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## DU BOIS PROPERTY OF LOG CENTERS

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The analytic aspects of multiplier ideals, log canonical thresholds and log canonical centers played an important role in several papers of Demailly, including [DEL00, Dem01, DK01, Dem12, DP14, Dem16, CDM17, Dem18].

Log canonical centers are seminormal by [Amb03, Fuj17], even Du Bois by [KK10, KK20]. This has important applications to birational geometry and moduli theory; see [KK10, KK20] or [Kol22, Sec.2.5].

We recall the concept of Du Bois singularities in Definition–Theorem 4. An unusual aspect is that this notion makes sense for complex spaces that have irreducible components of different dimension. This is crucial even for the statement of our theorem.

In this note we generalize the results of [KK10, KK20] by showing that if a closed subset  $V \subset X$  is ‘close enough’ to being a union of log canonical centers, then it is Du Bois.

The minimal log discrepancy—denoted by  $\text{mld}(V, X, \Delta)$  as in Definition 2—is a nonnegative rational number, that measures the deviation from being a union of log canonical centers. The log canonical gap in dimension  $n$ —denoted by  $\text{lcg}(n)$  as in Definition 3—gives the precise notion of ‘closeness.’

**Theorem 1.** *Let  $(X, \Delta)$  be a log canonical pair of dimension  $n$ , and  $V \subset X$  a closed subset such that  $\text{mld}(V, X, \Delta) < \text{lcg}(n)$ . Then  $V$  has Du Bois singularities.*

The theorem applies to algebraic varieties and algebraic spaces of finite type over a field of characteristic 0.

By [Kol14], if  $\text{mld}(V, X, \Delta) < \frac{1}{6}$ , then  $V$  is seminormal, and  $\frac{1}{6}$  is optimal in every dimension  $\geq 2$ . The bound  $\text{lcg}(n)$  is also optimal for every  $n$ , but its value is not known for  $n \geq 4$ , and  $\text{lcg}(n)$  converges to 0 very rapidly, see (3.3) and (3.4).

We follow the terminology of [KM98] and of [Kol13].

**Definition 2** (Minimal log discrepancy). Let  $(X, \Delta)$  be a log canonical pair and  $V \subset X$  an irreducible subvariety. The *minimal log discrepancy* of  $V$  is the infimum of the numbers  $1 + a(E, X, \Delta)$ , where  $E$  runs through all divisors over  $X$  that dominate  $V$ , where  $a(E, X, \Delta)$  denotes the discrepancy as in [KM98, 2.25]. It is denoted by  $\text{mld}(V, X, \Delta)$ . Thus if  $V = D$  is a divisor on  $X$ , then  $\text{mld}(D, X, \Delta) = 1 - \text{coeff}_D \Delta$ .

$V$  is a *log canonical center* or *lc center* of  $(X, \Delta)$  iff  $\text{mld}(V, X, \Delta) = 0$ . It is sometimes convenient to view  $X$  itself as a log canonical center.

Let  $V \subset Z$  be a closed subset with irreducible components  $V_i$ . We define its minimal log discrepancy as  $\text{mld}(V, X, \Delta) := \max_i \{\text{mld}(V_i, X, \Delta)\}$ .

**Definition 3** (Log canonical gap). The log canonical gap in dimension  $n$ , denoted by  $\text{lcg}(n)$ , is the largest (real) number  $\epsilon$  with the following property.

(3.1) Let  $(X, dD)$  be a  $\mathbb{Q}$ -factorial, log canonical pair with  $\dim X = n$  and  $D$  a  $\mathbb{Z}$ -divisor. Assume that  $d > 1 - \epsilon$ . Then  $(X, D)$  is also log canonical.

Note that  $1 - \text{lcg}(n)$  is the largest log canonical threshold in dimension  $n$  (that is  $< 1$ ). By replacing all  $d_i$  by the smallest one we see that it is also the largest  $\epsilon$  with the following property.

(3.2) Let  $(X, \sum d_i D_i)$  be a  $\mathbb{Q}$ -factorial, log canonical pair with  $\dim X = n$ , where  $D_i$  are  $\mathbb{Z}$ -divisors and  $d_i \in \mathbb{Q}$ . Assume that  $d_i > 1 - \epsilon$  for every  $i$ . Then  $(X, \sum D_i)$  is also log canonical.

