THE RELATIVE DU BOIS COMPLEX — ON A QUESTION OF S. ZUCKER

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To Vyacheslav Shokurov on the occasion of his $70+2^{nd}$ birthday

1. Introduction

Rational singularities form an extremley useful class and have provided a powerful tool in the study of higher dimensional algebraic varieties. A prominent example of rational singularities is provided by the main class of singularities used in the minimal model program, that of *klt singularities* [Elk81]. The other pillar of classification theory, besides the minimal model program, is moduli theory. In fact, the minimal model program is a very useful tool for constructing moduli spaces. In order to keep track of degenerations, or in other words to work with compact moduli spaces, one needs to enlarge the class of singularities allowed from klt to slc singularities.

Unfortunately, slc singularities are not always rational, so one needs to enlarge the class of rational singularities as well. This enlargement is provided by the class of Du Bois singularities. In fact, the relationships between these two pairs of classes of singularities are very similar. For instance, as mentioned above, klt singularities are rational, and furthermore, for normal Gorenstein singularities being rational and klt are equivalent. Similarly, slc singularities are Du Bois, and for normal Gorenstein singularities being Du Bois is equivalent to slc.

Du Bois singularities were introduced by Steenbrink [Ste83]. Unfortunately, their definition is rather complicated and so the reader is referred to [Kol13, §6] for details. For our purposes the relevant detail is that the definition starts with the construction of the Du Bois complex, which is an analogue of the de Rham complex for not necessarily smooth varieties.

From a cohomological point-of-view this complex behaves very much like the usual de Rham complex, including the existence and degeneration at E_1 of the Hodge-to-de Rham spectral sequence. Just like the de Rham complex, the Du Bois complex may be used to acquire a Hodge structure and this actually agrees with Deligne's Hodge structure on singular complex varieties.

In moduli theory we study families of varieties. An interesting question for smooth families is how their Hodge structure is changing. In other words, for each smooth family we obtain a variation of Hodge structure (of geometric origin). An effective way to produce this is given by higher direct images of the relative de Rham complex of the family. As Du Bois complexes provide a way to obtain Deligne's Hodge structure on singular varieties, it is a reasonable question to ask whether there is a way to produce an object that would be analogous to the relative de Rham complex of a smooth family and which could be used to produce a

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variation of Hodge structure for a not necessarily smooth family. This is probably too much to ask, but one may ask whether there are some reasonable restrictions on the singularities of the fibers under which this is possible.

In fact, this is a question that was posed by Steven Zucker at a JAMI conference at Johns Hopkins University in 1996, organized by Vyacheslav Shokurov. Zucker's question was motivated by a result of the first named author of the present paper in which an analogue of the sheaf of relative p-forms was constructed. This construction has been used successfully to prove various results for not necessarily smooth families [Kov96, Kov97, Kov02, KT21, KT22], however it fell short of a positive answer to Zucker's question.

In this paper we make the first step towards filling that hole. More precisely we prove the existence of a relative Du Bois complex for families parametrized by a smooth curve. This is done in Section 2. We also establish some expected properties of this complex in Section 3 and Section 4. This work still leaves some questions open, most notably a similar construction over arbitrary bases. We discuss this and other open questions in Section 5.

2. The relative Du Bois complex

In this section we will initially work in the abelian category of complexes of abelian sheaves on a complex variety, instead of the corresponding derived category.

Definition 2.1. Let M be a complex manifold and let $\mathscr{A}^{p,q}$ denote the sheaf of complex valued C^∞-forms of type (p,q) and let $\mathscr{A}^m_M = \bigoplus_{p+q=m} \mathscr{A}^{p,q}$. Next, let X be a complex variety of dimension n and $\varepsilon_\bullet : X_\bullet \to X$ a hyperresolution. Further let

$$\underline{\mathscr{A}}^m_X := \bigoplus_{i=0}^{n} (\varepsilon_i)_* \mathscr{A}^{m-i}_{X_i} = \bigoplus_{i=0}^{n} \bigoplus_{p=0}^{m-i} (\varepsilon_i)_* \mathscr{A}^{p,m-i-p}_{X_i},$$

with the filtration

$$(F^p \underline{\mathscr{A}}^\bullet_X)^m := \bigoplus_{i=0}^{n} \bigoplus_{r=p}^{m-i} (\varepsilon_i)_* \mathscr{A}^{r,m-i-r}_{X_i}.$$

