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ON THE VALLEYS OF THE STOCHASTIC HEAT EQUATION

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We consider a generalization of the parabolic Anderson model driven by space-time white noise, also called the stochastic heat equation, on the real line:

$$\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + \sigma\big(u(t,x)\big)\xi(t,x) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}.$$

High peaks of solutions have been extensively studied under the name of intermittency, but less is known about spatial regions between peaks, which we may loosely refer to as valleys. We present two results about the valleys of the solution.

Our first theorem provides information about the size of valleys and the supremum of the solution $u(t,x)$ over a valley. More precisely, when the initial function $u_0(x) = 1$ for all $x \in \mathbb{R}$, we show that the supremum of the solution over a valley vanishes as $t \to \infty$, and we establish an upper bound of $\exp\{-\text{const} \cdot t^{1/3}\}$ for $u(t,x)$ when x lies in a valley. We demonstrate also that the length of a valley grows at least as $\exp\{+\text{const} \cdot t^{1/3}\}$ as $t \to \infty$.

Our second theorem asserts that the length of the valleys are eventually infinite when the initial function $u(0,x)$ has subgaussian tails.

1. Introduction and main result. Our objects of study are stochastic heat equations driven by multiplicative space-time white noise, including the parabolic Anderson model, whose solutions are known to exhibit *intermittency*. Intuitively speaking, intermittency refers to the property that the solution tends to develop tall peaks distributed over small regions—these are the so-called *intermittent islands*—and those islands are separated by large areas where the solution is small—these are the so-called *valleys* or voids. There is an extensive literature about the peaks particularly when the driving noise does not depend on the time variable—see König [19] and its extensive references, for example—and many techniques have been developed for understanding the peaks. In the present context of space-time white noise, a macroscopic fractal analysis has been developed, in Khoshnevisan, Kim, and Xiao [17, 18], which characterizes how tall peaks are distributed over small islands. In the case of the parabolic Anderson model for space-time noise, much more detailed results have recently become available; see, for example, Corwin and Ghosal [5], Das and Ghosal [9], and Das, Ghosal, and Lin [10], together with their substantial combined references.

In contrast to this literature, the regions between peaks, which we call *valleys*, have received less attention. Our goal in this paper is to study the width of the valleys, how they grow over time, and to estimate the supremum of our solution over the valley that straddles a given point (here, the origin).

Now we describe our results in more detail. Let $\xi := \{\xi(t,x)\}_{t\geq 0, x\in\mathbb{R}}$ denote a two-parameter white noise. That is, ξ is a generalized mean-zero Gaussian random field with generalized covariance

$$\operatorname{Cov}\big[\xi(t,x), \xi(s,y)\big] = \delta_0(t-s)\delta_0(x-y) \quad \text{for all } s,t \geq 0 \text{ and } x,y \in \mathbb{R}.$$

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Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ designate the filtration of the white noise ξ; that is, for every $t > 0$, $\mathcal{F}_t$ denotes the σ-algebra generated by all Wiener integrals of the form $\int_{(0,t)\times\mathbb{R}} \varphi \, \mathrm{d}\xi$ as φ ranges over $L^2(\mathbb{R}_+ \times \mathbb{R})$. We assume without incurring loss of generality that the filtration $\mathcal{F}$ satisfies the usual conditions.

The function $\sigma : \mathbb{R} \to \mathbb{R}$ is assumed to be nonrandom and satisfies

$$\sigma(0) = 0 \quad \text{and} \quad 0 < \mathrm{L}_\sigma \leq \mathrm{Lip}_\sigma < \infty, \tag{1.1}$$

where

$$\mathrm{L}_\sigma := \inf_{a\in\mathbb{R}\setminus\{0\}} \left|\frac{\sigma(a)}{a}\right| \quad \text{and} \quad \mathrm{Lip}_\sigma := \sup_{\substack{a,b\in\mathbb{R}:\\ a\neq b}} \left|\frac{\sigma(b)-\sigma(a)}{b-a}\right|.$$

Note, in particular, that

$$\mathrm{L}_\sigma |a| \leq \left|\sigma(a)\right| \leq \mathrm{Lip}_\sigma |a| \quad \text{for every } a \in \mathbb{R}. \tag{1.2}$$

We say that u is a solution to the stochastic heat equation if

$$\begin{aligned} &\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + \sigma\big(u(t,x)\big)\xi(t,x) \quad \text{for } t > 0, x \in \mathbb{R}, \\ &\text{subject to} u(0,x) = u_0(x) \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{1.3}$$

where $u_0 : \mathbb{R} \to \mathbb{R}_+$ is assumed to be continuous and bounded and $0 < \|u_0\|_{L^1(\mathbb{R})} \leq \infty$.

Because the solution u is not expected to be differentiable in either of its two variables, (1.3) must be interpreted in the generalized sense. Therefore, we follow the treatment of Walsh [24] and regard the SPDE (1.3) as shorthand for the random integral equation

$$u(t,x) = (S_t u_0)(x) + \int_{(0,t)\times\mathbb{R}} p_{t-s}(x-y)\sigma\big(u(s,y)\big)\xi(\mathrm{d}s\,\mathrm{d}y), \tag{1.4}$$

solved pointwise for every nonrandom choice of $t > 0$ and $x \in \mathbb{R}$. Here $(t,x) \mapsto p_t(x)$ represents the fundamental solution to the heat equation on $\mathbb{R}$; that is,

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right),$$

and $\{S_t\}_{t\geq 0}$ denotes the heat semigroup, which acts on $f : \mathbb{R} \to \mathbb{R}$ as

$$(S_t f)(x) = \int_{-\infty}^{\infty} p_t(x-y) f(y)\,\mathrm{d}y. \tag{1.5}$$

Finally, the double integral in (1.4) is a white noise integral in the sense of Walsh [24]. We refer to this solution as the *mild solution* to (1.3); see Walsh (ibid.).

The existence and uniqueness of a mild solution to (1.4) is well known; see Walsh [24], Chapter 3, for similar statements, and Dalang [8] for the general theory. Based on this general theory, we conclude that there is unique mild solution that is continuous in the variables (t,x). Moreover, the method of Mueller [21] shows that

$$\mathrm{P}\big\{u > 0 \text{ on } (0,\infty) \times \mathbb{R}\big\} = 1. \tag{1.6}$$

Our main theorem follows.

THEOREM 1.1. *If u solves (1.3) subject to $u_0 \equiv 1$, then there exist nonrandom numbers $\Lambda_i = \Lambda_i(\mathrm{L}_\sigma, \mathrm{Lip}_\sigma) > 0$ $[i = 1, 2]$ and an a.s.-finite random variable $T > 0$ such that*

$$\sup_{|x|<\exp(\Lambda_1 t^{1/3})} u(t,x) \leq \exp\big(-\Lambda_2 t^{1/3}\big) \quad \textit{for all } t > T.$$

Theorem 1.1 states that the solution is very small over a large region, approximately of length $\exp(\Lambda_1 t^{1/3})$. On the other hand, when considering $u(t,x)$ over a larger region, approximately of length $\exp(ct)$ for some constant $c>0$, we observe *tall peaks* with heights approximately exponentially large. These are characterized by the concept of macroscopic Hausdorff dimension (see below), as introduced by Barlow and Taylor [1, 2].

Barlow and Taylor [1, 2] introduced a notion of macroscopic Hausdorff dimension of a subset E of $\mathbb{R}^d$. We can appeal to their dimension in order to shed some light on the content of Theorem 1.1. In order to do that, let us first define $\mathcal{V}_n = (-\mathrm{e}^n, \mathrm{e}^n]^d$, $\mathcal{S}_0 = \mathcal{V}_0$, and $\mathcal{S}_{n+1} = \mathcal{V}_{n+1} \setminus \mathcal{V}_n$ for all $n \in \mathbb{Z}_+$, and refer to (for $x \in \mathbb{R}^d$ and $r>0$) $Q = [x_1, x_1 + r) \times \cdots \times [x_d, x_d + r)$ as an upright box with southwest corner x and sidelength $\mathrm{side}(Q) = r$. Let $\nu_\rho^n(E) = \inf \sum_{i=1}^m (\mathrm{side}(Q_i)/\mathrm{e}^n)^\rho$, where the infimum is taken over all upright boxes $Q_, \ldots, Q_m$ of side ≥ 1 that cover $E \cap \mathcal{S}_n$. The *Barlow–Taylor macroscopic Hausdorff dimension* of E is defined as the quantity

$$\mathrm{Dim}_{\mathrm{H}} E = \inf\left\{\rho > 0 : \sum_{n=1}^\infty \nu_\rho^n(E) < \infty\right\},$$

where $\inf \varnothing = 0$.

Khoshnevisan, Kim, and Xiao [18] have shown that the tall peaks of u form complex macroscopic space-time multifractals in the sense that there exist nonrandom numbers $A > a > 0$ and $b, \varepsilon > 0$ such that

$$2 - A\beta^{3/2} \leq \mathrm{Dim}_{\mathrm{H}}\{(\mathrm{e}^{t/\vartheta}, x) \in \mathbb{R}_+ \times \mathbb{R} : u(t,x) \geq \mathrm{e}^{\beta t}\} \leq 2 - a\beta^{3/2} \quad \text{a.s.},$$

for every $\beta > b$ and $\vartheta \in (0, \varepsilon\beta^{-3/2})$. In the above, we are using the convention that "$\mathrm{Dim}_{\mathrm{H}} E < 0$" means that E is bounded.

Among other things, this fact and the definition of the macroscopic Hausdorff dimension together imply that there a.s. exist tall peaks of height $\mathrm{e}^{\beta t}$ over an interval of size $\asymp \mathrm{e}^{t/\vartheta}$ for some $\vartheta > 0$ and $\beta > b > 0$ on an unbounded set of times $t \gg 1$; more precisely,

$$\left\{t > 0 : \sup_{|x| \leq \exp(1+[t/\vartheta])} u(t,x) \geq \mathrm{e}^{\beta t}\right\} \quad \text{is a.s. unbounded.} \tag{1.7}$$

Consider a small but fixed number $h_0 \in (0,1)$ and define for every $t > 0$,

$$\mathscr{L}(t) := \sup\left\{\ell > 0 : \sup_{|x| \leq \ell} u(t,x) \leq h_0\right\},$$

where $\sup \varnothing = 0$. We may think of the interval $(-\mathscr{L}(t), \mathscr{L}(t)) \subset \mathbb{R}$ as the *valley at time t* that straddles the origin. Note that this valley might be empty at some time $t > 0$, in which case $\mathscr{L}(t) = 0$. Because

$$\left\{t > 0 : \sup_{|x| \leq \exp(1+[t/\vartheta])} u(t,x) \geq \mathrm{e}^{\beta t}\right\} \subseteq \left\{t > 0 : \sup_{|x| \leq \exp(1+[t/\vartheta])} u(t,x) > h_0\right\},$$

it follows from (1.7) that $\mathscr{L}(t) \leq \exp(1 + [t/\vartheta])$ for an unbounded set of times $t \gg 1$. Since $\vartheta \in (0, \varepsilon\beta^{-3/2})$ and $\beta > b > 0$ are arbitrary, we learn from this endeavor that

$$\liminf_{t\to\infty} t^{-1} \log \mathscr{L}(t) \leq b^{3/2}\varepsilon^{-1} < \infty \quad \text{a.s.} \tag{1.8}$$

This is the best-known upper bound to date, but is likely not sharp. In the case of the parabolic Anderson model $[\sigma(z) = z]$ Das and Tsai [11] have developed much sharper large-deviations estimates than those in [18], Proposition 3.1. It might be possible to combine the Das–Tsai estimates instead of Proposition 3.1 of Khoshnevisan, Kim, and Xiao [18], together with the remaining arguments of [18], in order to improve this to prove that the constant b can be chosen arbitrarily. If this were so, then it would imply that (1.8) might hold for every $b > 0$ and hence $\liminf_{t\to\infty} t^{-1}\log\mathscr{L}(t) = 0$ a.s. when $\sigma(z) \equiv z$.