It is easy to see that  $\text{lcg}(2) = \frac{1}{6}$ , and  $\text{lcg}(3) = \frac{1}{42}$  by [Kol94, 5.5]. A difficult theorem [HMX14, Thm.1.1] says that  $\text{lcg}(n)$  is positive for every  $n$ , but no explicit lower bound is known for  $n \geq 4$ .

*Remark 3.3.* Let  $(X, dD)$  be a  $\mathbb{Q}$ -factorial, log canonical pair such that  $(X, D)$  is not log canonical. As in [KM98, 2.49–53], it has a quasi-étale cover  $\pi : (X', D') \rightarrow (X, D)$  such that  $K_{X'}$  and  $D'$  are both Cartier. By Reid's lemma,  $(X', dD')$  is log canonical and  $(X', D')$  is not log canonical; see [KM98, 5.20]. Since  $K_{D'}$  is Cartier, log canonical coincides with Du Bois by [Kov99]. This shows that the bound  $\text{mld}(V, X, \Delta) < \text{lcg}(n)$  is optimal in Theorem 1.

*Example 3.4.* Set  $c_1 = 2$  and let  $c_{k+1} := c_1 \cdots c_k + 1$ ; it is called Euclid's or Sylvester's sequence, see [Slo03, A00058]. It starts as 2, 3, 7, 43, 1807, 3263443, ... Then  $D_n := (z_1^{c_1} + \cdots + z_n^{c_n} = 0) \subset \mathbb{C}^n$  is not log canonical, but  $(\mathbb{A}^n, (1 - \frac{1}{c_1 \cdots c_n})D)$  is log canonical; see [Kol13, 8.6] for details.

Thus  $\text{lcg}(n) \leq \frac{1}{c_1 \cdots c_n}$ , and the latter goes to 0 doubly exponentially.

### Du Bois property.

Let  $M$  be a compact Kähler manifold. One of the useful consequences of the Hodge decomposition is the surjectivity of the natural map

$$H^i(M, \mathbb{C}) \twoheadrightarrow H^i(M, \mathcal{O}_M).$$

Roughly speaking, projective varieties with Du Bois singularities form the largest class that is stable under natural operations (small deformations, products, general hyperplane sections) where the above surjectivity still holds. For our current purposes the Du Bois property can be handled as a black box. We list in Paragraph 5 all the properties that we use. We give references to the original papers; [Kol13, Chap.6] is a suitable general introduction.

The original and most useful definition is rather complicated; see the papers [DB81, Sch07, Kov12a] or [Kol13, Sec.6.1]. The following characterization emphasizes that Du Bois is a generalization of seminormality. We can take (4.2) as our definition. (We use 'X is Du Bois' and 'X has Du Bois singularities' as synonyms.) We state the version given in [KS11, 6.4].

**Definition–Theorem 4.** [Sch07] Let  $X$  be reduced and  $Y \supset X$  a smooth space containing it. Let  $\pi : Y' \rightarrow Y$  be an embedded log resolution of  $X$ , that is,  $Y'$  is smooth and  $E := \text{red } \pi^{-1}(X)$  is a simple normal crossing divisor. Then  $X$  is

(4.1) seminormal iff  $\pi_* \mathcal{O}_E = \mathcal{O}_X$ , and

(4.2) Du Bois iff  $\pi_* \mathcal{O}_E = \mathcal{O}_X$  and  $R^i \pi_* \mathcal{O}_E = 0$  for  $i > 0$ . □

In particular, if  $X$  is Du Bois, then it is reduced and seminormal.

If  $X$  is smooth then the blow-up  $Y' := B_X Y \rightarrow Y$  shows that  $X$  is Du Bois.

**5** (Properties of Du Bois singularities that we use). We work either with algebraic spaces of finite type over a field of characteristic 0. Note that we allow them to have irreducible components of different dimensions.

*Property 5.0.* Smooth implies Du Bois. Du Bois implies reduced and seminormal.

