greater than j, evaluated at a point q,

$$D^j(q) = \operatorname{span}\{ \big[X_{i_1}(q), \dots [X_{i_{s-1}}, X_{i_s}](q) \dots\big] : (i_1, \dots, i_s) \in [1:m]^s, s \in [1:j]\} \tag{1.5}$$

(here, given a positive integer n, we denote by $[1:n]$ the set $\{1, \dots, n\}$). A distribution D is called *bracket-generating* (or *completely nonholonomic*) if for any q there exists $\mu(q) \in \mathbb{N}$ such that $D^{\mu(q)}(q) = T_qU$. The number $\mu(q)$ is called the *degree of nonholonomy* of D at a point q. If the degree of nonholonomy is equal to a constant μ at every point, one says that D is *step μ distribution.*

Since we work locally, the assumption of bracket-genericity is not too restrictive: if a distribution is not bracket-generating, then in a neighborhood U of a generic point there exists a positive integer

μ such that $D^{\mu+1} = D^{\mu} \subsetneq TM$. So, D^{μ} is a proper involutive subbundle of TU and the distribution D is bracket-generating on each integral submanifold of D^{μ} in U. So, we can restrict ourselves to these integral submanifolds instead of U.

What are sub-Riemannian geodesics? There are at least two different approaches to this concept. One approach is variational: a geodesic is seen as an extremal trajectory, i. e., a candidate for the "shortest" or "energy-minimizing" path connecting its endpoints, with respect to the corresponding length or energy functional. The other approach is differential-geometric: geodesic is the "straightest path", i. e., the curves for which the vector field of velocities is parallel along the curve, with respect to a natural connection. While in Riemannian geometry these two approaches lead to the same set of trajectories, in proper sub-Riemannian geometry (i. e., when $D \neq TM$) they lead to different sets of trajectories (see [3] for details), and in general, for the second approach, the natural connection only exists under additional (and rather restrictive) assumptions of constancy of sub-Riemannian symbol [8].

In the present paper, we consider the geodesic defined by the variational approach. A horizontal curve $\gamma : [a, b] \to M$ is an absolutely continuous curve tangent to D, i. e., $\gamma'(t) \in D(\gamma(t))$. In the sequel, the manifold M is assumed to be connected. By the Rashevskii – Chow theorem the assumption that D is bracket-generating guarantees that the space of horizontal curves connecting two given points q_0 and q_1 is not empty. The following energy-minimizing problem:

$$\begin{aligned} &E(\gamma) = \int_a^b g\big(\gamma'(t), \gamma'(t)\big)\, dt \to \min, \\ &\gamma'(t) \in D\big(\gamma(t)\big) \quad \text{a.e.} \quad t, \\ &\gamma(a) = q_0, \quad \gamma(b) = q_1 \end{aligned} \tag{1.6}$$

can be solved using the Pontryagin Maximum Principle [2, 9] in optimal control theory that defines special curves in the cotangent bundle T^*M, called the *Pontryagin extremals*, so that a minimizer of the optimal control problem (1.6) is a projection from T^*M to M of some *Pontryagin extremal* (for a more explicit description of Pontryagin extremals, see the beginning of Section (3) below).

Definition 1. The (*variational*) *sub-Riemannian geodesics* are projections of the Pontryagin extremals of the optimal control problem (1.6).

Note that in the Riemannian case the geodesics given by Definition 1 coincides with the usual Riemannian geodesics. We thus extend the definitions of projective and affine equivalences of Riemannian metrics to the general sub-Riemannian case in the following way.

Definition 2. Let M be a manifold and D be a bracket-generating distribution on M. Two sub-Riemannian metrics g_1 and g_2 on (M, D) are called *projectively equivalent* at $q_0 \in M$ if they have the same geodesics, up to a reparameterization, in a neighborhood of q_0. They are called *affinely equivalent* at q_0 if they have the same geodesics, up to affine reparameterization, in a neighborhood of q_0.

Again, we will write $g_1 \overset{p}{\sim} g_2$ and $g_1 \overset{a}{\sim} g_2$ in the case of projective and affine equivalence, respectively. By complete analogy with the Riemannian case, for a sub-Riemannian metric g on (M, D) and a positive constant c the metrics cg and g are affinely equivalent. The metric cg will be said a *constantly proportional metric* to the metric g.

Definition 3. A sub-Riemannian metric g on (M, D) is called *affinely rigid* if the sub-Riemannian metrics constantly proportional to it are the only sub-Riemannian metrics on (M, D) that are affinely equivalent to g.

As in the Riemannian case, examples of affinely nonrigid sub-Riemannian structures can be constructed with the help of an appropriate notion of product structure. For this we first have to define distributions admitting product structure as follows:

Definition 4. A distribution D on a manifold M *admits a product structure* if there exist two manifolds M_1 and M_2 of positive dimension endowed with two distributions D_1 and D_2 of positive rank (on M_1 and M_2, respectively) such that the following two conditions holds:

1) $M = M_1 \times M_2$;

2) If $\pi_i : M \to M_i$, $i = 1, 2$, are the canonical projections and $\pi_i^* D_i$ denotes the pullback of the distribution D_i from M_i to M, i. e.,

$$\pi_i^* D_i(q) = \{v \in T_qM : d\pi_i(q)v \in D_i\bigl(\pi_i(q)\bigr)\},$$

then

$$D(q) = \pi_1^* D_1(q) \cap \pi_2^* D_2(q). \tag{1.7}$$

In this case, we will write that $(M, D) = (M_1, D_1) \times (M_2, D_2)$.

Definition 5. A sub-Riemannian structure (M, D, g) *admits a product structure* if there exist (nonempty) sub-Riemannian structures (M_1, D_1, g_1) and (M_2, D_2, g_2) such that $(M, D) = (M_1, D_1) \times (M_2, D_2)$ and, if $\pi_i : M \mapsto M_1$ are the canonical projections, then identity (1.1) holds. In this case we will write that $(M, D, g) = (M_1, D_1, g_1) \times (M_2, D_2, g_2)$.

It is easy to see that, if $(M, D, g) = (M_1, D_1, g_1) \times (M_2, D_2, g_2)$, then this sub-Riemannian metric is affinely equivalent to

$$(M_1, D_1, c_1 g_1) \times (M_2, D_2, c_2 g_2) \tag{1.8}$$

for every two positive constants c_1 and c_2, but the latter metric is not constantly proportional to (M, D, g) if $c_1 \neq c_2$, i. e., a *sub-Riemannian metric admitting product structure* is not affinely rigid. The main question is whether or not the converse of this statement, at least in a local setting, i. e., the analog of the Eisenhart theorem (Theorem 1) holds.

Conjecture 1 ([6]). *If a sub-Riemannian metric g is not affinely rigid near a point q_0, i. e., admits a locally affinely equivalent nonconstantly proportional sub-Riemannian metric in a neighborhood of a point q_0, then the metric g is the direct product of two sub-Riemannian metrics in a neighborhood of q_0.*

In this paper, we prove this conjecture for sub-Riemannian metrics on a class of step 2 distributions, see Theorem 3.

2. THE ROLE OF TANAKA SYMBOL/NILPOTENT APPROXIMATION AND THE MAIN RESULT

Conjecture 1 is still widely open. In the present paper, we prove it for sub-Riemannian metrics on a particular, but still rather large class of distributions (see Theorem 3 below). To formulate our main result (Theorem 3) we need to introduce some terminology.

2.1. Direct Product Structure on the Level of Tanaka Symbol/Nilpotent Approximation

In [6] we proved, among other things, a weaker product structure result for affinely nonrigid sub-Riemannian structures, in which the product structure necessarily occurs on the level of Tanaka symbol/ nilpotent approximation of the the sub-Riemannian structure.

To define the Tanaka symbol of the distribution D at a point q, we need another assumption on D near q, called *equiregularity*. A distribution D is called *equiregular* at a point q if there is a neighborhood U of q in M such that for every $j > 0$ the dimensions of subspaces $D^j(y)$ are constant for all $y \in U$, where D^j is in (1.4) (equivalently, as in (1.5)). Note that a bracket-generating distribution is equiregular at a generic point.

From now on we assume that D is an equiregular bracket-generating distribution with the degree of nonholonomy μ. Set

$$\mathfrak{m}_{-1}(q) := D(q), \quad \mathfrak{m}_{-j}(q) := D^j(q)/D^{j-1}(q), \ \forall j > 1 \tag{2.1}$$

and consider the graded space

$$\mathfrak{m}(q) = \bigoplus_{j=-\mu}^{-1} \mathfrak{m}_j(q), \tag{2.2}$$

associated with the filtration (1.3).

The space $\mathfrak{m}(q)$ is endowed with the natural structure of a graded Lie algebra, i. e., with the natural Lie product $[\cdot,\cdot]$ such that

$$[\mathfrak{m}_{i_1}(q), \mathfrak{m}_{i_2}(q)] \subset \mathfrak{m}_{i_1+i_2} \tag{2.3}$$

defined as follows:

Let $\mathfrak{p}_j : D^j(q) \mapsto \mathfrak{m}_{-j}(q)$ be the canonical projection to a factor space. Take $Y_1 \in \mathfrak{m}_{-i_1}(q)$ and $Y_2 \in \mathfrak{m}_{-i_2}(q)$. To define the Lie bracket $[Y_1, Y_2]$ take a local section $\widetilde{Y}_1$ of the distribution D^{i_1} and a local section $\widetilde{Y}_2$ of the distribution D^{i_2} such that

$$\mathfrak{p}_{i_1}\big(\widetilde{Y}_1(q)\big) = Y_1, \quad \mathfrak{p}_{i_2}\big(\widetilde{Y}_2(q)\big) = Y_2. \tag{2.4}$$

It is clear from the definitions of the spaces D^j that $[\widetilde{Y}_1, \widetilde{Y}_2] \in D^{i_1+i_2}$. Then set

$$[Y_1, Y_2] := \mathfrak{p}_{i_1+i_2}\big([\widetilde{Y}_1, \widetilde{Y}_2](q)\big). \tag{2.5}$$

It can be shown [10, 12] that the right-hand side of (2.5) does not depend on the choice of sections $\widetilde{Y}_1$ and $\widetilde{Y}_2$. By constructions, it is also clear that (2.3) holds.

Definition 6. The graded Lie algebra $\mathfrak{m}(q)$ from (2.2) is called the *symbol* of the distribution D at the point q.

By constructions, it is clear that the Lie algebra $\mathfrak{m}(q)$ is nilpotent. The Tanaka symbol is the infinitesimal version of the so-called *nilpotent approximation* of the distribution D at q, which can be defined as the left-invariant distribution $\widehat{D}$ on the simply connected Lie group with the Lie algebra $\mathfrak{m}(q)$ and the identity e, such that $\widehat{D}(e) = \mathfrak{m}_{-1}(q)$.

Further, since D is bracket-generating, its Tanaka symbol $\mathfrak{m}(q)$ at any point is generated by the component $\mathfrak{m}_{-1}(q)$.

Definition 7. A (nilpotent) $\mathbb{Z}_-$-graded Lie algebra

$$\mathfrak{m} = \bigoplus_{j=-\mu}^{-1} \mathfrak{m}_j \tag{2.6}$$

is called a *fundamental* graded Lie algebra (here $\mathbb{Z}_-$ denotes the set of all negative integers) if it is generated by $\mathfrak{m}_{-1}$.

The following notion will be crucial in the sequel:

Definition 8. A fundamental graded Lie algebra $\mathfrak{m}$ is called *decomposable* if it can be represented as a direct sum of two nonzero fundamental graded Lie algebras $\mathfrak{m}^1$ and $\mathfrak{m}^2$ and it is called *indecomposable* otherwise. Here the jth component of $\mathfrak{m}$ is the direct sum of the jth components of $\mathfrak{m}^1$ and $\mathfrak{m}^2$.

Obviously, if a distribution D admits product structure, then its Tanaka symbol at any point is decomposable.

Example 1 (contact and even contact distributions). Assume that D is a corank 1 distributions, $\dim D(q) = \dim M - 1$, and assume α is its defining 1-form, i. e., $D = \ker \alpha$.

- Recall that the distribution D is called *contact* if $\dim M$ is odd and the form $d\alpha|_D$ is nondegenerate. In this case the Tanaka symbol at a point q is isomorphic to the Heisenberg algebra $\mathfrak{m}_{-1}(q) \oplus \mathfrak{m}_{-2}(q)$ of dimension equal to $\dim M$, where $\mathfrak{m}_{-2}(q)$ is the (one-dimensional) center and the brackets on $\mathfrak{m}_{-1}(q)$ ($\cong D(q)$) are given by $[X, Y] := d\alpha(X, Y)Z$, where Z is the generator of $\mathfrak{m}_{-1}$ so that $\alpha(Z) = 1$. Note that the Heisenberg algebra is indecomposable as the fundamental graded Lie algebra. Otherwise, since $\dim \mathfrak{m}_{-2}(q) = 1$ one of the components in the nontrivial decomposition of $\mathfrak{m}(q)$ will be commutative and belong to $\mathfrak{m}_{-1}(q)$ and hence to the kernel of $d\alpha|_D$, which contradicts the condition of contactness.

- Recall that the distribution D is called *quasi-contact* (in some literature *even contact*) if $\dim M$ is even and the form $d\alpha|_D$ has a one-dimensional kernel (i.e., of the minimal possible dimension for a skew-symmetric form on an odd-dimensional vector space). In this case by the arguments similar to the previous item the Tanaka symbol is the direct some of the Heisenberg algebra (of dimension $\dim M - 1$) and $\mathbb{R}$ (the kernel of $d\alpha|_D$), i. e., the Tanaka symbol is decomposable.

Remark 1. It is easy to show that the decomposition of a fundamental graded $\mathfrak{m}$ Lie algebra into indecomposable fundamental Lie algebras is unique modulo the center of $\mathfrak{m}$ and a permutation of components.

The following theorem is a consequence of the results proved in [6] and it is a weak version of Conjecture 1:

Theorem 2 ([6, a consequence of Theorem 7.1, Proposition 4.7, and Corollary 4.9 there]). *If a sub-Riemannian metric on an equiangular distribution D is not affinely rigid near a point q_0, then its Tanaka symbol at q_0 is decomposable.*

In other words, *the problem of affine equivalence is nontrivial only on the distributions with decomposable Tanaka symbols* (at points where the distribution is equiregular).

2.2. Ad-Surjective Tanaka Symbols and the Main Result

Now we are almost ready to formulate the main result of the paper. We restrict ourselves here to step 2 distributions, i. e., when $D^2 = TM$. Such distributions are automatically equiregular (at any point). Then it is clear that the components in the decomposition of the Tanaka symbols of such distribution are of step not greater than 2 (i. e., with $\mu \leqslant 2$ in (2.6)). So, they are either of step 2 or commutative.

Definition 9. We say that a step 2 fundamental graded Lie algebra $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ is ad-*surjective* if there exists $X \in \mathfrak{m}_{-1}$ such that the map $\mathrm{ad}X : \mathfrak{m}_{-1} \to \mathfrak{m}_{-2}$,

$$Y \mapsto [X, Y], \quad Y \in \mathfrak{m}_{-1},$$

is surjective. An element $X \in \mathfrak{m}_{-1}$ for which $\mathrm{ad}X$ is surjective is called an ad-*generating element* of the algebra $\mathfrak{m}$.

Remark 2. Note that the direct sum $\mathfrak{m}^1 \oplus \mathfrak{m}^2$ of two ad-surjective Lie algebras $\mathfrak{m}^i = \mathfrak{m}^i_{-1} \oplus \mathfrak{m}^i_{-2}$, $i = 1, 2$, is ad-surjective. Indeed, if $X_i \in \mathfrak{m}^i_{-1}$, $i = 1, 2$, are such that $\mathrm{ad}\, X_i : \mathfrak{m}^i_{-1} \to \mathfrak{m}^i_{-2}$ is surjective, then

$$\mathrm{ad}(X_1 + X_2) : \mathfrak{m}^1_{-1} \oplus \mathfrak{m}^2_{-1} \to \mathfrak{m}^1_{-2} \oplus \mathfrak{m}^2_{-2}$$

is surjective as well. And vice versa, if a step 2 fundamental Lie algebra $\mathfrak{m}$ is ad-surjective, then any component of its decomposition into fundamental graded Lie algebra is ad-surjective: the projection of an ad-generating element of $\mathfrak{m}$ to any component is ad-generating element of this component.

Remark 3. Nilpotent Lie algebras of H-type, introduced by A. Kaplan [7] in 1980 and extensively studied since then, are ad-surjective because, among other properties of Lie algebras of H-type, it is required that every element of $\mathfrak{m}_{-1}$ be ad-generating.

The following proposition will be proved in Appendix A:

Proposition 1. *Any step 2 fundamental graded Lie algebra $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ such that the following three conditions hold:*

1) $\dim \mathfrak{m}_{-2} \leqslant 3$*;*

2) $\dim \mathfrak{m}_{-2} < \dim \mathfrak{m}_{-1}$*;*

3) *the intersection of $\mathfrak{m}_{-1}$ with the center of $\mathfrak{m}$ is trivial*

is ad-*surjective.*

Note that, if $\dim \mathfrak{m}_{-2} \leqslant 2$, then item (2) of the previous proposition holds automatically. Besides, item (2) is obviously a necessary condition for ad− surjectivity.

Corollary 1. *The only non*-ad-*surjective step 2 fundamental graded Lie algebra with $\mathfrak{m}_{-2} \leqslant 3$ is the truncated step 2 free Lie algebra with 3 generators.*

Note that Proposition 1 does not hold if one drops item (1), see Appendix A, Example 2, for a family of counterexamples with $\dim \mathfrak{m}_{-2} = 4$ and $\dim \mathfrak{m}_{-1} = 5$. These counterexamples are semidirect sums of the truncated step 2 free Lie algebras with three generators and the 3-dimensional Heisenberg algebra.

Nevertheless, following [6, Section 8], given integers $m > 0$ and $d \geqslant 0$, if we denote by $\mathrm{GLNA}(m, m + d)$ the set of all fundamental graded nilpotent Lie algebras $\mathfrak{m}$ of step not greater than 2 satisfying

$$\dim \mathfrak{m}_{-1} = m, \quad \dim \mathfrak{m}_{-2} = d, \tag{2.7}$$

we have the following genericity results:

Lemma 1. *If $m > d$, the subset of all* ad-*surjective graded nilpotent Lie algebras belonging to* $\mathrm{GLNA}(m, m + d)$ *is generic in* $\mathrm{GLNA}(m, m + d)$.