In any case, Theorem 1.1 assures us of the following complementary result:

$$\liminf_{t\to\infty} t^{-1/3}\log\mathscr{L}(t) \geq \Lambda_1^{1/3} > 0 \quad \text{a.s.},$$

and moreover tells us the solution is $\leq \exp(-\Lambda_2 t^{1/3})$ everywhere in that valley at all sufficiently large times.

Recently, Ghosal and Yi [13] have shown that, in the case of the parabolic Anderson model $[\sigma(z) \equiv z]$, $\mathrm{Dim}_{\mathrm{H}}\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}; u(t,x) \leq \mathrm{e}^{-\alpha t}\} = 2$ a.s. provided that α is sufficiently small. Their result is not about the length of the valleys. Rather, it tells us that there are many points (t,x) where the solution is exponentially small. Intermittency could in principle imply that the supremum of the solution over a valley is much larger than its smallest, or even typical, value over the same valley. We currently do not know whether or not this is true however.

The strategy for the proof of Theorem 1.1 is as follows. We first decompose the initial profile $u_0 \equiv 1$ as

$$u_0(x) = \sum_{i=-M}^{M} v_0^{(i)}(x) + v_0^{(M+1)}(x),$$

where $v_0^{(i)}$ and $v_0^{(M+1)}$ are continuous and nonnegative functions such that the support of $v_0^{(i)}$ is in $[i-1, i+1]$ and the support of $v_0^{(M+1)}$ is in $\mathbb{R} \setminus (-M, M)$. We prove that the solution u to (1.3) with $u_0 \equiv 1$ can in turn be decomposed as

$$u(t,x) = \sum_{i=-M}^{M} v^{(i)}(t,x) + v^{(M+1)}(t,x),$$

where $v^{(i)}$ and $v^{(M+1)}$ satisfy parabolic Anderson models driven by certain worthy martingale measures—see Section 2 and especially (2.4)—and starting from respective initial functions $v_0^{(i)}$ and $v_0^{(M+1)}$. Once we establish this, we freeze the time variable t and appeal to the preceding decomposition with $M := M(t) = 2R(t)$, where

$$R(t) = \exp\bigl(\Lambda_1 t^{1/3}\bigr) \quad \text{for all } t > 0.$$

On one hand, since $v_0^{(M+1)}(x) = 0$ for $|x| \leq 2R(t)$, we show that $\sup_{|x|\leq R(t)} v^{(M+1)}(t,x)$ is extremely small with very high probability. On the other hand, when $i < M$, the initial profile of $v^{(i)}$ has a compact support, and we can use the following theorem in order to prove that the global supremum of $v^{(i)}(t)$ tends rapidly to zero as $t \to \infty$.

THEOREM 1.2. *Let v solve the SPDE* (2.4) *below with a continuous and nonnegative initial function v_0 that satisfies $\limsup_{|x|\to\infty} x^{-2}\log v_0(x) < 0$, keeping in mind the convention $\log 0 = -\infty < 0$. Then, there exists a nonrandom number $\Lambda_3 = \Lambda_3(\mathrm{L}_\sigma, \mathrm{Lip}_\sigma) > 0$ and an a.s.-finite random time T such that*

$$\sup_{x\in\mathbb{R}} v(t,x) \leq \exp\bigl(-\Lambda_3 t^{1/3}\bigr) \quad \textit{for all } t > T.$$

Lemma 2.1 below ensures that the SPDE (2.4) is a generalization of our original SPDE (1.3). Therefore, Theorem 1.2 implies that if the initial data of the SPDE (1.3) is nonnegative and has subgaussian tails, then the global supremum of the solution to (1.3) vanishes at least as rapidly as $\exp(-\Lambda_2 t^{1/3})$ as $t \to \infty$. That is, a specialization of Theorem 1.2 implies the second announced result in the abstract of the paper: With probability one, $\mathscr{L}(t) = \infty$ for all sufficiently large t.

In order to prove Theorem 1.2 we first show that $\sup_{x\in\mathbb{R}} v(t,x)$ can be controlled by the total mass $\|v(t)\|_{L^1(\mathbb{R})}$ of v; this is done in Section 3. One may see a similar result in our earlier paper [16] where we consider a stochastic heat equation driven by space-time white noise on the one-dimensional torus $\mathbb{R}/\mathbb{Z}$ rather than on $\mathbb{R}$. Because $\mathbb{R}$ is not compact, we need to make significant modifications to the method of [16] especially when we estimate moments; see Section 3.1. Once we are able to prove that $\sup_{x\in\mathbb{R}} v(t,x)$ can be controlled by $\|v(t)\|_{L^1(\mathbb{R})}$, we appeal to a known result about dissipation of the total mass of the solution (see Chen, Cranston, Khoshnevisan, and Kim [4]) to prove Theorem 1.2; this is done in Section 4. Finally, we combine the results from Sections 2–4 in order to verify Theorem 1.1 in Section 5.

We conclude the Introduction by setting forth some notation that will be used throughout the paper. In order to simplify some of the formulas, we distinguish between the spaces L^k and $L^k(\mathrm{P})$ by writing the former as

$$L^k := L^k[\mathbb{R}] \quad [1 \le k < \infty].$$

Thus, for example, if $f \in L^k[\mathbb{R}]$ for some $1 \le k < \infty$, then

$$\|f\|_{L^k} := \left[\int_{-\infty}^{\infty} |f(x)|^k \,\mathrm{d}x\right]^{1/k}.$$

We will abuse notation slightly and write

$$\|f\|_{L^\infty} := \sup_{x\in\mathbb{R}} |f(x)|,$$

in place of the more customary essential supremum.

The $L^k(\mathrm{P})$-norm of a random variable $Z \in L^k(\mathrm{P})$ is denoted by

$$\|Z\|_k := \{\mathrm{E}(|Z|^k)\}^{1/k} \quad \text{for all } 1 \le k < \infty.$$

On multiple occasions we refer to $C_b(\mathbb{R})$ as the collection of bounded and continuous real-valued functions on $\mathbb{R}$, and to $C_b^+(\mathbb{R})$ as the cone of all nonnegative elements of $C_b(\mathbb{R})$. Finally, we follow Shiga [23] and define $C_{rap}^+(\mathbb{R})$ to be the set of functions in $C_b^+(\mathbb{R})$ that satisfy the rapid decrease condition $\limsup_{|x|\to\infty} |x|^{-1} \log v_0(x) = -\infty$.

2. A partition of the stochastic heat equation. Suppose we write the initial function $u_0 \in L^\infty$ as $u_0 = v_0 + w_0$ and call v and w the solutions to (1.3) with respective initial functions v_0 and w_0. Since (1.3) may not be linear, except when $\sigma(u) = u$, there is no reason to believe that $u = v + w$ in general. Instead we show in this section that $u = v + w$ where v and w solve the closely related stochastic heat equations (2.4) below with respective initial functions v_0 and w_0. In other words, we plan to show that, to a certain extent, the semilinear SPDE (1.3) always has a kind of "linear dependence on the initial data." We will see later that this kind of linear dependence on initial data suffices for our needs thanks to condition (1.1).

To implement our splitting, we first rewrite (1.3) so that it looks more like the linear parabolic Anderson equation; that is, we write

$$\begin{aligned} &\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + u(t,x)\tilde{\xi}(t,x) \quad \text{for } t > 0, x \in \mathbb{R}, \\ &\text{subject to} u(0) = u_0 \quad \text{on } \mathbb{R}, \end{aligned} \tag{2.1}$$

where

$$\tilde{\xi}(t,x) = \tilde{\xi}(t,x;u) := \tilde{\sigma}(t,x)\xi(t,x) \quad \text{for } \tilde{\sigma}(t,x) = \tilde{\sigma}(t,x;u) := \frac{\sigma(u(t,x))}{u(t,x)}. \tag{2.2}$$

Thanks to (1.6), the random function $\tilde{\sigma}$, and hence the random distribution $\tilde{\xi}$, are well defined.

From (1.2) we may conclude that, with probability one,

$$\mathrm{L}_\sigma \le \left|\tilde{\sigma}(t,x)\right| \le \mathrm{Lip}_\sigma \quad \text{for every } (t,x)\in[0,\infty)\times\mathbb{R}.$$

Therefore, in the sense of Walsh [24], we can regard the new noise $\tilde{\xi}$ as a worthy martingale measure with a dominating measure that is bounded below and above by constant multiples of Lebesgue measure. More precisely, if

$$M_t(\varphi) := \int_{(0,t)\times\mathbb{R}} \varphi(s,y)\tilde{\xi}(\mathrm{d}s\,\mathrm{d}x) = \int_{(0,t)\times\mathbb{R}} \varphi(s,y)\frac{\sigma(u(s,y))}{u(s,y)}\xi(\mathrm{d}s\,\mathrm{d}y),$$

then (1.2) implies that for all $t>0$ and for all nonnegative $\varphi,\psi\in C_c(\mathbb{R}_+\times\mathbb{R})$,

$$\begin{aligned}\mathrm{L}_\sigma^2\int_0^t\int_{-\infty}^\infty \varphi(s,y)\psi(s,y)\,\mathrm{d}y\,\mathrm{d}s &\le \langle M(\varphi),M(\psi)\rangle_t\\ &\le \mathrm{Lip}_\sigma^2\int_0^t\int_{-\infty}^\infty \varphi(s,y)\psi(s,y)\,\mathrm{d}y\,\mathrm{d}s.\end{aligned} \tag{2.3}$$

We emphasize that the upper and lower bounds on $\langle M(\varphi),M(\psi)\rangle_t$ are not random and, in particular, do not depend on u.

Now choose and fix some $v_0\in C_b^+(\mathbb{R})$ and consider solutions v to the following parabolic Anderson model forced by the martingale measure $\tilde{\xi}$:

$$\begin{aligned}&\partial_t v(t,x) = \frac{1}{2}\partial_x^2 v(t,x) + v(t,x)\tilde{\xi}(t,x) \quad \text{for } t>0, x\in\mathbb{R},\\ &\text{subject to} v(0)=v_0 \quad \text{on } \mathbb{R}.\end{aligned} \tag{2.4}$$

As in (1.4), we can define a mild solution to (2.4) as

$$v(t,x) = \int_{-\infty}^\infty p_t(x-y)v_0(y)\,\mathrm{d}y + \int_{(0,t)\times\mathbb{R}} p_{t-s}(x-y)v(s,y)\tilde{\xi}(\mathrm{d}s\,\mathrm{d}y). \tag{2.5}$$

As was mentioned before, $\tilde{\xi}$ is a worthy martingale measure thanks to (2.3), and therefore the stochastic integral in (2.5) can be understood in the sense of Walsh [24]. In addition, since (2.4) is linear in v, we may use Walsh's theory [24]—see also Shiga [23], Theorems 2.2 and 2.3,—in order to conclude that (2.4) has a unique mild solution v that is a.s. nonnegative and continuous on $[0,\infty)\times\mathbb{R}$. Therefore, the uniqueness theorem for such SPDEs implies the following.