*Property 5.1.* [KK10, KK20] Let  $(X, \Delta)$  be a log canonical pair and  $V \subset X$  a union of some of its log canonical centers. Then  $V$  is Du Bois. More generally, this holds for log canonical centers of crepant log structures, as in Definition 11.

*Property 5.2.* [Kov11, 2.11-12] Let  $X_1, X_2 \subset X$  be closed subspaces. If 3 of  $\{X_1 \cap X_2, X_1, X_2, X_1 \cup X_2\}$  are Du Bois, then so is the 4th.

*Property 5.3.* [KK10, 1.6], [Kov12b, 3.3] and [Kol13, 6.27]. Let  $p : Y \rightarrow X$  be a proper surjective morphism,  $V \subset X$  a closed, reduced subscheme, and  $D := \text{Supp } p^{-1}(V)$ . Assume that  $\mathcal{O}_X(-V) \rightarrow Rp_*\mathcal{O}_Y(-D)$  has a left inverse and  $Y, D$  are Du Bois. Then  $X$  is Du Bois  $\Leftrightarrow V$  is Du Bois.

In applications the key is to find examples where Property 5.3 applies. The following gives most known cases.

**Theorem 6.** *Let  $f : Y \rightarrow Z$  be a projective morphism with connected fibers between normal spaces. Assume that  $(Y, \Delta)$  is  $\mathbb{Q}$ -factorial, dlt and  $K_Y + \Delta \sim_{f, \mathbb{R}} 0$ . Let  $D$  be an effective  $\mathbb{Z}$ -divisor such that  $\lfloor \Delta \rfloor \subset \text{Supp } D \subset \text{Supp } \Delta$  and  $-D$  is  $f$ -semiample. Set  $V = f(\text{Supp } D)$ . Then  $\mathcal{O}_Z(-V) \rightarrow Rf_*\mathcal{O}_Y(-D)$  has a left inverse*

Note that, since  $-D$  is  $f$ -semiample,  $D$  does not dominate  $Z$ . Thus  $V \subsetneq Z$  and  $\mathcal{O}_Z(-V)$  makes sense.

*Proof.* If  $f$  is birational then the proof is much simpler, and worth doing separately.

Choose  $\epsilon > 0$  such that  $\Theta := \Delta - \epsilon D$  is effective. Note that

$$-D \sim_{f, \mathbb{R}} K_Y + \Theta + (1 - \epsilon)(-D), \quad (6.1)$$

$(Y, \Theta)$  is klt and  $(1 - \epsilon)(-D)$  is  $f$ -semiample.

In the birational case, the general form of Grauert-Riemenschneider vanishing gives that  $R^i f_*\mathcal{O}_Y(-D) = 0$  for  $i > 0$ ; see [KM98, 2.68]. Thus  $Rf_*\mathcal{O}_Y(-D) \simeq_{qis} f_*\mathcal{O}_Y(-D) = \mathcal{O}_Z(-V)$ .

If  $D$  does not dominate  $Z$ , then the assumption  $\lfloor \Delta \rfloor \subset \text{Supp } D$  implies that the generic fiber is klt. Also,  $D = \text{Supp } f^{-1}(V)$ , since  $-D$  is  $f$ -nef and the fibers are connected. We can now use [Kol86, 3.1], more precisely the form given in [Kol13, 10.41], to get the required left inverse.  $\square$

### The klt case of Theorem 1.

7. The proof is short and follows [Kol14].

First we show the special case when  $\text{Supp } V$  is a divisor; see Lemma 8.

In general, we find a dlt modification  $g : (Y, \Delta_Y) \rightarrow (X, \Delta)$  such that  $D := g^{-1}(V)$  is a divisor and  $\text{mld}(D, Y, \Delta_Y) = \text{mld}(V, X, \Delta)$ ; see Proposition 9. Choose  $\epsilon > 0$  such that  $\Theta := \Delta_Y - \epsilon D$  is effective. Note that

$$-D \sim_{g, \mathbb{R}} K_Y + \Theta + (1 - \epsilon)(-D), \quad (7.1)$$

$(Y, \Theta)$  is klt. If  $-D$  is  $g$ -nef, then Grauert-Riemenschneider vanishing applies to  $R^i g_*\mathcal{O}_Y(-D)$ . We can achieve these after running a suitable MMP; see Lemma 10. Thus we may assume that  $\mathcal{O}_X(-V) \cong Rg_*\mathcal{O}_Y(-D)$ .  $V$  is now Du Bois by (5.3).  $\square$