The differential of $\underline{\mathscr{A}}^\bullet_X$ is a combination of the differentials of the complexes $\mathscr{A}^\bullet_{X_i}$ and the pull-back morphisms between the pieces of the hyperresolution $\varepsilon_\bullet$. For more details see [Ste85]. Note that the complex $\underline{\mathscr{A}}^\bullet_X$ is an incarnation of the filtered de Rham-Du Bois complex, i.e., its image in $D_{\text{filt}}(X)$, the filtered derived category of $\mathscr{O}_X$-modules, is isomorphic to $\underline{\Omega}^\bullet_X$. Further note that because $n = \dim X$, $F^p \underline{\mathscr{A}}^\bullet_X = 0$ for $p > n$.

Now, let $f : X \to C$ be a flat morphism from a complex variety X to a smooth complex curve C. Following the construction in [Kov96], we will construct a Koszul filtration on $\underline{\mathscr{A}}^\bullet_X$. We define the map on simple tensors, and then extend by linearity to the tensor product pre-sheaf and take the induced map on the sheafification:

$$\begin{aligned} \wedge : \underline{\mathscr{A}}^\bullet_X \otimes f^*\omega_C &\longrightarrow \underline{\mathscr{A}}^\bullet_X[1] \\ (\varepsilon_i)_*\eta_i \otimes \xi &\longmapsto (\varepsilon_i)_*(\eta_i \wedge \varepsilon_i^*\xi) \end{aligned}$$

This is a filtered map:

$$\begin{aligned} \wedge_p := \wedge|_{F^p\underline{\mathscr{A}}^\bullet_X\otimes f^*\omega_C} : F^p\underline{\mathscr{A}}^\bullet_X \otimes f^*\omega_C &\longrightarrow F^p(\underline{\mathscr{A}}^\bullet_X[1]) = (F^{p+1}\underline{\mathscr{A}}^\bullet_X)[1] \\ (\varepsilon_i)_*\eta_i\otimes\xi &\longmapsto (\varepsilon_i)_*(\eta_i\wedge\varepsilon_i^*\xi). \end{aligned}$$

Next, we will define a series of objects recursively. In order to simplify the notation we will use the following shorthand: $F^p := F^p\underline{\mathscr{A}}^\bullet_X$, $\wedge' := \wedge\otimes \mathrm{id}_{f^*\omega_C}$, and $\wedge'_p := \wedge_p\otimes\mathrm{id}_{f^*\omega_C}$. Because ω_C is a line bundle, $\wedge\circ\wedge' = 0$ and $\wedge_p\circ\wedge'_{p-1} = 0$.

We define E^p as follows.

For $p \geq n = \dim X$, let $E^p := 0$, $\mathsf{w}''_p := 0 \in \mathrm{Hom}_{C(X)}(F^p\otimes f^*\omega_C, E^p\otimes f^*\omega_C)$, and $\mathsf{w}'_p := 0\in\mathrm{Hom}_{C(X)}(E^p\otimes f^*\omega_C, F^{p+1}[1])$. Note that then $\mathsf{w}'_p\circ\mathsf{w}''_p = 0$, $\mathsf{w}''_p\circ\wedge'_{p-1} = 0$, because $\mathsf{w}''_p = 0$, and $\wedge_p = 0$, because $F^{p+1} = 0$.

For $p < n$, assume that the following objects are defined for all $r > p$: $E^r \in \mathrm{Obj}(C(X))$, $\mathsf{w}''_r \in \mathrm{Hom}_{C(X)}(F^r\otimes f^*\omega_C, E^r\otimes f^*\omega_C)$, and $\mathsf{w}'_r\in\mathrm{Hom}_{C(X)}(E^r\otimes f^*\omega_C, F^{r+1}[1])$ such that $\mathsf{w}'_r\circ\mathsf{w}''_r = \wedge_r$ and $\mathsf{w}''_r\circ\wedge'_{r-1} = 0$. Set $\mathsf{w}_r := \mathsf{w}''_r\otimes\mathrm{id}_{f^*\omega_C^{-1}}\in\mathrm{Hom}_{C(X)}(F^r, E^r)$, and define

$$E^p := \mathrm{Cone}(\mathsf{w}_{p+1})\otimes f^*\omega_C^{-1},$$

i.e.,

$$(E^p)^m = \Big(\big(F^{p+1}\big)^{m+1}\oplus\big(E^{p+1}\big)^m\Big)\otimes f^*\omega_C^{-1},$$

with differential

$$d^m_{E^p} = \begin{pmatrix} -d^{m+1}_{F^{p+1}} & 0 \\ w^{m+1}_{p+1} & d^m_{E^{p+1}}\end{pmatrix}\otimes\mathrm{id}_{f^*\omega_C^{-1}}.$$