Proof. Indeed, for such Lie algebras, the Lie algebra structure is encoded by the Levi operator $\mathcal{L}_q \in \mathrm{Hom}\big(\bigwedge^2 \mathfrak{m}_{-1}, \mathfrak{m}_{-2}\big)$ which is defined as follows:

$$\mathcal{L}(X, Y) = [X, Y], \quad \forall X, Y \in \mathfrak{m}_{-1}, \tag{2.8}$$

and the fundamentality assumption implies that $\mathcal{L}$ is surjective. Equivalently, one can consider the dual operator $\mathcal{L}^* \in \mathrm{Hom}\big((\mathfrak{m}_{-2})^*, \bigwedge^2(\mathfrak{m}_{-1})^*\big)$,

$$\mathcal{L}^*(p)(X, Y) = p([X, Y]) \quad X, Y \in \mathfrak{m}_{-1},\ p \in (\mathfrak{m}_{-2})^*. \tag{2.9}$$

Here we use the natural identification $\left(\bigwedge^2 \mathfrak{m}_{-1}\right)^* \cong \bigwedge^2(\mathfrak{m}_{-1})^*$, which in turn is naturally identified with the space of skew-symmetric bilinear forms on $\mathfrak{m}_{-1}$. Note that, again from the surjectivity of $\mathcal{L}$, its dual $\mathcal{L}^*$ is injective and is described by its image, which is a d-dimensional space. So, the space of all fundamental graded nilpotent Lie algebras of step not greater than 2 satisfying (2.7) is isomorphic to the Grassmannian of d-dimensional subspaces in the space of skew-symmetric forms of an m-dimensional vector space, modulo the natural action of the general linear group on this space. In particular, the latter Grassmannian is a connected algebraic variety and the subset of ad-surjective graded nilpotent Lie algebras of step not greater than 2 satisfying (2.7) with $m > d$ corresponds to a nonempty Zariski open subset of it, therefore it is generic. □

Remark 4. By [6, Proposition 8.1], the subset of indecomposable graded nilpotent Lie algebras in $\mathrm{GLNA}(m, m+d)$ is generic $\mathrm{GLNA}(m, m+d)$ for all pairs (m, d) with the exception of the following three cases:

1) $d = 0$, $m > 1$ (Riemannian case of dimension greater than 1);

2) $d = 1$, $m > 1$ and odd (the quasi-contact case);

3) $d = 2$, $m = 4$.

Moreover, in cases (1) and (2) all graded Lie algebras in $\mathrm{GLNA}(m, m+d)$ are decomposable, while in case (3) the set of decomposable fundamental symbols is nonempty open and corresponds to symbols for which the set of solutions of the equation $\mathcal{L}^*(p) \wedge \mathcal{L}^*(p) = 0$ considered as the equation with respect to $p \in (\mathfrak{m}_{-2})^*$, where $L^*(p)$ is as in (2.9), consists of two distinct (real) lines.

The main result of the present paper is the following

Theorem 3. *Assume that D is a step 2 distribution such that its Tanaka symbol is* ad*-surjective. If a sub-Riemannian metric (M, D, g_1) is not affinely rigid near a point q_0, then it admits a product structure in a neighborhood of q_0.*

Remark 5. First note that by Theorem 2.1 under the hypothesis of the previous theorem the Tanaka symbol of D must be decomposable and by the second sentence of Remark 2 all components of this decomposition are ad-surjective. Second, by Remark 1, if such a decomposition consists of indecomposable components only, the number of these components is independent of the decomposition. Let us denote this number by $\hat{k}$. Then the sub-Riemannian metric in Theorem 3 is a product of at least two and at most $\hat{k}$ sub-Riemannian structures each of which is affinely rigid (in the neighborhood of the projection of q_0 to the corresponding manifold).

The rest of the paper is devoted to the proof of Theorem 3. This theorem confirms Conjecture 1 for sub-Riemannian metrics on step 2 distributions with ad-surjective Tanaka symbol. As a direct consequence of Theorem 3 and Corollary 1, we get the following

Corollary 2. *Assume that D is a step 2 distribution such that its Tanaka symbol is decomposed into $\hat{k} \geqslant 2$ nonzero indecomposable fundamental graded Lie algebras with degree -2 components of dimension not greater than 3 and such that among them there is no truncated step 2 free Lie algebra with 3 generators. If a sub-Riemannian metric (M, D, g_1) is not affinely rigid near a point q_0, then it admits a product of of at least two and at most $\hat{k}$ sub-Riemannian structures each of which is affinely rigid (in the neighborhood of the projection of q_0 to the corresponding manifold).*

The assumption of ad-surjectivity of the Tanaka symbol is crucial for our proof of Theorem 3 because we strongly use a natural quasi-normal form for ad-surjective Lie algebras, see (4.8). We hope that analogous quasi-normal forms can be found for more general graded nilpotent Lie algebras so that Conjecture 1 can be proved similarly for a more general class of sub-Riemannian metrics.

3. ORBITAL EQUIVALENCE AND FUNDAMENTAL ALGEBRAIC SYSTEM

In general, there are two types of Pontryagin extremals for optimal control problems, *normal* and *abnormal* [1, 2]: for the former, the Lagrange multiplier near the functional is not zero, and for the latter, it is zero. In particular, abnormal extremals, as unparameterized curves, depend on the distribution D only and not on a metric g on it. This indicates that only normal extremals are essential for the considered problems of affine/projective equivalence (see Proposition 2 for the precise formulation). Therefore, we give an explicit description only of normal extremals. They are the integral curves of the Hamiltonian vector field $\vec{h}$ on T^*M corresponding to the Hamiltonian

$$h(p, q) = ||\, p|_{D(q)}||^2, \quad q \in M, p \in T_q^*M, \tag{3.1}$$

and lying on a nonzero level set of h. Here $||\,p|_{D(q)}||$ denotes the operator norm of the functional $p|_{D(q)}$, i. e.,

$$||\,p|_{D(q)}|| = \max\{p(v) : v \in D(q), g(q)(v,v) = 1\}.$$

The Hamiltonian h defined by (3.1) is called the *Hamiltonian associated with the metric* g or shortly the *sub-Riemannian Hamiltonian*.

In [6], following [11], the problems of projectively and affine equivalence of sub-Riemannian metric were reduced to the study of the orbital equivalence of the corresponding sub-Riemannian Hamiltonian systems for normal Pontryagin extremals of the energy minimizing problem (1.6), which in turn is reduced to the study of solvability of a special linear algebraic system with coefficients being polynomial in the fibers, called the *fundamental algebraic system* [6, Proposition 3.10]. In this section we summarize all information from [6] we need for the proof of Theorem 3.

As before, fix a connected manifold M and a bracket-generating equiregular distribution D on M, and consider two sub-Riemannian metrics g_1 and g_2 on (M, D). We denote by h_1 and h_2 the respective sub-Riemannian Hamiltonians of g_1 and g_2, as defined in (3.1). Let the annihilator $D^\perp$ of D in T^*M be defined as follows:

$$D^\perp = \{(p,q) \in T^*M : p|_{D(q)} = 0\}. \tag{3.2}$$

It coincides with the zero level set of the sub-Riemannian Hamiltonian h from (3.1).

Definition 10. We say that $\vec{h}_1$ and $\vec{h}_2$ are *orbitally diffeomorphic* on an open subset V_1 of $T^*M\backslash D^\perp$ if there exists an open subset V_2 of $T^*M\backslash D^\perp$ and a diffeomorphism $\Phi : V_1 \to V_2$ such that Φ is fiber-preserving, i. e., $\pi\big(\Phi(\lambda)\big) = \pi(\lambda)$, and Φ sends the integral curves of $\vec{h}_1$ to the reparameterized integral curves of $\vec{h}_2$, i. e., there exists a smooth function $s = s(\lambda, t)$ with $s(\lambda, 0) = 0$ such that $\Phi\big(e^{t\vec{h}_1}\lambda\big) = e^{s\vec{h}_2}\big(\Phi(\lambda)\big)$ for all $\lambda \in V_1$ and $t \in \mathbb{R}$ for which $e^{t\vec{h}_1}\lambda$ is well defined. Equivalently, there exists a smooth function $\alpha(\lambda)$ such that

$$d\Phi\,\vec{h}_1(\lambda) = \alpha(\lambda)\vec{h}_2\big(\Phi(\lambda)\big). \tag{3.3}$$

The map Φ is called an *orbital diffeomorphism* between the extremal flows of g_1 and g_2.

The reduction of projective (respectively, affine) equivalence of sub-Riemannian metrics to the orbital (respectively, a special form of orbital) equivalence of the corresponding sub-Riemannian Hamiltonian systems is given by the following:

Proposition 2 ([6, a combination of Proposition 3.4 and Theorem 2.10 there]). *Assume that the sub-Riemannian metrics g_1 and g_2 are projectively equivalent in a neighborhood $U \subset M$ and let $\pi : T^*M \to M$ be the canonical projection. Then, for a generic point $\lambda_1 \in \pi^{-1}(U)\backslash D^\perp$, $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic on a neighborhood V_1 of λ_1 in T^*M. Moreover, if g_1 and g_2 are affinely equivalent in a neighborhood $U \subset M$, then the function $\alpha(\lambda)$ in* (3.3) *satisfies $\vec{h}_1(\alpha) = 0$.*

Further, the differential equation (3.3) can be written [6, Lemma 3.8] and transformed to the algebraic system via a sequence of prolongations [6, Proposition 3.9] in a special moving frame adapted to the sub-Riemannian structures g_1 and g_2. For this, first, we need the following

Definition 11. The *transition operator* at a point $q \in M$ of the pair of metrics (g_1, g_2) is the linear operator $S_q : D(q) \to D(q)$ such that

$$g_2(q)(v_1, v_2) = g_1(q)(S_q v_1, v_2), \quad \forall v_1, v_2 \in D(q). \tag{3.4}$$

Obviously, S_q is a positive g_1-self-adjoint operator and its eigenvalues $\alpha_1^2(q), \ldots, \alpha_m^2(q)$ are positive real numbers (we choose $\alpha_1(q), \ldots, \alpha_m(q)$ as positive numbers as well). The field S of transition operators is a $(1,1)$-tensor field that will be called the *transition tensor*.

The important simplification in the case of the affine equivalence compared to the projective equivalence is given in the following

Proposition 3 ([6, Propostion 4.7]). *If two sub-Riemannian metrics g_1, g_2 on (M, D) are affinely equivalent on an open connected subset $U \subset M$, then all the eigenvalues $\alpha_1^2, \ldots, \alpha_m^2$ of the transition operator are constant.*

This proposition implies that the number of the distinct eigenvalues $k(q)$ of the transition operators S_q is independent of $q \in U$, i. e., $k(q) \equiv k$ on U for some positive integer k. Also, there are k distributions D_i such that

$$D(q) = \bigoplus_{i=1}^{k} D_i(q) \tag{3.5}$$

is the eigenspace decomposition of $D(q)$ with respect to the eigenspaces of the operator S_q. Now let

$$\mathfrak{m}_{-1}^i(q) = D_i(q), \quad \mathfrak{m}_{-j}^i(q) = (D_i)^j(q)/\left((D_i)^j(q) \cap D^{j-1}(q)\right), \ \forall j > 1.^{1)} \tag{3.6}$$

Set

$$\mathfrak{m}^i(q) = \bigoplus_{j=1}^{\mu} \mathfrak{m}_{-j}^i(q). \tag{3.7}$$

By construction $\mathfrak{m}^i$, $i = 1, \ldots, k$, are fundamental graded Lie algebras.

Remark 6. Note that in general $\mathfrak{m}^i(q)$ is not equal/isomorphic to the Tanaka symbol of the distribution D_i at q, as when defining the components $\mathfrak{m}_{-j}^i(q)$ with $j > 1$ we also make the quotient by the powers of D. In fact, the proof that $\mathfrak{m}^i(q)$ is isomorphic to the Tanaka symbol of the distribution D_i under the assumption of affine nonrigidity is one of the main steps in the proof of Theorem 3.

Proposition 4 ([6, Theorem 6.2 and Theorem 7.1]). *If sub-Riemannian metrics g_1, g_2 are affinely equivalent and not constantly proportional to each other in a connected open set U, then for every $q \in U$ the Tanaka symbol $\mathfrak{m}(q)$ of the distribution D at q is the direct sum of the fundamental graded Lie algebras $\mathfrak{m}^i(q)$, $i = 1, \ldots, k$ defined by* (3.6) *and* (3.7)*, i. e.,*

$$\mathfrak{m}(q) = \bigoplus_{i=1}^{k} \mathfrak{m}^i(q), \tag{3.8}$$

as the direct sum of Lie algebras.

Further, in a neighborhood U_1 of any point $q_0 \in U$ we can choose a g_1-orthonormal local frame $X_1, \ldots, X_m$ of D whose values at any $q \in U_1$ diagonalize S_q, i. e., $X_i(q)$ is an eigenvector of S_q associated with the eigenvalues $\alpha_i^2(q)$, $i = 1, \ldots, m$. Note that $\frac{1}{\alpha_1}X_1, \ldots, \frac{1}{\alpha_m}X_m$ form a g_2-orthonormal frame of D. We then complete $X_1, \ldots, X_m$ into a frame $\{X_1, \ldots, X_n\}$ of TM adapted to the distribution D near q_0, i. e., such that for every positive integer j this frame contains a local frame of D^j. We call such a set of vector fields $\{X_1, \ldots, X_n\}$ a (*local*) *frame adapted to the* (*ordered*) *pair of metrics* (g_1, g_2). The *structure coefficients* of the frame $\{X_1, \ldots, X_n\}$ are the real-valued functions c_{ij}^k, $i, j, k \in \{1, \ldots, n\}$ defined near q by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k. \tag{3.9}$$

Let $u = (u_1, \ldots, u_n)$ be the coordinates on the fibers T_q^*M induced by this frame, i. e.,

$$u_i(q, p) = p\big(X_i(q)\big). \tag{3.10}$$

1) Since $(D_i)^j \subset D^j$, the space $\mathfrak{m}_{-j}^i$ is a subspace of $\mathfrak{m}_{-j}$.

These coordinates in turn induce a basis $\partial_{u_1},\dots,\partial_{u_n}$ of $T_\lambda(T_q^*M)$ for any $\lambda\in\pi^{-1}(q)$. For $i=1,\dots,n$, we define the lift Y_i of X_i as the (local) vector field on T^*M such that $\pi_*Y_i=X_i$ and $du_j(Y_i)=0\ \ \forall 1\leqslant j\leqslant n$. The family of vector fields $\{Y_1,\dots,Y_n,\partial_{u_1},\dots,\partial_{u_n}\}$ obtained in this way is called a *frame of* $T(T^*M)$ *adapted at* q_0. By a standard calculation, we obtain the expression for the sub-Riemannian Hamiltonian h_1 of the metric g_1 and the corresponding Hamiltonian vector field $\vec{h}_1$:

$$h_1=\frac{1}{2}\sum_{i=1}^m u_i^2 \tag{3.11}$$

$$\vec{h}_1=\sum_{i=1}^m u_i\vec{u}_i=\sum_{i=1}^m u_iY_i+\sum_{i=1}^m\sum_{j,k=1}^n c_{ij}^k u_iu_k\partial_{u_j}. \tag{3.12}$$

Indeed, to prove (3.12), recall that, if for a vector field Z in M, we denote

$$H_Z(p,q)=p\big(Z(q)\big),\quad q,p\in T_q^*M,$$

then for any two vector fields Z_1 and Z_2 on M we have the following identities:

$$\overrightarrow{H_{Z_1}}(H_{Z_2})=d\,H_{Z_2}\big(\overrightarrow{H_{Z_1}}\big)=H_{[Z_1,Z_2]}. \tag{3.13}$$

From this and (3.9) it follows immediately that

$$\vec{u_i}=Y_i+\sum_{j=1}^n\vec{u_i}(u_j)\partial_{u_j}=Y_i+\sum_{j=1}^n\sum_{k=1}^n c_{ij}^k u_k\partial_{u_j}, \tag{3.14}$$

which immediately implies (3.12).

Assume now that $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic near $\lambda_0\in H_1\cap\pi^{-1}(q_0)$ and let Φ be the corresponding orbital diffeomorphism. Let us denote by $\Phi_i,\ i=1,\dots,n$, the coordinates u_i of Φ on the fiber, i. e., $u\circ\Phi(\lambda)=\big(\Phi_1(\lambda),\Phi_2(\lambda),\dots,\Phi_n(\lambda)\big)$. Then first it is easy to see [11, Lemma 1] that the function α from (3.3) satisfies

$$\alpha=\sqrt{\frac{\sum_{i=1}^m\alpha_i^2u_i^2}{\sum_{i=1}^m u_i^2}} \tag{3.15}$$

and

$$\Phi_k=\frac{\alpha_k^2u_k}{\alpha},\forall 1\leqslant k\leqslant m. \tag{3.16}$$

In [6], in order to find the equations for the rest of the components $\Phi_{m+1},\dots\Phi_n$ of Φ we first plugged into (3.3) the expression (3.12) for $\vec{h}_1$ and a similar expression for $\vec{h}_2$ and then we made the "prolongation" of the resulting differential equation by recursively differentiating it in the direction of $\vec{h}_1$ and replacing the derivatives of Φ_i's in the direction of $\vec{h}_1$ by their expressions from the first step. The resulting system of algebraic equations for $\Phi_{m+1},\dots,\Phi_n$ is summarized in the following

Proposition 5[2)] **([6, a combination of Proposition 3.4, Proposition 4.3, Proposition 3.10 applied to the case of affine equivalence]).** *Assume that the sub-Riemannian metrics g_1 and g_2 are projectively equivalent in a neighborhood $U\subset M$ and let Φ be the corresponding orbital diffeomorphism between the normal extremal flows of g_1 and g_2 with coordinates $(\Phi_1,\dots,\Phi_n)$. Set*

$$\widetilde{\Phi}=\alpha(\Phi_{m+1},\dots,\Phi_n).$$

[2)] Since in the present paper we mainly work with the affine equivalence only, for which α_i's are constant and $\vec{h}_1(\alpha)=0$, the expressions in (3.18) and (3.22) are significantly simpler than in [6], where the more general case of the projective equivalence is considered.

Let also

$$q_{jk} = \sum_{i=1}^{m} c_{ij}^{k} u_i \tag{3.17}$$

and

$$R_j = \alpha_j^2 \vec{h}_1(u_j) - \sum_{1 \leqslant i,k \leqslant m} c_{ij}^{k} \alpha_k^2 u_i u_k. \tag{3.18}$$

Then $\widetilde{\Phi}$ *satisfies a linear system of equations,*

$$A\widetilde{\Phi} = b, \tag{3.19}$$

where A *is a matrix with* $(n-m)$ *columns and an infinite number of rows, and* b *is a column vector with an infinite number of rows. These infinite matrices can be decomposed in layers of* m *rows, each as*

$$A = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^s \\ \vdots \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^s \\ \vdots \end{pmatrix}, \tag{3.20}$$

where the coefficients a_{jk}^s *of the* $(m \times (n-m))$ *matrix* A^s, $s \in \mathbb{N}$, *are defined by induction as*

$$\begin{cases} a_{j,k}^1 = q_{jk}, & 1 \leqslant j \leqslant m,\ m < k \leqslant n, \\ a_{j,k}^{s+1} = \vec{h}_1(a_{j,k}^s) + \displaystyle\sum_{l=m+1}^{n} a_{j,l}^s q_{lk}, & 1 \leqslant j \leqslant m,\ m < k \leqslant n, \end{cases} \tag{3.21}$$

(*note that the columns of* A *are numbered from* $m+1$ *to* n *according to the indices of* $\widetilde{\Phi}$) *and the coefficients* b_j^s, $1 \leqslant j \leqslant m$, *of the vector* $b^s \in \mathbb{R}^m$ *are defined by*

$$\begin{cases} b_j^1 = R_j, \\ b_j^{s+1} = \vec{h}_1(b_j^s) - \displaystyle\sum_{k=m+1}^{n} a_{j,k}^s \sum_{i=1}^{m} u_i \alpha_i^2 q_{ki} \end{cases} \tag{3.22}$$

Definition 12. The system (3.19) with A and b defined recursively by (3.21) and (3.22) is called the *fundamental algebraic system* for the affine equivalence of the sub-Riemannian metrics g_1 and g_2[3]. The subsystem

$$A^i\widetilde{\Phi} = b^i \tag{3.23}$$

with A^i and b^i as in (3.20) is called the *ith layer of the fundamental algebraic system* (3.19).