**Lemma 8.** *Theorem 1 holds if  $X$  is  $\mathbb{Q}$ -factorial and  $V$  has pure codimension 1.*

*Proof.* Write  $V = \cup_{i \in I} D_i$  where the  $D_i \subset X$  are irreducible divisors. Note that  $\text{mld}(D, X, \Delta) = 1 - \text{coeff}_D \Delta$  for any irreducible divisor  $D$ . Thus we can write  $\Delta = \sum_{i \in I} d_i D_i + \Delta'$  where  $d_i > 1 - \text{lcm}(n)$  and  $D_i \not\subset \text{Supp } \Delta'$ . Since  $X$  is  $\mathbb{Q}$ -factorial,  $(X, \sum_{i \in I} d_i D_i)$  is also lc, hence so is  $(X, \sum_{i \in I} D_i)$  by Definition 3. Note that each  $D_i$  is a log canonical center of  $(X, \sum_{i \in I} D_i)$ , so  $\cup_i D_i$  is Du Bois by (5.1).  $\square$

**Proposition 9.** [Kol13, 1.38] *Let  $(X, \Delta)$  be log canonical, and  $\{E_i : i \in I\}$  finitely many exceptional divisors over  $X$  such that  $-1 \leq a(E_i, X, \Delta) < 0$ . Then there is a  $\mathbb{Q}$ -factorial, dlt modification  $g : (Y, \Delta_Y) \rightarrow (X, \Delta)$  such that*

(9.1) *the  $\{E_i : i \in I\}$  are among the exceptional divisors of  $g$ , and*

(9.2) *every other exceptional divisor  $F$  of  $g$  has discrepancy  $-1$ .*  $\square$

**Lemma 10.** *Let  $f : Y \rightarrow Z$  be a projective morphism between normal spaces. Assume that  $(Y, \Delta)$  is  $\mathbb{Q}$ -factorial, dlt and  $K_Y + \Delta \sim_{f, \mathbb{Q}} 0$ . Let  $D$  be an effective  $\mathbb{Z}$ -divisor such that  $\text{Supp } D \subset \text{Supp } \Delta$ . Then the  $(-D)$ -MMP runs and terminates in a good minimal model if  $D$  does not dominate  $Z$ .*

*Proof.* The  $(-D)$ -MMP is the same as the  $(-\epsilon D)$ -MMP, which in turn agrees with the  $(K_Y + \Delta - \epsilon D)$ -MMP since  $K_Y + \Delta \sim_{f, \mathbb{Q}} 0$ .

If  $f$  is birational, the  $(K_Y + \Delta - \epsilon D)$ -MMP runs and terminates by [Bir12, HX13].

If  $f$  is not birational, then the generic fiber of  $(Y, \Delta - \epsilon D) \rightarrow X$  is the same as the generic fiber of  $(Y, \Delta) \rightarrow X$ , and the latter is a good minimal model by assumption. Thus the MMP for  $(Y, \Delta - \epsilon D) \rightarrow X$  runs and terminates by [HX13].

The above references work for varieties; see [VP21, Kol21] for algebraic spaces of finite type, [Fuj22] for analytic spaces and [LM22] for the most general settings.  $\square$

### Crepant log structures.

For the general case of Theorem 1, we first study what the above proof gives. Keeping in mind the inductive arguments of [KK10], we do this for crepant log structures. The end result is Lemma 14. Then induction and repeated use of (5.2) completes the proof in Proposition 16.

**Definition 11.** A *crepant log structure* is a dominant, projective morphism with connected fibers  $g : (Y, \Delta) \rightarrow Z$ , where  $(Y, \Delta)$  is lc,  $Z$  is normal and  $K_Y + \Delta \sim_{g, \mathbb{R}} 0$ .

If  $(X, \Delta)$  is lc, then the identity  $(X, \Delta) \rightarrow X$  is a crepant log structure.

As a generalization of (5.1),  $Z$  is Du Bois [Kol13, 6.31].

