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Author for correspondence:

Gregory D. Scholes

e-mail: gscholes@princeton.edu

Quantum-like states on complex synchronized networks

Gregory D. Scholes

Department of Chemistry, Princeton University, Princeton, NJ 08544, USA

GDS, 0000-0003-3336-7960

Recent work has exposed the idea that interesting quantum-like (QL) probability laws, including interference effects, can be manifest in classical systems. Here, we propose a model for QL states and QL bits. We suggest a way that huge, complex systems can host robust states that can process information in a QL fashion. Axioms that such states should satisfy are proposed. Specifically, it is shown that building blocks suited for QL states are networks, possibly very complex, that we defined based on k -regular random graphs. These networks can dynamically encode a lot of information that is distilled into the emergent states we can use for QL processing. Although the emergent states are classical, they have properties analogous to quantum states. Concrete examples of how QL functions are possible are given. The possibility of a ‘QL advantage’ for computing-type operations and the potential relevance for new kinds of function in the brain are discussed and left as open questions.

1. Introduction

Quantum states characterize the microscopic world, where they convey the dual wave-particle basis for quantum theory. Of special interest is the way probability laws for quantum states allow for interference effects and other, more exotic, correlations evidenced in the effect and outcome of measurements. We thus find fascinating non-classical phenomena that can be exploited in quantum information science [1]. Here, a goal is to leverage quantum correlations at the molecular scale for function at the human scale. A well-known challenge, however, is that quantum

states, at least the most interesting ones, tend to be fragile—easily destroyed by environmental noise and fluctuations (the process of decoherence [2]). Researchers therefore focus on stabilizing simple states using techniques such as very low temperature. Here, we explore a way to produce robust states, comprising highly complex systems, that exhibit ‘quantum like’ properties.

Let us start by defining what we mean, in the present work, by a quantum-like (QL) state. An important requirement is that these states allow amplitude-level interferences that are explained by a measurement formulation like that required by the quantum theory. This is a fairly broad criterion that can be fulfilled by either: (i) states in vector spaces, where we think of amplitude (vector) additions followed by measurement of probability as the amplitude squared; or (ii) we can work only with probabilities, but formulate a vector-like model for measurement sequences (this is the essence of the Växjö model). In the present work, we would like to take the QL state framework further, and show how eigenstates of classical systems can be combined (through interactions between the systems serving as QL-bits) to produce states in a state space that represents the tensor product of vector spaces or each QL-bit. Since we are dealing with inner product spaces here, the vector spaces are Hilbert spaces.

Why would complex QL systems be interesting? There are two major motivations for this work. First, by exhibiting a concrete example of robust QL states, we can suggest ways to design circuits with quantum like functions. By considering such circuits, we can conceive of experiments designed to seek QL correlations in a range of different systems. Moreover, this platform could suggest a way to perform QL computing using simplified hardware that is a step towards true quantum computing. Second, we establish a hypothesis for what QL states could ‘look like’ in settings as complex as biology. The key point we will establish is the feasibility of these states.

The possibility that some kinds of quantum effects provide function in biological systems has intrigued researchers for over a century. Yet, it is easy to argue against such effects because the kinds of quantum states that we are familiar with are fragile, and would rapidly decohere in a noisy environment. This point is well recognized, yet, as enunciated by Penrose [3], calls for discovery of new kinds of quantum systems that are not only robust in a noisy environment, but ‘retain their manifest quantum nature at a much larger scale’. Even if such states are not present, or functional, in living systems, elucidating the principles that enable them would open up new ways of engaging quantum (like) correlations for technology.

In quantum states, classical correlations are prominent [4]. For example, in the entangled state $|\Psi_{-}\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B \right]$, it is obvious that if a measurement of qubit A indicates it is in state $|0\rangle_A$, the qubit B (measured in the same basis) must be in the state $|1\rangle_B$. In addition, we note that interference phenomena—a key contributor to quantum function—are not confined to quantum systems. Related ideas have been previously discussed [5,6]. Thus, there is reason to believe that classical systems possessing the right kinds of correlations and with the potential to exhibit interference effects might replicate many (obviously not all) kinds of quantum phenomena. It is thus proposed that it may not always be necessary to produce truly quantum devices that exploit quantum correlations. It might often be sufficient to develop a QL mimic constructed from classical components. Not only does this open new avenues for engineering systems that can capture some of the properties of quantum systems, but it also provides a framework for thinking about QL function in nature. The essential advantage would be that more sophisticated probability laws can be leveraged.

Here, we combine ideas from graph theory and operator theory to demonstrate how complex networks can serve as a basis for generating complex QL states and producing functions using those states. We formulate axioms that guide the construction of the states and demonstrate examples of such states. Importantly, we describe how these states are resilient in noisy environments. In the discussion, we speculate about functions unique to these states and how they provide a basis for testable hypotheses for exotic phenomena, including how

QL processes might augment classical processes in neural networks like the brain. Some of this might sound far-fetched, but keep in mind that the central finding of this work is to define and analyse the QL states themselves. From this foundation, perhaps we can begin asking questions that could not previously be well posed.

One inspiration for the work was to define physical realizations of QL states that can show properties described by QL probabilities, as proposed by Khrennikov in the framework of a contextual probabilistic model [7], termed the Växjö model. The focus of the Växjö model is on non-Kolmogorov probabilities—that is models where probability can be thought of as a kind of vector quantity. This allows probability amplitude to interfere, like in quantum theory. However, there is no requirement of an underlying microscopic quantum state, so the QL model can be used to analyse problems as far afield as financial markets, psychology, or even the mind. Our goal in this paper is to propose a concrete model for QL states that are complex enough to accommodate the idea of ‘context’ in measurement theory.

The main focus of the paper is to formulate examples of QL-states in a vector space, closely analogous to quantum states, that can be realized in classical set-ups. To do that, we notice that emergent states of very complex systems are incredibly robust and isolated in the spectrum, so they can potentially serve as building blocks for QL states [8]. An emergent phenomenon is manifest when numerous small interactions add coherently to produce a stunning collective effect that appears above some threshold of the number of coherently interacting systems. In quantum systems, an emergent state is signalled by a single eigenvalue that splits away from the many other eigenvalues in the spectrum, opening up an energy gap [9]. The questions that we solve in the present work include how to design networks that produce stable superposition states of two basis states, and how to combine those networks to produce a larger state space that can be mapped onto a state space comprising linear combinations of product states.

Emergent states are characteristic of synchronized complex networks of phase oscillators. There are many examples of synchronization in a wide variety of systems [10–18]. A few illustrative examples of synchronization include power networks [19], bacterial networks interacting via signalling [20], human networks of violin players [21] and mechanisms underpinning memory processes in the brain [22]. We will discuss synchronization in neural networks in more detail later in the paper. All these kinds of systems attain a stable synchronized steady state by feedback loops.