The matrix A has $n-m$ columns and infinitely many rows and b is the infinite-dimensional column vector. So, the fundamental algebraic system (3.19) is an over-determined linear system on $(\Phi_{m+1}, \dots, \Phi_n)$, and all entries of A and b are polynomials (3.21) and (3.22) in u_j's. Therefore,

[3] The column vector b in (3.19) here corresponds to αb in the notation of the fundamental algebraic system in [6]. It is more convenient in the context of affine equivalence as all components of b become polynomial in u_j's, i. e., the polynomials on the fibers of T^*M.

all $(n-m+1)\times(n-m+1)$ minors of the augmented matrix $[A|b]$ must be equal to zero. Since all of these minors are polynomials in u_j's, the coefficient of every monomial of these polynomials is equal to zero. It results in a huge collection of constraints on the structure coefficient c_{ij}^k. By discovering and analyzing the monomials with the "simplest" coefficients we were able to prove our main theorem, Theorem 3. This analysis is given, for example, in Lemmas 4, 5, and 7. Quasi-normal forms (4.8) for ad-surjective Lie algebras were crucial for this analysis.

4. PROOF OF THEOREM 3

Let (M, D, g_1) be a sub-Riemannian metric satisfying the assumptions of Theorem 3. Assume that g_2 is a sub-Riemannian metric that is affinely equivalent and nonconstantly proportional to g_1 in a neighborhood U of a point q_0. Let k be the number of distinct eigenvalues of the transition operators of the pair of metrics (g_1, g_2). As mentioned after Proposition 3, k is constant on U and by Remark 5 we have $2 \leqslant k \leqslant \hat{k}$. Consider the sub-distributions D_i, $i \in [1:k]$ defined by (3.5), and the algebras $\mathfrak{m}^i(q)$ as in (3.6)–(3.7). Hence,

$$D = D_1 \oplus D_2 \oplus \ldots \oplus D_k. \tag{4.1}$$

By Remark 2 the algebra $\mathfrak{m}^i$ for every $i \in [1:k]$ is ad-surjective.

4.1. Main Steps in the Proof of Theorem 3

Observe that in general

$$D_i(q) \subset D(q) \cap (D_i)^2(q), \quad \forall i \in [1:\tilde{k}]. \tag{4.2}$$

One can define the canonical projection of quotient spaces

$$\mathrm{pr}_i : (D_i)^2(q)/D_i(q) \to (D_i)^2(q)/\left(D(q) \cap (D_i)^2(q)\right). \tag{4.3}$$

Further, given $X \in D_i(q)$, we can define two different operators

$$\begin{aligned} &(\mathrm{ad}\, X)_{\mathrm{mod}\, D} : D(q) \to D^2(q)/D(q), \\ &(\mathrm{ad}\, X)_{\mathrm{mod}\, D_i} : D_i(q) \to (D_i)^2(q)/D_i(q), \end{aligned} \tag{4.4}$$

where in the first case we apply the Lie brackets with X as in the Tanaka symbol of the distribution D at q and in the second case we apply the Lie brackets with X as in the Tanaka symbol of the distribution D_i at q.

The main steps in the proof of Theorem 3 are described by the following five propositions together with the final step in Section 4.7 below:

Proposition 6. *Assume that $X \in D_i(q)$ is such that the restriction of the map $(\mathrm{ad}\, X)_{\mathrm{mod}\, D}$ to $D_i(q)$ is onto $(D_i)^2(q)/\left(D(q) \cap (D_i)^2(q)\right)$. Then the projection pr_i as in* (4.3) *defines the bijection between the image of the map $(\mathrm{ad}\, X)_{\mathrm{mod}\, D_i}$ and the image of the restriction of the map $(\mathrm{ad}\, X)_{\mathrm{mod}\, D}$ to $D_i(q)$.*

Proposition 7. *Assume that $X \in D_i(q)$ satisfies the assumption of the previous lemma. Then $(D_i)^2(q)/D_i(q)$ coincides with the image of the map $(\mathrm{ad}\, X)_{\mathrm{mod}\, D_i}$.*

Proposition 8. *The following identity holds:*

$$D(q) \cap (D_i)^2(q) = D_i(q) \quad \forall i \in [1:\tilde{k}]. \tag{4.5}$$

Proposition 9. *The following identity holds:*

$$(D_i)^3(q) = (D_i)^2(q) \quad \forall i \in [1:\tilde{k}] \tag{4.6}$$

and therefore the distribution D_i^2 is involutive.

Proposition 10. *For every two distinct r and t from $[1:k]$ the distribution $D_r^2 + D_t^2$ is involutive.*

First, let us show that Propositions 6 and 7 imply Proposition 8. Indeed, we have the following chain of inequalities/equalities:

$$\dim(D_i)^2(q)/D_i(q) \overset{(4.2)}{\leqslant} \dim(D_i)^2(q)/\left(D(q)\cap(D_i)^2(q)\right) \overset{\text{Prop. 7}}{=}$$

$$\operatorname{rank}\left((\operatorname{ad}X)_{\operatorname{mod}D_i}\right) \overset{\text{Prop.6}}{=} \operatorname{rank}\left((\operatorname{ad}X)_{\operatorname{mod}D}|_{D_i(q)}\right) \leqslant \dim(D_i)^2(q)/D_i(q).$$

Hence, $\dim(D_i)^2(q)/D_i(q) = \dim(D_i)^2(q)/\left(D(q)\cap(D_i)^2(q)\right)$, which implies (4.5).

Further, as a direct consequence of Propositions 8 and 9 and the assumption that D is a step 2 distribution one gets the following

Corollary 3. *The Tanaka symbol of D_i at q is isomorphic to $\mathfrak{m}^i(q)$ and the distribution D_i^2 is of rank equal to* $\dim\mathfrak{m}^i$.

To guide the reader, the rest of the proof of Theorem 3 is organized as follows: Proposition 6 is proved in Section 4.2, Proposition 10 is proved in Sections 4.3 and 4.6, Proposition 9 is proved in Section 4.5, and Proposition 7 is proved in Section 4.4. The final step in the proof of Theorem 3 is done in Section 4.7.

4.2. *Proof of Proposition* 6

Let $\mathfrak{m}^i(q)$, $i=1,2$ be as in (3.6)–(3.7). Note that by the paragraph after Proposition 3 and the fact that the graded algebras $\mathfrak{m}^i(q)$ are of step not greater than 2, $\dim\mathfrak{m}^i_j(q)$ are independent of q. Let

$$\begin{aligned} &m_i := \dim\mathfrak{m}^i_{-1}(q), \quad d_i := \dim\mathfrak{m}^i_{-2}(q);\\ &n_i := \sum_{j=1}^{i} m_j, \quad e_i := \sum_{j=1}^{i} d_j; \\ &\mathcal{I}_i^1 = \left[(n_{i-1}+1):n_i\right], \quad \mathcal{I}_i^2 = \left[(m+e_{i-1}+1):(m+e_i)\right]. \end{aligned} \tag{4.7}$$

Note that $n_0 = e_0 = 0$.

Since $\mathfrak{m}^i$ is ad-surjective for every $i \in [1:k]$, we can choose a local g_1-orthonormal basis $(X_1,\dots,X_m)$ of D such that the following conditions hold for every $i\in[1:k]$:

1) $D_i = \operatorname{span}\{X_j\}_{j\in\mathcal{I}_i^1}$;

2) $X_{n_{i-1}+1}(q)$ is an ad-generating (in a sense of Definition 9) element of the algebra $\mathfrak{m}^i(q)$.

Then one can complete $(X_1,\dots,X_m)$ to the local frame $(X_1,\dots,X_n)$ of TM by setting

$$X_{m+e_{i-1}+j} := [X_{n_{i-1}+1}, X_{n_{i-1}+j+1}], \quad \forall i\in[1:k], j\in[1:d_i]. \tag{4.8}$$

A local frame $(X_1,\dots,X_n)$ of TM constructed in this way will be called a *quasi-normal frame adapted to the tuple* $\{X_{n_{i-1}+1}\}_{i=1}^k$ of ad-generating elements (one for each $\mathfrak{m}^i$).

By construction, quasi-normality implies the following conditions for the structure functions of the frame:

$$c^l_{n_{i-1}+1,j} = \delta_{l,m+j+e_{i-1}-n_{i-1}-1}, \quad \forall i\in[1:k], j\in[n_{i-1}+2:n_{i-1}+d_i+1], l\in[1:n], \tag{4.9}$$

where $\delta_{s,t}$ stands for the Kronecker symbol.

In the sequel, we will work with quasi-normal frames: we start with one quasi-normal frame and, if necessary, perturb it to other quasi-normal frames adapted to the same tuple of ad-generating elements. The statement of Proposition 6 is true if one shows that pr_i restricted to $\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_i}$ is injective, while the surjectivity follows automatically from the definition of the projection. Without loss of generality, we can assume that $i = 1$, as the proof for $i \neq 1$ is completely analogous. The injectivity of $\mathrm{pr}_1|_{\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_1}}$ is equivalent to

$$\ker(\mathrm{pr}_1|_{\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_1}}) = 0. \tag{4.10}$$

Clearly,

$$\ker(\mathrm{pr}_1) = \left((D_1)^2(q) \cap D(q)\right)/D_1(q). \tag{4.11}$$

Set $X = X_1$ Then by (4.11)

$$\ker(\mathrm{pr}_1|_{\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_1}}) = \left(\left((D_1)^2(q) \cap D(q)\right)/D_1(q)\right) \cap \left(\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_1}\right). \tag{4.12}$$

So, the desired relation (4.10) is equivalent to

$$c^k_{1l} = 0, \text{ for } l \in \mathcal{I}^1_1, k \in \bigcup_{j=2}^{k} \mathcal{I}^1_j. \tag{4.13}$$

In the sequel, we will use the following proposition many times:

Proposition 11 ([11, Proposition 6]). *If g_1 and g_2 are affinely equivalent but not constantly proportional to each other, then the following properties hold:*

1) $c^j_{ji} = 0$, *for any $i \in \mathcal{I}^1_s$, $j \in \mathcal{I}^1_v$ with $s \neq v$;*

2) $c^i_{jk} = -c^k_{ji}$, *for any $i \neq k \in \mathcal{I}^1_s$, $j \in \mathcal{I}^1_v$ with $s \neq v$;*[4)]

3) $(\alpha^2_j - \alpha^2_i)c^k_{ji} + (\alpha^2_j - \alpha^2_k)c^i_{jk} + (\alpha^2_i - \alpha^2_k)c^j_{ik} = 0$ *for every pairwise distinct i, j, k from $[1 : m]$.*

Note that item (2) above is the consequence of item (3) applied to the case when one pair in the triple $\{i, j, k\}$ belongs to the same $\mathcal{I}^1_s$. In all subsequent lemmas, we will assume that the relations given in items (1)–(3) of Proposition 11 hold.

Now let us give more explicit expressions for the vector b in the fundamental algebraic system (3.19), which will be helpful in the sequel:

Lemma 2. *The entries b^1_j in (3.22) with $j \in \mathcal{I}^1_i$ are given by*

$$b^1_j = (\alpha_{n_{i-1}+1})^2 \sum_{l \in \mathcal{I}^2_i} q_{jl}u_l + \sum_{s \neq i} \left((\alpha_{n_{i-1}+1})^2 - (\alpha_{n_{s-1}+1})^2\right) \sum_{l \in \mathcal{I}^1_s} q_{jl}u_l, \tag{4.14}$$

where q_{jk} are defined by (3.17).

Proof. Using (3.12), (3.13), and (3.9), we get

$$\vec{h}_1(u_j) = \sum_{i=1}^{m} u_i \vec{u}_i(u_j) = \sum_{k=1}^{n}\sum_{i=1}^{m} u_i u_k c^k_{ij} = \sum_{k=1}^{n} q_{jk}u_k. \tag{4.15}$$

[4)] In more detail, one of the conclusions of [11, Proposition 6], formulated for projective equivalence, is that $X_i\left(\frac{\alpha_j^2}{\alpha_i^2}\right) = 2c^j_{ji}\left(1 - \frac{\alpha_j^2}{\alpha_i^2}\right)$, but in the case of affine equivalence α_i^2 and α_j^2 are constant and we find that $c^j_{ji} = 0$.

From (3.22) and (3.18), using (3.17) and the fact that by our constructions $(\alpha_l)^2 = (\alpha_{n_{i-1}+1})^2$ for every $l \in \mathcal{I}_i^1$, we have

$$\begin{aligned} b_j^1 &= (\alpha_{n_{i-1}+1})^2 \vec{h}_1(u_j) - \sum_{1 \leqslant i,l \leqslant m} (\alpha_l)^2 u_i u_l c_{ij}^l \\ &= (\alpha_{n_{i-1}+1})^2 \sum_{k=1}^{n} q_{jk} u_k - \sum_{i=1}^{k} (\alpha_{n_{i-1}+1})^2 \sum_{l \in \mathcal{I}_i^1} q_{jl} u_l. \end{aligned} \tag{4.16}$$

Note that from (3.8) and (3.17) we have $q_{jk} = 0$ for $k \in \bigcup_{s \neq i} \mathcal{I}_s^2$, so

$$\sum_{k=1}^{n} q_{jk} u_k = \sum_{s=1}^{k} \sum_{l=\mathcal{I}_s^1}^{m} q_{jl} u_l + \sum_{l \in \mathcal{I}_i^2} q_{jl} u_l. \tag{4.17}$$

Substituting (4.17) into (4.16), we get (4.14). □

Lemma 2 implies that, to analyze maximal minors of the augmented matrix $[A|b]$, it is convenient to perform the following column operation by setting

$$\tilde{b} = b - \sum_{i=1}^{k} (\alpha_{n_{i-1}+1})^2 \sum_{t=e_{i-1}}^{e_i} (A)_t u_{m+t}, \tag{4.18}$$

where $(A)_j$ represents the jth column of A and e_i are defined in (4.7), so the corresponding maximal minors of $[A|b]$ and $[A|\tilde{b}]$ coincide.

Remark 7. Using the first line of (3.21), one gets that the first term in (4.14) is canceled by the column operations, namely,

$$\tilde{b}_j^1 = \sum_{v \neq i} \left((\alpha_{n_{i-1}+1})^2 - (\alpha_{n_{v-1}+1})^2 \right) \sum_{l \in \mathcal{I}_v^1} q_{jl} u_l, \quad j \in \mathcal{I}_i^1, \tag{4.19}$$

where $\tilde{b}_j^1$ is the jth component (from the top) of the column vector $\tilde{b}^1$. Hence, $\tilde{b}^1$ has no term of u_i's, with $i > m$.

Lemma 3. *Let $i, v \in [1:k]$, $i \neq v$, $j \in \mathcal{I}_i^1$ and $r, l \in \mathcal{I}_v^1$. Then $\tilde{b}_j$ does not contain a monomial $u_r u_l$ and, in particular, it does not contain squares u_r^2.*

Proof. Indeed, by (4.19), using (3.17), the coefficient of the monomial $u_r u_l$ is equal to

$$\left((\alpha_{n_{i-1}+1})^2 - (\alpha_{n_{v-1}+1})^2 \right) \left(c_{rj}^l + c_{lj}^r \right), \tag{4.20}$$

which is equal to zero by items (1) and (2) of Proposition 11. □

Now we will make a long analysis of coefficients of specific monomials in the specific $(n - m + 1) \times (n - m + 1)$ minor of the augmented matrix $[A|b]$ (equivalently, $[A|\tilde{b}]$).

First, to achieve (4.13), given $i_0 \in [1:k]$, we consider submatrices M_{i_0} of $[A|\tilde{b}]$ consisting of rows with indices from the set

$$S_{i_0} := [n_{i_0-1} + 1 : n_{i_0-1} + d_{i_0} + 1] \cup \bigcup_{i \in [1:k] \setminus \{i_0\}} [n_{i-1} + 1 : n_{i-1} + d_i]. \tag{4.21}$$

From (3.8) it follows that M_{i_0} is a block-diagonal matrix,

$$M_{i_0} = \begin{pmatrix} M_{i_0,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & M_{i_0,k} \end{pmatrix}, \tag{4.22}$$

where M_{i_0,i_0} is of size $(d_{i_0}+1)\times d_{i_0}$ and $M_{i_0,i}$, $i\neq i_0$ is of size $d_{i_0}\times d_{i_0}$.

Remark 8. Note that, given $j\in[1:k]$, the blocks $M_{i,j}$ are the same for all $i\neq j$.

Let $b^1|_{S_{i_0}}$ (resp. $\tilde{b}^1|_{S_{i_0}}$) be the subcolumn of the column b^1 (reps. $\tilde{b}^1$) consisting of the same rows as in M_{i_0}, i.e., the rows of b (resp. $\tilde{b}$) from the set S_{i_0} as in (4.21). Since the fundamental algebraic system is an overdetermined linear system admitting a solution, the determinant $\det([M_{i_0}|b^1|_{S_{i_0}}])$ must vanish, as a polynomial with respect to u_i's. It implies that the coefficients of each monomial w.r.t u_i's in $\det(M_1|b^{1,1})$ must vanish as well. We have the following

Lemma 4. *Given $i_0\in[1:k]$, if the coefficients of all monomials of the form*

$$u_l u_s\left(\prod_{i\neq i_0} u_{n_{i-1}+d_i+1}\right)(u_{n_{i_0-1}+1})^{d_{i_0}}\left(\prod_{i\neq i_0}(u_{n_{i-1}+1})^{d_i-1}\right),\quad l\in\mathcal{I}^1_{i_0}, s\in[1:m]\backslash\mathcal{I}^1_{i_0} \tag{4.23}$$

in $\det([M_{i_0}|b^1|_{S_{i_0}}])$ *vanish, then*

$$c^s_{n_{i_0-1}+1\,l}=0,\quad l\in\mathcal{I}^1_{i_0}, s\in[1:m]\backslash\mathcal{I}^1_{i_0}. \tag{4.24}$$

Proof. In the sequel, we will refer to the classical formula for determinants in terms of permutations of the matrix elements as the *Leibniz formula for determinants*. Without loss of generality, we can assume that $i_0=1$ and $s\in\mathcal{I}^1_2$. Using (3.17), the first line of (3.21), and (4.9), it is easy to conclude that the variable $u_{n_{i-1}+1}$ appears only in the following columns of the augmented matrix $[M_1|b^1|_{S_1}]$:

(A1$_i$) The columns containing all columns of the matrix $M_{1,i}$ if $i=1$ or all columns of $M_{1,i}$ except the last one, if $i\in[2:k]$. Moreover, in each of these columns, $u_{n_{i-1}+1}$ appears exactly in the entry $(M_{1,i})_{j+1j}$, i.e., in the entry situated right below the diagonal of the block $M_{1,i}$. Besides, the coefficient of $u_{n_{i-1}+1}$ in this entry is equal to 1;

(A2$_i$) the last column of the matrix $[M_1|b^1|_{S_1}]$.

Applying the normalization conditions (4.9) to (4.19), one gets

(A3$_i$) The components of the column vector $b^1|_{S_1}$ in the rows corresponding to the rows of the block $M_{1,i}$ of M_1 do not contain $u_{n_{i-1}+1}$.