For an irreducible  $V \subset Z$  we define  $\text{mld}(V, Y, \Delta)$  as the infimum of the numbers  $1 + a(E, Y, \Delta)$  where  $E$  runs through all divisors over  $Y$  that dominate  $V$ .

As in Definition 2, if  $V \subset Z$  is a closed subset with irreducible components  $V_i$ , then we set  $\text{mld}(V, Y, \Delta) := \max_i \{\text{mld}(V_i, Y, \Delta)\}$ .

We will use the following property proved in [Kol14], see also [Kol13, 7.5].

$$(11.1) \quad \text{mld}(V_1 \cap V_2, Y, \Delta) \leq \text{mld}(V_1, Y, \Delta) + \text{mld}(V_2, Y, \Delta).$$

The following generalization of Theorem 1 is better suited for induction.

**Theorem 12.** *Let  $g : (Y, \Delta) \rightarrow Z$  be a crepant log structure. Set  $n = \dim Y$  and let  $V \subset Z$  be a closed subset such that  $\text{mld}(V, Y, \Delta) < \text{lcm}(n)$ . Then  $V$  has Du Bois singularities.*

Next we see what the method of (7) gives for crepant log structures.

**Notation 13.** Let  $g : (Y, \Delta) \rightarrow Z$  be a crepant log structure. For a closed subset  $Z_1 \subset Z$ , let  $Z_1^\diamond \subset Z_1$  denote the union of those log canonical centers of  $(Y, \Delta)$  that are contained in  $Z_1$ , but are nowhere dense in it.

Note that  $Z^\diamond$  is the non-klt locus of  $g : (Y, \Delta) \rightarrow Z$ .

If  $Z_1$  itself is the union of log canonical centers of  $(Y, \Delta)$ , then  $Z_1$  is seminormal and  $Z_1 \setminus Z_1^\diamond$  is normal by [Amb03, Fuj17] and [Kol13, 4.32].

**Lemma 14.** *Let  $g : (Y, \Delta) \rightarrow Z$  be a crepant log structure with klt generic fiber. Set  $n = \dim Y$  and let  $V \subset Z$  be a closed subset such that  $\text{mld}(V, Y, \Delta) < \text{lcm}(n)$ . Then  $V \cup Z^\diamond$  has Du Bois singularities.*

*Proof.* By Proposition 9, we may assume that  $(Y, \Delta)$  is  $\mathbb{Q}$ -factorial, dlt, and there is a divisor  $D = \sum D_i \subset Y$  such that  $[\Delta] \subset D$ ,  $\text{mld}(D, Y, \Delta) < \text{lcm}(n)$ , and  $g(D) = V \cup Z^\diamond$ . After running the MMP for  $K_Y + \Delta - \epsilon D$  for some  $\epsilon > 0$  as in Lemma 10, we may also assume that  $-D$  is  $g$ -semiample. As in (6.1),

$$-D \sim_{g, \mathbb{R}} K_Y + (\Delta - \epsilon D) + (1 - \epsilon)(-D), \quad (14.1)$$

where  $(Y, \Delta - \epsilon D)$  is klt and  $(1 - \epsilon)(-D)$  is  $g$ -semiample. Also note that  $D = \text{Supp } g^{-1}(V)$ , since  $-D$  is  $g$ -nef and the fibers are connected. We can now use Theorem 6 to get that

$$\mathcal{O}_Z(-(V \cup Z^\diamond)) \rightarrow Rg_* \mathcal{O}_Y(-D) \quad (14.2)$$

has a left inverse. As we noted in Definition 11,  $Z$  is Du Bois. By (5.3) these imply that  $V \cup Z^\diamond$  is Du Bois.  $\square$

**Corollary 15.** *Let  $g : (Y, \Delta) \rightarrow X$  be a crepant log structure of dimension  $n$ , and  $Z \subset X$  a union of some of its log canonical centers. Let  $V \subset Z$  be a closed subset such that  $\text{mld}(V, Y, \Delta) < \text{lcm}(n)$ . Then  $V \cup Z^\diamond$  is Du Bois.*

*Proof.* We may assume that  $(Y, \Delta)$  is  $\mathbb{Q}$ -factorial and dlt.