Synchronization is usually analysed with the Kuramoto Model [11]; a system of coupled differential equations that predict how initial conditions and feedback acting in a set of phase oscillators allow synchronization to emerge. Here, we assume that the initial conditions (phase offsets) allow the system to be synchronizable. We have studied the connections between network synchronization and homomorphic quantum eigenstates in other work [23]. This approach allows us to focus on the eigenstates rather than dynamics. In particular, we are interested in an ordered eigenstate whose eigenvalue is well separated from the eigenvalues of the many other random states of a system. We construct the basis for these emergent states in terms of networks (or graphs).

The main outcome of this paper is to combine these ideas—QL states and emergent states of networks—to propose a network structure that serves as a building block for the full tensor product representation of QL states. That is, we show how to mimic linear combinations of product states. A schematic of the construction that will be developed in the paper is shown in [figure 1](#). We then discuss how those states might be exploited for function in ways that have not been considered previously for typical network structures.

Detailed reviews of quantum information science can be found in various books and reviews [1,24–28]. In recent review type papers [4,29], we describe the idea of entanglement and qubits and summarize the conceptual ideas behind quantum information science for a broader readership.

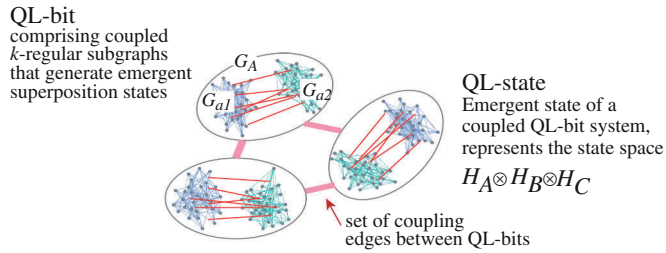


Figure 1. Example of the way three QL-bits couple to produce any state in the tensor product space. The QL-bits are comprised of graphs, labelled here for later reference. Connections (edges) between the QL-bits are shown schematically. The emergent state of the network produces a state that can be mapped to the state in the tensor product of Hilbert spaces of each QL-bit, as indicated. In reality, there should be many edges between each pair of subgraphs.

2. Graphs and emergent states

A network can be represented as a graph. A graph $G(n, m)$, that we will often write simply as G , comprises n vertices and a set of m edges that connect pairs of vertices. The size of a graph or subgraph, that is the number of vertices is written $|G|$. In the present work, the edges are assigned semi-randomly—that is there is an overarching ‘rule’ that guides how the edges are placed randomly. In the present work, we only consider that ‘rule’ to be that we construct k -regular random graphs (we explain these graphs below). In some cases, we also delete some fraction of the edges randomly after building the graph [8]. Rather than assigning a specific number of edges, sometimes we assign edges between vertices with probability p , so the graph is formally $G(n, p)$. Here, we discuss only graphs with no loops and no multiple edges. For background, see [30–32].

The adjacency matrix of a graph A is the $n \times n$ matrix containing entries $a_{ij} = a_{ji} = 1$, with $i, j \in \{1, 2, \dots, n\}$, when an *undirected* edge joins vertices i and j . The adjacency matrix of a *directed* graph contains entries $a_{ij} = 1$ when an edge connects from vertex i to vertex j . We consider also *signed* graphs, where the edge entry in the adjacency matrix is designated from $\{0, 1, -1\}$.

Undirected graphs can be thought of as representing coupling between entities located at the vertices—a kind of coupling map of the system. For instance, below we will discuss networks of interacting oscillating dipoles, so the interaction is the dipole-dipole coupling. That coupling can be negative, indicating that the coupled oscillators are locked out-of-phase, then we use the signed graph. Directed graphs indicate a flow, timing, or causal coupling from one vertex to another. That is often found in physical systems [33], including transportation networks, aspects of social networks, downstream influence (common in biology) and communication.

The spectrum of a graph G is defined as the spectrum (i.e. eigenvalues) of its adjacency matrix A . Here, we use the spectrum to detect and characterize emergent states by their eigenvalues. The spectrum can tell us a lot about the graph, and *vice versa*, see [34]. The graph can also be used as a template to construct an ensemble of random matrices, whose spectrum is analysed [23].

In earlier work we noticed that certain graph types have a spectrum where the lowest (or highest, depending on the signs of the edges) eigenvalue (or few eigenvalues) are sufficiently separated from the rest of the spectrum so that those states are resilient to decoherence [23]. Indeed, it is known that there are families of graphs with exceptional properties of this kind—expander graphs [35–40]. In the present work, expander graphs of a particular type will underlie our prototypical networks. Expander graphs involve high connectivity throughout the graph, and are therefore optimal structures for communication or random walks.

Definition 2.1. (Expander graph) Let G be the n -vertex graph with vertex set V and take $0 < \epsilon \in \mathbb{R}$. G is an ϵ -expander if for every vertex subset Y of V with $|Y| \leq \frac{1}{2}|V|$, we have

$$|\partial Y| \geq \epsilon |Y|, \quad (2.1)$$

where ∂Y is the boundary of Y , which means the set of edges in V that have one endpoint in Y and one endpoint in $V \setminus Y$.

Definition 2.2. (Isoperimetric constant) The isoperimetric constant of G , $h(G)$, indicates the minimum ϵ for the graph:

$$h(G) = \min \left\{ \frac{|\partial Y|}{|Y|} \right\}, \quad (2.2)$$

with $|Y| \leq \frac{1}{2} |V|$.

The specific expander graphs we study here are the k -regular random graphs.

Definition 2.3. (k -regular graph) A graph G is k -regular if every vertex has degree (valency) k . That is, every vertex connects to k edges.

Definition 2.4. (Expander family, def 1.74 of [36]) Let k be a positive integer. Let (G_n) be a sequence of k -regular graphs such that $|G_n| \rightarrow \infty$ as $n \rightarrow \infty$. (G_n) is an expander family if the sequence $(h(G_n))$ is bounded away from zero.

Let the eigenvalues of G be $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1}$. When G is a k -regular graph, then $\lambda_0 = k$. This fact is easily demonstrated by noting that the eigenvector associated with λ_0 is $(1, 1, 1, \dots)/\sqrt{n}$. The second-largest eigenvalue λ_1 is bounded by the Alon–Boppana theorem [40,41]. It can then be shown that the smaller λ_1 is, the larger $h(G)$ is.

Definition 2.1. (prop 1.84 of [36]) Let G be a k -regular graph, then

$$\frac{k - \lambda_1}{2} \leq h(G) \leq \sqrt{2k(k - \lambda_1)}. \quad (2.3)$$

Conversely, the larger $h(G)$, the larger the spectral gap $\lambda_0 - \lambda_1$ (recall that $\lambda_0 = k$). This is the reason that expander graphs are good candidates for emergent states.

The isoperimetric constant is not so easy to calculate. It is closely related to the property of topological groups called weak containment, specifically a quantity called Kazhdan's property (T) [42], which can sometimes be more easily bounded. However, for the present purposes, we simply point out that the expansion property of a family of graphs is best estimated from graph spectra. This is demonstrated for k -regular graphs by the Alon–Boppana bound [40,41].