LEMMA 2.1. *If u denotes the solution to* (1.3) *with the initial function $v_0\in C_b^+(\mathbb{R})$ and v denotes the solution to* (2.4) *with the same initial function v_0, then $u=v$ almost surely.*

Before we move on, let us pause to summarize the philosophy of the construction of this section up to this point: u solves the original SPDE (1.3) starting from nonnegative $u_0\in L^\infty$. With u fixed in our minds, we may solve (2.4) for every $v_0\in C_b^+(\mathbb{R})$. Lemma 2.1 assures us that $v=u$ if $v_0=u_0$. However, it should be clear also that u and v can differ when $u_0\neq v_0$. The following remark describes how we intend to use this observation in conjunction with Lemma 2.1.

REMARK 2.2. Since (2.4) is linear in v, we can partition the solution u to (1.3) with the initial function $u_0\equiv 1$. Indeed, a partition of unity enables to write $u_0\equiv 1$ as

$$u_0 = \sum_{i=-M}^{M} v_0^{(i)} + v_0^{(M+1)},$$

where $v_0^{(i)}, v_0^{(M+1)} : \mathbb{R} \to \mathbb{R}_+$ are continuous and nonnegative functions such that $v_0^{(i)}$ is supported in $[i-1, i+1]$ for $i = -M, \ldots, M$, and $v_0^{(M+1)}$ is supported in $\mathbb{R} \setminus (-M, M)$. Because the SPDE (2.4) is linear, we may superimpose solutions and appeal to Lemma 2.1 in order to decompose u as follows:

$$u(t, x) = \sum_{i=-M}^{M} v^{(i)}(t, x) + v^{(M+1)}(t, x) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R},$$

where $v^{(-M)}, \ldots, v^{(M+1)}$ satisfy the SPDE (2.4) with respective initial functions $v_0^{(-M)}, \ldots, v_0^{(M+1)}$.

Let us conclude this section with a remark about the strong Markov property (henceforth, denoted by SMP). Let u denote the solution to (1.3) subject to $u_0 \in L^\infty$. It is well known that $\{u(t)\}_{t\geq 0}$ is a diffusion with values in the space $C(\mathbb{R})$. In order to write down exactly what this means, we need to first introduce some measure-theoretic notation: For every $t \geq 0$ and $x \in \mathbb{R}$ define

$$\omega(t, x) = \int_{(0,t)\times(-x_-, x_+)} \mathrm{d}\xi, \tag{2.6}$$

where $x_- = -\min(0, x)$ and $x_+ = \max(0, x)$, and the integral is defined in the sense of Wiener. Elementary properties of the Wiener integral show that ω is a Brownian sheet indexed by $\mathbb{R}_+ \times \mathbb{R}$ and, as such, $\omega \in C(\mathbb{R}_+ \times \mathbb{R})$ a.s.; see Walsh [24], Chapter 1. Moreover, the Brownian filtration $\mathcal{F}$ is nothing but the filtration generated by the infinite-dimensional Brownian motion $\{\omega(t)\}_{t\geq 0}$.

We use a standard relabeling from measure theory in order to be able to assume, without any loss in generality, that the underlying probability space is $\Omega = C(\mathbb{R}_+ \times \mathbb{R})$ on which ω acts as a coordinate function. In this way, every random variable on Ω is a Borel function of the coordinate functions ω. We omit the remaining measure-theoretic details. Instead we observe that the distribution-valued random variable ξ is therefore also a function of ω, as is shown in (2.6).

We may define a shift operator $\theta(t) : \Omega \to \Omega$ for every $t \geq 0$ as follows: $(\theta(t)\omega)(s, x) = \omega(s + t, x)$ for all $s \geq 0$ and $x \in \mathbb{R}$. This induces a shift $\theta(t)X$ on every random variable X via $\theta(t)X(\omega) = X(\theta(t)\omega)$. If τ is a stopping time with respect to the filtration $\mathcal{F}$, then the random shift $\theta \circ \tau$ is well defined: We simply define $(\theta \circ \tau)(t)(\omega) = \theta(\tau(t)(\omega))$ for every $t \geq 0$ and $\omega \in \Omega$. We might write $\theta(\tau)$ in place of $\theta \circ \tau$.

It is not hard to check that if τ is a finite stopping time with respect to $\mathcal{F}$, then $\theta(\tau)\xi$ is a copy of ξ that is independent of $\mathcal{F}_\tau$, where the latter σ-algebra is defined in the usual sense. The SMP of u can now be cast as the slightly stronger assertion that the space-time random field $\theta(\tau)u$ solves (1.3) where the space-time white noise ξ is replaced by the space-time white noise $\theta(\tau)\xi$.

Let now $v_0 \in C_b^+(\mathbb{R})$ be nonrandom, and define v to be the solution to (2.4) starting from v_0. Because $\tilde{\sigma}$ in (2.2) is random, more specifically it is a mapping from Ω to Ω, the process $\{v(t)\}_{t\geq 0}$ does not satisfy the SMP (though it is an adapted random field). We have introduced the measure-theoretic notation above in order to discuss how the lack of the SMP of $\{v(t)\}_{t\geq 0}$ can be mostly salvaged.

Direct inspection leads to the following whose proof is omitted as it follows a well-known argument; see Da Prato and Zabczyk [7], Section 9.2.

LEMMA 2.3. *Let v denote the solution to (2.4) starting from a nonrandom $v_0 \in C_b^+(\mathbb{R})$. Then, $\{(u(t), v(t))\}_{t\geq 0}$ is a diffusion with values in the space $C(\mathbb{R}, \mathbb{R}^2)$. Moreover, for every finite stopping time τ, $\theta(\tau)v$ solves (2.4) with $(v_0, \tilde{\xi})$ replaced by $(\theta(\tau)v, \theta(\tau)\tilde{\xi})$ and the underlying Brownian filtration replaced by $\mathcal{F}_{\tau+\bullet}$.*

In order to see how this lemma salvages a portion of the SMP of the process $t \mapsto v(t)$, let us define for all nonrandom functions $\varphi \in C_c(\mathbb{R}_+ \times \mathbb{R})$ and all $t > 0$,

$$\tilde{M}_t(\varphi) = \int_{(0,t)\times\mathbb{R}} \varphi(s, y)\tilde{\xi}(\mathrm{d}s\,\mathrm{d}y) = \int_{(0,t)\times\mathbb{R}} \frac{\varphi(s, y)\sigma(u(s, y))}{u(s, y)} \xi(\mathrm{d}s\,\mathrm{d}y),$$

the second identity being a consequence of the definition in (2.2). The basic properties of the Walsh stochastic integral ensure that $\tilde{M}$ defines a worthy martingale measure whose dominating measure satisfies (2.3) with exactly the same constants as does M. The basic properties of Walsh stochastic integrals show that $\theta(\tau)\tilde{M}$ is the martingale measure that correspond to the noise $\theta(\tau)\tilde{\xi}$ that arose in Lemma 2.3. The martingale measure $\theta(\tau)\tilde{M}$ is worthy and satisfies (2.3) as well, also with exactly the same constants as does M. Because our work with $\tilde{\xi}$ does not involve knowing the law of $\tilde{\xi}$, rather its property (2.3) only, it follows that many of the properties of $\tilde{\xi}$ that we study here are (typically "up to constants") the same as those properties for $\theta(\tau)\tilde{\xi}$. And therefore the same can be said of v and $\theta(\tau)v$: They do not always have the same law (if this were the case, then this would be the SMP), rather they have the same properties (typically "up to constants").

3. Control of tall peaks by total mass. In this section, we consider the solution v to the stochastic heat equation (2.4) that is driven by the worthy martingale measure $\tilde{\xi}$. We write the mild formulation for (2.4) in the same manner as in Walsh [24]. Namely, for all $t > 0$ and $x \in \mathbb{R}$,

$$v(t, x) = (S_t v_0)(x) + \mathcal{I}(t, x), \tag{3.1}$$

where $\{S_t\}_{t\geq 0}$ designates the heat semigroup—see (1.5)—and

$$\mathcal{I}(t, x) := \int_{(0,t)\times\mathbb{R}} p_{t-s}(x - y)v(s, y)\tilde{\xi}(\mathrm{d}s\,\mathrm{d}y). \tag{3.2}$$

The principal aim of this section is to prove the following proposition which basically says that the tallest peak height $\|v(t)\|_{L^\infty}$ of v at time t can be controlled by the total mass $\|v(t)\|_{L^1}$ of v; this fact will play a role in the proof of Theorem 1.2. It might help to recall from the Introduction that $C^+_{rap}(\mathbb{R})$ denotes the set of all functions in $C^+_b(\mathbb{R})$ that decay at least exponentially rapidly at $\pm\infty$.

PROPOSITION 3.1. *Assume that $v_0 \in C^+_{rap}(\mathbb{R})$. For every $\gamma \in (\frac{4}{3}, 2]$ and $\beta \geq 6/(3\gamma - 4)$ there exist numbers $c_1, c_2 > 0$—that only depend on $\gamma, \beta, \mathrm{Lip}_\sigma$—such that*

$$\mathrm{P}\left\{\sup_{0\leq t\leq n} \frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}} \geq n^\beta\right\} < c_1 \exp(-c_2 n) \quad \textit{for every } n \geq 1. \tag{3.3}$$

A similar result can be found in our earlier paper [16], Theorem 3.1, valid in the case that the spatial domain is the torus $\mathbb{R}/\mathbb{Z}$ instead of $\mathbb{R}$.[1] Although the proof of Proposition 3.1 borrows liberally from the ideas of our earlier paper (ibid.), there also are several significant differences. Perhaps the first obvious difference is that the spatial domain is now $\mathbb{R}$, which is not compact. The change from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$ requires making several nontrivial modifications to our earlier arguments, especially when we estimate the moments of the solution. Those modifications involve "factorization" ideas from semigroup theory; see Da Prato, Kwapień, and Zabczyk [6], and in particular, Cerrai [3] and Salins [22].

[1]Actually, Theorem 3.1 of [16] is about SPDEs over $\mathbb{R}/(2\mathbb{Z})$, but it is clear from the proof that any other torus would also work.

There is another difference between the proof of Proposition 3.1 and the earlier methods of [16]. Namely, the proof of Theorem 3.1 of [16] hinged on the SMP of the solution u to (1.3). In the following, those arguments will be applied to v by appealing to Lemma 2.3. Among other things, if τ is a stopping time for the Brownian filtration $\mathcal{F}$, then we condition on $\mathcal{F}_\tau$ to see that $\tilde{v} = v(\tau(\omega) + \bullet)$ solves the parabolic Anderson model,

$$\partial_t \tilde{v} = \frac{1}{2}\partial_x^2 \tilde{v} + \tilde{v}\theta(\tau)\tilde{\xi}, \qquad \text{subject to } \tilde{v}(0) = v(\tau). \tag{3.4}$$

The point is that $\theta(\tau)\tilde{\xi}$ defines a worthy martingale measure with a dominating measure that is bounded below and above by the same constant multiples of Lebesgue measure as did $\tilde{\xi}$; see (2.3).