For each irreducible component  $Z_i \subset Z$ , let  $Y_i \subset Y$  be a minimal dimensional log canonical center of  $(Y, \Delta)$  that dominates  $Z_i$ . Set  $\Theta_i := \text{Diff}_{Y_i}^* \Delta$  as in [Kol13, 4.18.4].

Let  $\pi_i : \tilde{Z}_i \rightarrow Z_i$  denote the normalization. Stein factorization of  $Y_i \rightarrow Z_i$  gives  $g_i : Y_i \rightarrow \tilde{Z}_i$  and  $\tau_i : \tilde{Z}_i \rightarrow Z_i$ . The  $g_i : (Y_i, \Theta_i) \rightarrow \tilde{Z}_i$  are crepant log structures with klt general fibers.

Precise inversion of adjunction [Kol13, 7.10] shows that  $(\pi_i \circ \tau_i)^{-1}(Z^\diamond) = (\tilde{Z}_i)^\diamond$  and  $\text{mld}(\tilde{V}_i, Y_i, \Theta_i) \leq \text{mld}(V, Y, \Delta)$ , where  $\tilde{V}_i := (\pi_i \circ \tau_i)^{-1}(V)$ .

The  $\tilde{Z}_i$  are Du Bois by (5.1), and the  $\tilde{V}_i \cup \tilde{Z}_i^\diamond$  are Du Bois by Lemma 14.

Set  $\tilde{V}_i := \pi_i^{-1}(V)$ ,  $\tilde{Z}_i^\diamond := \pi_i^{-1}(Z^\diamond)$  and  $\tilde{Z}^\diamond := \cup_i \tilde{Z}_i^\diamond$ .

The normalized trace map splits  $\mathcal{O}_{\tilde{Z}_i} \hookrightarrow (\tau_i)_* \mathcal{O}_{\tilde{Z}_i}$ , and hence also splits

$$\mathcal{O}_{\tilde{Z}_i}(-(\tilde{V}_i \cup \tilde{Z}_i^\diamond)) \hookrightarrow (\tau_i)_* \mathcal{O}_{\tilde{Z}_i}(-(\tilde{V}_i \cup \tilde{Z}_i^\diamond)).$$

Using the first splitting and applying (5.3) to  $(\tilde{Z}_i, \emptyset) \rightarrow (\tilde{Z}_i, \emptyset)$  shows that  $\tilde{Z}_i$  is Du Bois. Applying (5.3) and the second splitting now gives that  $\tilde{V}_i \cup \tilde{Z}_i^\diamond$  is Du Bois.

As we noted in (13),  $Z$  is seminormal and normal outside  $Z^\diamond$ , thus  $\mathcal{O}_Z(-Z^\diamond) = \pi_* \mathcal{O}_{\tilde{Z}}(-\tilde{Z}^\diamond)$ , hence

$$\mathcal{O}_Z(-(V \cup Z^\diamond)) = \pi_* \mathcal{O}_{\tilde{Z}}(-(\tilde{V} \cup \tilde{Z}^\diamond)).$$

Since  $Z^\diamond$  is Du Bois by induction on the dimension, the first splitting shows that  $Z$  is Du Bois. Using (5.3) and the second splitting gives that  $V \cup Z^\diamond$  is Du Bois.  $\square$

**Proof of Theorems 1 and 12.**

Since Theorem 12 implies Theorem 1, all that remains is to formulate a variant of Theorem 12 that allows for induction on the dimension. The strongest version would use the language of quasi-log structures as in [Fuj17]. They appear implicitly in the proof of Proposition 16, but our approach works well enough.

Note that Theorem 12 is the  $Z = X$  special case of Proposition 16. Thus the proof of Proposition 16 yields Theorem 12.

**Proposition 16.** *Let  $g : (Y, \Delta) \rightarrow X$  be a crepant log structure of dimension  $n$ , and  $Z \subset X$  a union of some of its log canonical centers, allowing  $Z = X$ . Let  $V \subset Z$  be a closed subset such that  $\text{mld}(V, Y, \Delta) < \text{lcm}(n)$ . Then  $V$  is Du Bois.*

*Proof.* The proof is by induction on  $\dim Z$ . If  $\dim Z = 0$  then  $V$  is a union of smooth points, hence Du Bois.