In recent work, we have shown that k -regular random graphs retain their characteristic spectral properties even when a substantial fraction of edges have been removed at random [8]. We explain those numerical results on a more formal level, with proofs, in another paper. We found that the ensemble of graphs has the spectrum of an ensemble of d -regular random graphs, where $d \leq k$ is the average valency of vertices for the ensemble of graphs. Therefore, these graphs can be quite disordered, but still display a prominent eigenvalue that is well-separated from those of the remaining 'random' states. This particular point, and the reasons that the emergent state is spontaneously stable, will be developed in another paper.

In other work, it has been shown that complex networks where nodes are coupled with the structure of k -regular random graphs are highly synchronizable [43]. In that work, the authors prove that for the case of regular graphs, spectral expansion implies global synchrony. That is, a network of phase oscillators coupled in the fashion of a k -regular graph will synchronize regardless of their initial phases. In the present work, we do not necessarily need perfectly synchronizable networks. In other words, it may be advantageous to suffer some initial phase arrangements that inhibit synchronization with the benefit of less-stringent construction requirements for the graph. After all, these networks will be much more robust to disorder and implementable if they can sustain a significant variance in structure. Our main requirement is that the graphs are on average d -regular.

Some examples of k -regular random graphs on $n = 40$ vertices, together with their spectra, are shown in figure 2. Here, $k = 20$, so the number of edges in the graph without edge disorder

is $kn/2 = 400$. We plot edge-disordered graphs, with various numbers of edges removed. The deleted edges are indicated by green lines, whereas the retained edges are light blue. We compare cases where 20, 100 and 300 edges have been randomly removed. The emergent state—evident as the isolated, largest eigenvalue—is qualitatively stable (that is, separated from the rest of the spectrum) until more than half the edges have been deleted. The eigenvector of the emergent state is $(1, 1, 1, \dots)/\sqrt{n}$.

It is useful to compare these graphs and spectra to those of signed graphs. Let's say the largest eigenvalue of the unsigned graph is d , then, if the signs of all vertices are reversed, the smallest eigenvalue of the signed graph is $-d$. It is associated with the eigenvector $(1, 1, 1, \dots)/\sqrt{n}$. We show in figure 3 examples where some fraction of the edges are randomly given negative sign, according to probability p_n . The positive edges are blue, while the negative edges are red. We compare case of $p_n = 0.2$ with $p_n = 0.8$. It is evident that signed graphs also preserve the emergent state, but only when the number of edges with one sign is sufficiently greater than those of the other sign.

3. Quantum-like states on networks

Here, we define properties that QL states should possess by giving some brief definitions of the state space we use for quantum systems, and then proposing some axioms that QL states should satisfy. After that, we build concrete examples of systems that exhibit such states.

(a) Background and axioms

Our aim is to demonstrate stable states of a complex network that are defined in some basis, $\{|a_1\rangle, |a_2\rangle\}$, such that we can produce arbitrary states by arbitrary convex combinations of the basis states,

$$|A\rangle = \alpha_1 |a_1\rangle + \alpha_2 |a_2\rangle, \quad (3.1)$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ or $\mathbb{R}$ and by normalization $\alpha_1^2 + \alpha_2^2 = 1$. These vectors will comprise a Hilbert space $\mathcal{H}_A$.

It might be worth defining Hilbert space here, and for more detail see [44]. A Hilbert space $\mathcal{H}$ is a (complete) vector space over a field $\mathbb{K}$ (which can include $\mathbb{C}$ or $\mathbb{R}$, for example) with an inner product $\langle x, y \rangle$ which generalizes the idea of a dot product of vectors in Cartesian space. For any vectors x, y, z in $\mathcal{H}$ and scalars $\alpha \in \mathbb{K}$, we have

$$\begin{aligned} \langle x + y, z \rangle &= \langle x, z \rangle + \langle y, z \rangle \\ \langle \alpha x, y \rangle &= \alpha \langle x, y \rangle \\ \langle x, y \rangle &= \overline{\langle y, x \rangle} \\ \langle x, x \rangle &= \alpha \geq 0, \end{aligned}$$

where the overbar means complex conjugate (or its equivalent for the relevant field). Note also that the 'vectors' can be functions satisfying the properties listed above. The norm is defined in terms of the inner product:

$$|x| = \langle x, x \rangle^{\frac{1}{2}}.$$

We will define in the following sections QL states, constructed using networks that have emergent states satisfying these properties. Here, our vector spaces are defined explicitly over $\mathbb{R}$. If we want to define them over $\mathbb{C}$, we should use biased graphs [45–47], with edges taking values from $\{0, 1, -1, i, -i\}$.