For our purpose, it is enough to set variables u_i not appearing in (4.23) to be equal to 0. From item (A1$_i$) above, the fact that $u_{n_{i-1}+1}$ appears in the power not less than d_i-1 in the monomial (4.23), and that by Lemma 3 and (A3$_i$) the participating entries do not contain $(u_{n_{i-1}+1})^2$, it follows that in the Leibniz formula for the determinant of the matrix $(M_1|\tilde{b}^1|_{S_1})$ the contribution to the monomial (4.23) from the block $M_{1,i}$ comes only from the following terms:

(B1$_i$) *Terms containing all factors of the form $(M_{1,i})_{j+1\,j}$*. Moreover, from the facts that all rows of $M_{1,i}$ except the first one are used, that M_1 has block diagonal structure, and that

 (a) for $i=1$ all columns appearing in $M_{1,i}$ are used, it follows that in this case the terms giving the desired contribution must contain the factor b^1_1 as the only possible factor from the first row of the augmented matrix $(M_1|b^1|_{S_1})$.

(b) for $i \in [2:k]$ all columns of $M_{1,i}$ except the last one are used, it follows that in this case the terms giving the desired contribution must contain the entry $(M_{1,i})_{1\,d_2}$. Note that, if we take into account only those u_i's which appear in (4.23) (or, equivalently, set all other u_i's equal to zero) and that, by (4.9), $c^{m+e_1}_{n_1+1,s} = \delta_{n_1+d_2+1,s}$, we have

$$(M_{1,i})_{1\,d_i} = -u_{n_{i-1}+d_i+1}. \tag{4.25}$$

($B2_i$) (possible only if either $i = 1$ or $i = 2$ and $s \neq n_1 + 1$[5]) *Terms containing all factors of the form* $(M_{1,i})_{j+1\,j}$, $j \in [1 : d_2 - 1]$ *except one.* Then these terms also contain a factor from the column $b^1|_{S_1}$ depending on the variable $u_{n_{i-1}+1}$.

Now consider four possible cases separately:

(C1) Assume that ($\mathbf{B2_1}$) and ($\mathbf{B2_2}$) occur simultaneously. Since for every $i \in [3 : k]$ item $(B1_i)$ holds, the participating factor b^1_j must satisfy $j \in [1 : d_1 + 1] \cup [n_1 + 1 : n_1 + d_2]$. On the other hand, it must contain the monomial $u_1 u_{n_1+1}$, which by $(A3_1)$ and $(A3_2)$ implies that $j \notin [1 : d_1 + 1] \cup [n_1 + 1 : n_1 + d_2]$, so we got the contradiction. So, the considered term does not contribute to the monomial (4.23).

(C2) Assume that ($\mathbf{B1}_i$) for all $i \in [2 : k]$ and ($\mathbf{B2_1}$) occur simultaneously. Then the participating factor b^1_j from $b^1|_{S_1}$ must, on the one hand, satisfy $j \in [1 : d_1]$ and, on the other hand, must contain u_1, which contradicts $(A3_1)$. So, the considered term does not contribute to the monomial (4.23).

(C3) Assume that ($\mathbf{B1}_i$) for $i \neq 2$ and ($\mathbf{B2_2}$) occur simultaneously. In this case the contribution to the monomial (4.23) is from the coefficient of the monomial $u_l u_{n_1+1}$ in the factor b^1_1, which is equal to $((\alpha_{n_1+1})^2 - \alpha_1^2)c^{n_1+1}_{1l}$.

(C4). Assume that ($\mathbf{B1}_i$) occur simultaneously for every $i \in [1 : k]$. In this case, the coefficient of the monomial (4.23) is equal, up to a sign, to the coefficient of the monomial $u_l u_s u_{n_1+d_2+1}$ in the polynomial $b^1_1(M_{1,2})_{1,d_2}$, because the coefficients of the relevant monomials in all other factors in the corresponding term of Leibniz formula are equal to 1. From (4.25) it follows that the coefficient of the monomial $u_l u_s u_{n_1+d_2+1}$ in the polynomial $b^1_1(M_{1,2})_{1,d_2}$ is equal to the coefficient of $u_l u_s$ in b^1_1, which by (4.19) is equal to

$$\big((\alpha_{n_1+1})^2 - \alpha_1^2\big)c^s_{1l}. \tag{4.26}$$

Now we are ready to complete the proof of the lemma. First, assume that $s = n_1 + 1$. Then the case $(B2_2)$ and therefore (C3) is impossible, so (C4) holds and the coefficient of the monomial (4.23) is equal, up to a sign, to the expression in (4.26) with $s = n_1 + 1$, so vanishing of this coefficient implies (4.24) for $s = n_1 + 1$.

Further, the last paragraph implies that the case (C3) does not contribute to the coefficients of the monomials (4.23), so the only contribution is from the case (C4), which is equal to the expression in (4.26). Vanishing of the latter implies (4.24), which completes the proof of Lemma 4. □

Now given $i \in [1 : k]$, assume that $X_{n_{i-1}+1}$ is a local section of D_i such that $X_{n_{i-1}+1}(q)$ is an ad-generating element of $\mathfrak{m}^i(q)$ in the sense of Definition (9) for every q. Let

$$K_{X_{n_{i-1}+1}} := \ker\big((\mathrm{ad}\, X_{n_{i-1}+1})_{\mathrm{mod} D_i}\big) \tag{4.27}$$

$$H_{X_{n_{i-1}+1}} := (\mathrm{span}\{X_{n_{i-1}+1}\})^{\perp} \cap D_i, \tag{4.28}$$

where $^{\perp}$ stands for the g_1-orthogonal complement. As a direct consequence of Lemma 4, we get the following

[5] Otherwise the desired power of $u_{n_{i-1}+1}$ in (4.23) cannot be achieved.

Corollary 4. *For any d_i-dimensional subspace F_i of $H_{X_{n_{i-1}+1}}$ with*

$$F_i \cap K_{X_{n_{i-1}+1}} = 0 \tag{4.29}$$

the image of the restriction of the map $(\operatorname{ad} X_{n_{i-1}+1})_{\operatorname{mod} D_1}$ to the subspace F_i coincides with the entire image of the map $(\operatorname{ad} X_{n_{i-1}+1})_{\operatorname{mod} D_i}$,

$$\operatorname{Im}\Big((\operatorname{ad} X_{n_{i-1}+1})_{\operatorname{mod} D_i}\Big|_{F_i}\Big) = \operatorname{Im}\Big((\operatorname{ad} X_{n_{i-1}+1})_{\operatorname{mod} D_i}\Big). \tag{4.30}$$

In particular, this image of the restriction is independent of F_i.

Proof. Indeed, previously we used $\operatorname{span}\{X_{n_{i-1}+1}, \ldots, X_{n_i}\}$ as a subspace F_i, but relation (4.29) was the only property we actually used to get the conclusion of Lemma 4. □

We will denote the space in (4.30) by $\mathcal{L}_{X_{n_{i-1}+1}}$,

$$\mathcal{L}_{X_{n_{i-1}+1}} := \operatorname{Im}\big((\operatorname{ad} X_{n_{i-1}+1})_{\operatorname{mod} D_i}\big). \tag{4.31}$$

Remark 9. Note that, if $d_i = 0$, then any element of D_i is trivially an ad-generating element of $\mathfrak{m}_i$.

Remark 10. If $d_i = 1$, then $\dim K_{X_{n_{i-1}+1}} = \dim H_{X_{n_{i-1}+1}} = m_i - 1$ but $K_{X_{n_{i-1}+1}} \neq H_{X_{n_{i-1}+1}}$, because $X_{n_{i-1}+1}$ belongs to $K_{X_{n_{i-1}+1}}$ but does not belong to $H_{X_{n_{i-1}+1}}$, so $K_{X_{n_{i-1}+1}} \cap H_{X_{n_{i-1}+1}}$ is a codimension 1 subspace $H_{X_{n_{i-1}+1}}$. Therefore, we can perturb the original quasi-normal frame $(X_1, \ldots, X_n)$ to a quasi-normal frame $(\tilde{X}_1, \ldots, \tilde{X}_n)$ adapted to the same tuple of ad-generating elements as the original frame such that for every $i \in [1:k]$ and $j \in \mathcal{I}_i^1$ the line generated by $\tilde{X}_j$ is transversal to $K_{X_{n_{i-1}+1}}$. Since $d_i = 1$ and by Corollary 4 $[X_{n_{i-1}+1}, \tilde{X}_j] \neq 0 \operatorname{mod} D$ (otherwise $\tilde{X}_j \in K_{X_{n_{i-1}+1}}$), we get that $\tilde{X}_j$ is also an ad-generating element of $\mathfrak{m}_i$.

4.3. Proof that $[D_r, D_t] \in \mathcal{L}_{X_{n_{r-1}+1}} + \mathcal{L}_{X_{n_{t-1}+1}} \mod D_r + D_t$ $r \neq t$ (*Toward the Proof of Proposition* 10)

The claim in the title of the subsection is equivalent to

$$\begin{aligned} &c_{jl}^s = 0 \quad \forall j \in \mathcal{I}_r^1, s \in \mathcal{I}_t^1, l \in \mathcal{I}_i^1, \\ &\text{where}\{i, r, t\} \in [1:k] \text{ are pairwise distinct.} \end{aligned} \tag{4.32}$$

In particular, for $k \leqslant 2$ Eq. (4.32) is void, so it is relevant to assume that $k \geqslant 3$.

Given $i_0 \in [1:k]$, consider submatrices P_{i_0} of $[A|\tilde{b}]$ consisting of rows with indices from the set

$$R_{i_0} := \left(\bigcup_{i \in [1:k] \setminus \{i_0\}} [n_{i-1}+2 : n_{i-1}+d_i+1] \right) \cup [n_{i_0-1}+1 : n_{i_0-1}+d_{i_0}+1]. \tag{4.33}$$

From (3.8) it follows that P_{i_0} is a block-diagonal matrix,

$$P_{i_0} = \begin{pmatrix} P_{i_0,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P_{i_0,k} \end{pmatrix}, \tag{4.34}$$

where P_{i_0,i_0} is of size $(d_{1_0}+1) \times d_{i_0}$ and $P_{i_0,i}$, $i \neq i_0$ is of size $d_{i_0} \times d_{i_0}$. Note that by construction

$$P_{i_0,i_0} = M_{i_0,i_0}, \tag{4.35}$$

where M_{i_0,i_0} is defined in (4.22).

Similar to the previous subsection, let $\tilde{b}^1|_{P_{i_0}}$ be the subcolumn of the column $\tilde{b}^1$ consisting of the same rows as in P_{i_0},

Lemma 5. *If for every triple of pairwise distinct integers* $\{i_0, r, t\} \in [1:k]$ *the coefficient of all monomials of the form*

$$u_j u_l u_s u_{n_{i_0-1}+1}^{d_{i_0}-1} \prod_{i\in[1:k]\setminus\{i_0\}} u_{n_{i-1}+1}^{d_i}, \quad \text{where } j \in [n_{r-1}+1 : n_{r-1}+d_r+1], \\ l \in [n_{t-1}+1 : n_{t-1}+d_t+1], s \in [n_{i_0-1}+1 : n_{i_0-1}+d_{i_0}+1] \tag{4.36}$$

in the determinant $\det[P_{i_0}|\tilde{b}|_{R_{i_0}}]$ *vanishes, then*

$$c^s_{jl} = 0, \text{where } j \in [n_{r-1}+1 : n_{r-1}+d_r+1], \\ l \in [n_{t-1}+1 : n_{t-1}+d_t+1], s \in [n_{i_0-1}+1 : n_{i_0-1}+d_{i_0}+1] \tag{4.37}$$

or, equivalently,

$$[X_j, X_l] \in \mathcal{L}_r + \mathcal{L}_t \mod(D_r + D_t), \\ j \in [n_{r-1}+1 : n_{r-1}+d_r+1], l \in [n_{t-1}+1 : n_{t-1}+d_t+1]. \tag{4.38}$$

Proof. Without loss of generality, we can assume that $r = 1$, $t = 2$, and $i_0 = k$. Further, using (4.35), similarly to the proof of Lemma 4 (statements of items $(A1_i)$ and $(A2_i)$ there), one can conclude that the variable $u_{n_{i-1}+1}$, $i \in [1:k]$ appears only in the following columns of the augmented matrix $[N_1|\tilde{b}|_{T_1}]$:

(D1$_i$) The columns containing all columns of $P_{k,i}$ if $i \in [1:k-1]$, and all columns of $P_{k,i}$ except the last one, if $i = k$. Moreover, in each of these columns $u_{n_{i-1}+1}$ appears exactly in the diagonal entry $(P_{k,i})_{jj}$ if $i \in [1:k-1]$, and in the entry $(P_{k,i})_{j+1j}$, i. e., in the entry situated right below the diagonal of the block $P_{k,i}$, if $i = k$. Besides, the coefficient of $u_{n_{i-1}+1}$ in this entry is equal to 1;

(D2$_i$) the last column of the matrix $[P_k|\tilde{b}|_{R_k}]$;

As before, we can set all irrelevant variables u's (which are not in (4.36)) to zero. From this and the normalization condition (4.9) it follows that

(D3) The only nonzero entry of the first row of $P_{k,k}$ is the entry in the $s - n_{k-1} - 1$ column, i. e., $P_{k,k}|_{1,s-n_{k-1}1}$ and it is equal to $-u_s$.

Finally, the statement of $(A3_i)$ with M_1 replaced by P_k, $M_{1,i}$ replaced by $P_{1,i}$, and S_1 replaced by R_k holds true, i. e.,

(D4$_i$) The components of the column vector $b^1|_{R_1}$ in the rows corresponding to the rows of the block $P_{k,i}$ of P_k do not contain $u_{n_{i-1}+1}$.

From item (D1$_i$) above, the fact that $u_{n_{i-1}+1}$ appears in the monomial (4.36) in the power d_i for $i \in [1:k-1]$ and in the power not less than $d_i - 1$ if $i = k$, and that by Lemma 3 and $(D4_i)$ the participating entries do not contain $(u_{n_{i-1}+1})^2$ it follows that in the Leibniz formula for the determinant of the matrix $(P_k|\tilde{b}^1|_{R_k})$ the contribution to the monomial (4.36) from the block $P_{k,i}$ comes only from the following terms:

(E1) *Terms containing all factors of the form* $(P_{k,i})_{j\,j}$, $j \in [1:d_i]$, *if* $i \in [1:k-1]$. Otherwise, if one of them is omitted, then a nondiagonal entry of $P_{k,i}$ has to be used as well, therefore, another diagonal term of $P_{k,i}$ has to be omitted, however, in this way there is no way to reach $u_{n_{i-1}}$ in the power d_i in the resulting monomial.

(E2) *Terms containing all factors of the form* $(P_{k,i})_{j\,j+1}$, $j \in [1 : d_k]$, *if* $i = k$ *and* $s = n_{k-1} + 1$. In this case, the desired contribution must contain the component of $\tilde{b}^1$ which belongs to the row, corresponding to the first row of $P_{k,k}$, i. e., $\tilde{b}_{n_{k-1}+1}$.

(E3) *Terms containing all factors of the form* $(P_{k,k+1})_{j+1\,j}$, $j \in [1 : d_k$ *except one, if* $i = k$ *and* $s \neq n_{k-1} + 1$ .Then by (D3) and the fact that P_k has the block-diagonal form as in (4.34), it follows that the omitted factor is $(Pk, k_{s-n_{k-1}, s-n_{k-1}-1})$. Hence, the desired contribution must contain the component of $\tilde{b}^1$ which belongs to the row, corresponding to the $s - n_{k-1}$th row of $P_{k,k}$, i. e., the component $\tilde{b}_s$.

So, we have two cases:

(F1) $s = n_{k-1} + 1$. In this case, by combining (E1) and (E2), we conclude that the coefficient of the monomial (4.36) in $\det[P_{i_0}|\tilde{b}|_{R_{i_0}}]$ is, up to a sign, equal to the coefficient of the monomial $u_j u_l$ in $\tilde{b}_{n_{k-1}+1}$, which by (4.19) is equal to

$$\left(\alpha_k^2 - \alpha_1^2\right) c^j_{l,n_{k-1}+1} + \left(\alpha_k^2 - \alpha_2^2\right) c^l_{j,n_{k-1}+1}. \tag{4.39}$$

(F2) $s \neq n_{k-1} + 1$. In this case, by combining (E1) and (E3) and using (4.19), we conclude that the coefficient of the monomial (4.36) in $\det[P_{i_0}|\tilde{b}|_{R_{i_0}}]$ is, up to a sign, equal to the coefficient of the monomial $u_j u_l$ in $\tilde{b}_s$, which by (4.19) is equal to (4.36)

$$\left(\alpha_k^2 - \alpha_1^2\right) c^j_{l,s} + \left(\alpha_k^2 - \alpha_2^2\right) c^l_{j,s} = 0. \tag{4.40}$$

So, by the assumptions of the lemma, expressions in (4.39) and (4.40) vanish.

Repeating the same arguments for an arbitrary pairwise distinct triple $\{r, s, i_0\} \in [1 : k]$ instead of $\{1, 2, k\}$, we will get that

$$\begin{aligned} &\left(\alpha_{i_0}^2 - \alpha_r^2\right) c^j_{l,s} + \left(\alpha_{i_0}^2 - \alpha_t^2\right) c^l_{j,s} = 0, \quad \forall j \in [n_{r-1} + 1 : n_{r-1} + d_r + 1], \\ &l \in [n_{t-1} + 1 : n_{t-1} + d_t + 1], s \in [n_{i_0-1} + 1 : n_{i_0-1} + d_{i_0} + 1]. \end{aligned} \tag{4.41}$$

Permuting the indices in (4.41), we can get

$$\left(\alpha_t^2 - \alpha_r^2\right) c^j_{s,l} + \left(\alpha_t^2 - \alpha_{i_0}^2\right) c^s_{j,l} = \left(\alpha_r^2 - \alpha_t^2\right) c^j_{l,s} + \left(\alpha_t^2 - \alpha_{i_0}^2\right) c^s_{j,l} = 0. \tag{4.42}$$

Finally, item (3) of Proposition 11 implies

$$\left(\alpha_{i_0}^2 - \alpha_t^2\right) c^j_{s,l} + \left(\alpha_r^2 - \alpha_{i_0}^2\right) c^l_{j,s} + \left(\alpha_r^2 - \alpha_t^2\right) c^s_{j,l} = 0. \tag{4.43}$$

The linear homogeneous system with respect to $c^j_{l,s}$, $c^l_{j,s}$, and $c^s_{j,s}$, consisting of Eqs. (4.41)–(4.43), has the matrix

$$\begin{pmatrix} \alpha_{i_0}^2 - \alpha_r^2 & \alpha_{i_0}^2 - \alpha_t^2 & 0 \\ \alpha_r^2 - \alpha_t^2 & 0 & \alpha_t^2 - \alpha_{i_0}^2 \\ \alpha_{i_0}^2 - \alpha_t^2 & \alpha_r^2 - \alpha_{i_0}^2 & \alpha_r^2 - \alpha_t^2 \end{pmatrix}, \tag{4.44}$$

whose determinant is equal to

$$(\alpha_t^2 - \alpha_{i_0}^2)\Big((\alpha_{i_0}^2 - \alpha_j^2)^2 + (\alpha_r^2 - \alpha_t^2)^2 + (\alpha_{i_0}^2 - \alpha_t^2)^2\Big)$$

and is not zero as by our assumption that $\alpha_{i_0}^2$, α_r^2, and α_t^2 are pairwise distinct. This implies (4.37). □

Lemma 5 proves only a subset of relations from (4.32). Using the flexibility given by Corollary 4, one can show that (4.32) holds not for the original quasi-normal frame but for its perturbation adapted to the same tuple of ad-generating vector fields $\{X_{n_{i-1}+1}\}_{i=1}^k$.