Notice that according to this definition we have,

$$E^{n-1} = F^n[1]\otimes f^*\omega_C^{-1}. \tag{2.1}$$

Next, let

$$\mathsf{w}''_p : F^p\otimes f^*\omega_C \longrightarrow E^p\otimes f^*\omega_C$$

be defined by

$$F^p\otimes f^*\omega_C \xrightarrow{(\wedge_p,0)} F^{p+1}[1]\oplus E^{p+1}.$$

This is a morphism of complexes, because by assumption $\mathsf{w}''_{p+1}\circ\wedge_p = 0$. It follows directly from the definition that

$$\mathsf{w}''_p\circ\wedge'_{p-1} = \wedge_p\circ\wedge'_{p-1} = 0.$$

Finally, let

$$\mathsf{w}'_p : E^p\otimes f^*\omega_C\longrightarrow F^{p+1}[1]$$

be defined by

$$F^{p+1}[1]\oplus E^{p+1}\xrightarrow{(\mathrm{id},0)} F^{p+1}[1].$$

Again, directly by the definition, we see that

$$\mathsf{w}'_p\circ\mathsf{w}''_p = \wedge_p.$$

Iterating this construction we obtain that for each $p \in \mathbb{Z}$ there exist short exact sequences in $C(X)$:

$$0 \longrightarrow E^{p+1} \longrightarrow E^p \otimes f^*\omega_C \longrightarrow F^{p+1}[1] \longrightarrow 0. \tag{2.2}$$

The above constructions works even for negative p's, although we will primarily be interested in the case of $p \geq -1$. The following statement will be crucial later:

Lemma 2.2. *For each $p \in \mathbb{Z}$, $E^{p+1} \subseteq E^p$ is a subcomplex.*

Remark 2.2.1. Notice that the construction of $E^{\bullet}$ exhibits $E^{p+1} \subseteq E^p \otimes f^*\omega_C$ as a subcomplex (cf. (2.2)), so one might wonder if that conflicts with the statement of Lemma 2.2. There is no conflict, because the two maps giving the two embeddings are independent of each other and they map to parts of the complexes E^p and $E^p \otimes f^*\omega_C$ that do not correspond to each other when twisting with $f^*\omega_C$.

Proof. We use descending induction on p. If $p \geq n-1$, then $E^{p+1} = 0$ and hence the statement is trivially true. Let p be fixed and assume that $E^{r+1} \subseteq E^r$ is a subcomplex for each $r > p$. Consider the following diagram:

$$\begin{array}{ccc}
F^{p+2} & \xrightarrow{\ \mathsf{w}_{p+2}\ } & E^{p+2} \\
\big\downarrow & & \big\downarrow \\
F^{p+1} & \xrightarrow{\ \mathsf{w}_{p+1}\ } & E^{p+1}
\end{array}$$

The left vertical arrow is an injection by definition, the right vertical arrow is an injection by the inductive hypothesis. The diagram is commutative because the non-trivial part of w_r is $\wedge_r$ (for both $r = p+1$ and $r = p+2$) and $\wedge_{p+1}|_{F^{p+2}} = \wedge_{p+2}$ by definition. It follows that then there is a commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc}
F^{p+2} & \xrightarrow{\ \mathsf{w}_{p+2}\ } & E^{p+2} & \longrightarrow & E^{p+1} \otimes f^*\omega_C & \xrightarrow{\ +1\ } & \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \delta'_p} & & \\
F^{p+1} & \xrightarrow{\ \mathsf{w}_{p+1}\ } & E^{p+1} & \longrightarrow & E^p \otimes f^*\omega_C & \xrightarrow{\ +1\ } &
\end{array}$$

The broken arrow, denoted by δ'_p, exists by the basic properties of mapping cones. Therefore turning the distinguished triangles around we obtain a morphism of short exact sequences in $C(X)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^{p+2} & \longrightarrow & E^{p+1} \otimes f^*\omega_C & \longrightarrow & F^{p+2}[1] & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \delta'_p} & & \big\downarrow & & \\
0 & \longrightarrow & E^{p+1} & \longrightarrow & E^p \otimes f^*\omega_C & \longrightarrow & F^{p+1}[1] & \longrightarrow & 0
\end{array} \tag{2.2}$$