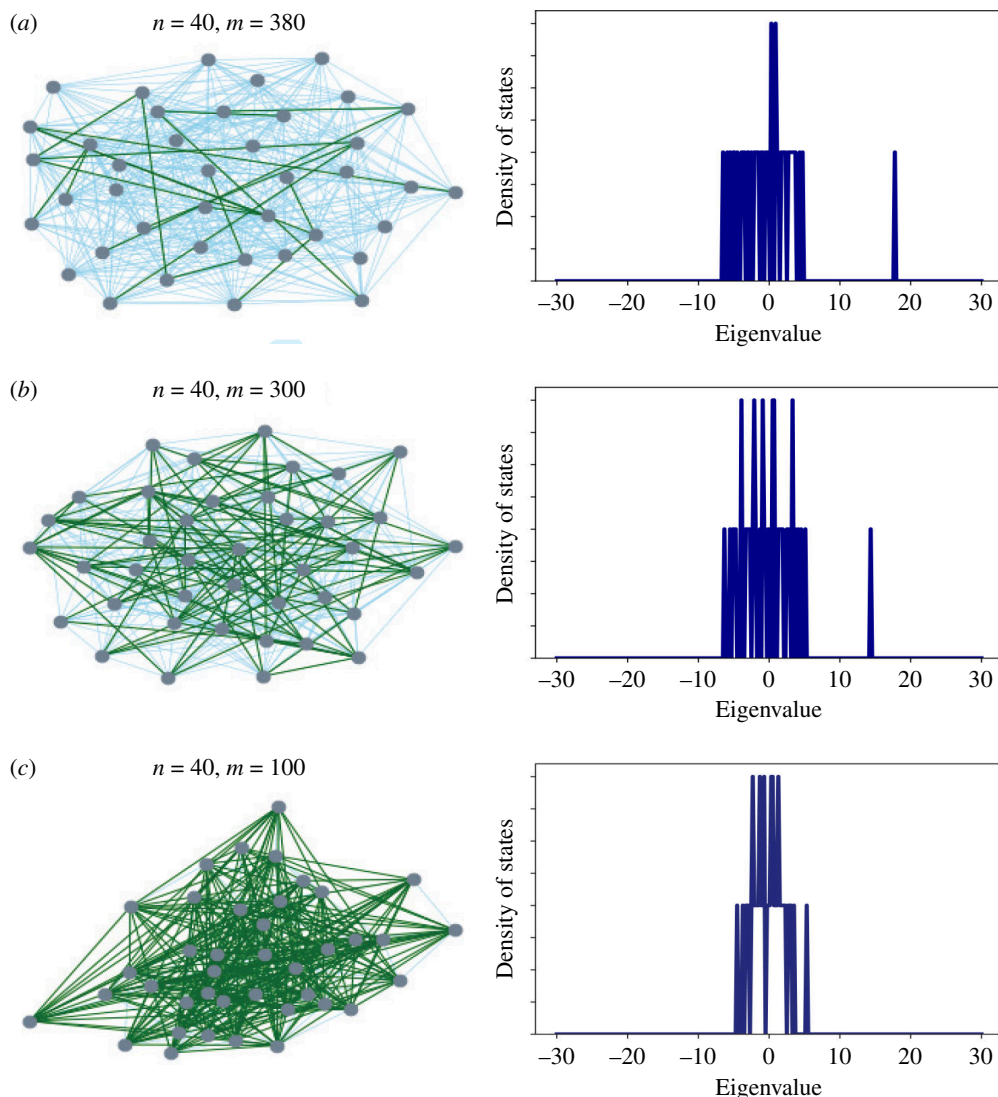


Figure 2. Examples of k -regular random graphs on $n = 40$ vertices with $k = 20$, together with their spectra. See text. (a) The number of edges is $m = 380$ (note that the maximum possible number of edges is 400). Here, the emergent state is clearly evident as the isolated state with greatest eigenvalue. (b) The number of edges is $m = 300$. (c) The number of edges is $m = 100$. Here, the average valency (degree) of the vertices is too small for the emergent state to separate clearly from the spectrum of random states.

We propose that those QL states and/or their underlying graphs should satisfy the following axioms:

- (1) The graphs will exhibit an emergent state that is distinguished in the eigenvalue spectrum.
- (2) Each graph will be robust and stable to disorder in its precise construction as well as respect to disorder in the frequency of the oscillators represented by its nodes (vertices).
- (3) The basis QL bit is a two-state system, but one of the two states is not 'off' because the QL bit is defined by turning 'on' a network of oscillators. Therefore, we need the QL bit to comprise a basis of two distinguishable 'on' states. What we mean by 'on' and 'off' distinguishes QL states from quantum states, where we are happy to have a two-level system comprising the vacuum, or ground, state and another level. In the case of QL

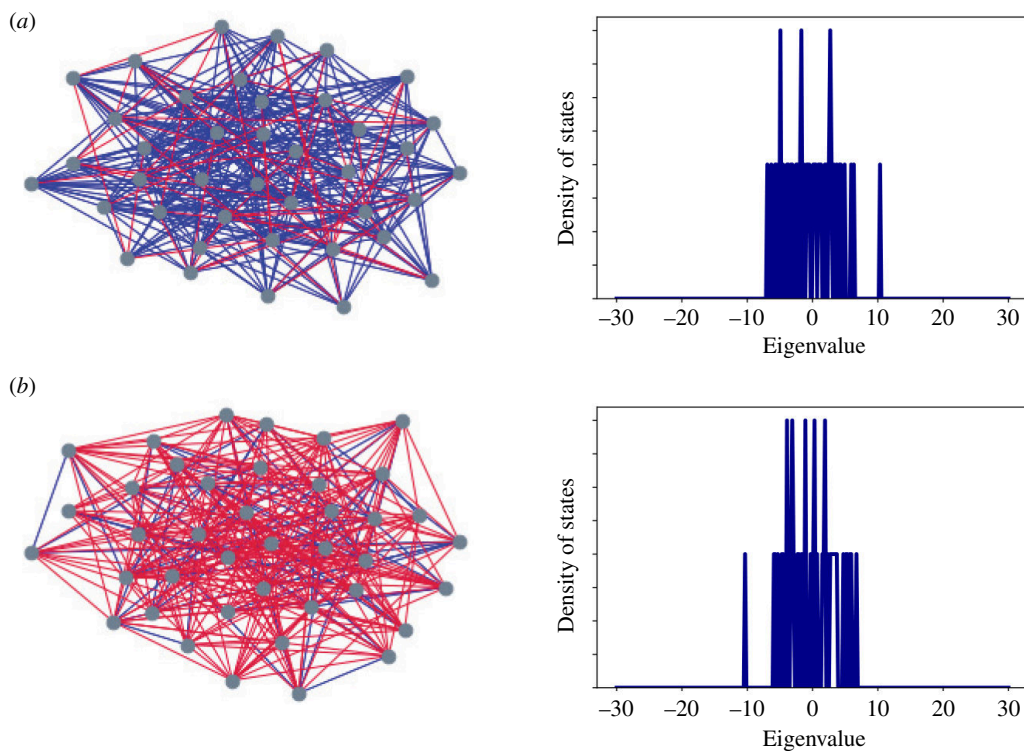


Figure 3. Examples of k -regular random graphs with $n = 40$, $k = 20$ and $m = 300$ with signed edges. (a) The probability that an edge is negative is $p_n = 0.2$. Here, the positive edges dominate, so the emergent state has the greatest eigenvalue in the spectrum. (b) The probability that an edge is negative is $p_n = 0.8$. Now, negative edges dominate, so the emergent state has the least (most negative) eigenvalue in the spectrum.

states, the network needs to be activated (i.e. ‘on’) to function as a network and to be able to produce superpositions by interacting with other networks of oscillating dipoles.

- (4) Unitary operations on graphs and subgraphs should be defined.
- (5) We need a construction that couples graphs or subgraphs in such a way that allows production of a state space homomorphic to superpositions of product states.

(b) The quantum-like bit

Khrennikov *et al.* initiated a program for identifying how classical systems could produce probabilistic outcomes like those peculiar to quantum mechanical systems. Progress has largely concerned how probability laws are modified, and how probability is measured, as we review in a later section. In the course of many examples, they refer to suitable QL states. In one paper [48], a specific example is proposed for a single neuron. Here, we take a different approach by exploring the possibility that QL states can be composed from interacting networks. A main motivation is that these states can be surprisingly robust to decoherence—whereas typical quantum states are likely to decohere quickly in disordered environments, which is the main objection to the idea that quantum phenomena play any functional role in biological systems. It is worth reiterating, too, that these states are not strictly quantum mechanical states; they are QL states.

Building blocks proposed for QL states are networks, possibly very complex, that we define using graphs. Networks have been extensively studied [49] and can have the properties of

Axioms 1 and 2, as discussed earlier in this paper. Networks are known to provide insights into social networks, neural networks, biochemical and biological networks, communications systems and so on. They can be functionally sophisticated and encode a lot of information. Networks can evolve through feedback to shift their balance and condense consensus. In short, these building blocks are rich and powerful structures, although in the present work, we solely consider their emergent eigenstates.

The emergent states of networks are classical. They arise from nonlinear interactions among the vertices so that synchronization—the key characteristic of the emergent state—arises from feedback [11,12]. This produces a stable eigenstate of the network of oscillators, which for n oscillators is the state $(1, 1, 1, \dots)/\sqrt{n}$. Our task is to develop a way to build from these classical states new states that have properties analogous key characteristics of quantum states. Specifically, we first need to define a QL bit $|A\rangle$ that ‘lives in’ a Hilbert space $\mathcal{H}_A$.