3.1. *Moment estimates.* We first estimate the moments of the solution $v(t, x)$ to (2.4) at a fixed point $(t, x) \in (0, \infty) \times \mathbb{R}$. It might help to recall that $\{S_t\}_{t\ge 0}$ denotes the heat semigroup; see (1.5).

LEMMA 3.2. *There exists a real number* $A := A(\mathrm{Lip}_\sigma) > 0$ *such that*

$$\|v(t,x)\|_k^2 \le A\|v_0\|_{L^\infty}(S_t v_0)(x)\exp\bigl(Ak^2 t\bigr), \tag{3.5}$$

uniformly for all $t > 0$, $x \in \mathbb{R}$, *all nonnegative functions* $v_0 \in L^\infty$, *and* $k \ge 2$.

PROOF. We develop some of the ideas of Foondun and Khoshnevisan [12]. Let us consider the Picard iteration: Define a sequence $\{v^{(n)}\}_{n\ge 0}$ as

$$v^{(0)}(t,x) := (S_t v_0)(x) \quad \text{and} \quad v^{(n)}(t,x) := (S_t v_0)(x) + \mathcal{I}^{(n)}(t,x),$$

for all $t > 0$, $x \in \mathbb{R}$, and $n \in \mathbb{N}$, where

$$\mathcal{I}^{(n)}(t,x) := \int_{(0,t)\times\mathbb{R}} p_{t-s}(x,y) v^{(n-1)}(s,y)\tilde{\xi}(\mathrm{d}s\,\mathrm{d}y).$$

The random field $v^{(n)}$ is the nth-stage Picard-iteration approximation of v. Next we follow Walsh [24], Chapter 3 (see also [15], Chapter 5) and obtain the following: For every $T > 0$ and $k \in [2, \infty)$,

$$\lim_{n\to\infty} \sup_{t\in(0,T]} \sup_{x\in\mathbb{R}} \mathrm{E}\bigl(|v^{(n)}(t,x) - v(t,x)|^k\bigr) = 0. \tag{3.6}$$

We appeal to a Burkholder–Davis–Gundy type inequality for stochastic convolutions [15], Proposition 4.4, p. 36, and the worthy condition (2.3) on the underlying noise $\tilde{\xi}$ in order to see that

$$\|v^{(n)}(t,x)\|_k^2 \le 2|(S_t v_0)(x)|^2 + 8k\mathrm{Lip}_\sigma^2 \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y [p_{t-s}(y-x)]^2 \|v^{(n-1)}(s,y)\|_k^2. \tag{3.7}$$

It might help to pause and observe that $(S_t v_0)(x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$. Let $\beta > 0$ be a fixed parameter, and divide both sides of the preceding display by $\exp(\beta t)(S_t v_0)(x)$ in order to find that

$$\begin{aligned}&\frac{\mathrm{e}^{-\beta t}\|v^{(n)}(t,x)\|_k^2}{(S_t v_0)(x)}\\ &\quad\le 2(S_t v_0)(x) + \frac{8k\mathrm{Lip}_\sigma^2}{(S_t v_0)(x)}\end{aligned}$$

$$\begin{aligned}
&\times \int_0^t \mathrm{d}s \mathrm{e}^{-\beta(t-s)} \int_{-\infty}^{\infty} \mathrm{d}y \big[p_{t-s}(y-x)\big]^2 (S_s v_0)(y) \frac{\mathrm{e}^{-\beta s} \|v^{(n-1)}(s,y)\|_k^2}{(S_s v_0)(y)} \\
&\leq 2\|v_0\|_{L^\infty} + \frac{8k\mathrm{Lip}_\sigma^2}{(S_t v_0)(x)} \int_0^t \mathrm{d}s \frac{\mathrm{e}^{-\beta(t-s)}}{\sqrt{2\pi(t-s)}} \\
&\times \int_{-\infty}^{\infty} \mathrm{d}y p_{t-s}(y-x)(S_s v_0)(y) \frac{\mathrm{e}^{-\beta s} \|v^{(n-1)}(s,y)\|_k^2}{(S_s v_0)(y)}.
\end{aligned}$$

In the last inequality above, we use the fact that $|S_t v_0(x)| \leq \|v_0\|_{L^\infty}$ and $p_t(z) \leq (2\pi t)^{-1/2}$ for all $z \in \mathbb{R}$. Define extended real numbers $\Phi_0, \Phi_1, \dots$ as

$$\Phi_n := \sup_{t>0} \sup_{x \in \mathbb{R}} \frac{\mathrm{e}^{-\beta t} \|v^{(n)}(t,x)\|_k^2}{(S_t v_0)(x)} \quad \text{for every } n \in \mathbb{Z}_+.$$

The semigroup property of $\{S_t\}_{t \geq 0}$ implies that, for all $n \in \mathbb{N}$, $t > 0$, and $x \in \mathbb{R}$,

$$\begin{aligned}
&\frac{\mathrm{e}^{-\beta t} \|v^{(n)}(t,x)\|_k^2}{(S_t v_0)(x)} \\
&\leq 2\|v_0\|_{L^\infty} + \frac{8k\mathrm{Lip}_\sigma^2}{(S_t v_0)(x)} \Phi_{n-1} \int_0^t \mathrm{d}s \frac{\mathrm{e}^{-\beta s}}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} \mathrm{d}y p_s(y-x)(S_{t-s} v_0)(y) \\
&\leq 2\|v_0\|_{L^\infty} + 8k\mathrm{Lip}_\sigma^2 \Phi_{n-1} \int_0^\infty \mathrm{d}s \frac{\mathrm{e}^{-\beta s}}{\sqrt{2\pi s}} \\
&= 2\|v_0\|_{L^\infty} + \frac{8k\mathrm{Lip}_\sigma^2}{\sqrt{2\beta}} \Phi_{n-1}.
\end{aligned}$$

Since $v^{(0)}(t,x) = (S_t v_0)(x)$, we have $\Phi_0 \leq \|v_0\|_{L^\infty}$. This implies that all of the Φ_ns are finite. In addition, if we choose $\beta := Ak^2$ for some constant $A := A(\mathrm{Lip}_\sigma)$, then we get that

$$\Phi_n \leq 2\|v_0\|_{L^\infty} + \frac{1}{2}\Phi_{n-1} \quad \text{for every } n \in \mathbb{N}.$$

We now apply induction to the preceding in order to find that $\Phi_n \leq 4\|v_0\|_{L^\infty}$ for all $n \in \mathbb{Z}_+$. Therefore, (3.6) and Fatou's lemma together yield

$$\sup_{t \geq 0} \sup_{x \in \mathbb{R}} \frac{\mathrm{e}^{-Ak^2 t} \|v(t,x)\|_k^2}{(S_t v_0)(x)} \leq 4\|v_0\|_{L^\infty},$$

and hence complete the proof. □

We now have the following moment estimate of the noise term [see (3.2)]:

LEMMA 3.3. *For every* $0 < \theta < \frac{1}{4}$ *there exists a number* $c = c(\theta, \mathrm{Lip}_\sigma) > 0$ *such that*

$$\mathrm{E}\Big(\sup_{s \in [0,t]} \|\mathcal{I}(s)\|_{L^\infty}^k\Big) \leq (ck)^{k/2} \exp(ck^3 t) \|v_0\|_{L^\infty}^{k-1} \|v_0\|_{L^1} t^{k\theta}, \tag{3.8}$$

uniformly for all $t \in (0,1]$, *nonnegative functions* $v_0 \in L^1 \cap L^\infty$, *and real numbers* $k \in [2,\infty)$ *that satisfy* $k(1-4\theta) > 2$.

PROOF. The proof of Lemma 3.3 is similar to results in Cerrai [3] and Salins [22], Appendix A, that consider SPDE on bounded domains. In order to adapt to the present setting where the spatial domain is $\mathbb{R}$, we modify the ideas of [3, 22] and also keep track of the dependence of constants on k and t.

First, we use the factorization method of Da Prato, Kwapień, and Zabczyk [6] to write

$$\begin{aligned}\mathcal{I}(t,x) &= \int_{(0,t)\times\mathbb{R}} p_{t-s}(x,y)v(s,y)\tilde{\xi}(\mathrm{d}s\,\mathrm{d}y)\\ &= \frac{\sin(\alpha\pi)}{\pi}\int_0^t (t-s)^{\alpha-1}\big(S_{t-s}[\mathcal{I}_\alpha(s)]\big)(x)\,\mathrm{d}s,\end{aligned}$$

where $\alpha \in (0,1)$ is arbitrary but fixed, and

$$\mathcal{I}_\alpha(t,x) := \int_{(0,t)\times\mathbb{R}} (t-s)^{-\alpha} p_{t-s}(y-x)v(s,y)\tilde{\xi}(\mathrm{d}s\,\mathrm{d}y).$$

In this way, we find that

$$\|\mathcal{I}(t)\|_{L^\infty} \le \frac{1}{\pi}\int_0^t (t-s)^{\alpha-1}\|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{L^\infty}\,\mathrm{d}s. \tag{3.9}$$

By the Sobolev embedding theorem—see, for example, Grafakos [14], Theorem 6.2.4—if we choose $\delta \in (0,1)$ and $k \ge 2$ such that

$$\delta k > 1, \tag{3.10}$$

then

$$\|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{L^\infty} \le C(\delta)\|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{H^{\delta,k}(\mathbb{R})},$$

where $H^{\delta,k}(\mathbb{R})$ denotes the space of Bessel potentials and $C(\delta) > 0$ is a number that depends only on δ. Since $H^{\delta,k}(\mathbb{R})$ coincides with the complex interpolation space $[L^k, W^{1,k}(\mathbb{R})]_\delta$—see, for example, Lunardi [20], Example 2.12—we may appeal to Corollary 2.8 of Lunardi (ibid.) in order to be able to say that

$$\|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{H^{\delta,k}(\mathbb{R})} \le \|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{L^k}^{1-\delta}\|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{W^{1,k}(\mathbb{R})}^{\delta}.$$

The heat semigroup is a contraction mapping on L^k. Therefore,

$$\|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{L^k} \le \|\mathcal{I}_\alpha(s)\|_{L^k},$$

and a direct calculation shows that there exists some constant $C > 0$—independent of the parameters (s,t,α,k,δ)—such that

$$\|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{W^{1,k}(\mathbb{R})} \le C(t-s)^{-1/2}\|\mathcal{I}_\alpha(s)\|_{L^k}.$$

We can combine our efforts so far in order to obtain the following:

$$\|S_{t-s}[\mathcal{I}_\alpha(s)]\|_{L^\infty} \le C(\delta)(t-s)^{-\delta/2}\|\mathcal{I}_\alpha(s)\|_{L^k}. \tag{3.11}$$