Then δ'_p is also injective by the five lemma. Twisting δ'_p with $\mathrm{id}_{f^*\omega_C^{-1}}$ exhibits E^{p+1} as a subcomplex of E^p. ☐

Definition 2.3. Let $\underline{\Omega}^{\bullet}_{X/C}$ denote the object in $D_{\mathrm{filt}}(X)$ represented by E^0, the filtration given by $E^{\bullet}$, i.e., $E^p\underline{\Omega}^{\bullet}_{X/C} := E^p$ for each $p \in \mathbb{N}$, and call it the *relative Du Bois complex* of the morphism $f : X \to C$. For later use, let us also set $E^{-1}\underline{\Omega}^{\bullet}_{X/C} := E^{-1}$.

Recall that a similar object, $\underline{\Omega}^p_{X/C}$ was defined in [Kov96, 1.3]. This is the singular analogue of $\Omega^p_{X/C}$ for smooth morphisms. In the next theorem we show the connection between these objects.

Theorem 2.4. *$\underline{\Omega}^\bullet_{X/C}$ is a complex filtered by $E^{\bullet \geq 0} = E^\bullet \underline{\Omega}^\bullet_{X/C}$ such that its associated graded quotients satisfy the following in $D(X)$:*

$$Gr^p_E \underline{\Omega}^\bullet_{X/C}[p] \simeq_{qis} \underline{\Omega}^p_{X/C}.$$

Furthermore, $\underline{\Omega}^\bullet_{X/C}$ is a bounded complex, i.e., $\underline{\Omega}^\bullet_{X/C} \in \operatorname{Obj} D^b_{\text{filt}}(X)$ and if f is proper, then the cohmology sheaves of its associated graded quotients are coherent, i.e., in that case $\underline{\Omega}^\bullet_{X/C} \in \operatorname{Obj} D^b_{\text{filt,coh}}(X)$

Remark 2.4.1. Note that the original construction/definition of the Du Bois complex of X uses simplicial or cubic hyperresolution and then it must be verified that the isomorphism class (in the appropriate derived category) of the obtained complex does not depend on the chosen hyperresolution. In the construction of $\underline{\Omega}^\bullet_{X/C}$ we did not make direct use of a hyperresolution of X, only a particular representative of the Du Bois complex. It is easy to see that the construction yields another complex in the same isomorphism class even if one uses a different representative. Hence, $\underline{\Omega}^\bullet_{X/C}$, the isomorphism class (in the appropriate derived category) of the relative Du Bois complex of X is also independent from the choice of hyperresolutions.

Proof. Recall that by definition E^0 is a complex representing the isomorphism class of $\underline{\Omega}^\bullet_{X/C}$ and its corresponding filtration is given by E^p for $p \in \mathbb{N}$. Similarly for F^0 representing $\underline{\Omega}^\bullet_X$, where E^p and F^p are as defined above. Consider the following commutative diagram in $C(X)$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^{p+2}[p] & \longrightarrow & E^{p+1}[p] \otimes f^*\omega_C & \longrightarrow & F^{p+2}[p+1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^{p+1}[p] & \longrightarrow & E^{p}[p] \otimes f^*\omega_C & \longrightarrow & F^{p+1}[p+1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Gr^{p+1}_E E^0[p] & \longrightarrow & Gr^{p}_E E^0[p] \otimes f^*\omega_C & \longrightarrow & Gr^{p+1}_F F^0[p+1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.2}$$

The columns are exact by the definition of the associated graded quotients, the first two rows are exact by (2.2), and then the last row is exact by the nine lemma.