Write  $V = V_1 \cup V_2$ , where  $V_2 \subset Z^\circ$  and none of the irreducible components of  $V_1$  is contained in  $Z^\circ$ .

Note that  $V_1 \cup Z^\circ$  is Du Bois by (15), and so is  $Z^\circ$ . Furthermore,  $\text{mld}(V_1 \cap Z^\circ, X, \Delta) < \text{lcm}(n)$  by (11.1), hence  $V_1 \cap Z^\circ$  is Du Bois by induction since  $\dim Z^\circ < \dim Z$ . Thus  $V_1$  is Du Bois by (5.2).

Next  $\text{mld}(V \cap Z^\circ, X, \Delta) < \text{lcm}(n)$  by (11.1), hence  $V \cap Z^\circ$  is Du Bois by induction. We already checked that  $V_1$  and  $V_1 \cap Z^\circ = V_1 \cap (V \cap Z^\circ)$  are Du Bois. Thus  $V = V_1 \cup (V \cap Z^\circ)$  is Du Bois by (5.2).  $\square$

**Conjectures and comments.**

In the proof of Theorem 1, instead of  $\text{mld}(V, Y, \Delta) < \text{lcm}(n)$ , we only use the assumption that  $\text{mld}(V_i, Y, \Delta) < \text{lcm}(n - 1)$  if  $V_i$  is contained in a log canonical center, and  $\text{mld}(V_i, Y, \Delta) < \text{lcm}(n)$  otherwise. This suggests that the following should be true.

**Question 17.** Let  $g : (Y, \Delta) \rightarrow Z$  be a crepant log structure. Let  $V \subset Z$  be a closed subset with irreducible components  $V_i$ . Let  $Z_i \supset V_i$  be the minimal log canonical center that contains  $V_i$  (we allow  $Z_i = X$ ). Assume that  $\text{mld}(V_i, Y, \Delta) < \text{lcm}(\dim(Z_i))$  for every  $i$ . Is  $V$  necessarily Du Bois?

A related question is the following.

**Conjecture 18.** *Let  $(X, \Delta)$  be a quasi-projective, log canonical pair of dimension  $n$ . Let  $V \subset X$  be a closed subset such that  $\text{mld}(V, X, \Delta) < \text{lcm}(n)$  and  $V$  contains all log canonical centers of  $(X, \Delta)$ .*

*Then there is a log canonical pair  $(X, \Theta)$  such that  $V$  is the union of all log canonical centers of  $(X, \Theta)$ .*

Note that usually one can not choose  $\Theta \geq \Delta$ , as shown by the 2-dimensional example  $(\mathbb{A}^2, (1 - \eta)(x = 0) + (1 - \eta)(y = 0) + \eta(x = y))$ . Also, if  $Z$  is a log canonical center of  $(X, 0)$ , then it is also a log canonical center of any  $(X, \Theta)$ , so the assumption that  $V$  contain all log canonical centers of  $(X, \Delta)$  is necessary in many cases.

If  $(X, \Delta)$  is klt, then a proof of Conjecture 18 is given in [KK22]. Together with [KK10, KK20], this gives another proof of the klt case of Theorem 1. However, even the full conjecture does not seem to imply Theorem 1, since we get no information about those  $V_i$  that are contained in a log canonical center of  $(X, \Delta)$ .

A positive answer to the following stronger version would imply Theorem 1.

**Question 19.** Let  $(X, \Delta)$  be a quasi-projective, log canonical pair of dimension  $n$ . Is there a log canonical pair  $(X, \Theta)$  such that every irreducible subvariety satisfying  $\text{mld}(V, X, \Delta) < \text{l cg}(n)$  is a log canonical center of  $(X, \Theta)$ ?

Note that usually we can not achieve that the log canonical centers are exactly the  $\{V : \text{mld}(V, X, \Delta) < \text{l cg}(n)\}$ . Indeed, any intersection of log canonical centers is a union of log canonical centers, but this does not hold for the  $\text{mld}(V, X, \Delta) < \text{l cg}(n)$  condition.

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