Therefore, if we assume additionally that $\delta < 2\alpha$ and $k(\alpha - \frac{\delta}{2}) > 1$, then we deduce from (3.9) that

$$\begin{aligned}\|\mathcal{I}(t)\|_{L^\infty}^k &\le \left|C(\delta)\int_0^t (t-s)^{\alpha-1-\frac{\delta}{2}}\|\mathcal{I}_\alpha(s)\|_{L^k}\,\mathrm{d}s\right|^k\\ &\le [C(\delta)]^k\left|\int_0^t (t-s)^{k(\alpha-1-\frac{\delta}{2})/(k-1)}\,\mathrm{d}s\right|^{k-1}\int_0^t \|\mathcal{I}_\alpha(s)\|_{L^k}^k\,\mathrm{d}s \qquad (3.12)\\ &= [C(\delta)]^k\left[\frac{(k-1)}{k(\alpha-\frac{\delta}{2})-1}\right]^{k-1} t^{k(\alpha-\frac{\delta}{2})-1}\int_0^t \|\mathcal{I}_\alpha(s)\|_{L^k}^k\,\mathrm{d}s.\end{aligned}$$

In the second inequality of (3.12) we used Hölder's inequality. We now choose $\alpha = \alpha(\delta)$ judiciously—for instance, $\alpha = 2\delta$—in order to deduce from the above that

$$\frac{(k-1)}{k(\alpha - \frac{\delta}{2}) - 1} \le C_1,$$

for a constant $C_1 = C_1(\delta) > 0$. With our present choice in mind, we find that

$$\mathrm{E}\Big[\sup_{0\le s\le t} \|\mathcal{I}(s)\|_{L^\infty}^k\Big] \le \big[C(\delta)\big]^k t^{k(\alpha-\frac{\delta}{2})-1} \int_0^t \mathrm{E}\big[\|\mathcal{I}_\alpha(s)\|_{L^k}^k\big]\,\mathrm{d}s. \tag{3.13}$$

It remains to estimate $\mathrm{E}[\|\mathcal{I}_\alpha(s)\|_{L^k}^k]$ for $0 < s < t$.

We can combine the Burkolder–Davis–Gundy inequality, (2.3), and the Minkowski inequality, in order to see that there exists a number $C := C(\mathrm{Lip}_\sigma) > 0$ such that

$$\begin{aligned}
\mathrm{E}\big[|\mathcal{I}_\alpha(t,x)|^k\big] &\le C^k k^{k/2} \mathrm{E}\left|\int_0^t \mathrm{d}s\,(t-s)^{-2\alpha} \int_{-\infty}^\infty \mathrm{d}y\big[p_{t-s}(y-x)v(s,y)\big]^2\right|^{k/2} \\
&\le C^k k^{k/2} \left|\int_0^t \mathrm{d}s\,(t-s)^{-2\alpha} \int_{-\infty}^\infty \mathrm{d}y\big[p_{t-s}(y-x)\big]^2 \|v(s,y)\|_k^2\right|^{k/2} \\
&\le C^k k^{k/2} \Big|\int_0^t \mathrm{d}s\,(t-s)^{-2\alpha-\frac12} \int_{-\infty}^\infty \mathrm{d}y p_{t-s}(y-x) \|v(s,y))\|_k^2\Big|^{k/2},
\end{aligned}$$

where we used the elementary inequality $\sup_{x\in\mathbb{R}} p_r(x) \le r^{-1/2}$, valid for all $r > 0$, in the last line. We now can use Lemma 3.2 and the semigroup property of $\{S_t\}_{t\ge0}$ to see that

$$\begin{aligned}
&\mathrm{E}\int_{-\infty}^\infty \big[|\mathcal{I}_\alpha(t,x)|^k\big]\,\mathrm{d}x \\
&\quad\le C^k k^{k/2} \int_{-\infty}^\infty \mathrm{d}x \Big|\int_0^t \mathrm{d}s\,(t-s)^{-2\alpha-\frac12} \int_{-\infty}^\infty \mathrm{d}y p_{t-s}(y-x)\|v(s,y))\|_k^2\Big|^{k/2} \\
&\quad\le C^k A^{k/2} k^{k/2} \|v_0\|_{L^\infty}^{k/2} \exp\big(Ak^3 t\big) \int_{-\infty}^\infty \mathrm{d}x \\
&\qquad\times \left|\int_0^t \mathrm{d}s\,(t-s)^{-2\alpha-\frac12} \int_{-\infty}^\infty \mathrm{d}y p_{t-s}(y-x) S_s v_0(y)\right|^{k/2} \\
&\quad\le C^k A^{k/2} k^{k/2} \|v_0\|_{L^\infty}^{k/2} \exp\big(Ak^3 t\big) \int_{-\infty}^\infty |S_t v_0(x)|^{k/2}\,\mathrm{d}x \left|\int_0^t \frac{\mathrm{d}s}{(t-s)^{2\alpha+\frac12}}\right|^{k/2}.
\end{aligned}$$

We can currently employ the contraction property of the heat semigroup on L^k in order to see that if $\alpha < 1/4$, then

$$\begin{aligned}
\left(\mathrm{E}\int_{-\infty}^\infty |\mathcal{I}_\alpha(t,x)|^k\,\mathrm{d}x\right)^{1/k} &\lesssim \sqrt{k}\|v_0\|_{L^\infty}^{1/2} \exp\big(Ak^2 t\big) \left|\int_0^t \frac{\mathrm{d}s}{(t-s)^{2\alpha+\frac12}}\right|^{1/2} \|S_t v_0\|_{L^{k/2}}^{1/2} \\
&\lesssim \sqrt{k}\|v_0\|_{L^\infty}^{1/2} \exp\big(Ak^2 t\big)(1-4\alpha)^{-1/2} t^{(\frac12-2\alpha)/2} \|v_0\|_{L^{k/2}}^{1/2} \\
&\lesssim \sqrt{k} \exp\big(Ak^2 t\big)(1-4\alpha)^{-1/2} t^{(\frac12-2\alpha)/2} \|v_0\|_{L^\infty}^{(k-1)/k} \|v_0\|_{L^1}^{1/k},
\end{aligned}$$

where the implicit constants do not depend on (t, k). Therefore, (3.13) yields real numbers $C_i := C_i(\delta, \alpha) > 0$ $[i = 1, 2]$ such that

$$\begin{aligned}
\mathrm{E}\Big[\sup_{0\le s\le t} \|\mathcal{I}(s)\|_{L^\infty}^k\Big] &\le C_1^k k^{k/2} \exp\big(Ak^3 t\big) t^{k(\alpha-\frac{\delta}{2})-1} \|v_0\|_{L^\infty}^{k-1} \|v_0\|_{L^1} \int_0^t s^{(\frac12-2\alpha)k/2}\,\mathrm{d}s \\
&\le C_2^k k^{k/2} \exp\big(Ak^3 t\big) t^{k(\frac14-\frac{\delta}{2})} \|v_0\|_{L^\infty}^{k-1} \|v_0\|_{L^1}.
\end{aligned}$$

We now choose $\theta := \frac{1}{4} - \frac{\delta}{2}$ and $\alpha := 2\delta$ in order to obtain (3.8) for $\theta \in (3/16, 1/4)$. The lower bound $3/16$ comes from the assumption that $\alpha < 1/4$. For smaller values of θ, we merely observe that (3.8) holds automatically; this is because $t \in (0, 1]$. Because the condition (3.10) is equivalent to $k(1 - 4\theta) > 2$, this completes the proof of Lemma 3.3. □

LEMMA 3.4. *For every* $0 < \theta < \frac{1}{4}$ *there exists* $c = c(\theta, \mathrm{Lip}_\sigma) > 0$ *such that*

$$\big\| \|v(t)\|_{L^\infty} \big\|_k \lesssim \sqrt{k} \exp(ck^2 t) \|v_0\|_{L^\infty}^{1-1/k} t^\theta + t^{-1/2},$$

$$\Big\| \sup_{s \in (0,t)} \|v(s)\|_{L^\infty} \Big\|_k \lesssim \sqrt{k} \exp(ck^2 t) \|v_0\|_{L^\infty}^{1-1/k} t^\theta + \|v_0\|_{L^\infty},$$

uniformly for all $t \in (0, 1]$, *all nonnegative functions* $v_0 \in L^1 \cap L^\infty$ *that satisfy* $\|v_0\|_{L^1} = 1$, *and all real numbers* $k > 2/(1 - 4\theta)$.

PROOF. Since $\int_{-\infty}^{\infty} v_0(x)\,\mathrm{d}x = 1$ and $\sup_{x \in \mathbb{R}} p_t(x) \le t^{-1/2}$, it follows that $\|S_t v_0\|_{L^\infty} \lesssim \|v_0\|_{L^\infty} \wedge t^{-1/2}$. The first portion of the lemma follows from this observation, Lemma 3.3, and (3.1). The second portion follows similarly. □

3.2. *Control of tall peaks and total mass.* In this section we assume that v denotes the unique solution to (2.4), and show that the tall peaks and total mass of v do not move much for a short time.

PROPOSITION 3.5. *For* $\frac{4}{3} < \gamma < 2$, *there exist* $C = C(\gamma, \mathrm{Lip}_\sigma) > 0$ *such that*

$$\mathrm{P}\{\|v(N^{-\gamma})\|_{L^\infty} \ge N\} \le C \exp(-N^{(3\gamma-4)/2}),$$

$$\mathrm{P}\Big\{\sup_{0 \le s \le N^{-\gamma}} \|v(s)\|_{L^\infty} \ge 2N\Big\} \le C \exp\Big(-\frac{1}{2} N^{(3\gamma-4)/2}\Big),$$

uniformly for all real numbers $N \ge 1$, *and all nonnegative functions* v_0 *that satisfy* $\|v_0\|_{L^1} = 1$ *and* $\|v_0\|_{L^\infty} \le N$.

PROOF. For each fixed $\gamma \in (4/3, 2)$, we can choose and fix $0 < \theta < \frac{1}{4}$ that satisfies $\gamma(3 - 4\theta) < 4$. We now apply Lemma 3.4 with $t = N^{-\gamma}$ and $k = N^{(3\gamma-4)/2}$ to see that there exists a real number $C_1 = C_1(\gamma, \theta, \mathrm{Lip}_\sigma) > 0$ such that

$$(3.14)\qquad \begin{aligned} \big\| \|v(N^{-\gamma})\|_{L^\infty} \big\|_{N^{(3\gamma-4)/2}} &\le \frac{1}{2} C_1 N^{\gamma(3-4\theta)/4} + N^{\gamma/2}, \\ \Big\| \sup_{0 \le s \le N^{-\gamma}} \|v(s)\|_{L^\infty} \Big\|_{N^{(3\gamma-4)/2}} &\le \frac{1}{2} C_1 N^{\gamma(3-4\theta)/4} + N, \end{aligned}$$

uniformly for all large N, and all nonnegative functions $v_0 \in L^1 \cap L^\infty$ that satisfy $\|v_0\|_{L^1} = 1$ and $\|v_0\|_{L^\infty} \le N$. By the first bound in (3.14) and Chebyshev's inequality, we have

$$\begin{aligned} \mathrm{P}\{\|v(N^{-\gamma})\|_{L^\infty} \ge N\} &\le \mathrm{E}\bigg(\bigg|\frac{\|v(N^{-\gamma})\|_{L^\infty}}{N}\bigg|^k\bigg) \\ &\le \exp(-N^{(3\gamma-4)/2}) \quad \text{for all } N \ge \mathrm{e}\Big(\frac{1}{2} C_1 N^{\gamma(3-4\theta)/4} + N^{\gamma/2}\Big). \end{aligned}$$

Note that since $\gamma(3 - 4\theta) < 4$ and $\gamma < 2$, there exists $N_0 > 0$ such that $N \ge \mathrm{e}(\frac{1}{2} C_1 N^{\gamma(3-4\theta)/4} + N^{\gamma/2})$ for all $N \ge N_0$. Thus, we find that there exists $C_2 = C_2(\gamma, \theta, \mathrm{Lip}_\sigma) > 0$ such that

$$\mathrm{P}\{\|v(N^{-\gamma})\|_{L^\infty} \ge N\} \le C_2 \exp(-N^{(3\gamma-4)/2}) \quad \text{for all } N \ge 1,$$

which results in the first assertion of Proposition 3.5.