Recall that by definition $\underline{\Omega}^p_X = Gr^p_F \underline{\Omega}^\bullet_X[p]$ and that for each p there exists a distinguished triangle in $D(X)$ (cf.[Kov96, (1.3.2)]),

$$\underline{\Omega}^p_{X/C} \otimes f^*\omega_C \longrightarrow \underline{\Omega}^{p+1}_X \longrightarrow \underline{\Omega}^{p+1}_{X/C} \xrightarrow{+1} . \tag{2.3}$$

Because $F^{n+1} = 0$, we have that $F^n \simeq Gr^n_F \underline{\Omega}^\bullet_X$ and hence $\underline{\Omega}^n_X \simeq_{\text{qis}} F^n[n]$. Then by (2.1), (2.3) (applied for $p = n-1$), and because $E^n = 0$ and $\underline{\Omega}^n_{X/C} \simeq_{\text{qis}} 0$,we have the following quasi-isomorphisms in $D(X)$:

$$Gr^{n-1}_E \underline{\Omega}^\bullet_{X/C}[n-1] := E^{n-1}[n-1] \simeq_{\text{qis}} \underline{\Omega}^n_X \otimes f^*\omega_C^{-1} \simeq_{\text{qis}} \underline{\Omega}^{n-1}_{X/C}. \tag{2.4}$$

Comparing (2.2) and (2.3) and using (2.4) and descending induction on p proves the first statement.

Boundedness and coherence in the case when f is proper follows from the construction and [Kov96, (1.3.5)]. ☐

Corollary 2.5. *There exists a morphism $\underline{\Omega}^\bullet_X \to \underline{\Omega}^\bullet_{X/C}$ in $D^b_{\text{filt}}(X)$, (and if f is proper, then in $D^b_{\text{filt,coh}}(X)$), which induces morphisms, for each p, that fit into the following distinguished triangles:*

$$E^{p-1}\underline{\Omega}^\bullet_{X/C}[-1] \otimes f^*\omega_C \longrightarrow F^p\underline{\Omega}^\bullet_X \longrightarrow E^p\underline{\Omega}^\bullet_{X/C} \xrightarrow{+1}, \tag{2.1}$$

$$E^{-1}\underline{\Omega}^\bullet_{X/C}[-1] \otimes f^*\omega_C \longrightarrow \underline{\Omega}^\bullet_X \longrightarrow \underline{\Omega}^\bullet_{X/C} \xrightarrow{+1}, \text{ and} \tag{2.2}$$

$$\underline{\Omega}^{p-1}_{X/C} \otimes f^*\omega_C \longrightarrow \underline{\Omega}^p_X \longrightarrow \underline{\Omega}^p_{X/C} \xrightarrow{+1}, \tag{2.3}$$

Proof. (2.1) follows directly from the construction of E^p, (2.2) is the special case of (2.1) with $p = -1$, and (2.3) is [Kov96, (1.3.2)]. The only new information in this statement is that the second morphisms in each of these distinguished triangles are induced by a single filtered morphism $\underline{\Omega}^\bullet_X \to \underline{\Omega}^\bullet_{X/C}$. This follows from Theorem 2.4. ☐

Next, we compare our construction to the existing relative de Rham complex in the case of a smooth family.

Theorem 2.6. *Let $f : X \to C$ be a smooth morphism from a (smooth) complex variety X to a smooth complex curve C. There exists a natural filtered isomorphism in $D^b_{\text{filt}}(X)$, (and if f is proper, then in $D^b_{\text{filt,coh}}(X)$),*

$$\underline{\Omega}^\bullet_{X/C} \simeq \Omega^\bullet_{X/C}.$$

Proof. If f is smooth, then $\omega_{X/C}$ is a locally free sheaf and $\underline{\Omega}^p_{X/C} \simeq \Omega^p_{X/C}$ by [Kov96, (1.3.4)]. Using the diagram (2.2) and descending induction on p shows that the filtration $E^\bullet$ constructed earlier is simply the *filtration bête* of the relative de Rham complex $\Omega^\bullet_{X/C}$, which in particular shows that desired statement. ☐

Remark 2.7. Note that if f is smooth, then by Theorem 2.6 the filtration $E^\bullet$ becomes stationary downwards from $p = 0$, i.e., $E^0 = E^{-1} = E^{-2} = \dots$. This implies that in this case $E^{-1}\underline{\Omega}^\bullet_{X/C}[-1] \simeq \underline{\Omega}^\bullet_{X/C}[-1]$ and hence (2.2) recovers the well-known short exact sequence for smooth morphisms, cf.[PS08, (X-11), p.250]:

$$0 \longrightarrow \Omega^\bullet_{X/C}[-1] \otimes f^*\omega_C \longrightarrow \Omega^\bullet_X \longrightarrow \Omega^\bullet_{X/C} \longrightarrow 0.$$

3. Open covers

The construction of the relative Du Bois complex is invariant under restricting to an open set:

Proposition 3.1. *Let $f : X \to C$ be a flat morphism from a complex variety X to a smooth complex curve C and $U \subseteq X$ an open set. Then there exists a natural filtered isomorphism in $D^b_{\mathrm{filt}}(U)$, (and if $f|_U : U \to C$ is proper, then in $D^b_{\mathrm{filt,coh}}(U)$),*

$$\underline{\Omega}^{\bullet}_{X/C}|_U \simeq \underline{\Omega}^{\bullet}_{U/C}.$$

Proof. This follows directly from the construction and [DB81, 3.10]. □

Corollary 3.2. *The construction of the relative Du Bois complex is invariant under an open base change. In other words, using the notation from Proposition 3.1, further let $\jmath : V \hookrightarrow C$ be an open embedding and let $\imath : X_V = V \times_C X \hookrightarrow X$ the corresponding open embedding into X and $f_V : X_V \to V$ the base change of f. Then there exists a natural filtered isomorphism in $D^b_{\mathrm{filt}}(X_V)$, (and if f is proper, then in $D^b_{\mathrm{filt,coh}}(X_V)$),*

$$\imath^*\underline{\Omega}^{\bullet}_{X/C} \simeq \underline{\Omega}^{\bullet}_{X_V/V}.$$

Proof. It is straightforward from Proposition 3.1 that $\imath^*\underline{\Omega}^{\bullet}_{X/C} \simeq \underline{\Omega}^{\bullet}_{X_V/C}$. Next, note that as $f_V : X_V \to C$ factors through V, in the construction of $\underline{\Omega}^{\bullet}_{X_V/C}$, whenever $f_V^*\omega_C$ appears, it may be replaced by $f_V^*\omega_V$, because the two are actually naturally isomorphic. Hence $\underline{\Omega}^{\bullet}_{X_V/C} \simeq \underline{\Omega}^{\bullet}_{X_V/V}$, which proves the statement. □

Finally, we note that the relative Du Bois complex can be computed on an open cover in the following sense:

Theorem 3.3. *Let $f : X \to C$ be a flat morphism from a complex variety X to a smooth complex curve C and let $\{\nu_i : U_i \hookrightarrow X\}$ be a finite open cover of X. Then there exists a natural filtered isomorphism in $D^b_{\mathrm{filt}}(X)$, (and if $f|_{U_i}$ is proper for each i, then in $D^b_{\mathrm{filt,coh}}(X)$),*

$$\underline{\Omega}^{\bullet}_{X/C} \simeq \mathcal{R}\nu_{\bullet *}\underline{\Omega}^{\bullet}_{U_\bullet/C}.$$

Remark 3.4. The reader not familiar with the notation used on the right hand side of this isomorphism, should consult [DB81, p.46] or [Kov96, 1.1].

Proof. This follows from the construction and [Kov96, (1.3.6)]. □

4. Functoriality

In this section we explore the functorial properties of the construction of the relative Du Bois complex.

Notation 4.1. We will consider a commutative diagram,

$$\begin{array}{ccc} Y & \xrightarrow{\ \phi\ } & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & C, & \end{array}$$

where f and g are flat morphisms, X and Y are complex varieties and C is a smooth complex curve.

Theorem 4.2. *Under Notation 4.1 there exists a commutative diagram of natural filtered morphisms in $D^b_{\text{filt}}(X)$, (and if f is proper, then in $D^b_{\text{filt,coh}}(X)$),*

$$\begin{array}{ccc} \underline{\Omega}^{\bullet}_X & \longrightarrow & \mathcal{R}\phi_*\underline{\Omega}^{\bullet}_Y \\ \downarrow & & \downarrow \\ \underline{\Omega}^{\bullet}_{X/C} & \longrightarrow & \mathcal{R}\phi_*\underline{\Omega}^{\bullet}_{Y/C}, \end{array}$$

where the vertical morphisms are the ones obtained in Corollary 2.5.