It is proposed that QL states can be built from QL bits defined by partitioning an edge-disordered k -regular random graph G into blocks, giving the subgraphs G_{a_1} and G_{a_2} , and the tensor sum structure for the graph’s adjacency matrix, $A = A_{a_1} \oplus A_{a_2}$. We need to ensure that there is a lower probability of edges interconnecting the subgraphs than within each subgraph (for reasons discussed below). Thus, we should delete some fraction of the interconnecting edges (those spanning G_{a_1} and G_{a_2}). In the demonstrations below, we start from unconnected subgraphs (no edges between G_{a_1} and G_{a_2}), then randomly add edges with some probability.

The edges within subgraphs, or those edges connecting subgraphs, can be positive or negative—that is these are signed graphs. The physical interpretation of signed graphs is discussed below. The QL bit thus defined has a bipartite construction with respect to the vertex sets of the constituent pair of subgraphs G_{a_1} and G_{a_2} (but is not a bipartite graph). It has intrinsic k -regular random structure, ensuring that the emergent states are robust.

Some examples of QL bits are shown in figure 4a,b. Here, each subgraph is on 30 vertices, $k = 20$ and the probability of an edge between any two vertices within a subgraph is approximately 0.23. The two subgraphs, G_{a_1} and G_{a_2} are shown coloured blue and green, respectively. Coupling edges, that connect the subgraphs, are drawn in red. The left-hand drawing shows the graph and all vertices, whereas the middle drawing shows the separated subgraphs, including all their edges, and the coupling edges are drawn schematically. In figure 4a, the probability of adding a coupling edge between any two vertices, one in G_{a_1} and the other in G_{a_2} , is 0.01. That probability is 0.1 in figure 4b. In these examples, the edges within the subgraphs are all positive, whereas the coupling edges are negative. Note also that the value of the coupling edge (the coupling strength) is set to 1 because we are plotting the spectra of the adjacency matrices. This configuration makes the largest emergent eigenstate, shown in the spectra, $\frac{1}{\sqrt{2}}(a_1 - a_2)$ and the second largest $\frac{1}{\sqrt{2}}(a_1 + a_2)$, as indicated. Here, a_i means the emergent eigenstate of subgraph G_{a_i} . The order of these eigenstates can be reversed by changing the signs of the coupling edges.

The emergent superposition state can be tuned to a mixed state—that is where it could be a_1 or a_2 with classical probability—by reducing the number of coupling edges (or their coupling strength), so that the subgraphs are not synchronized and the eigenvalues of $\frac{1}{\sqrt{2}}(a_1 - a_2)$ and $\frac{1}{\sqrt{2}}(a_1 + a_2)$ converge. This is seen by comparing the graph spectra in figure 4a,b. Conversely, when the number of coupling edges is increased, the graph G becomes overall k -regular; it is no longer described as a pair of coupled k -regular random subgraphs. Then, the second largest eigenvalue converges into the density of random states (figure 4c). This is why we want to balance the density of edges within each of G_{a_1} and G_{a_2} relative to those coupling edges that connect the subgraphs.

To summarize, the keys to the proposal of these QL bits are:

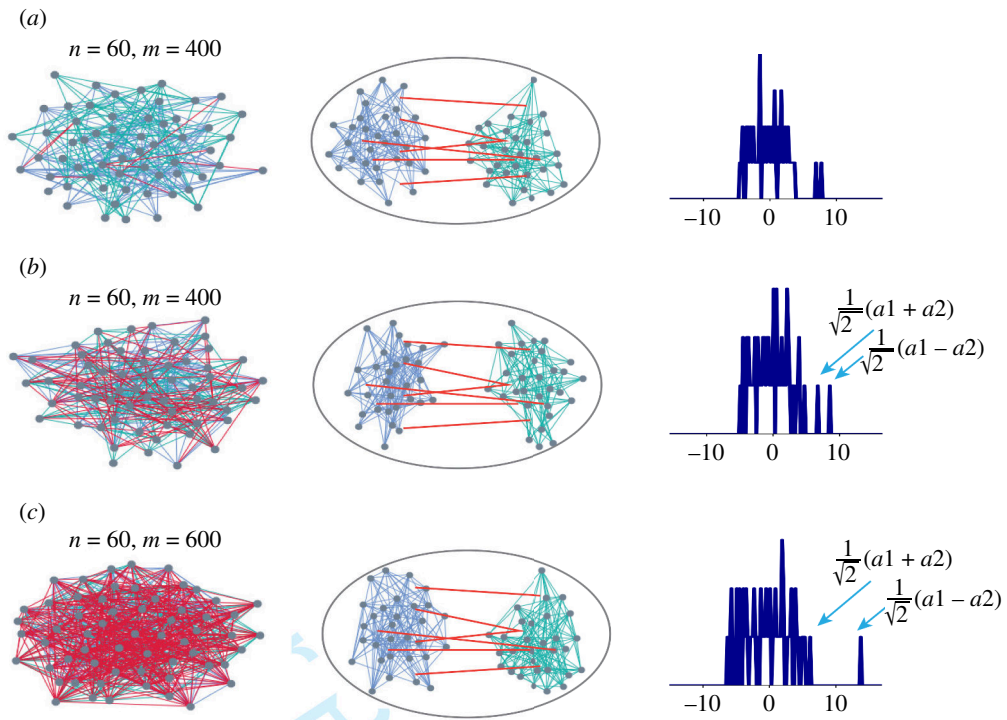


Figure 4. Examples of networks that produce QL states, as described in the text. The left diagrams show the network. The middle pictures shows the subgraphs G_{a_1} and G_{a_2} separated in space, and a schematic indications of the coupling edges between them. On the right, the spectra of the graphs are shown. The eigenstates are written as linear combinations of the eigenstates of the subgraphs, that is α_1 is the emergent eigenstate of G_{a_1} , and so on. In the graphs, the red edges are negative. Other edges are taken to be positive in these examples.

- (i) By coupling a pair of k -regular random subgraphs G_{a_1} and G_{a_2} , we can produce convex combinations of the eigenstates of the subgraphs. These states therefore serve as basis states that can be generated as superposition states.
- (ii) The subgraphs are k -regular random graphs, which ensures that the basis states and their superpositions are very robust for networks of arbitrary complexity and are identified clearly by their emergent eigenvalue.
- (iii) The k -regular random subgraph construction also ensures that, regardless of the number of nodes (vertices) in each subgraph, the emergent eigenvalues are found at the same frequencies, that is d , where d is the average degree of vertices in the subgraph. d is less than k when the graphs are disordered and have randomly missing edges [8].

How do we think about what these graphs represent physically? Our working model is that each vertex is an oscillator. The concept of an oscillator is very broad, but the idea is that the vertex can flip periodically between two states or phases. It might be the phase of a neuron firing, it might represent the ‘for or against’ an opinion given some information specific to a social sub-network, it might represent the phase of an electrical pulse in a circuit, or perhaps the concentration—above or below a threshold—of a biochemical marker. The coupling between vertices maps out interactions among these states. In this paper, we describe the networks within each QL bit as perfectly correlated (all the edges within each subgraph have the same sign). That is not necessary for these constructions, as suggested by figure 3. We simply need consensus within the network.