For the second portion of Proposition 3.5, since $\gamma(3-4\theta)/4 \in (0,1)$, it follows from (3.14) that for every $q > 1$ there exists $N_0 = N_0(q, \gamma, \theta, \mathrm{Lip}_\sigma) > 0$ such that

$$\Big\| \sup_{0 \le s \le N^{-\gamma}} \|v(s)\|_{L^\infty} \Big\|_{N^{(3\gamma-4)/2}} \le qN \quad \text{for all } N \ge N_0.$$

We now choose $q = 2\exp(-1/2)$, and then use Chebyshev's inequality to get that

$$\mathrm{P}\Big\{ \sup_{0 \le s \le N^{-\gamma}} \|v(s)\|_{L^\infty} \ge 2N \Big\} \le \exp\Big(-\frac{1}{2} N^{(3\gamma-4)/2} \Big) \quad \text{for all } N \ge N_0.$$

This implies that there exists $C_3 = C_3(\gamma, \theta, \mathrm{Lip}_\sigma) > 0$ such that

$$\mathrm{P}\Big\{ \sup_{0 \le s \le N^{-\gamma}} \|v(s)\|_{L^\infty} \ge 2N \Big\} \le C_3 \exp\Big(-\frac{1}{2} N^{(3\gamma-4)/2} \Big) \quad \text{for all } N \ge 1,$$

which completes the proof of Proposition 3.5. □

PROPOSITION 3.6. *For every* $\frac{4}{3} < \gamma < 2$ *there exists* $L = L(\gamma, \mathrm{Lip}_\sigma) > 1$ *such that*

$$\mathrm{P}\Big\{ \inf_{0 \le t \le N^{-\gamma}} \|v(t)\|_{L^1} \le \frac{1}{2} \textit{ or } \sup_{0 \le t \le N^{-\gamma}} \|v(t)\|_{L^1} \ge 2 \Big\} \le L \exp\Big(-\frac{N^{(3\gamma-4)/3}}{L} \Big),$$

uniformly for all real numbers $N \ge 1$ *and all nonnegative functions* $v_0 \in L^1 \cap L^\infty$ *that satisfy* $\|v_0\|_{L^1} = 1$ *and* $\|v_0\|_{L^\infty} \le N$.

PROOF. Since $\tilde{\xi}$ can be regarded as a worthy martingale measure, we can conclude from (2.3) that $\|v(t)\|_{L^1}$ is a continuous $L^2(\Omega)$-martingale whose quadratic variation at time $t > 0$ is bounded by $\mathrm{Lip}_\sigma^2 \int_0^t \|v(s)\|_{L^2(\mathbb{R})}^2 \, \mathrm{d}s$. We now simply follow the proof of Proposition 3.7 of [16]. The only difference is that the present Lemma 3.2 can be used for the estimates of the moments of v; one can easily see that this change does not affect the validity of our claim and deduce the proposition. The details of the proof are left to the interested reader. □

3.3. *Proof of Proposition* 3.1. We now prove Proposition 3.1. Suppose v denotes the unique solution to (2.4) subject to initial data $v_0 \in C^+_{\mathrm{rap}}(\mathbb{R})$. Since $v_0 \in C^+_{\mathrm{rap}}(\mathbb{R})$, it follows that $v(t) \in C^+_{\mathrm{rap}}(\mathbb{R})$ for all $t \ge 0$ almost surely. For details; see, for example, Shiga [23], Theorem 2.5.[2] Thus, we can see that $v(t) \in L^1 \cap L^\infty$ for all $t \ge 0$ almost surely. We are ready to proceed with the proof.

PROOF OF PROPOSITION 3.1. This proof is similar to the proof of [16], Theorem 1.3. We first define a stopping time $\tau(n)$ for every $n \ge 1$ as

$$\tau(n) = \inf\{ t > 0 : \|v(t)\|_{L^\infty} \ge n^{\beta/2} \|v(t)\|_{L^1} \} \quad [\inf \varnothing = \infty].$$

Then, since $\mathrm{P}(A \cap B) \le \mathrm{P}(A|B)$,

$$\begin{aligned} \mathrm{P}\Big\{ \sup_{0 \le t \le n} \frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}} \ge n^\beta \Big\} &= \mathrm{P}\Big\{ \tau(n) < n, \sup_{\tau(n) \le t \le n} \frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}} \ge n^\beta \Big\} \\ &\le \mathrm{P}\Big(\sup_{\tau(n) \le t \le n} \frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}} \ge n^\beta | \tau(n) < \infty \Big). \end{aligned}$$

[2]In Shiga's work [23], Theorem 2.5, the noise part is $\tilde{\sigma}(t, x, u, \omega)\xi(t, x)$ where ξ denotes space-time white noise and $\tilde{\sigma}$ is a random function that satisfies certain regularity conditions. Here, we can regard $v(t,x)\tilde{\xi}(t,x) := v(t,x)\frac{\sigma(u(t,x))}{u(t,x)}\xi(t,x)$ as $\tilde{\sigma}(t,x,v,\omega)\xi(t,x)$ and immediately conclude that $\tilde{\sigma}$ satisfies the certain regularity conditions of Shiga [23].

Since $v(t) \in C^+_{\mathrm{rap}}(\mathbb{R})$ for all $t \geq 0$ almost surely, and because $t \mapsto v(t)$ is continuous in time,

$$\|v(\tau(n))\|_{L^\infty} = n^{\beta/2}\|v(\tau(n))\|_{L^1} \quad \text{a.s. on } \{\tau(n) < \infty\}.$$

Thus, by the strong Markov property of (u, v) [see Lemma 2.3],

$$\mathrm{P}\Big(\sup_{\tau(n)\leq t\leq n} \frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}} \geq n^\beta \,|\, \mathcal{F}_{\tau(n)}\Big) \leq \sup_{\substack{\tilde{v}_0\in C_b^+(\mathbb{R}):\\ \|\tilde{v}_0\|_{L^\infty}=n^{\beta/2}\|\tilde{v}_0\|_{L^1}}} \mathrm{P}\Big\{\sup_{0\leq t\leq n} \frac{\|\tilde{v}(t)\|_{L^\infty}}{\|\tilde{v}(t)\|_{L^1}} \geq n^\beta\Big\},$$

where $\tilde{v}$ denotes the solution to (3.4) subject to initial data $\tilde{v}_0$ that is being optimized under "$\sup_{\tilde{v}_0\in C_b^+(\mathbb{R})}\cdots$." Suppress the dependence of the following on the parameter n, and define

$$\tilde{V}(t,x) = \frac{\tilde{v}(t,x)}{\|\tilde{v}_0\|_{L^1}} \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}.$$

The random field $\tilde{V} = \{\tilde{V}(t,x)\}_{t\geq 0, x\in\mathbb{R}}$ solves the SPDE,

$$\partial_t \tilde{V}(t,x) = \frac{1}{2}\partial_x^2 \tilde{V}(t,x) + \tilde{V}(t,x)\hat{\xi}(t,x) \quad \text{on } (0,\infty)\times\mathbb{R}, \tag{3.15}$$

subject to $\tilde{V}(0,x) = \tilde{V}_0(x)$, where $\hat{\xi}(t,x) = \theta_{\tau(n)}\tilde{\xi}(t,x)$, $\|\tilde{V}_0\|_{L^\infty} = n^{\beta/2}$, and $\|\tilde{V}_0\|_{L^1} = 1$.

Note that $\hat{\xi}$ can be considered as a worthy martingale measure whose dominating measure is bounded by constant multiple of the Lebesgue measure (as in (2.3)). Thus, we can use all the results from the previous sections.

Define

$$N := n^{\beta/2}.$$

Then,

$$\begin{aligned}
\mathrm{P}\Big\{\sup_{0\leq t\leq n} \frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}} \geq n^\beta\Big\} &\leq \sup_{\substack{\tilde{V}_0\in C_b^+(\mathbb{R}):\\ \|\tilde{V}_0\|_{L^\infty}=n^{\beta/2}, \|\tilde{V}_0\|_{L^1}=1}} \mathrm{P}\Big\{\sup_{0\leq t\leq n} \frac{\|\tilde{V}(t)\|_{L^\infty}}{\|\tilde{V}(t)\|_{L^1}} \geq n^\beta\Big\} \\
&= \sup_{\substack{\tilde{V}_0\in C_b^+(\mathbb{R}):\\ \|\tilde{V}_0\|_{L^\infty}=N, \|\tilde{V}_0\|_{L^1}=1}} \mathrm{P}\Big\{\sup_{0\leq t\leq N^{2/\beta}} \frac{\|\tilde{V}(t)\|_{L^\infty}}{\|\tilde{V}(t)\|_{L^1}} \geq N^2\Big\} \\
&\leq A_1 + A_2,
\end{aligned}$$

where

$$A_1 := \sup_{\substack{\tilde{V}_0\in C_b^+(\mathbb{R}):\\ \|\tilde{V}_0\|_{L^\infty}=N, \|\tilde{V}_0\|_{L^1}=1}} \mathrm{P}\Big\{\sup_{0\leq t\leq N^{-\gamma}} \frac{\|\tilde{V}(t)\|_{L^\infty}}{\|\tilde{V}(t)\|_{L^1}} \geq N^2\Big\},$$

$$A_2 := \sup_{\substack{\tilde{V}_0\in C_b^+(\mathbb{R}):\\ \|\tilde{V}_0\|_{L^\infty}=N, \|\tilde{V}_0\|_{L^1}=1}} \mathrm{P}\Big\{\sup_{N^{-\gamma}\leq t\leq N^{2/\beta}} \frac{\|\tilde{V}(t)\|_{L^\infty}}{\|\tilde{V}(t)\|_{L^1}} \geq N^2\Big\}.$$

We first bound A_1 as follows:

$$A_1 \leq \sup_{\substack{\tilde{V}_0\in C_b^+(\mathbb{R}):\\ \|\tilde{V}_0\|_{L^\infty}=N, \|\tilde{V}_0\|_{L^1}=1}} \mathrm{P}\Big\{\sup_{0\leq t\leq N^{-\gamma}} \|\tilde{V}(t)\|_{L^\infty} \geq 2N\Big\}$$

$$
\begin{aligned}
&+ \sup_{\substack{\tilde V_0 \in C_b^+(\mathbb{R}):\\ \|\tilde V_0\|_{L^\infty}=N, \|\tilde V_0\|_{L^1}=1}} \mathrm{P}\left\{ \sup_{0\le t\le N^{-\gamma}} \|\tilde V(t)\|_{L^\infty} < 2N, \inf_{0\le t\le N^{-\gamma}} \|\tilde V(t)\|_{L^1} \le \frac{2}{N}\right\}\\
&\le \sup_{\substack{\tilde V_0 \in C_b^+(\mathbb{R}):\\ \|\tilde V_0\|_{L^\infty}=N, \|\tilde V_0\|_{L^1}=1}} \left[\mathrm{P}\left\{ \sup_{0\le t\le N^{-\gamma}} \|\tilde V(t)\|_{L^\infty} \ge 2N\right\} + \mathrm{P}\left\{\inf_{0\le t\le N^{-\gamma}} \|\tilde V(t)\|_{L^1} \le \frac{2}{N}\right\}\right]\\
&\le C_1 \exp\bigl(-C_1^{-1} N^{(3\gamma-4)/3}\bigr) \quad \text{for all } N \ge 1.
\end{aligned}
$$

In the last inequality, we used Propositions 3.5 and 3.6, and the constant C_1 only depends on γ and Lip_σ.