Proof. Let $F^{\bullet}_X$ and $F^{\bullet}_Y$ denote the filtrations of $\underline{\Omega}^{\bullet}_X$ and $\underline{\Omega}^{\bullet}_Y$ and $E^{\bullet}_X$ and $E^{\bullet}_Y$ the filtrations of $\underline{\Omega}^{\bullet}_{X/C}$ and $\underline{\Omega}^{\bullet}_{Y/C}$ constructed above. By [DB81, (3.2.1)] there exists a morphism $\gamma : F^{\bullet}_X \to \mathcal{R}\phi_* F^{\bullet}_Y$ in $D^b_{\text{filt}}(X)$. We choose and fix a complex in $C(X)$ representing the class of $\mathcal{R}\phi_* F^{\bullet}_Y$ such that this morphism is represented by a morphism of complexes. By abuse of notation we will use the same symbols for these complexes. Next we choose complexes representing $\mathcal{R}\phi_* E^p_Y$ by descending induction for each $p \in \mathbb{N}$, starting with the analogue of (2.1) and following the same steps as on Page 3 until we get to the analogue of (2.2):

$$0 \longrightarrow \mathcal{R}\phi_* E^{p+1}_Y \longrightarrow \mathcal{R}\phi_* E^p_Y \otimes f^*\omega_C \longrightarrow \mathcal{R}\phi_* F^{p+1}_Y[1] \longrightarrow 0$$

We want to compare this short exact sequence to the one on X, i.e., consider the following commutative diagram of short exact sequences in $C(X)$ (we are using the above chosen complexes to represent the derived image objects). The morphisms will be explained below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^{p+1}_X & \longrightarrow & E^p_X \otimes f^*\omega_C & \longrightarrow & F^{p+1}_X[1] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{p+1}} & & \downarrow{\scriptstyle \beta_p} & & \downarrow{\scriptstyle \gamma_p} & & \\ 0 & \longrightarrow & \mathcal{R}\phi_* E^{p+1}_Y & \longrightarrow & \mathcal{R}\phi_* E^p_Y \otimes f^*\omega_C & \longrightarrow & \mathcal{R}\phi_* F^{p+1}_Y[1] & \longrightarrow & 0 \end{array} \tag{4.1}$$

The morphism γ_p (in $C(X)$!) is induced by the morphism γ above. Using descending induction on p we assume that a morphism α_{p+1} as indicated exists in $C(X)$. Then the diagram (4.1) shows that there exists a β_p that makes it commutative and hence we may define $\alpha_p := \beta_p \otimes \text{id}_{f^*\omega_C^{-1}}$ for the next inductive step.

Comparing this with the construction of these complexes shows that the induced morphisms on the associated graded quotients recover the similar compatible morphisms from [Kov96, (1.3.3)]. ☐

5. Open questions

As it is obvious from the rest of this article, this construction is presently only carried out when the base of a family is a smooth curve. The next step would be to extend this construction to families over arbitrary bases.

Problem 5.1. Is there a similar construction for families over arbitrary bases? More precisely, let $f : X \to S$ be a flat morphism from a complex variety X to a smooth complex variety S. Is there an object $\underline{\Omega}^{\bullet}_{X/S}$ with properties similar to the ones proved here in the case $\dim S = 1$?

Remark 5.1.1. It seems likely that the methods of [Kov05] and [Kov97] would allow this construction, but there are a few obstacles to be removed.

Perhaps the most important questions with respect to applications are with regard to base change properties of this construction. We have established base change for open embeddings. However, the interesting question is arbitrary base change, especially restriction to a point.

Problem 5.2. Let $f : X \to C$ be a flat morphism from a complex variety X to a smooth complex curve C and let $t \in C$. Under what conditions does the following hold?

$$\mathcal{L}\jmath^*\underline{\Omega}^{\bullet}_{X/C} \simeq \underline{\Omega}^{\bullet}_{X_t},$$

where $\jmath : \{t\} \hookrightarrow C$ is the embedding of t in C.

Remark 5.2.1. Of course, if f is smooth, then $\underline{\Omega}^{\bullet}_{X/C} \simeq \Omega^{\bullet}_{X/C}$ consists of locally free sheaves and hence $\mathcal{L}\jmath^*\underline{\Omega}^{\bullet}_{X/C} \simeq \jmath^*\Omega^{\bullet}_{X/C} \simeq \Omega^{\bullet}_{X_t}$.

In case Problem 5.1 has a positive solution, then one can ask the more general question:

Problem 5.3. let $f : X \to S$ be a flat morphism from a complex variety X to a smooth complex variety S and let $\tau : T \to S$ be a morphism from a smooth complex variety T. Under what conditions does the following hold?

$$\mathcal{L}\jmath^*\underline{\Omega}^{\bullet}_{X/S} \simeq \underline{\Omega}^{\bullet}_{X_T},$$

where $X_T = X \times_S T$.

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