The QL bit can be synchronized to a superposition state of the constituent subgraphs, or it can display either subgraph probabilistically, thus satisfying Axiom 3. The physical concept of the graph is that it represents a network of oscillating dipoles, one positioned at each vertex. Edges indicate the dipole–dipole coupling between vertices. Negative edges denote coupling between oppositely phased dipoles.

Unitary operations, including QL gates, can be imagined for these networks as graph homomorphisms. In particular, we simply need operations that flip subgraph or coupling edge phases. We will explain this, with examples, in a later section. The QL bits thus satisfy Axiom 4.

Lastly, these states are presented as spectra and associated states of graphs, that is of the graphs' adjacency matrices. Therefore, the non-zero off-diagonal terms are all set to unity. In a realistic system, these off-diagonal terms have energy values reflecting the coupling and the diagonal entries can have frequency disorder [23]. Here, we do not need to examine various parameters because we are attempting to establish 'existence' of the prescribed QL states—are they feasible with the right kind of network? The key to the argument in favour is that the eigenvalue gap between the emergent state and the next state is fixed by the mean valency of the disordered k -regular graph, d . For large networks, then, we simply need to ensure that d is 'large enough' so that the QL-bit superposition state is emergent, that is, distinct in the spectrum.

(c) Quantum-like states

Two or more QL bits can be suitably coupled so that QL states are constructed. By imagining the QL bit as an oscillating dipole (or transition density [50,51]), we can propose how 'wiring diagrams' can lock the relative phases of subgraphs of each QL bit in lock-step across the multi-QL bit network. Then the coupling between two QL bits labelled A and B, with basis states a_1, a_2 and b_1, b_2 , respectively, can be thought of, conceptually, as a dipole–dipole coupling [51]. The coupling Hamiltonian is given by:

$$H = \hbar\nu_A |A\rangle\langle A| + \hbar\nu_B |B\rangle\langle B| + \sigma J(|A\rangle\langle B| + |B\rangle\langle A|), \quad (3.2)$$

where ν_A is the oscillation frequency of QL bit A, ν_B is the oscillation frequency of QL bit B and J is the coupling. We have $\sigma = \pm 1$ depending on whether the QL bit pairs are coupled so that the oscillations $a_1 \rightarrow a_2$ and $b_1 \rightarrow b_2$ are in-phase or anti-phased, respectively. In our calculations of the graph adjacency matrices, we have $\nu_A = \nu_B = 0$ and $J = 1$. The basis states are emergent states of the subgraphs, for example by a_1 , we mean the emergent state of subgraph G_{a_1} .

If we think of each QL bit conceptually as an oscillator in a superposition of states of its subgraphs $|x_1\rangle$ and $|x_2\rangle$, where $x \in \{a, b, c, \dots\}$, then we can envision how suitable connections among QL bits can produce a variety of synchronized states reminiscent of the set of states in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \dots$. To do this, we introduce signed edge sets that lock subgraphs of the QL bits in- or out-of-phase with each other, so that the entire graph is locked relative to a global phase.

What we need to establish is a representation of the basis of the relevant tensor product of Hilbert spaces as a pattern of phases (relative phases) of the eigenstates of network subgraphs in the emergent eigenstate of the entire network of coupled QL-bits. The product basis for quantum states has inbuilt correlations that we can represent as linear combinations in a free vector space that will serve as the QL representation, physically laying out the phase correlations across the network. In future work, we report a significant further development, enabling us to exhibit a one-to-one map from graphs to arbitrary tensor product states. The particular advance was to consider graph products. While the method proposed here enables us to produce superpositions of states within any excitation subspace, generating more complex states requires our more recent developments. The idea, for the present, is that a set of

appropriate linear combinations of QL-bit states (naturally produced by joining QL-bits by edges) allows us to construct a QL state.

Here, we only consider the Bell states in order to explain the concept (figure 5). Figure 5a shows how the superposition states are produced by coupling the QL bits. Any of these four states can be selected as the emergent state by suitable choice of signs of the couplings within each QL bit and between the QL bits. The case of positive-signed edges between the QL bits is shown here. The phases of the blocks of coefficients in the resulting eigenstates are indicated, with ordering a_1, a_2, b_1, b_2 . In figure 5b,c, we show schematics of how to construct networks that emulate two of the Bell states, $|\Psi_-\rangle$ and $|\Psi_+\rangle$. The pictures of the networks shows the idea of how the coupling between subgraphs controls their relative phases. The analogy of coupled dipoles shows the same concept and the phase map sketches indicate how the subgraph phases are phase-locked to each other. Calculated spectra are shown in the other panels (figure 5d–g). First, for reference, in figure 5d, we show the spectrum of $G_{a_1} \oplus G_{a_2} \oplus G_{b_1} \oplus G_{b_2}$, where none of the subgraphs are coupled. This is the perfectly mixed state. In figure 5e, the spectrum of the network where the subgraphs within each QL-bit are coupled (with random edges between subgraphs within each QL-bit added with probability $p = 0.2$), but the QL-bits are not coupled to each other. So, here we have QL-bits but no entangled states. The adjacency matrix is therefore represented as two blocks, $G_A \oplus G_B$.

In figure 5f,g, we show spectra where QL-bits are coupled, removing the block form of the adjacency matrix, to produce an emergent state corresponding to the Bell state $|\Psi_-\rangle$. The inter-QL-bit coupling edges are added with $p = 0.05$ and with signs as indicated in figure 5b. In figure 5g, we show the spectrum when all coupling edges are added with $p = 0.05$, that is both the intra- and inter-QL-bit edges. For each calculation, the composition of the emergent eigenstate is indicated.

We can characterize the resulting states according to the overall ‘excitation number’ of the $|x_2\rangle$ states, that is just like the excitation subspaces of $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \dots$. Synchronization locks the QL bit array into a particular global state (which can also be a mixed state, caused by insufficient synchronization). Just like in the coupled dipole analogy, when N QL bits are coupled, we can specify N states that map to superpositions of product states comprising one $|x_2\rangle$ state and $N - 1$ of the $|x_1\rangle$ states—the single excitation subspace. We have $\binom{N}{2}$ double excitation states, and so on. A frequency difference between $|x_2\rangle$ and $|x_1\rangle$ is allowed, but not essential, depending on how the states are detected.

The QL bits are networks that can be spatially separated from each other (like shown in figure 5), or they can be completely entwined. They can be different sizes (different numbers of nodes) and can be slightly detuned from one another (thus changing how the states mix). There are a rich variety of ways that the system can be composed. The important rule is that the expander subgraphs are approximately d -regular, so that the eigenvalues of each QL bit are close to resonance with the others.

4. Context and the Växjö model

A model for QL measurements has been developed to apply to a wide range of classical systems, the Växjö model [7]. These developments complement our definition of QL states, which give concrete examples of how to engineer or identify complex systems that yield QL probability laws.