Regarding A_2, we have

$$
\begin{aligned}
A_2 &\le \sup_{\substack{\tilde V_0 \in C_b^+(\mathbb{R}):\\ \|\tilde V_0\|_{L^\infty}=N, \|\tilde V_0\|_{L^1}=1}} \mathrm{P}\{\|\tilde V(N^{-\gamma})\|_{L^\infty} \ge N\}\\
&\quad + \sup_{\substack{\tilde V_0 \in C_b^+(\mathbb{R}):\\ \|\tilde V_0\|_{L^\infty}=N, \|\tilde V_0\|_{L^1}=1}} \mathrm{P}\left\{\|\tilde V(N^{-\gamma})\|_{L^\infty} \le N \text{ and } \sup_{N^{-\gamma}\le t\le N^{2/\beta}} \frac{\|\tilde V(t)\|_{L^\infty}}{\|\tilde V(t)\|_{L^1}} \ge N^2\right\}\\
&\le K_2 \exp\bigl(-N^{(3\gamma-4)/2}\bigr) + \sup_{\substack{\tilde V_0 \in C_b^+(\mathbb{R}):\\ \|\tilde V_0\|_{L^\infty}=N, \|\tilde V_0\|_{L^1}=1}} \mathrm{P}\left\{\sup_{0\le t\le N^{2/\beta}-N^{-\gamma}} \frac{\|\tilde V(t)\|_{L^\infty}}{\|\tilde V(t)\|_{L^1}} \ge N^2\right\}
\end{aligned}
$$

for all $N \ge 1$.

We used Proposition 3.5 in the last inequality, and we note that the constant K_2 depends only on γ and Lip_σ. In addition, we also conditioned on $\mathcal{F}_{\tau(N^{2/\beta})+N^{-\gamma}}$ and consider a new SPDE of the form (3.15) using the SMP (Lemma 2.3).

Combine the preceding bounds for A_1 and A_2 and repeat the above process to obtain a real number $C = C(\gamma, \mathrm{Lip}_\sigma) > 1$ such that

$$
\begin{aligned}
&\sup_{\substack{\tilde V_0 \in C_b^+(\mathbb{R}):\\ \|\tilde V_0\|_{L^\infty}=N, \|\tilde V_0\|_{L^1}=1}} \mathrm{P}\left\{\sup_{0\le t\le N^{2/\beta}} \frac{\|\tilde V(t)\|_{L^\infty}}{\|\tilde V(t)\|_{L^1}} \ge N^2\right\}\\
&\le C \exp\bigl(-C^{-1} N^{(3\gamma-4)/3}\bigr) + \sup_{\substack{\tilde V_0 \in C_b^+(\mathbb{R}):\\ \|\tilde V_0\|_{L^\infty}=N, \|\tilde V_0\|_{L^1}=1}} \mathrm{P}\left\{\sup_{0\le t\le N^{2/\beta}-N^{-\gamma}} \frac{\|\tilde V(t)\|_{L^\infty}}{\|\tilde V(t)\|_{L^1}} \ge N^2\right\}\\
&\le 2C \exp\bigl(-C^{-1} N^{(3\gamma-4)/3}\bigr) + \sup_{\substack{\tilde V_0 \in C_b^+(\mathbb{R}):\\ \|\tilde V_0\|_{L^\infty}=N, \|\tilde V_0\|_{L^1}=1}} \mathrm{P}\left\{\sup_{0\le t\le N^{2/\beta}-2N^{-\gamma}} \frac{\|\tilde V(t)\|_{L^\infty}}{\|\tilde V(t)\|_{L^1}} \ge N^2\right\}\\
&\;\;\vdots\\
&\le \ell_N C \exp\bigl(-C^{-1} N^{(3\gamma-4)/3}\bigr),
\end{aligned}
$$

where $\ell_N = \lfloor N^{\gamma+2/\beta}\rfloor + 1$. Since $N = n^{\beta/2}$,

$$
\mathrm{P}\left\{\sup_{0\le t\le n} \frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}} \ge n^\beta\right\} \le C\bigl(1 + n^{1+\beta\gamma/2}\bigr)\exp\bigl(-C^{-1} n^{(3\gamma-4)\beta/6}\bigr).
$$

Our choices of γ and $\beta \ge 6/(3\gamma - 4)$ yield the proof of Proposition 3.1. □

4. Proof of Theorem 1.2: Uniform dissipation. In this section, we show the solution v to (2.4) dissipates uniformly in x when $v_0(x)$ decays rapidly as $|x| \to \infty$ (see Theorem 4.1). This also provides the proof of Theorem 1.2.

THEOREM 4.1. *Suppose v denotes the unique solution to* (2.4) *with initial data $v_0 \in C_b^+(\mathbb{R})$ that satisfies $\limsup_{|x|\to\infty} x^{-2} \log v_0(x) < 0$ and $\|v_0\|_{L^1} > 0$. Then, there exists a constant $\Lambda_3 > 0$ that only depends on L_σ and Lip_σ, and an almost surely finite random number $T > 0$ such that*

$$\sup_{x\in\mathbb{R}} v(t, x) \le \exp(-\Lambda_3 t^{1/3}) \quad \textit{for all } t > T.$$

Before we proceed with a proof, let us make two quick remarks.

REMARK 4.2.

(a) Every initial function $v_0 \in C_b^+(\mathbb{R})$ that satisfies $\limsup_{|x|\to\infty} x^{-2} \log v_0(x) < 0$ is automatically an element of $C_{\mathrm{rap}}^+(\mathbb{R})$.

(b) Theorem 4.1 is about uniform dissipation. We establish dissipation by first investigating the long-time behavior of the total mass process $t \mapsto \|v(t)\|_{L^1}$.

The proof of Theorem 4.1 hinges on the following quantitative probability estimate.

PROPOSITION 4.3. *There exist constants $\Lambda_4 > 0$ and $\Lambda_5 > 0$ that only depend on L_σ and Lip_σ and satisfy*

$$\mathrm{P}\Big\{\sup_{s\ge t} \|v(s)\|_{L^1} \ge \exp(-\Lambda_4 t^{1/3})\Big\} \le \exp(-\Lambda_5 t^{1/3}) \quad \textit{for every } t > 0. \tag{4.1}$$

PROOF. Recall that, off a single null set, $v(t, x) \ge 0$ for all $t \ge 0$ and $x \in \mathbb{R}$. In fact, because of condition (1.2), the argument of [21] can be used virtually unchanged in order to prove that, off a single null set, $v(t, x) > 0$ simultaneously for every $t > 0$ and $x \in \mathbb{R}$. Thus, just as we did in the proof of Proposition 3.6, we may conclude that $t \mapsto \|v(t)\|_{L^1}$ is a strictly positive continuous $L^2(\Omega)$-martingale whose quadratic variation at every time $t > 0$ is bounded below and above by constant multiples of $\int_0^t \|v(s)\|_{L^2}^2 \,\mathrm{d}s$ that do not depend on t; see also (2.3). We now follow the proof of Theorem 4.1 of Chen, Cranston, Khoshnevisan, and Kim [4], (4.20), essentially line by line, in order to learn that there exists a number $C = C(\mathrm{L}_\sigma, \mathrm{Lip}_\sigma) > 0$ such that

$$\mathrm{E}\sqrt{\|v(t)\|_{L^1}} \le C \exp(-Ct^{1/3}) \quad \text{for every } t > 0.$$

Proposition 4.3 follows from this and Doob's maximal inequality for supermartingales. □

We now use Proposition 4.3, along with Proposition 3.1, in order to prove Theorem 4.1.

PROOF OF THEOREM 4.1. Let $\Lambda_4 > 0$ and $\Lambda_5 > 0$ denote the constants that were given in Proposition 4.3, and let $\alpha > 0$ be a real number that will be chosen later.

By the sub-additivity of the probability measure, we have

$$\mathrm{P}\Big\{\sup_{n-1\le t\le n} \|v(t)\|_{L^\infty} \ge \exp(-\alpha n^{1/3})\Big\} \le A_1 + A_2, \tag{4.2}$$

where

$$A_1 := \mathrm{P}\left\{\sup_{0\le t\le n}\frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}}\ge n^6\right\}, \tag{4.3}$$

$$A_2 := \mathrm{P}\left\{\sup_{n-1\le t\le n}\|v(t)\|_{L^\infty}\ge \exp(-\alpha n^{1/3}) \text{ and } \sup_{0\le t\le n}\frac{\|v(t)\|_{L^\infty}}{\|v(t)\|_{L^1}}\le n^6\right\}. \tag{4.4}$$

Recall Proposition 3.1 and choose $\gamma = 5/3$ and $\beta = 6$ in order to find that there exists a constant $c = c(\mathrm{Lip}_\sigma) > 0$ such that

$$A_1 \le \exp(-cn) \quad \text{for all large } n \ge 1. \tag{4.5}$$

As regards A_2, let us choose $\alpha < \Lambda_4$ so that for all large $n \ge 2$,

$$n^{-6}\exp(-\alpha n^{1/3}) \ge \exp(-\Lambda_4(n-1)^{1/3}).$$

In this way, we can find that

$$\begin{aligned} A_2 &\le \mathrm{P}\left\{\sup_{n-1\le t\le n}\|v(t)\|_{L^1}\ge n^{-6}\exp(-\alpha n^{1/3})\right\} \\ &\le \mathrm{P}\left\{\sup_{t\ge n-1}\|v(t)\|_{L^1}\ge \exp(-\Lambda_4(n-1)^{1/3})\right\}\le \exp(-\Lambda_5(n-1)^{1/3}), \end{aligned} \tag{4.6}$$

thanks to Proposition 4.3. Therefore, (4.5) and (4.6) together imply the existence of a number $\beta = \beta(\mathrm{L}_\sigma, \mathrm{Lip}_\sigma) > 0$ such that for all large $n \ge 1$

$$\mathrm{P}\left\{\sup_{n-1\le t\le n}\|v(t)\|_{L^\infty}\ge \exp(-\alpha n^{1/3})\right\}\le \exp(-\beta n^{1/3}). \tag{4.7}$$

This and the Borel–Cantelli lemma yield Theorem 4.1. □

5. Proof of Theorem 1.1. Finally, in this section we use the results from Sections 2–4 in order to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let $\eta_1, \eta_2 > 0$ be two real numbers whose numerical values will be determined later on in (5.1).