Corollary 5. *Let $K_{X_{n_{i-1}+1}}$ and $H_{X_{n_{i-1}+1}}$ be as in (4.27) and (4.28), respectively. For every $i \in [1:k]$ let Y_i be either equal to $X_{n_{i-1}+1}$ or a normalized (i. e., $g(Y_i, Y_i) = 1$) local section of $H_{X_{n_{i-1}+1}}$ so that*

$$Y_i \notin K_{X_{n_{i-1}+1}}. \tag{4.45}$$

Then for every $r \neq t$ from $[1:k]$

$$[Y_r, Y_t] \in \mathcal{L}_r + \mathcal{L}_t \mod(D_r + D_t). \tag{4.46}$$

Proof. Indeed, (4.45) implies that we can find a quasi-normal frame $(X_1, \dots, X_n)$ such that either $X_{n_{i-1}+1} = Y_i$ or $X_{n_{i-1}+2} = Y_i$. Then (4.46) follows from (4.38) of Lemma 5. □

If $d_i \geqslant 1$, then $\dim K_{X_{n_{i-1}+1}} \geqslant \dim H_{X_{n_{i-1}+1}}$ and $K_{X_{n_{i-1}+1}} \neq H_{X_{n_{i-1}+1}}$. The latter holds because $X_{n_{i-1}+1}$ is in $K_{X_{n_{i-1}+1}}$ but not in $H_{X_{n_{i-1}+1}}$. So, in this case the complement of $K_{X_{n_{i-1}+1}} \cap H_{X_{n_{i-1}+1}}$ to $H_{X_{n_{i-1}+1}}$ is open and dense in $H_{X_{n_{i-1}+1}}$. Therefore, by a finite number of consecutive small perturbations of the original quasi-normal frame $(X_1, \dots, X_n)$, one can build a quasi-normal frame $(\tilde{X}_1, \dots, \tilde{X}_n)$ adapted to the same tuple of ad-generating elements as the original frame such that for every $r \in [1:k]$

- if $d_r \geqslant 1$, then every $j \in \mathcal{I}_r^1 \backslash \{n_{r-1}+1\}$, the vector field $\tilde{X}_j$ is in the complement of $K_{X_{n_{i-1}+1}} \cap H_{X_{n_{i-1}+1}}$ to $H_{X_{n_{i-1}+1}}$;
- if $d_r = 0$, then no additional conditions on $\tilde{X}_j$ with $j \in \mathcal{I}_r^1$ are imposed and by permutation any $\tilde{X}_j$ can be seen as $X_{n_{r-1}+1}$ (note that in this case any element of D_r is trivially an ad-generating element of $\mathfrak{m}_r$).

So, by (4.46) this frame satisfies

$$[X_j, X_l] \in \mathcal{L}_r + \mathcal{L}_t \mod(D_r + D_t), j \in \mathcal{I}_r^1, l \in \mathcal{I}_t^1, \quad r \neq t \in [1:k], \tag{4.47}$$

which is equivalent to (4.32).

Remark 11. Note that, if (4.47) holds for some quasi-normal frame, then it holds for any quasi-normal frame adapted to the same tuple of ad-generating elements as the original one.

As the direct consequence of (4.19) and (4.32), we get that for any triple $\{i, r, t\}$ of pairwise distinct integers from $[1:k]$ and any $j \in \mathcal{I}_i^1$, $j \in \mathcal{I}_r^1$, and $l \in \mathcal{I}_t^1$, the polynomial $\tilde{b}_j$ does not contain a monomial $u_r u_t$. Combining this with Lemma 3, we get

Lemma 6. *Given $j \in \mathcal{I}_i^1$, $\tilde{b}_j$ does not contain monomials $u_r u_l$ with $r, l \in [1:m] \backslash \mathcal{I}_i^1$, or, equivalently, every monomial of $\tilde{b}_j$ must contain a variable u_r with $r \in \mathcal{I}_i^1$.*

Moreover, from (4.19) and Lemma 4 it follows that

Corollary 6. *For every $i \in [1:k]$*

$$\tilde{b}_{n_{i-1}+1} = 0. \tag{4.48}$$

4.4. Proof of Proposition 7

The statement of Proposition 7 is equivalent to showing that

$$D_i^2(q) \cap \bigoplus_{j\in[1:k]\backslash\{i\}} D_j(q) = \{0\}. \tag{4.49}$$

In terms of structure functions of a quasi-normal frame, it is equivalent to showing that

$$c_{lr}^s = 0, \text{ for } l, r \in \mathcal{I}_i^1,\ s \in [1:m]\backslash\mathcal{I}_i^1. \tag{4.50}$$

Remark 12. Note that by Remarks 9 and 10, if $d_i = 0$ or $d_i = 1$ for some $i \in [1:k]$, one can perturb the original quasi-normal frame to a quasi-normal frame $(\tilde{X}_1, \dots, \tilde{X}_n)$ for which all $\tilde{X}_j$ are ad-generating elements of $\mathfrak{m}_i$ (for $d_i = 0$ any quasi-normal frame satisfies this property). This implies that for this (perhaps perturbed) frame relation (4.50) will follow from Lemma 4 because we can apply this lemma for the frame obtained from the original quasi-normal frame by an appropriate permutation.

To show (4.50), first, given $i_0 \in [1:k]$, we construct a submatrix N_i of A with row indices

$$T_{i_0} := \left(\bigcup_{i=1}^{k}[n_{i-1}+1 : n_{i-1}+d_i]\right) \cup \{m + e_{i_0-1}+1\}. \tag{4.51}$$

Then N_{i_0} has the following form:

$$N_{i_0} = \left(\begin{array}{ccc} N_{i_0,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & N_{i_0,k} \\ \hline & a^2_{n_{i_0-1}+1} & \end{array}\right), \tag{4.52}$$

where $N_{i_0,j}$ is of size $d_i \times d_i$, and $a^2_{n_{i_0-1}+1}$ is the $(n_{i_0-1}+1)$th row of the matrix A^2 from the second layer of the fundamental algebraic system (3.19). Moreover,

$$N_{i_0,j} = \begin{cases} M_{i_0,j} & \text{if } j \neq i_0 \\ M_{s,i_0} & \text{if } j = i_0 \text{ (here } s \neq i_0), \end{cases} \tag{4.53}$$

where $M_{i,j}$ are as in (4.22). Note that by Remark 8 the right hand-side in the second line of (4.53) is independent of $s \neq i_0$.

Denote by $b|_{T_{i_0}}$ the subcolumn of the column b of the fundamental algebraic system (3.19) consisting of the same rows as in matrix N_{i_0}, i. e., the rows of b indexed by the set T_{i_0} as in (4.51).

Lemma 7. *Assume that* (4.24) *holds and that, given* $i_0 \in [1:k]$, *the coefficients of all monomials of the form*

$$u_l u_s u_j \left(\prod_{i=1}^{k} u_{n_{i-1}+d_i+1}\right)\left(\prod_{i=1}^{k}(u_{n_{i-1}+1})^{d_i-1}\right) \quad \textit{with}$$
$$l \in [n_{i_0-1}+2 : n_{i_0-1}+d_{i_0}+1], s \in [1:m]\backslash\mathcal{I}^1_{i_0}, \textit{ and } j \in \mathcal{I}^2_{i_0} \tag{4.54}$$

in $\det([N_{i_0}|b|_{T_{i_0}}]$ *vanish. Then*

$$c_{lr}^s = 0, \quad l, r \in [n_{i_0-1}+2 : n_{i_0-1}+d_{i_0}+1], \textit{ and } s \in [1:m]\backslash\mathcal{I}^1_{i_0}. \tag{4.55}$$

Proof. Without loss of generality we can assume that $i_0 = 1$ and that $s \in \mathcal{I}_2^1$. Let $\tilde{b}_{T_1}$ be the subcolumn of the column $\tilde{b}$ consisting of the rows of $\tilde{b}$ indexed by the set T_1, where $\tilde{b}$ is as in (4.19).

Further, using (4.53), similarly to the proof of Lemma 4 (statements of items $(A1_i)$ and $(A2_i)$ there), one can conclude that the variable $u_{n_{i-1}+1}$, $i \in [1:k]$ appears only in the following columns of the augmented matrix $[N_1|\tilde{b}|_{T_1}]$:

(F1$_i$) The columns containing all columns of $N_{1,i}$ except the last one. Moreover, in each of these columns $u_{n_{i-1}+1}$ appears exactly in the entry $(N_{1,i})_{j+1j}$, i.e., in the entry situated right below the diagonal of the block $N_{1,i}$. Besides, the coefficient of $u_{n_{i-1}+1}$ in this entry is equal to 1;

(F2$_i$) the last column of the matrix $[N_1|\tilde{b}|_{T_1}]$;

(F3$_i$) the last row of the matrix $[N_1|\tilde{b}|_{T_1}]$.

Besides, applying the normalization conditions (4.9) to (4.19), one gets

(F4$_i$) The components of the column vector $\tilde{b}|_{T_1}$ in the rows corresponding to the rows of the block $N_{1,i}$ of N_1 do not contain $u_{n_{i-1}+1}$.

The following sublemma is important in the sequel, but its proof consists of tedious computations and is postponed to Appendix B:

Sublemma 1. *The entry from the last column and the last row of $[N_1|\tilde{b}|_{T_1}]$, i. e., $\tilde{b}_{m+1}$, is independent of u_j with $j \in \mathcal{I}_1^2$.*

From Sublemma 1 and the fact that the elements of the blocks $N_{1,v}$ defined by (4.52) do not depend on variables u_j with $j > m$, it follows that a term $\mathcal{L}$ in the Leibniz formula for $\det[N_1|\tilde{b}|_{T_1}]$, which contributes to the monomial (4.54), cannot contain as a factor the entry from the last column and the last row of the augmented matrix $[N_1|\tilde{b}|_{T_1}]$. Therefore, the term $\mathcal{L}$ must contain one factor of the form $\tilde{b}_r$, where $r \in \bigcup_{i=1}^{k} [n_{i-1}+2 : n_{i-1}+d_i]$ (here we also use Corollary 6) and one factor from the last row of $[N_1|\tilde{b}|_{T_1}]$, i. e., $a^2_{1,m+t}$. Moreover, by Remark 7, since the second line of (B.5) is independent of u_j with $j > m$, we must have

$$t \in [1 : d_1]. \tag{4.56}$$

Now, for definiteness, assume that $r \in [n_{i_1-1}+2 : n_{i_1-1}+d_{i_1}]$ for some $i_1 \in [1:k]$. By property $(F4_i)$, $\tilde{b}_r$ does not contain $u_{n_{i_1-1}+1}$. Besides, the entry $(N_{1,i_1})r - n_{i_1-1}, r - n_{i_1-1} - 1$ cannot be a factor in $\mathcal{L}$. So, using properties (F1$_i$)–(F3$_i$) and the fact that $u_{n_{i_1-1}+1}$ appears in the monomial (4.54) in the power not less than $d_{i_1} - 1$, we get that the factor $a^2_{1,m+t}$ must contain $u_{n_{i_1-1}+1}$. Hence, the desired contribution of $a^2_{1,m+t}$ to the monomial (4.54) is equal to the coefficient of the monomial $u_{n_{i_1-1}+1}u_j$ in $a^2_{1,m+t}$.

To find this coefficient, note that $a^2_{1,m+d_1}$ is given by (B.5). From (3.17) it follows that the second term of (B.5) depends only on u_i's with $i \in [1:m]$, so it does not contribute to the required monomial. Therefore, we have to find the contribution of the first term, i. e., of $\vec{h}_1(q_{1,m+t})$. From (3.17), the decomposition of the Tanaka symbol (3.8), and the normalization conditions (4.9) it follows that

$$q_{1,m+t} = -u_{t+1} - \sum_{x=d_1+2}^{n_1} c_{1x}^{m+d_1} u_x. \tag{4.57}$$

Then, using (4.15),

$$\vec{h}_1(q_{1m+t}) = -\vec{h}_1(u_{t+1} + \sum_{x=d_1+2}^{n_1} c_{1x}^{m+t} u_x)$$
$$= -\left(\sum_{v=1}^{n} q_{t+1v} u_v + \sum_{x=d_1+2}^{n_1} \sum_{v=1}^{n} c_{1x}^{m+t} q_{xv} u_v + \sum_{x=d_1+2}^{n_1} \vec{h}_1(c_{1x}^{m+t}) u_x \right). \tag{4.58}$$

The last term in (4.58) depends only on u_w's with $w \in [1:m]$ and does not contribute to the coefficient of the monomial $u_{n_{i_1-1}+1}u_j$ with $j \in \mathcal{I}_1^2$ in $\vec{h}_1(q_{1,m+d_1})$. As q_{ik} depends on u_i's with $i \in [1:m]$ only (see (3.17)), in the first two terms of (4.58) only summands with $v = j$ contribute to the coefficient of the monomial $u_{n_{i_1-1}+1}u_j$ in $\vec{h}_1(q_{1,m+d_1})$. So, again by (3.17), this coefficient is equal to

$$-c_{n_{i_1-1}t+1}^{j} - \sum_{x=d_1+2}^{n_1} c_{n_{i_1-1}x}^{j} c_{1x}^{m+t}. \tag{4.59}$$

Since $j \in \mathcal{I}_1^2$, by the decomposition of the Tanaka symbol (3.8), the expression in (4.59) is equal to zero for $i_1 \in [2:k]$. So a nonzero contribution is obtained only if

$$i_1 = 1 \Rightarrow r \in [2:d_1]. \tag{4.60}$$

In this case the coefficient of the monomial u_1u_j in $\vec{h}_1(q_{1,m+d_1})$ is equal to

$$-c_{1t+1}^{j} - \sum_{x=d_1+2}^{n_1} c_{1x}^{j} c_{1x}^{m+t} = -\sum_{x=2}^{n_1} c_{1x}^{j} c_{1x}^{m+t}, \tag{4.61}$$

where the last equality follows from the normalization conditions (4.9). If we set

$$v^s = (c_{1,2}^s, c_{1,3}^s, \dots, c_{1,m_1}^s) \quad s \in [m+1:m+d_1], \tag{4.62}$$

then (4.61) can be rewritten in terms of the standard inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^{d_1}$ as follows:

$$-\langle v^j, v^{m+t}\rangle. \tag{4.63}$$

Further, as before, for our purpose, it is enough to set variables u_i not appearing in (4.54) to be equal to zero. Let us check what happens in the rows of $[N_1|\tilde{b}|_{T_1}]$ containing the first rows of each block $N_{1,i}$, Consider the cases $i > 2$ and $i \in \{1, 2\}$ separately:

(G1) The case $i > 2$. *Any term* $\mathcal{L}$ *in the Leibniz formula for* $\det[N_1|\tilde{b}|_{T_1}]$ *contributing to the monomial* (4.54) *must contain as factors all entries of the form* $(N_{1,i})_{x+1x}$*,* $x \in [1:d_i-1]$ *and also the entry* $(N_{1,i})_{1,d_i}$. Indeed, in this case, the only nonzero entry in the first row of $N_{1,i}$ is $(N_{1,i})_{1,d_i}$ and, using Corollary 6, $(N_{1,i})_{1,d_i}$ must appear as a factor in the term $\mathcal{L}$. Furthermore, if for some $x_0 \in [1:d_i-1]$ the entry $(N_{1,i})_{x_0+1x_0}$ is omitted, then by (4.52) and (4.56) the term $\mathcal{L}$ will contain a factor of the form $(N_{1,i})_{y,x_0}$ for some $y \in [2:d_i]\backslash\{x_0+1\}$. Hence, the entry $(N_{1,i})_{yy-1}$ does not appear as a factor in the term $\mathcal{L}$ either. Then from properties (F1$_i$)–(F3$_i$) and the fact that by (4.60) the factor a_{1m+t}^2 from the last row of N_1 does not contain $u_{n_{i-1}+1}$, it follows that the power of $u_{n_{i-1}+1}$ in the term considered is less than $d_i - 1$, but in the monomial (4.54) it is at least $d_i - 1$ and we get the contradiction. Note also that in the case at hand, using the normalization conditions (4.9), we have

$$(N_{1,i})_{x+1x} = u_{n_{i-1}+1}, \quad (N_{1,i})_{1,d_i} = -u_{n_{i-1}+d_i+1}, \tag{4.64}$$

so the total contribution of these factors to the coefficient of the monomial (4.54) is ±1, i. e., trivial.

(G2) The cases $i = 1$ **and** $i = 2$. From Corollary 6 it follows that

- the only nonzero entries in the first row of $[N_1|\tilde{b}|_{T_1}]$ are

$$(N_{1,1})_{1\,l-1} = -u_l, \quad (N_{1,1})_{1\,d_1} = -u_{d_1+1}; \tag{4.65}$$

- the only nonzero entries in the row of $[N_1|\tilde{b}|_{T_1}]$ containing the first row of $N_{1,2}$ are

$$(N_{1,2})_{1\,s-n_1-1} = -u_s, \quad \text{if } s \in [n_1+2 : n_1+d_1+1] \tag{4.66}$$

$$(N_{1,2})_{1\,d_1} = -u_{n_1+d_2+1}. \tag{4.67}$$

Consequently, we have the following four possibilities:

(G2a) The entries $(N_{1,1})_{1\,d_1}$ and $(N_{1,2})_{1\,d_2}$ are factors in $\mathcal{L}$. This and (4.60) implies that the entries $(N_{1,2})_{x+1,x}$ for all $x \in [1 : d_2-1]$ are factors in $\mathcal{L}$, and there exists $r \in [2 : d_1]$ such that $\tilde{b}_r$ is a factor in $\mathcal{L}$. The latter implies that the entry $(N_{1,1})_{r,r-1}$ is not a factor in $\mathcal{L}$ and therefore $a^2_{1,m+r-1}$ is a factor in $\mathcal{L}$, otherwise more than one of the entries of the form $(N_{1,1})_{x+1,x}$, $x \in [1 : d_1-1]$ will not appear as factors in $N_{1,1}$ and it will contradict the fact that u_1 appears in the monomial (4.54) in the power not less than $d_1 - 1$.

Recall that in $a^2_{1,m+r-1}$ we are interested in the coefficient of the monomial u_1u_j, which by (4.63) is equal to

$$-\langle v^j, v^{m+r-1}\rangle. \tag{4.68}$$

Taking into account all factors we revealed in $\mathcal{L}$, relations (4.64), (4.65), (4.67), and (4.64), and also that in $a^2_{1,m+r-1}$ we are interested in the monomial u_1u_j, we conclude that in $\tilde{b}_r$ we are interested in the coefficient of u_lu_s, as u_l and u_s are the only variables in the monomial (4.54) that have still not been used. By (4.19) the latter coefficient is equal to $(\alpha_1^2-\alpha_2^2)c^s_{lr}$. So, this, together with (4.68) and the last sentence of (G1), implies that the total contribution of all possible terms satisfying the assumptions of (G2a) (i. e., for every $r \in [2, d_1]$) can be written as

$$(\alpha_1^2-\alpha_2^2)\left\langle \sum_{r=2}^{d_2} \operatorname{sgn}(\sigma_r) c^s_{lr} v^{m+r-1}, v^j \right\rangle, \tag{4.69}$$

where $\operatorname{sgn}(\sigma_r)$ stands for the sign of the permutation related to the corresponding term in the Leibniz formula for $\det[N_1|\tilde{b}|_{T_1}]$. Since the value of $\operatorname{sgn}(\sigma_r)$ is not important to the final conclusions, its explicit expression is not written out here.