The Växjö model does not require specification of systems and their states. Instead, it focuses on measurement and probability laws. Specifically, how probability laws might be formulated so that interference-like phenomena are accounted for. By appending our observables with extra structure that relates to conditions of the measurement and other information potentially gathered through prior measurements—context—Khrennikov shows how probabilistic laws

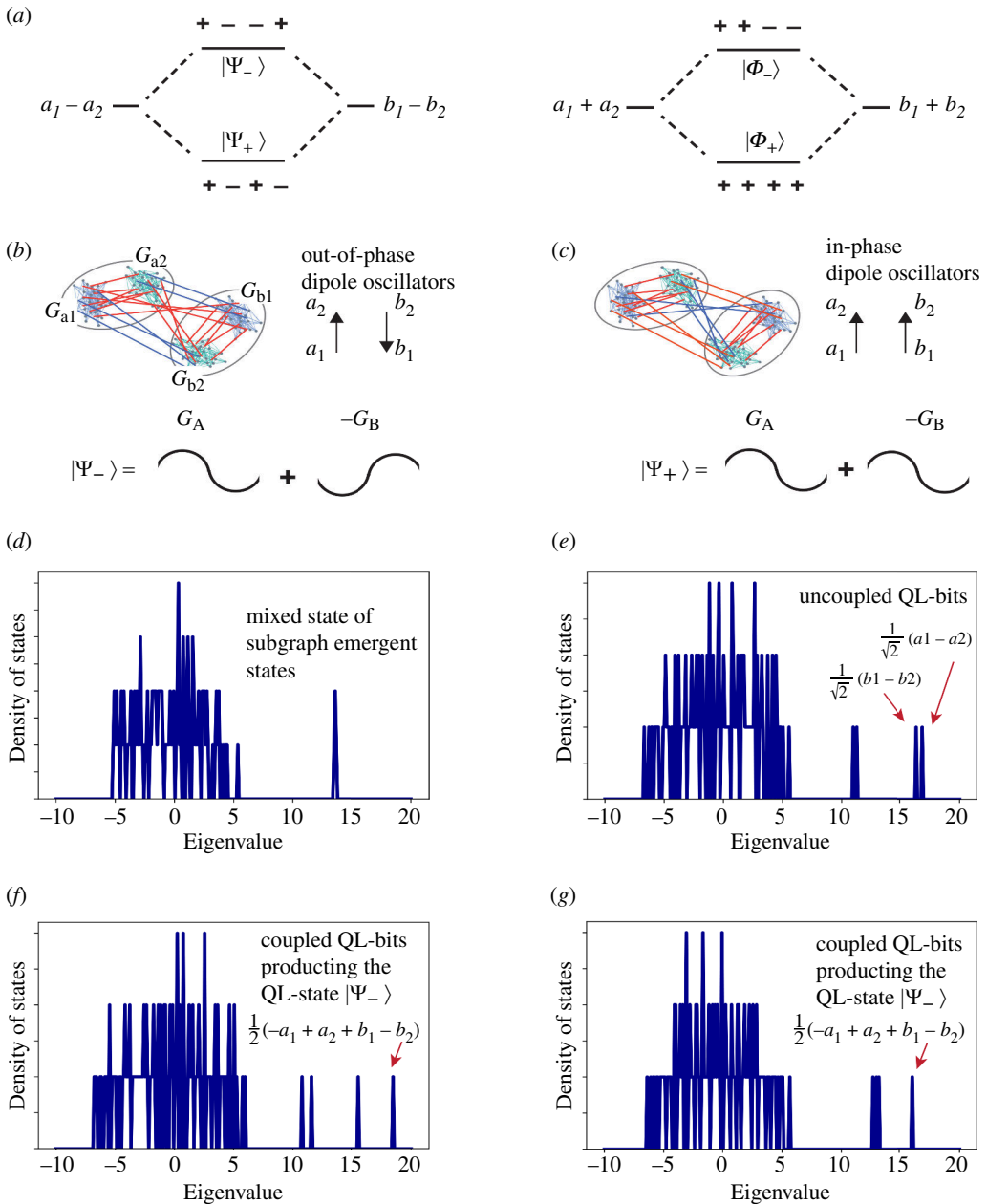


Figure 5. (a) The superposition states are produced by coupling the QL bits. Here, constructions of the four Bell states are shown. (b, c) Diagrams that show how the QL-bit is analogous to a pair of coupled oscillating dipoles. To change the state of the QL-bit from out-of-phase oscillations to in-phase oscillations, the signs of the edges need to be reversed, as indicated. Red edges are negative, blue edges are positive. The corresponding phase map sketches are also shown for reference. (d) The spectrum of $G_{a_1} \oplus G_{a_2} \oplus G_{b_1} \oplus G_{b_2}$ that is none of the subgraphs are coupled. (e) The spectrum of the direct sum of QL-bit graphs, that is the subgraphs within each QL-bit are coupled (with random edge added with probability $p = 0.2$), but the QL-bits are not coupled to each other. (f) The QL-bits are coupled to produce an emergent state corresponding to the Bell state $|\Psi_{-}\rangle$. The coupling edges are added with $p = 0.05$ and with signs as indicated in part (b). (g) The QL-bits are coupled to produce an emergent state corresponding to the Bell state $|\Psi_{-}\rangle$, like in part (e). The difference in this calculation all coupling edges are added with $p = 0.05$, see text.

can incorporate interference effects. He defines the probability for obtaining $a = \alpha$ by observing a under context C as:

$$p_C^a(\alpha) \equiv \mathbb{P}(a = \alpha \mid C), \quad (4.1)$$

where each contextual probability satisfies $p_C^a(\alpha) \geq 0$ and,

$$p_C^a(\alpha_1) + \dots + p_C^a(\alpha_n) + \dots = 1, \quad (4.2)$$

for any context C .

Definition 4.1. (Khrennikov) Contextual expectation $\mathbb{E}[a \mid C]$ of an observable $a \in \mathcal{O}$ with respect to context C is given by:

$$\bar{a}C = \mathbb{E}[a \mid C] = \alpha_1 p_C^a(\alpha_1) + \dots + \alpha_n p_C^a(\alpha_n) + \dots \quad (4.3)$$

Definition 4.2. (Växjö model) A contextual probability model is a contextual probability space $\mathcal{P} = (C, \mathcal{O}, \pi)$ such that C contains a special subfamily of contexts $\{C_\alpha^a\}_{a \in \mathcal{O}, \alpha \in X_a}$ which are interpreted as $[a = \alpha]$ -selection contexts. Context C_α^a corresponds to the selection with respect to the result $a = \alpha$. Moreover, it is assumed that C_α^a satisfies the condition:

$$\mathbb{P}(a = \alpha \mid C_\alpha^a) = 1. \quad (4.4)$$

It is assumed that, for each observable $a \in \mathcal{O}$ and its value α , the selection context C_α^a is uniquely determined in the class of contexts C of the model.

We also need to define what Khrennikov terms ‘transition probabilities’ using a conditional probability construction:

$$p_{\beta \mid \alpha} \equiv \mathbb{P}(b = \beta \mid a = \alpha) = \mathbb{P}(b = \beta \mid C_\alpha^a), \quad (4.5)$$

because QL probability laws will be evidenced by measurements on two observables, a and b . The reader is referred to Khrennikov [7] for more details.