Let us first fix an integer $n \ge 1$ and define

$$L(t) := \exp(\eta_1 t^{1/3}) \quad \text{and} \quad M := M_n := 2\lfloor L(n)\rfloor.$$

As in Remark 2.2, we may write $u(t,x)$—with the initial function $u_0 \equiv 1$ — as

$$u(t,x) = \sum_{i=-M}^{M} v^{(i)}(t,x) + v^{(M+1)}(t,x),$$

where $v^{(i)}$ and $v^{(M+1)}$ satisfy (2.4) with respective initial data $v_0^{(i)}$ and $v_0^{(M+1)}$ that are continuous and nonnegative functions such that the support of $v_0^{(i)}$ is in $[i-1, i+1]$ for every $i = -M, \dots, M$ and the support of $v_0^{(M+1)}$ is in $\mathbb{R}\setminus(-M, M)$.

For every $\rho \in (0,1)$,

$$\mathrm{P}\left\{\sup_{n-1\le t\le n}\sup_{|x|\le L(t)}\frac{u(t,x)}{\mathrm{e}^{-\eta_2 t^{1/3}}}\ge\rho\right\}\le A_n^{(1)} + A_n^{(2)},$$

where

$$A_n^{(1)} := \sum_{i=-M}^{M} \mathrm{P}\left\{ \sup_{n-1\le t\le n} \sup_{x\in\mathbb{R}} \frac{v^{(i)}(t,x)}{\mathrm{e}^{-\eta_2 t^{1/3}}} \ge \frac{\rho}{2(M+1)} \right\},$$

$$A_n^{(2)} := \mathrm{P}\left\{ \sup_{n-1\le t\le n} \sup_{|x|\le L(t)} \frac{v^{(M+1)}(t,x)}{\mathrm{e}^{-\eta_2 t^{1/3}}} \ge \frac{\rho}{2(M+1)} \right\}.$$

We first consider $A_n^{(1)}$. Since each $v^{(i)}(0)$ has a compact support, we may use (4.7). Let $\alpha > 0$ and $\beta > 0$ denote the constants given in (4.7). We now choose and fix $\eta_1, \eta_2 > 0$ so that

$$\eta_1 < \beta \quad \text{and} \quad \eta_1 + \eta_2 \le \frac{\alpha}{2}. \tag{5.1}$$

Any such choice of η_1 and η_2 ensures that for all large $n \ge 1$,

$$\begin{aligned} A_n^{(1)} &\le \sum_{i=-M}^{M} \mathrm{P}\left\{ \sup_{n-1\le t\le n} \|v^{(i)}(t)\|_{L^\infty} \ge \frac{\rho}{8} \exp\left[-(\eta_1+\eta_2)n^{1/3}\right] \right\} \\ &\le \sum_{i=-M}^{M} \mathrm{P}\left\{ \sup_{n-1\le t\le n} \|v^{(i)}(t)\|_{L^\infty} \ge \exp\left[-2(\eta_1+\eta_2)n^{1/3}\right] \right\} \\ &\le 4\exp\left[-(\beta-\eta_1)n^{1/3}\right]. \end{aligned} \tag{5.2}$$

We now consider $A_n^{(2)}$. First, let us recall that $v_0^{(M+1)}(x) = 0$ when $|x| \le M \le 2L(n)$ and $v_0^{(M+1)}(x) \in [0,1]$ in general. Consequently, for every $t \in [n-1, n]$,

$$\begin{aligned} \sup_{|x|\le L(n)} \left(S_t v_0^{(M+1)}\right)(x) &= \sup_{|x|\le L(n)} \int_{-\infty}^{\infty} p_t(x-y) v_0^{(M+1)}(y)\,\mathrm{d}y \\ &\le \sup_{|x|\le L(n)} \int_{|y|\ge 2L(n)} p_t(x-y)\,\mathrm{d}y \\ &\le 2\exp\left(-\frac{L(n)}{4n}\right) = 2\exp\left(-\frac{\mathrm{e}^{\eta_1 n^{1/3}}}{4n}\right), \end{aligned} \tag{5.3}$$

for every $n \in \mathbb{Z}_+$. Next, we estimate

$$\begin{aligned} \mathrm{E}\left[\sup_{n-1\le t\le n} \sup_{|x|\le L(t)} \left|v^{(M+1)}(t,x)\right|^k \right] &\le \mathrm{E}\left[\sup_{n-1\le t\le n} \sup_{|x|\le L(n)} \left|v^{(M+1)}(t,x)\right|^k \right] \\ &\le \sum_{i=-L(n)}^{L(n)-1} \mathrm{E}\left[\sup_{n-1\le t\le n} \sup_{i\le x\le i+1} \left|v^{(M+1)}(t,x)\right|^k \right] \\ &\le 2^{k/2}\left(B_n^{(1)} + B_n^{(2)}\right), \end{aligned}$$

where

$$B_n^{(1)} := \sum_{i=-L(n)}^{L(n)-1} \mathrm{E}\left(\left|v^{(M+1)}(n,i)\right|^k\right),$$

$$B_n^{(2)} := \sum_{i=-L(n)}^{L(n)-1} \mathrm{E}\left[\sup_{\substack{n-1\le s_1,s_2\le n \\ i\le x_1,x_2\le i+1}} \left|v^{(M+1)}(s_1,x_1) - v^{(M+1)}(s_2,x_2)\right|^k \right].$$

In order to estimate $B_n^{(1)}$ we use Lemma 3.2 and (5.3) as follows:

$$
\begin{aligned}
B_n^{(1)} &\le A\|v_0^{(M+1)}\|_{L^\infty}^{k/2}\exp(Ak^3n)\sum_{i=-L(n)}^{L(n)-1}(S_n v_0^{(M+1)})^{k/2}(i) \\
&\le 4A\exp\left(\eta_1 n^{1/3}+Ak^3n-\frac{k\mathrm{e}^{\eta_1 n^{1/3}}}{8n}\right).
\end{aligned}
\tag{5.4}
$$

Next we consider $B_n^{(2)}$. Here, in order to estimate the expected value of the supremum, we first estimate the supremum of the expectation and then apply an appropriate form of the Kolmogorov continuity theorem (see, e.g., [15], Theorem C.6). This allows us to estimate the expected value of the supremum (see details below).

Since $v_0^{(M+1)}\in C_b^+(\mathbb{R})$ with $\|v_0^{(M+1)}\|_{L^\infty}=1$, we use the well-known fact (see, e.g., Khoshnevisan [15], Chapter 5) that there exists a constant $B>0$ that is independent of n and satisfies for all $k\ge 2$,

$$
\begin{aligned}
\sup_{\substack{n-1\le s_1\ne s_2\le n\\ -\infty<x_1\ne x_2<\infty}}\mathrm{E}\left(\left|\frac{v^{(M+1)}(s_1,x_1)-v^{(M+1)}(s_2,x_2)}{|s_1-s_2|^{1/4}+|x_1-x_2|^{1/2}}\right|^k\right) &\le B\mathrm{e}^{Bk^3n}\|v_0^{(M+1)}\|_{L^\infty}^k \\
&= B\mathrm{e}^{Bk^3n}.
\end{aligned}
\tag{5.5}
$$

On the other hand, Lemma 3.2 and (5.3) together imply that, for all $k\ge 2$,

$$
\begin{aligned}
&\sup_{\substack{n-1\le s_1,s_2\le n\\ -L(n)\le x_1,x_2\le L(n)}}\mathrm{E}\left(\left|v^{(M+1)}(s_1,x_1)-v^{(M+1)}(s_2,x_2)\right|^k\right) \\
&\quad\le 2^k\sup_{\substack{n-1\le s\le n\\ -L(n)\le x\le L(n)}}\mathrm{E}\left(\left|v^{(M+1)}(s,x)\right|^k\right) \\
&\quad\le (2A)^k\|v_0^{(M+1)}\|_{L^\infty}^{k/2}\exp(Ak^3n)\sup_{|x|\le L(n)}\left|(S_s v_0^{(M+1)})(x)\right|^{k/2} \\
&\quad\le 2(2A)^k\exp\left(Ak^3n-\frac{k\mathrm{e}^{\eta_1 n^{1/3}}}{8n}\right).
\end{aligned}
\tag{5.6}
$$

Since $\min\{a,b\}\le (ab)^{1/2}$ for every $a>0, b>0$, there exists a number $C_1>0$ that is independent of n and satisfies the following for all $n-1\le s_1\ne s_2\le n$ and all $i\le x_1\ne x_2\le i+1$ with $i\in[-L(n),L(n)-1]$:

$$
\begin{aligned}
&\mathrm{E}\left(\left|v^{(M+1)}(s_1,x_1)-v^{(M+1)}(s_2,x_2)\right|^k\right) \\
&\quad\le C_1^k\exp\left(C_1k^3n-\frac{k\mathrm{e}^{\eta_1 n^{1/3}}}{16n}\right)\left(|s_1-s_2|^{1/8}+|x_1-x_2|^{1/4}\right)^k.
\end{aligned}
$$

A suitable form of the Kolmogorov continuity theorem [15], Theorem C.6, implies that there exists a number $C_2>0$—independent of n—such that for all $k\ge 2$ and all $i\in[-L(n),L(n)-1]$,

$$
\mathrm{E}\left[\sup_{\substack{n-1\le s_1,s_2\le n\\ i\le x_1,x_2\le i+1\\ (s_1,x_1)\ne(s_2,x_2)}}\left|\frac{v^{(M+1)}(s_1,x_1)-v^{(M+1)}(s_2,x_2)}{|s_1-s_2|^{1/16}+|x_1-x_2|^{1/8}}\right|^k\right]\le C_2^k\exp\left(C_2k^3n-\frac{k\mathrm{e}^{\eta_1 n^{1/3}}}{16n}\right).
\tag{5.7}
$$

By (5.4) and (5.7) (here we fix $k=2$) we get that for all large $n\ge 1$

$$
\mathrm{E}\left[\sup_{n-1\le t\le n}\sup_{|x|\le L(t)}\left|v^{(M+1)}(t,x)\right|^2\right]\lesssim \mathrm{e}^{-n}.
$$

By Chebyshev's inequality, we have

$$A_n^{(2)} \le \mathrm{P}\Big\{\sup_{n-1\le t\le n}\sup_{|x|\le L(t)} v^{(M+1)}(t,x) \ge \exp\big[-2(\eta_1+\eta_2)n^{1/3}\big]\Big\} \tag{5.8}$$

$$\le \mathrm{E}\Big[\sup_{n-1\le t\le n}\sup_{|x|\le L(t)} \big|v^{(M+1)}(t,x)\big|^2\Big]\exp\big[4(\eta_1+\eta_2)n^{1/3}\big]$$

$$\le \exp\big(-n+4(\eta_1+\eta_2)n^{1/3}\big). \tag{5.9}$$

We combine (5.2) and (5.8) to see that

$$\sum_{n=1}^{\infty}\mathrm{P}\Big\{\sup_{n-1\le t\le n}\sup_{|x|\le L(t)}\frac{u(t,x)}{\mathrm{e}^{-\eta_2 t^{1/3}}}\ge\rho\Big\}<\infty.$$

An appeal to the Borel–Cantelli lemma completes the proof. □

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