(G2b) The entries $(N_{1,1})_{1\,l-1}$ and $(N_{1,2})_{1\,d_2}$ are factors in $\mathcal{L}$. First, regarding the entries from the block $N_{1,2}$ we can use the same arguments and the same conclusions as in (G1a). Regarding the entries of the block $N_{1,1}$, since the entry $(N_{1,1})_{1\,l-1}$ is already used in $\mathcal{L}$, the entry $(N_{1,1})_{l\,l-1}$ should not appear as a factor in $\mathcal{L}$, but then, by the same arguments about the lower bound for the power of u_1 that we have already used both in (G1) and (G2a), all other entries of the form $(N_{1,1})_{x,x-1}$ have to be in $\mathcal{L}$. Therefore, the entries $a^2_{1,m+d_1}$ and $\tilde{b}_l$ must appear as factors of $\mathcal{L}$. By (4.63), the former contributes to the monomial (4.54) the factor

$$-\langle v^j, v^{m+d_1}\rangle. \tag{4.70}$$

Next, similarly to (G2a), the contribution to the monomial (4.54) from $\tilde{b}_l$ is equal to the coefficient of $u_{d_1+1}u_s$ which by (4.19) is equal to $-(\alpha_1^2-\alpha_2^2)c^s_{ld_1+1}$, so that the contribution of the terms from (G2b) to the monomial (4.54) is equal to

$$(\alpha_1^2-\alpha_2^2)\langle \operatorname{sgn}(\sigma_{d_2+1}) c^s_{ld_2+1} v^{m+d_2}, v^j\rangle, \tag{4.71}$$

where $sgn(\sigma_{d_2+1})$ stands for the sign of the corresponding permutation. Combining (4.69) and (4.71), we get that the contribution of the terms satisfying the conditions of (G2a) or (G2b) can be written as

$$(\alpha_1^2-\alpha_2^2)\left\langle \sum_{r=2}^{d_2+1} \operatorname{sgn}(\sigma_r) c^s_{lr} v^{m+r-1}, v^j \right\rangle. \tag{4.72}$$

Further note that by (4.66) for $s \notin [n_1+2 : n_1+d_1+1]$ the cases (G2a) and (G2b) are the only possible cases, so in this case the expression in (4.72) is equal to the coefficient of the monomial (4.54). Vanishing of this coefficient for any $j \in \mathcal{I}_1^2$ implies that the vector $\sum_{r=2}^{d_1+1} \operatorname{sgn}(\sigma_r) c_{lr}^s v^{m+r-1}$, which belongs to $\operatorname{span}\{v^x : x \in \mathcal{I}_1^2\}$, is orthogonal to v^j for all $j \in \mathcal{I}_1^2$. Hence,

$$\sum_{r=2}^{d_1+1} \operatorname{sgn}(\sigma_r) c_{lr}^s v^{m+r-1} = 0. \tag{4.73}$$

From ad-surjectivity, and, more precisely, since X_1 is chosen as an ad-generating element, the tuple of vectors $(v^{m+1}, v^{m+2}, \dots, v^{m+d_1})$ forms a basis in $\mathbb{R}^{d^1}$. Therefore,

$$c_{lr}^s = 0, \text{ for } l, r \in [2 : d_1+1], \text{ and } s \notin [1 : d_1] \cup [n_1+2 : n_1+d_1+1]. \tag{4.74}$$

(G2c) The entries $(N_{1,1})_{1\,d_1}$ and $(N_{1,2})_{1\,s-n_1-1}$ are factors in $\mathcal{L}$ (relevant only for $s \in [n_1+2 : n_1+d_2+1]$). For the block $N_{1,2}$ let us apply the arguments similar to the ones we used for the block $N_{1,1}$ in (G2b):

Since the entry $(N_{1,2})_{1\,s-n_1-1}$ is already used in $\mathcal{L}$, the entry $(N_{1,2})_{s-n_1\,s-n_1-1}$ should not appear as a factor in $\mathcal{L}$, but then, by the same arguments about the lower bound for the power of u_1 that we have already used both in (G1) and (G2a), all other entries of the form $(N_{1,2})_{x,x-1}$, $x \in [2 : d_2]$ have to be in $\mathcal{L}$. Then the contribution to $\mathcal{L}$ of the row of the matrix $[N_1|\tilde{b}|_{T_1}]$ containing the $(s-n_1)$th row of $N_{1,2}$ must be equal to the entry $(N_{1,2})_{s-n_1,d_2}$. Here we also use that this contribution cannot be from the last column of $[N_1|\tilde{b}|_{T_1}]$, because of (4.60). Since by (4.66) U_s is already used, the term in $(N_{1,2})_{s-n_1,d_2}$ that may give a contribution to the monomial (4.54) is the one containing $u_{n_1+d_2+1}$. Further, passing to the rows of $[N_1|\tilde{b}|_{T_1}]$ containing the block $N_{1,1}$, since u_{d_1+1} is used in $(N_{1,1})_{d_1}$, the only possible contribution of $\tilde{b}_r$ comes from the monomial $u_l u_{n_1+1}$, the coefficient of which by (4.19) is equal to $(\alpha_1^2 - \alpha_2^2) c_{l,r}^{n_1+1}$. However, this coefficient is zero by (4.74). Hence, there is no contribution to the monomial (4.54) from the terms satisfying (G2c).

(G2d) The entries $(N_{1,1})_{1\,l-1}$ and $(N_{1,2})_{1\,s-n_1-1}$ are factors in $\mathcal{L}$ (relevant only for $s \in [n_1+2 : n_1+d_2+1]$). The only difference from (G2c) is that since $(N_{1,1})_{1\,l-1}$ in $\mathcal{L}$ is used, one has to use the term $\tilde{b}_l$, and since, by (4.65), it is already used in $(N_{1,1})_{1\,l-1}$, in $\tilde{b}_l$ we are interested in the coefficient of the monomial $u_{d_1+1}u_{n_1+1}$. By (4.19), this coefficient is equal to $(\alpha_1^- \alpha_2^2) c_{d_1+1\,l}^{n_1+1}$, which is equal to zero by (4.74). Hence, there is no contribution to the monomial (4.54) from the terms satisfying (G2d).

So, for $s \in [n_1+2 : n_1+d_1+1]$ also the coefficient of the monomial (4.54) in $\det[N_1|\tilde{b}|_{T_1}]$ is equal to (4.72). Repeating the same arguments as after (4.72), we will get that the equality in (4.74) holds also for $s \in [n_1+2 : n_1+d_1+1]$. This completes the proof of Lemma 7 in the case of $i_0 = 1$ and $s \in \mathcal{I}_2^1$. Since we can always permute the indices, it also proves this lemma in general. □

Lemma 7 proves only a subset of relations from (4.50). Using again the flexibility given by Corollary 4, one can show that (4.50) holds not for the original quasi-normal frame but for its perturbation adapted to the same choices of $X_{n_{i-1}+1}$, which will be enough to finish the proof of Proposition 7.

Corollary 7. *Let $K_{X_{n_{i-1}+1}}$ and $H_{X_{n_{i-1}+1}}$ be as in* (4.27) *and* (4.28)*, respectively. If Y_i and Z_i are sections of $H_{X_{n_{i-1}+1}}$ satisfying*

$$\dim \operatorname{span}\{Y_i, Z_i\} = 2, \quad \operatorname{span}\{Y_i, Z_i\} \cap K_{X_{n_{i-1}+1}} = 0, \tag{4.75}$$

then

$$[Y_i, Z_i] \in L_{X_{n_{i-1}+1}} \quad \operatorname{mod} D_i, \tag{4.76}$$

where $\mathcal{L}_{X_{n_{i-1}+1}}$ is defined as in (4.31).

Proof. Indeed, from (4.75) it follows that we can find a quasi-normal frame $(X_1, \ldots, X_n)$ such that $X_{n_{i-1}+2} = \dfrac{Y_i}{(g_1(Y_i, Y_i))^{1/2}}$ and $X_{n_{i-1}+3} \in \operatorname{span}\{Y_i, Z_i\}$. Then (4.76) follows from Lemmas 7 and Corollary 4. □

By Remark 12, we can assume that $d_i \geqslant 2$, as for $d_i = 0$ or 1, relations (4.50) have already been proved there, perhaps for a perturbed quasi-normal frame. Under this assumption the set of planes in $H_{X_{n_{i-1}+1}}$ having a trivial intersection with $K_{X_{n_{i-1}+1}}$ is generic. Therefore, by a finite number of consecutive small perturbations, one can build a quasi-normal frame $(X_1, \ldots, X_n)$ such that, for every $n_{i-1} + 2 \leqslant i < j \leqslant n_i$, the pair $(Y_i, Z_i) = (X_i, X_j)$ satisfies (4.75) and so by Corollary 7 this frame satisfies (4.50). Besides, by Remark 11, the new quasi-normal frame obtained in this step will automatically satisfy (4.47). This completes the proof of Proposition 7.

4.5. *Proof of Proposition* 9

From (4.50) and Lemma 6, using (3.17), one can get that

$$\tilde{b}_j = 0, \quad \forall j \in [1:m]. \tag{4.77}$$

This, together with (4.18), implies that, if for all $v \in [1:k]$ and $j \in I_v^2$ we set $\Psi_j := \Phi_j - \alpha^2_{n_{v-1}+1} u_j$ and $\Psi := (\Psi_1, \ldots, \Psi_m)^T$, then $A_1\Psi = 0$. This, again together with (4.18), implies that the tuple $\widetilde{\Phi} = (\Phi_{m+1}, \ldots, \Phi_n)^T$ with

$$\Phi_j = \alpha^2_{n_{v-1}+1} u_j, \quad j \in \mathcal{I}_v^2, \tag{4.78}$$

which corresponds to $\Psi = 0$, is the solution to the first layer $A^1\widetilde{\Phi} = b^1$ of the fundamental algebraic system (3.19).

Further, note that the $m \times (n-m)$-matrix A^1 has the maximal rank $n - m$ at a generic point, as from the normalization conditions (4.9) the coefficient of the monomial $\prod_{v=1}^{k} u^{d_v}_{n_{v-1}+1}$ in its maximal minor consisting of rows from the set $\bigcup_{v=1}^{k} [n_{v-1} + 2 : n_{v-1} + d_i + 1]$ is equal to 1. Hence, (4.78) defines the unique (rational in u's) solution of the the system $A^1\widetilde{\Phi} = b^1$ and therefore it must coincide with the solution of the whole fundamental algebraic system (3.19), i. e., it must satisfy other layers of it. In other words,

$$\sum_{v=1}^{k} \sum_{l \in \mathcal{I}_v^2} a^s_{t,l} \alpha^2_{n_{i-1}+1} u_l = b^s_t, \quad \forall s \geqslant 1, t \in [1:m], \tag{4.79}$$

where $a^s_{t,k}$ and b^s_t satisfy (3.21) and (3.22), respectively. Note that, by (4.18), relation (4.79) is equivalent to

$$\tilde{b} = 0. \tag{4.80}$$

Without loss of generality, it is enough to prove Proposition 9 for $i = 1$.

Lemma 8. *The coefficients of the monomials*

$$u_y u_l u_j, \quad y, l \in [2 : d_1 + 1], j \in \mathcal{I}_v^1 \cup \mathcal{I}_v^2, v \neq 1 \tag{4.81}$$

in $\tilde{b}_{m+1}$ *are equal to*

$$\begin{cases} -\left(\alpha_1^2 - \alpha^2_{n_{v-1}+1}\right)\left(c^j_{y,m+l-1} + c^j_{l,m+y-1}\right), & y \neq l \in [2 : d_1 + 1], j \in \mathcal{I}_v^1 \cup \mathcal{I}_v^2, v \neq 1 \\ -\left(\alpha_1^2 - \alpha^2_{n_{v-1}+1}\right) c^j_{y,m+y-1}, & y = l \in [2 : d_1 + 1], j \in \mathcal{I}_v^1 \cup \mathcal{I}_v^2, v \neq 1. \end{cases} \tag{4.82}$$

Proof. We use the expression for $\tilde{b}_{m+1}$ given by (B.10) in Appendix B. Consider the cases $j \in \mathcal{I}_v^2$ and $j \in \mathcal{I}_v^1$ separately.

Case 1: $j \in \mathcal{I}_v^2$**.** The first term of this expression does not contribute to the monomial (4.81), because every monomial in it contains u_w with $w \in [1:m]\backslash\mathcal{I}_1^1$. Also, since $w \in [1:m]\backslash\mathcal{I}_1^1$, by Lemma 4 $q_{1,w}$ are independent of u_z for $z \in \mathcal{I}_1^1$, hence the second term of (B.10) does not contain the monomial (4.81). Further, the third term of (B.10) does not depend on u_j with $j > m$ because by (3.17) all q_{jl} depend only on u_i's with $i \in [1:m]$. Therefore, in the case at hand the monomial (4.81) may appear in (B.10) only in the fourth term, i. e., in

$$\sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\sum_{r=m+1}^{m+d_1}\sum_{t\in\mathcal{I}_i^2}q_{1r}q_{rt}u_t. \tag{4.83}$$

Finally, by the normalization conditions (4.9) one can easily show that the coefficient of this monomial in (4.83) is equal to (4.82).

Case 2: $j \in \mathcal{I}_v^1$**.** In this case the fourth term of (B.10) does not contribute to the monomial (4.81) in $\tilde{b}_{m+1}$, because each monomial in this term contains factor u_t with $t > m$.

Let us analyze the second term of (B.10): by Lemma 4, the factor q_{1w} does not contain u_s with $s \in \mathcal{I}_1^1$, so it must contribute the coefficient of u_j, which is equal to c_{j1}^w. Hence, the index x, appearing in the second term of (B.10) must be in I_1^1, hence it must be equal to either y or l, where y and l are as in (4.81). So, the total contribution of the second term of (B.10) to the coefficient of the monomial (4.81) in $\tilde{b}_{m+1}$ is equal to

$$\sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\sum_{w\in\mathcal{I}_i^1}c_{j1}^w(c_{lw}^y+c_{yw}^l)=0, \tag{4.84}$$

because $c_{lw}^y + c_{yw}^l = 0$ by item (2) of Proposition 11. So, the second term of (B.10) does not contribute to the monomial (4.81) in $\tilde{b}_{m+1}$

Now let us analyze the first term of (B.10): First, the index w appearing there must be equal to j from (4.81). Second, as in (B.13),

$$\vec{h}_1(q_{1j})=\sum_{r=1}^{m}\sum_{x=1}^{n}c_{r1}^j q_{rx}u_x+\sum_{r=1}^{m}\vec{h}_1(c_{r1}^j)u_r. \tag{4.85}$$

In the second term of (4.85) we are interested in $r \in \mathcal{I}_1^1$, but in this case, by Lemma 4, $c_{r1}^j = 0$ and therefore the second term of (4.85) does not contribute to (4.89).

For the same reason, the index r in the first term of (4.85) can be taken from $[1:m]\backslash I_1^1$, Hence, the coefficient of $u_y u_l$ which is of interest to us in the first term of (4.85) for the monomial (4.81) is equal to

$$\sum_{r=m_1+1}^{m}c_{r1}^j(c_{lr}^y+c_{yr}^l)=0, \tag{4.86}$$

because $c_{lr}^y + c_{yr}^l = 0$ by item (2) of Proposition 11 again. Consequently, the first term of (B.10) does not contribute to the monomial (4.81) in $\tilde{b}_{m+1}$.

So, in the case at hand the monomial (4.81) may appear in (B.10) only in the third term, i. e., in

$$\sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\sum_{r=m+1}^{m+d_1}\sum_{w\in\mathcal{I}_i^1}q_{1r}q_{rw}u_w. \tag{4.87}$$

Finally, by the normalization conditions (4.9) one can easily show that the coefficient of this monomial in (4.87) is equal to (4.82). □

From condition (4.80) and Lemma 9 it follows that

$$\begin{cases} c^j_{y,m+l-1} + c^j_{l,m+y-1} = 0, & y \neq l \in [2:d_1+1], j \in [1:n]\backslash(\mathcal{I}^1_1 \cup \mathcal{I}^2_1); \\ c^j_{y,m+y-1} = 0, & y = l \in [2:d_1+1], j \in [1:n]\backslash(\mathcal{I}^1_1 \cup \mathcal{I}^2_1). \end{cases} \tag{4.88}$$

Lemma 9. *The coefficients of the monomials*

$$u_1^2 u_j, \quad j \in \mathcal{I}^1_v \cup \mathcal{I}^2_v, v \neq 1 \tag{4.89}$$

in $\tilde{b}_{m+f}$*,* $f \in [2:d_1+1]$ *are equal to*

$$\left(\alpha_1^2 - \alpha^2_{n_{v-1}+1}\right) c^j_{1,m+f-1}. \tag{4.90}$$

Proof. We use the expression for $\tilde{b}_{m+f}$ given by formula (B.10) from Appendix B. Consider the cases $j \in \mathcal{I}^2_v$ and $j \in \mathcal{I}^1_v$ separately.

Case 1: $j \in \mathcal{I}^2_v$**.** The first and the third term of (B.10) do not contribute to the monomial (4.89) by the same arguments as in the proof of case 1 of Lemma 8, while the same holds for the second term not only by Lemma 4 but also by Lemma 7. Hence, as in the proof of Lemma 8, the monomial (4.89) may appear only in the fourth term, i. e., in

$$\sum_{i=2}^{k} \left((\alpha_1)^2 - (\alpha_{n_{i-1}+1})^2\right) \sum_{r=m+1}^{m+d_1} \sum_{t \in \mathcal{I}^2_i} q_{fr} q_{rt} u_t. \tag{4.91}$$

Finally, by the normalization conditions (4.9) one can easily show that the coefficient of this monomial in (4.91) is equal to (4.90).

Case 2: $j \in \mathcal{I}^1_v$**.** In this case the fourth term of (B.10) does not contribute to the monomial (4.89) in $\tilde{b}_{m+1}$ for the same reason as in the proof of case 2 of Lemma 8.

The analysis of the first and second terms of (B.10) is also completely analogous to the one in the proof of case 2 of Lemma 8. The main differences are that here we use Lemma 7 together with Lemma 4 there and item (1) of Proposition 11 instead of item (2) of Proposition 11 there.

In more detail, for the second term of (B.10) by Lemma 7 the factor q_{fw} does not contain u_s with $s \in \mathcal{I}^1_1$, so it must contribute the coefficient u_j, which is equal to c^w_{j1}. Hence, the index x appearing in the second term of (B.10) must be in I^1_1, and hence it must be equal to 1. So, the total contribution of the second term of (B.10) to the coefficient of the monomial (4.89) in $\tilde{b}_{m+f}$ is equal to

$$\sum_{i=2}^{k} \left((\alpha_1)^2 - (\alpha_{n_{i-1}+1})^2\right) \sum_{w \in \mathcal{I}^1_i} c^w_{jf} c^1_{1w} = 0, \tag{4.92}$$

because $c^1_{1w} = 0$ by item (2) of Proposition 11. So, the second term of (B.10) does not contribute to the monomial (4.89) in $\tilde{b}_{m+f}$.

Now let us analyze the first term of (B.10): First, the index w appearing there must be equal to j from (4.89). Second, as in (B.13),

$$\vec{h}_1(q_{fj}) = \sum_{r=1}^{m} \sum_{x=1}^{n} c^j_{rf} q_{rx} u_x + \sum_{r=1}^{m} \vec{h}_1(c^j_{rf}) u_r. \tag{4.93}$$

In the second term of (4.86) we are interested in $r = 1$, but in this case, by Lemma 4, $c^j_{r1} = 0$ and therefore the second term of (4.85) does not contribute to (4.89).

By similar reasoning, using now also Lemma 7, the index r in the first term of (4.85) can be taken from $[1:m]\backslash I_1^1$. Hence, the coefficient of u_1^2 which is of interest to us in the first term of (4.93) for the monomial (4.89) is equal to

$$\sum_{r=m_1+1}^{m} c_{rf}^{j} c_{1r}^{r} = 0, \tag{4.94}$$

because $c_{1r}^1 = 0$ by item (1) of Proposition 11 again. Consequently, the first term of (B.10) does not contribute to the monomial (4.89) in $\tilde{b}_{m+f}$.

So, in the case at hand the monomial (4.81) may appear in (B.10) only in the third term, i. e., in (4.87). Finally, by the normalization conditions (4.9) one can easily show that the coefficient of this monomial in (4.87) is equal to (4.90). □

From condition (4.80) and Lemma 9 it follows that

$$c_{1,l}^{j} = 0, \quad l \in \mathcal{I}_1^2, j \in [1:n]\backslash(\mathcal{I}_1^1 \cup \mathcal{I}_1^2). \tag{4.95}$$

In other words,

$$\mathrm{ad}(X_1)(D_1^2) \subset D_1^2. \tag{4.96}$$

Lemma 10. *The following inclusion holds:*

$$\mathrm{ad}(X_y)(D_1^2) \subset D_1^2, \quad \forall y \in \mathcal{I}_1^1. \tag{4.97}$$

Proof. Take $y, l \in [2:d_1+1]$. Using the normalization conditions (4.8) and the Jacobi identity, we have

$$\begin{aligned}
[X_y, X_{m+l-1}] &\overset{(4.8)}{=} [X_y, [X_1, X_l]] = [[X_y, X_1], X_l] \\
+ [X_1, [X_y, X_l]] &\overset{(4.8)}{=} [-X_{m+y-1}, X_l] + [X_1, [X_y, X_l]].
\end{aligned} \tag{4.98}$$

Hence, using (4.96),

$$[X_y, X_{m+l-1}] - [X_l, X_{m+y-1}] = [X_1, [X_y, X_l]] \in \mathrm{ad}(X_1)(D_1^2) \subset D_1^2, \tag{4.99}$$

which implies

$$c_{y,m+l-1}^{j} - c_{l,m+y-1}^{j} = 0, \quad y, l \in [2:d_1+1], j \in [1:n]\backslash(\mathcal{I}_1^1 \cup \mathcal{I}_1^2). \tag{4.100}$$

Combining (4.88) and (4.100), we get

$$c_{y,m+l-1}^{j} = c_{l,m+y-1}^{j} = 0, \qquad y, l \in [2:d_1+1], j \in [1:n]\backslash(\mathcal{I}_1^1 \cup \mathcal{I}_1^2). \tag{4.101}$$

In other words, taking into account (4.96), we get

$$\mathrm{ad}(X_y)(D_1^2) \subset D_1^2, \quad \forall y \in [1:d_1+1]. \tag{4.102}$$

Furthermore, when $y \in [d_1+2:m_1]$ and $l \in [1:d_1]$, the Jacobi identity yields

$$\begin{aligned}
&[X_y, X_{m+l}] = [X_y, [X_1, X_{l+1}]] = [[X_y, X_1], X_{l+1}] + [X_1, [X_y, X_{l+1}]] \\
&\subset \mathrm{ad}(X_{l+1})(D_1^2) + \mathrm{ad}(X_1)(D_1^2) \overset{(4.102)}{\subset} (D_1^2),
\end{aligned} \tag{4.103}$$

which implies (4.97). □

Finally, observe that (4.97) implies $(D_1)^3(q) = (D_1)^2(q)$. This completes the proof of Proposition 9.

4.6. Completing the Proof of Proposition 10

Without loss of generality, it is enough to prove that $D_1^2 + D_2^2$ is involutive. Moreover, for this, by Proposition 9, it is enough to show that

$$[X_r, X_j] \in D_1^2 + D_2^2, \quad (r,j) \in (\mathcal{I}_1^1 \times \mathcal{I}_2^2) \cup (\mathcal{I}_1^2 \times \mathcal{I}_2^1). \tag{4.104}$$

Without loss of generality (swapping the indices), it is enough to show it for $(r,j) \in (\mathcal{I}_1^2 \times \mathcal{I}_2^1)$. For this, using the normalization conditions (4.8) and the Jacobi identity, we get

$$[X_r, X_j] \overset{(4.8)}{=} [[X_1, X_{r-m+1}], X_j] = [[X_1, X_j], X_{r-m+1}] + [X_1, [X_{r-m+1}, X_j]]. \tag{4.105}$$

Since 1 and $r-m+1$ belong to $\mathcal{I}_1^1$ and $j \in \mathcal{I}_2$, by the decomposition of the Tanaka symbol (3.8) and Lemma 5 it follows that

$$[X_1, X_j] \in D_1 + D_2, \quad [X_{r-m+1}, X_j] \in D_1 + D_2. \tag{4.106}$$

Consequently, using Lemma 5, from the right hand-side of (4.105) and (4.106) it follows that

$$[X_r, X_j] \in D_1^2 + D_2 \subset D_1^2 + D_2^2,$$

which completes the proof of Proposition 10.

4.7. Completing the Proof of Theorem 3 by Rotating the Frames on D_i's

By Proposition 9, we have that the distributions D_i^2, $i \in [1:k]$ are involutive. Moreover, by Proposition 10 the distributions $D_i^2 \oplus D_j^2$, $i,j \in [1:k]$ are involutive as well. The latter implies that for every subset $\mathcal{J} \subset [1:k]$ the distribution

$$\Delta_{\mathcal{J}} := \bigoplus_{i \in \mathcal{J}} D_i^2 \tag{4.107}$$

is involutive. Let $\mathcal{F}_{\mathcal{J}}(q)$ be the integral submanifolds of the distirbution $\Delta_{\mathcal{J}}$ passing through the point q.

By constructions, for any $i \in [1:k]$ and point $q \in M$, the manifold $\mathcal{F}_{\{i\}}(q)$ is transversal to $\mathcal{F}_{[1:k]\setminus\{i\}}(q)$. By the transversality and the inverse function theorem, for any $q_0 \in M$ there exists a neighborhood $\mathcal{U}$ of q_0 such that, for any $q_1 \in \mathcal{U}$ and $q_2 \in \mathcal{U}$, the leaf $\mathcal{F}_{\{i\}}(q_1)$ intersects with the leaf $\mathcal{F}_{[1:k]\setminus\{i\}}(q_2)$ at exactly one point. Then the following projection map $\pi_i^{q_0} : \mathcal{U} \to \mathcal{F}_{\{i\}}(q_0)$ is well defined:

$$\pi_i^{q_0}(q) = \mathcal{F}_{\{i\}}(q_0) \cap \mathcal{F}_{[1:k]\setminus\{i\}}(q) \tag{4.108}$$

for any $q \in \mathcal{U}$. Note that the map $\pi_i^{q_0}$ is a diffeomorphism between $\mathcal{F}_i(q) \cap U$ and $\mathcal{F}_i(q_0) \cap U$. As a consequence, for any $r \in \mathcal{I}_1^i$, there exists a unique vector field $\tilde{X}_r$ on $\mathcal{U}$ such that

$$\begin{aligned} &\tilde{X}_r \text{ is a section of } D_i^2 \text{ and} \\ &\mathrm{d}\pi_i^{q_0}\big(\tilde{X}_r(q)\big) = X_r(\pi_i^{q_0}(q)), \quad \text{where } r \in \mathcal{I}_1^i. \end{aligned} \tag{4.109}$$

Lemma 11. *$\tilde{X}_r$ is a section of D_i for every $r \in \mathcal{I}_i^1$.*

Proof. From the decomposition of the Tanaka symbol (3.8) and the involutivity of distributions $\Delta_{\mathcal{J}}$ it follows that that for any $i \in [1:k]$ and any section V of the distribution $\bigoplus_{j \in [1:k]\setminus\{i\}} D_j$

$$[V, D_i \oplus \Delta_{[1:k]\setminus\{i\}}] \subset D_i \oplus \Delta_{[1:k]\setminus\{i\}}, \tag{4.110}$$

which implies that the local flow e^{tV}, generated by the vector field V, consists of local symmetries of the distribution $D_i \oplus \Delta_{[1:k]\setminus\{i\}}$, i. e.,

$$(e^{tV})_*\Big(D_i \oplus \Delta_{[1:k]\setminus\{i\}}\Big) = D_i \oplus \Delta_{[1:k]\setminus\{i\}}. \tag{4.111}$$

By construction, for any section V of the distribution $\bigoplus_{j\in[1:k]\setminus\{i\}} D_j$ (and even for any section V of the distribution $\Delta_{[1:k]\setminus\{i\}}$) , we have

$$\pi_i^{q_0} \circ e^{tV} = \pi_i^{q_0}. \tag{4.112}$$

This, together with (4.111) and the fact that

$$\ker(d\pi_i^{q_0})(q) = \Delta_{[1:k]\setminus\{i\}}(q), \tag{4.113}$$

implies that

$$d\pi_i^{q_0}\big(D_i(e^{tV}q_0)\big) = D_i(q_0). \tag{4.114}$$

Consequently, by the definition of $\tilde{X}_r$ with $r \in \mathcal{I}_1^i$ given by (4.109), we get

$$X_r\big(e^{tV}q_0\big) \in D_i\big(e^{tV}q_0\big). \tag{4.115}$$

Finally, by the Rashevskii – Chow theorem, the point q_0 can be connected with any point of $\mathcal{F}_{\Delta_{[1:k]\setminus\{i\}}}(q_0) \cap U$ by a finite concatenation of integral curves tangent to the distribution $\bigoplus_{j\in[1:k]\setminus\{i\}} D_j$ and so we can apply relations (4.113) and (4.115) a finite number of times (with corners of the concatenation instead of q_0) to get the conclusion of the lemma. □

Lemma 12. *If $i \neq j \in [1:k]$, then for every $r \in \mathcal{I}_i^1$ and $l \in \mathcal{I}_j^1$ the vector fields $\tilde{X}_r$ and $\tilde{X}_l$ commute,*

$$[\tilde{X}_r, \tilde{X}_l] = 0, \quad \forall r \in \mathcal{I}_i^1, l \in \mathcal{I}_j^1, i \neq j. \tag{4.116}$$

Proof. As before, let $i \in [1:k]$ and V be a section of the distribution $\Delta_{[1:k]\setminus\{i\}}$. Then, using the standard properties of Lie derivatives and (4.112), one gets

$$\begin{aligned}(\pi_i^{q_0})_*[V, \tilde{X}_r](q) &= \frac{\mathrm{d}}{\mathrm{d}t}(\pi_i^{q_0})_*(e^{-tV})_*\tilde{X}_r(e^{tV}(q))\Big|_{t=0} \\ &\overset{(4.112)}{=} \frac{\mathrm{d}}{\mathrm{d}t}(\pi_i^{q_0})_*\tilde{X}_r(e^{tV}(q))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}X_r(\pi_i^{q_0}(q)) = 0,\end{aligned} \tag{4.117}$$

where $(\pi_i^{q_0})_*$ denotes the pushforward of the map $\pi_i^{q_0}$. Then, by (4.113), the above calculations show that

$$[V, \tilde{X}_r] \in \Delta_{[1:k]\setminus\{i\}}, \quad r \in \mathcal{I}_i^1,\ V \text{ is a section of } \Delta_{[1:k]\setminus\{i\}}. \tag{4.118}$$

Using (4.118), first for $V = \tilde{X}_l$ such that $l \in \mathcal{I}_j^1$ and $j \neq i$, then switching the roles of i and j, and finally using the distribution $\Delta_{\{i,j\}}$, we get

$$[\tilde{X}_r, \tilde{X}_l] \in \Delta_{[1:k]\setminus\{i\}} \cap \Delta_{[1:k]\setminus\{j\}} \cap \Delta_{\{i,j\}} = 0, \quad \forall r \in \mathcal{I}_i^1, l \in \mathcal{I}_j^1, i \neq j, \tag{4.119}$$

i. e., the vector fields $\tilde{X}_r$ and $\tilde{X}_l$ commute. □

Our final goal is to show the following

Lemma 13. *The frame $(\tilde{X}_1, \ldots, \tilde{X}_m)$ defined by (4.109) is g_1-orthonormal.*

Proof. By Lemma 11, the collection $\{\tilde{X}_r\}_{r\in\mathcal{I}_i^1}$ is a local frame of D_i, hence for every $q \in U$ there exists a $m_i \times m_i$ matrix $T_i(q) := \big(t_{rs}^i(q)\big)_{r,s\in\mathcal{I}_i^1} \in \mathrm{GL}(m_i)$, making the transition from the originally chosen local frame $(X_i)_{i\in\mathcal{I}_i^1}$ of D_i to $(\tilde{X}_i)_{i\in\mathcal{I}_1^1}$, i. e.,

$$\tilde{X}_r = \sum_{s\in\mathcal{I}_i^1} t_{rs}^i X_s, \quad i \in [1:k], r \in \mathcal{I}_i^1. \tag{4.120}$$

The lemma will be proved if we show that $T_i \in \mathrm{SO}(m_i)$ for every $i \in [1:k]$.

First, from the commutativity relation (4.116) by direct computations it follows that the entries of the transition matrix-valued function T_i satisfy the following system of partial equations 2:

$$X_l(t^i_{rs}) = -\sum_{v\in\mathcal{I}^1_i} c^s_{lv} t^i_{rv}, \quad \forall r, s \in \mathcal{I}^1_i, l \in [1:m]\backslash\mathcal{I}^1_i. \tag{4.121}$$

Now, using (4.121), we have

$$\begin{aligned} X_l\left(\sum_{s\in\mathcal{I}^1_i} t^i_{rs}t^i_{ws}\right) &= \sum_{s\in\mathcal{I}^1_i}\left(X_l(t^i_{rs})t^i_{ws} + t^i_{rs}X_l(t^i_{ws})\right) \\ &\overset{(4.121)}{=} -\sum_{s\in\mathcal{I}^1_i}\sum_{v\in\mathcal{I}^1_i}\left(c^s_{lv}t^i_{rv}t^i_{ws} + c^s_{lv}t^i_{wv}t^i_{rs}\right) = -\sum_{s,v\in\mathcal{I}^1_i}(c^s_{lv}+c^v_{ls})t^i_{rv}t^i_{ws} = 0, \end{aligned} \tag{4.122}$$

where the second expression of the chain of equalities in the second line of (4.122) is obtained by swapping indices s and v in the second term of the first expression of the chain of equalities in the second line of (4.122), and the last equality in (4.122) follows from items (1) and (2) of Proposition 11. Therefore, $\sum_{s\in\mathcal{I}^1_i} t^i_{rs}t^i_{ws}$ is constant on each piece of a leaf of $\mathcal{F}_{[1:k]\backslash\{i\}}$ lying in U. Besides, by (4.109),

$$\tilde{X}_r(q) = X_r(q), \quad \forall r \in \mathcal{I}^1_i, q \in \mathcal{F}_{\{i\}}(q_0)\cap U,$$

so by g_1-orthonormality of $\{X_r\}_{r\in\mathcal{I}^1_i}$ this implies that

$$\sum_{s\in\mathcal{I}^1_i} t^i_{rs}(q)t^i_{ws}(q) = \delta_{rw}, \quad \forall r, w \in \mathcal{I}^1_i, q \in \mathcal{F}_{\{i\}}(q_0)\cap U,$$

and hence, by (4.122),

$$\sum_{s\in\mathcal{I}^1_i} t^i_{rs}(q)t^i_{ws}(q) = \delta_{rw}, \quad \forall r, w \in \mathcal{I}^1_i, q \in U.$$

This proves the g_1-orthonormality of $\{\tilde{X}_r\}_{r\in\mathcal{I}^1_i}$. □

We complete the proof of the main theorem, Theorem 3, by noticing that, if g^i_1 denotes the sub-Riemannian metric on the distribution D_i on the leaf $F_i(q_0)$ defined by the condition that $\{X_r\}_{r\in\mathcal{I}^1_i}$ is orthogonal in this metric, then

$$(U, D|_U, g_1|_U) = \bigtimes_{i=1}^{k}\Big(\mathcal{F}_i(q_0)\cap U, D_i|_{\mathcal{F}_i(q_0)\cap U}, g^i_1|_{\mathcal{F}_i(q_0)\cap U}\Big).$$

APPENDIX A. PROOF OF PROPOSITION 1

Assume by contradiction that $\mathfrak{m}$ is not ad-surjective. Let $m := \dim\,\mathfrak{m}_{-1}$ and $d := \dim\,\mathfrak{m}_{-2}$. We start with consideration for general d. Assume that

$$r := \max_{X\in\mathfrak{m}_{-1}} \operatorname{rank}(\operatorname{ad}X). \tag{A.1}$$

Then from the non-ad-surjectivity assumption we have $r < d$. Take X_1 such that

$$\operatorname{rank}(\operatorname{ad}X_1) = r. \tag{A.2}$$

Then the rank-nullity theorem implies that

$$\dim\,\ker(\operatorname{ad}X_1) = m - r. \tag{A.3}$$

Obviously, $X_1 \in \ker(\operatorname{ad}X_1)$. Let us complete it to the basis $(X_1, \dots, X_m)$ of $\mathfrak{m}_{-1}$ such that

$$\ker(\operatorname{ad}X_1) = \operatorname{span}\{X_1, X_{r+2}, \dots, X_m\}. \tag{A.4}$$

Let

$$Y_l := [X_1, X_{l+1}], l \in [1 : r]. \tag{A.5}$$

By constructions, $Y_1, \ldots, Y_r$ are linearly independent, and

$$\mathrm{Im}(\mathrm{ad}X_1) = \mathrm{Im}(\mathrm{ad}X_1|_{\mathrm{span}\{X_2,\ldots,X_{r+1}\}}) = \mathrm{span}\{Y_1, \ldots, Y_r\}. \tag{A.6}$$

Since $\mathfrak{m}$ is step 2 and fundamental, there exists $i < j \in [2, m]$ such that

$$[X_i, X_j] \notin \mathrm{Im}(\mathrm{ad}X_1). \tag{A.7}$$

Set

$$Y_{r+1} := [X_i, X_j]. \tag{A.8}$$

Lemma 14. *The index j (and therefore also i) in (A.7) does not exceed $r + 1$.*

Proof. Assume by contradiction that $j \geqslant r + 2$. From maximality of r in (A.1), (A.2), and (A.6) it follows that for sufficiently small t

$$\mathrm{rank}(\mathrm{ad}(X_1 + tX_i)|_{\mathrm{span}\{X_2,\ldots,X_{r+1}\}}) = r \tag{A.9}$$

and the spaces $\mathrm{Im}(\mathrm{ad}(X_1 + tX_i)|_{\mathrm{span}\{X_2,\ldots,X_{r+1}\}})$ are sufficiently closed to $\mathrm{Im}(\mathrm{ad}X_1)$ so that

$$Y_{r+1} \notin \mathrm{Im}(\mathrm{ad}(X_1 + tX_i)|_{\mathrm{span}\{X_2,\ldots,X_{r+1}\}}). \tag{A.10}$$

On the other hand, from (A.4) and (A.8) it follows that

$$[X_1 + tX_i, X_j] = tY_{r+1}, \tag{A.11}$$

which implies that $\mathrm{rank}(\mathrm{ad}(X_1 + tX_i)) > r$ for sufficiently small $t \neq 0$. This contradicts the maximality of r in (A.1) and completes the proof of the lemma. □

In the proof of the previous lemma, based on (A.4) and (A.6) we actually have shown that

$$[\ker(\mathrm{ad}X_1), \mathfrak{m}_{-1}] \subset \mathrm{Im}(\mathrm{ad}X_1). \tag{A.12}$$

After permuting indices we can assume that $(i, j) = (2, 3)$, i. e., that

$$[X_2, X_3] \notin \mathrm{Im}(\mathrm{ad}X_1). \tag{A.13}$$

Now, given X and $\tilde{X}$ from $\mathfrak{m}_{-1}$, set

$$L_{X,\tilde{X}} := \ker(\mathrm{ad}X) \cap \ker(\mathrm{ad}\tilde{X}). \tag{A.14}$$

Let

$$k := \min_{X,\tilde{X}\in\mathfrak{m}_{-1}} \dim L_{X,\tilde{X}}. \tag{A.15}$$

By genericity of (A.2), (A.13) and (A.15) we can choose X_1, X_2, and X_3, maybe after a small perturbation, such that (A.2), (A.13), and

$$\dim\big(\ker(\mathrm{ad}X_1) \cap \ker(\mathrm{ad}X_2)\big) = k \tag{A.16}$$

hold simultaneously.

Now, by item 1 of Proposition 1 $d \leqslant 3$. Therefore, either $r = 1$ or $r = 2$. Consider these two cases separately.

Case 1: $r = 1$. By (A.4)

$$\ker(\mathrm{ad}X_1) = \mathrm{span}\{X_1, X_3, \ldots, X_m\} \tag{A.17}$$

and by this and (A.12)

$$[X_i, X_j] \in \mathrm{Im}(\mathrm{ad}X_1), \quad \forall i \in [2 : m], j \in [3, m] \tag{A.18}$$

or, equivalently, from the fundamentality of $\mathfrak{m}$,

$$\mathfrak{m}_{-2} = \mathrm{Im}(\mathrm{ad}X_1), \tag{A.19}$$

so in fact $d = r$ $(= 1)$ and this case is done.

Case 2: $r = 2$**.** By the previous constructions,

$$[X_1, X_2] = Y_1, \quad [X_2, X_3] = Y_3, \tag{A.20}$$

where

$$Y_3 \notin \mathrm{Im}(\mathrm{ad}X_1), \tag{A.21}$$

as a particular case of (A.10) for $r = 2$. Then from (A.20) and (A.21) by maximality of $r = 2$ in (A.1),

$$\mathrm{Im}(\mathrm{ad}X_2) = \mathrm{span}\{Y_1, Y_3\}. \tag{A.22}$$

From this, (A.6), and (A.12) it follows that

$$[X_2, X_i] \in \mathrm{Im}(\mathrm{ad}X_1) \cap \mathrm{Im}(\mathrm{ad}X_2) = \mathrm{span}\{Y_1\}, \quad \forall i \in [4 : m]. \tag{A.23}$$

Since $[X_1, X_2] = Y_1$ for any $i \in [4 : m]$ one can replace X_i by

$$\tilde{X}_i \equiv X_i \mod \mathrm{span}\{X_1\} \tag{A.24}$$

such that $[X_2, \tilde{X}_i] = 0$, i. e.,

$$\ker(\mathrm{ad}X_2) = \mathrm{span}\{X_2, \tilde{X}_4, \ldots, \tilde{X}_m\}.$$

This, together with (A.4) and (A.24), implies that

$$L_{X_1,X_2} = \mathrm{span}\{\tilde{X}_4, \ldots, \tilde{X}_m\}.$$

Therefore, by (A.16),

$$k = m - 3. \tag{A.25}$$

The following lemma will give a contradiction with item 3 of the assumptions of Proposition 1 and therefore will complete the proof of it in the case of $r = 2$:

Lemma 15. *The space L_{X,X_1} is the same for all $X \in \mathfrak{m}_{-1}$ for which* $\dim L_{X,X_1} = m - 3$*, and so the space L_{X,X_1} lies in the center of $\mathfrak{m}$* [6]*.*

Proof. Assume by contradiction that there exist X_2 and X_3 such that $\dim L_{X_i,X_1} = m - 3$, $i = 2, 3$ but

$$L_{X_2,X_1} \neq L_{X_3,X_1}. \tag{A.26}$$

Obviously, X_1, X_2 and X_3 are linearly independent. By openness of condition (A.26) we can always assume that

$$\mathrm{ad}X_1(\mathrm{span}X_2) \neq \mathrm{ad}X_1(\mathrm{span}X_3). \tag{A.27}$$

We claim that

$$L_{X_2,X_3} = L_{X_2,X_1} \cap L_{X_3,X_1}. \tag{A.28}$$

Before proving (A.28), note that, if it holds, then by (A.26) it will follow that $\dim L_{X_2 m X_3} < m - 3$, which will contradict the minimality of $k = m - 3$ in (A.15).

It remains to prove (A.28). First, it is clear that

$$L_{X_2,X_1} \cap L_{X_3,X_1} = \bigcap_{i=1}^{3} \ker(\mathrm{ad}X_i) \subset L_{X_2,X_3}. \tag{A.29}$$

[6] The latter conclusion follows from the fact that the set of such X is generic in $\mathfrak{m}_{-1}$.

On the other hand, from the dimension assumptions it follows that

$$\ker(\mathrm{ad}X_i) = \mathrm{span}\{X_i\} \oplus L_{X_i,X_1}, \quad i = 2, 3. \tag{A.30}$$

So if $v \in L_{X_2,X_3}$, then

$$v \equiv \alpha_2 X_2 \bmod \ker(\mathrm{ad}X_1) \equiv \alpha_3 X_3 \bmod \ker(\mathrm{ad}X_1). \tag{A.31}$$

Note that (A.27) means that X_2 and X_3 are linearly independent modulo $\ker(\mathrm{ad}X_1)$. Hence, (A.31) implies that $\alpha_2 = \alpha_3 = 0$, i. e., $v \in \ker(\mathrm{ad}X_1)$. This implies that $v \in L_{X_2,X_1} \cap L_{X_3,X_1}$, i.e.,

$$L_{X_2,X_3} \subset L_{X_2,X_1} \cap L_{X_3,X_1}.$$

This and inclusion (A.29) complete the proof of (A.28) and therefore that of Lemma 15. □

Finally, note that in the case of $\dim \mathfrak{m}_{-2} = 4$, even if assumptions 2 and 3 hold, Proposition 1 is wrong. Here is a counterexample:

Example 2. Let $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ be the step 2 graded Lie algebra such that

$$\begin{aligned} \mathfrak{m}_{-1} &= \mathrm{span}\{X_1, \dots, X_5\} \\ \mathfrak{m}_{-2} &= \mathrm{span}\{Y_1, \dots, Y_4\} \end{aligned} \tag{A.32}$$

so that, up to skew-symmetricity, the following brackets of the chosen basis are the only nonzero ones:

$$\begin{aligned} &[X_1, X_i] = Y_{i-1},\ \forall i \in [2:4], \\ &[X_2, X_3] = Y_4, [X_2, X_5] = \beta Y_3, \\ &[X_3, X_5] = \delta Y_3, [X_4.X_5] = \lambda Y_3, \end{aligned} \tag{A.33}$$

where β, δ, λ are nonzero constants. Note that $\mathrm{span}\{X_1, X_2, X_3, Y_1, Y_2, Y_4\}$ is a subalgebra of $\mathfrak{m}$ isomorphic to the truncated step 2 free Lie algebras with three generators and $\mathrm{span}\{X_4, X_5, Y_3\}$ is an ideal of $\mathfrak{m}$ isomorphic to the 3-dimensional Heisenberg algebra. Thus, $\mathfrak{m}$ *is a semidirect sum of the truncated step* 2 *free Lie algebras with three generators and the* 3*-dimensional Heisenberg algebra.*

It can be checked by straightforward computations that here r defined by (A.1) is equal to $3 < d = 4$ and that $\mathfrak{m}_{-1}$ meets the center trivially, i. e., it is indeed a counterexample.

Indeed, if $X \in \mathfrak{m}_{-1}$,

$$X = \sum_{i=1}^{5} C_i X_i, \tag{A.34}$$

then the map $\mathrm{ad}X$ has the following matrix with respect to the bases $(X_1, \dots, X_5)$ and $Y_1, \dots, Y_4)$ of $\mathfrak{m}_{-1}$ and $\mathfrak{m}_{-2}$:

$$\mathrm{ad}X = \begin{pmatrix} -C_2 & -C_3 & -C_4 & 0 \\ C_1 & 0 & -\beta C_5 & C_3 \\ 0 & C_1 & \delta C_5 & C_2 \\ 0 & 0 & C_1 - \lambda C_3 & 0 \\ 0 & 0 & \beta C_2 + \delta C_3 + \lambda C_4 & 0 \end{pmatrix}. \tag{A.35}$$

It is easy to check that the maximal rank of this matrix (as a function of C's) is equal to 3, which implies that $r = 3$. Also, this matrix is not equal to zero if $(C_1, \dots, C_5) \neq 0$, which means that $\mathfrak{m}_{-1}$ meets the center trivially.

Finally, note that $\mathfrak{m}$ is not decomposable. Assume the converse, i. e., that $\mathfrak{m} = \mathfrak{m}^1 \oplus \mathfrak{m}^2$ for some nonzero fundamental graded Lie algebra $\mathfrak{m}^1$ and $\mathfrak{m}^2$. Without loss of generality assume that

$$\dim\, \mathfrak{m}^1_{-1} \geqslant \dim\, \mathfrak{m}^2_{-1}. \tag{A.36}$$

Since $\mathfrak{m}_{-1}$ meets the center trivially, it is impossible that $\dim\, \mathfrak{m}^2_{-1} = 1$. Hence by (A.36) and the fact that $\dim \mathfrak{m}_{-1} = 5$, we have that $\dim\, \mathfrak{m}^2_{-1} = 2$ and the algebra $\mathfrak{m}_{-1}$ is nothing but the 3-dimensional Heisenberg algebra. Therefore, for every nonzero $X \in \mathfrak{m}^2_{-1}$, the rank of $\mathrm{ad}X$ is equal to 1. However, it is straightforward to show that, if the rank of the matrix (A.35) is not greater than 1, then $(C_1, \ldots, C_5) = 0$, which leads to a contradiction. So $\mathfrak{m}$ is indecomposable.

An alternative, more conceptual, way to prove indecomposability of $\mathfrak{m}$ is to observe that, otherwise, each component in its decomposition will have -2 degree part of dimension not greater than 3 (but not equal to 0) and by Proposition 1 each component is ad-surjective. Then by Remark 2, $\mathfrak{m}$ is ad-surjective, which is not the case.

APPENDIX B. PROOF OF SUBLEMMA 1

Let us derive an expression for $\tilde{b}_{m+f}$ with $f \in [1 : m_1]$. For the proof of Sublemma 1 we need only the case $f = 1$, but the cases of more general f are needed in Section 4.5, in particular, in Lemma 9. First, by column operations (4.18)

$$\tilde{b}_{m+f} = b^2_f - \sum_{i=1}^{k} (\alpha_{n_{i-1}+1})^2 \sum_{t=e_{i-1}}^{e_i} a^2_{f,m+t} u_{m+t}, \tag{B.1}$$

where by (3.21) and (3.22)

$$a^2_{f,m+j} = \vec{h}_1(q_{f,m+j}) + \sum_{r=m+1}^{n} q_{f,r} q_{r,m+j}, \tag{B.2}$$

$$b^2_f = \vec{h}_1(b^1_f) - \sum_{i=1}^{k} (\alpha_{n_{i-1}+1})^2 \sum_{r=m+1}^{n} \sum_{w \in \mathcal{I}^1_i} q_{1r} q_{rw} u_w, \tag{B.3}$$

and q_{jk} and b^1_f are as in (3.17) and (4.14), respectively. Note that from the decomposition of the Tanaka symbol (3.8) it follows that

$$q_{wr} = 0, \quad \forall w \in \mathcal{I}^1_i, r \in \mathcal{I}^2_v \text{ with } i \neq v. \tag{B.4}$$

Substituting (B.4) into (B.2), we get

$$a^2_{f,m+t} = \begin{cases} \vec{h}_1(q_{f,m+t}) + \displaystyle\sum_{r=m+1}^{m+d_1} q_{fr} q_{r,m+t}, & t \in [1 : d_1] \\ \displaystyle\sum_{r=m+1}^{m+d_1} q_{fr} q_{r,m+t}, & t \in [1 : n-m] \backslash [1 : d_1]. \end{cases} \tag{B.5}$$

Moreover, (B.4) implies that (B.3) can be rewritten as follows:

$$b^2_f = \vec{h}_1(b^1_f) - \sum_{i=1}^{k} (\alpha_{n_{i-1}+1})^2 \sum_{r=m+1}^{m+d_1} \sum_{v \in \mathcal{I}^1_i} q_{fr} q_{rv} u_v. \tag{B.6}$$

By (B.1), (B.5), and (B.6), we have

$$\begin{aligned} \tilde{b}_{m+f} = \vec{h}_1(b^1_f) &- \sum_{i=1}^{k} (\alpha_{n_{i-1}+1})^2 \sum_{r=m+1}^{m+d_1} \sum_{w \in \mathcal{I}^1_i} q_{fr} q_{rw} u_w \\ &- \sum_{i=1}^{k} (\alpha_{n_{i-1}+1})^2 \sum_{t=e_{i-1}+1}^{e_i} a^2_{f,m+t} u_{m+t}. \end{aligned} \tag{B.7}$$

Using (4.14) and (4.15), we get

$$\vec{h}_1(b_f^1) = \vec{h}_1\left((\alpha_1)^2\sum_{r=m+1}^{m+d_1} q_{fr}u_r + \sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\sum_{w\in\mathcal{I}_i^1} q_{fw}u_w\right)$$
$$= \alpha_1^2\sum_{r=m+1}^{m+d_1}\vec{h}_1(q_{fr})u_r + \alpha_1^2\sum_{r=m+1}^{m+d_1} q_{fr}\vec{h}_1(u_r) + \sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\sum_{w\in\mathcal{I}_i^1}\vec{h}_1(q_{fw})u_w$$
$$+\sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\sum_{w\in\mathcal{I}_i^1} q_{fw}\vec{h}_1(u_w) = \alpha_1^2\sum_{r=m+1}^{m+d_1}\vec{h}_1(q_{fr})u_r + \alpha_1^2\sum_{r=m+1}^{m+d_1}\sum_{x=1}^{n} q_{fr}q_{rx}u_x \tag{B.8}$$
$$+\sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\left(\sum_{w\in\mathcal{I}_i^1}\left(\vec{h}_1(q_{fw})u_w + \sum_{x=1}^{n} q_{fw}q_{wx}u_x\right)\right).$$

Substituting (B.5) and (B.8) into (B.7), and taking into account (B.5) again, we get the following cancellations:

$$\tilde{b}_{m+f} = \alpha_1^2\sum_{r=m+1}^{m+d_1}\vec{h}_1(q_{fr})u_r + \alpha_1^2\sum_{r=m+1}^{m+d_1}\sum_{w=1}^{n} q_{fr}q_{rw}u_w$$
$$+\sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\left(\sum_{w\in\mathcal{I}_i^1}\left(\vec{h}_1(q_{fw})u_w + \sum_{x=1}^{n} q_{fw}q_{wx}u_x\right)\right)$$
$$-\sum_{i=1}^{k}(\alpha_{n_{i-1}+1})^2\sum_{r=m+1}^{m+d_1}\sum_{w\in\mathcal{I}_i^1} q_{fr}q_{rw}u_w - \alpha_1^2\sum_{t=1}^{d_1}\vec{h}_1(q_{fm+t})u_{m+t} \tag{B.9}$$
$$-\sum_{i=1}^{k}(\alpha_{n_{i-1}+1})^2\sum_{t=e_{i-1}+1}^{e_i}\sum_{r=m+1}^{m+d_1} q_{fr}q_{r,m+t}u_{m+t}.$$

Applying the relation $[1:n] = \bigcup_{i=1}^{k}\mathcal{I}_i^1\cup\mathcal{I}_i^2$ to the sum in the second term of (B.9), we get

$$\tilde{b}_{m+f} = \sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\left(\sum_{w\in\mathcal{I}_i^1}\left(\vec{h}_1(q_{fw})u_w + \sum_{x=1}^{n} q_{fw}q_{wx}u_x\right)\right.$$
$$\left.+\sum_{r=m+1}^{m+d_1}\sum_{w\in\mathcal{I}_i^1} q_{fr}q_{rw}u_w + \sum_{r=m+1}^{m+d_1}\sum_{t\in\mathcal{I}_i^2} q_{fr}q_{rt}u_t\right). \tag{B.10}$$

From now on let $f=1$. Since by (3.17) all q_{jk} depend only on u_i's with $i\in[1:m]$, from expression (B.10) the variable u_j with $j\in\mathcal{I}_1^2$ may appear only in the following terms of (B.10):

$$\sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\sum_{w\in\mathcal{I}_i^1}\left(\vec{h}_1(q_{1w})u_w + q_{1w}q_{wj}u_j\right). \tag{B.11}$$

Moreover, $q_{wj}=0$ for $w\in[1:m]\backslash\mathcal{I}_1^1$ and $j\in\mathcal{I}_1^2$ by (B.4), so the variable u_j with $j\in\mathcal{I}_1^2$ may appear only in the following terms of (B.10):

$$\sum_{i=2}^{k}\left((\alpha_1)^2-(\alpha_{n_{i-1}+1})^2\right)\sum_{w\in\mathcal{I}_i^1}\vec{h}_1(q_{1w})u_w \tag{B.12}$$

or, more precisely, in terms $\vec{h}_1(q_{1w})$ with $w \in [1:m]\backslash\mathcal{I}_1^1$, $i \in [2:k]$. Let us analyze these terms. Using (3.17) and (4.15), we get

$$\begin{aligned}\vec{h}_1(q_{1w}) &= \sum_{r=1}^{m}\left(c_{r1}^{w}\vec{h}_1(u_r) + \vec{h}_1(c_{r1}^{w})u_r\right)\\ &= \sum_{r=1}^{m}\sum_{x=1}^{n} c_{r1}^{w} q_{rx} u_x + \sum_{r=1}^{m}\vec{h}_1(c_{r1}^{w})u_r.\end{aligned} \tag{B.13}$$

The second term in (B.13) does not depend on u_j with $j \in \mathcal{I}_1^2$, while to get this u_j in the first term the index x must be equal to j. So, the terms containing u_j in $\vec{h}_1(q_{1w})$ are

$$u_j \sum_{r=1}^{m} c_{r1}^{w} q_{rj}. \tag{B.14}$$

Recall that

$$\begin{cases} c_{r1}^{w} = 0, & \text{if } r \in \mathcal{I}_1^1\ w \in [1:m]\backslash\mathcal{I}_1^1;\\ q_{rj} = 0, & \text{if } r \in [1:m]\backslash\mathcal{I}_1^1.\end{cases} \tag{B.15}$$

Here the first line comes from (4.24) and the second line comes from (B.4). So, plugging (B.15) into (B.14), we get that u_j does not appear in $\tilde{b}_{m+1}$ and the proof of Sublemma 1 is completed.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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