The formula for total probability in the usual, Kolmogorov, model (i.e. without the structure of contexts) is:

$$p^b(\beta) = \sum_{\alpha} p^a(\alpha) p_{\beta \mid \alpha}. \quad (4.6)$$

But, as Khrennikov argues, this formula is not guaranteed to hold in the Växjö model, where probability distributions in the set of data for a, b, C may not be described by a single probability space. For instance, the context for observation a might vary depending on whether we previously learned something about b , through observation (or *vice versa*). That concept is exploited by the ‘question order effect’, discussed below.

The deviation of the contextual total probability from the Kolmogorov law serves as a kind of interference term, $\delta(\beta \mid a, C)$:

$$\mathbb{P}(b = \beta \mid C) \equiv p_C^b(\beta) = \sum_{\alpha} p_C^a(\alpha) p_{\beta \mid \alpha} + \delta(\beta \mid a, C). \quad (4.7)$$

The interference term is the key to connection with quantum systems, where interference of probability amplitudes is a defining feature. The interference term identified here is the classical counterpart to conditional expectation in quantum theory [52].

The QL states of classical synchronized systems proposed in the previous section can represent states encoded with context. That context essentially means that the QL bit networks or collective QL states respond and change as information is gathered by other QL bits—that

is we make measurements. This could be accomplished in various ways by connecting QL bits and/or adding additional ‘context’ networks.

5. Examples of quantum-like function

Here, we provide two simple, but very different, examples of how QL bits might serve as the basis for information processing. The first example integrates the QL bits into Khrennikov’s concept of the question order effect, that exploits interference effects that come from context, reflecting the order that information is gathered. It indicates how to think about measurement of QL states. The second example is a textbook protocol that exploits entanglement. It provides a reference point for assessing QL state functions.

It is important to note that quantum states and QL-states do not necessarily enable functions that are impossible to achieve by classical processes, but the different mechanisms that they allow to achieve the function could have advantages. The examples below illustrate different mechanisms of quantum functions.

(a) The question-order effect

Khrennikov *et al.* have studied how QL measurement theory can be useful for explaining psychological measurements [53–55]. That work provides an example of how the action of measurement matters when we are dealing with QL states. A prototypical example is the ‘question-order effect’. The question-order effect refers to the statistical observation that the answers to questions, for example in a survey, can depend on the order in which the questions are presented. Here, we describe the QL measurement model for the question-order effect qualitatively, using figure 6. We refer the reader to the paper by Ozawa and Khrennikov [54] for the details.

The system comprises a pair of ‘entangled’ QL states. These projections A and B on the Hilbert space of the system, $\mathcal{H}$, correspond to questions labelled A and B , respectively. For each question, we can think about eigenvalues corresponding to answers of yes or no. To measure the quantum state of our system, for example to measure the answer to question A , we need to interact the system with a measurement probe. This probe also comprises an entangled pair of QL states with projections on the Hilbert space $\mathcal{K}$. During measurement of A , for example the composite system evolves according to a unitary operator U_A acting on $\mathcal{H} \otimes \mathcal{K}$. The interaction encodes the probe with the eigenvalue of A that can be read out using a meter observable. The interaction concomitantly changes the state of the system, so that a subsequent measurement on B can give a different answer than if this measurement were carried out before the measurement on B . This interplay between system and measurement can model the question-order effect.

The example shows how an appropriate ‘read-out’ of states is important for the kinds of sophisticated functions enabled by networks, in particular when a sequence of evolving answers or iterations may be needed. A good example is interactions among the states to reach ‘consensus’.

(b) Superdense coding for sensing or processing

Superdense coding is a strategy whereby a quantum resource (entanglement) can mediate communication of two classical bits of information, potentially over a long distance. In an illustrative example, we send two classical bits (00, 10, 01 or 11) using a single qubit [1].

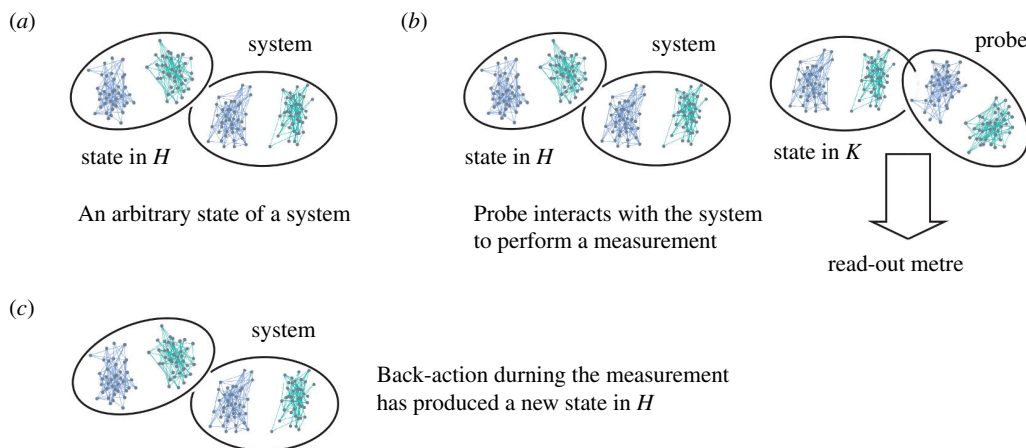


Figure 6. Diagram that shows the way networks interact for the steps needed to understand the question order effect described in the text.

Two QL bits can be ‘entangled’, that is synchronized, to produce a reference Bell-like state. Now, apply a quantum gate to QL-bit A in isolation. In practice, this means we switch phases appropriately of the subgraphs in the QL network A. This can be understood intuitively using the analogy where each QL bit comprises a pair of coupled oscillating dipoles. In practice, we simply need to flip the signs of the edges within subgraphs and/or of the coupling edges. The details for each quantum gate are shown in figure 7. Here, we use the usual notation [4] for the states and name QL-bit A Alice and QL-bit B Bob. We can associate $|0\rangle_A$ with $|a_1\rangle$ and $|1\rangle_A$ with $|a_2\rangle$, etc.

Bell-like states correspond to the four unique ways the two pairs of oscillating dipoles are locked in sync (figure 7). Relative to the fixed phases of QL bit B, then, we can rotate the phases in QL bit A. The information sent to QL bit B amounts to a single QL bit. However, a read-out of the total Bell-like state of the A–B QL-bit pair translates into two classical bits of information. Two qubits are needed to encode two classical bits. Therefore, we have the same total information supplied by four Bell states as four classical pair states. However, by making use of entanglement, QL bit A can communicate a pair of classical bits by sending only a single qubit. Notice that Alice’s QL-bit has four possible states, but pairs of them differ only by an overall phase, so they do not convey different information in isolation. It is only with the context of the QL state that the four QL-bit states translate to four different interpretations. The key is that context is provided by referencing phase mutually among the QL-bits.

The total Bell-like state of the A–B QL-bit pair could be detected by its spectrum. It is not obvious that the scheme would be useful for network communication when the entangled network states are close together, because we cannot exploit ‘spooky action at a distance’. However, the protocol might allow QL-bit A to serve as a quantum sensor, with read-out facilitated by QL-bit B. The proof of principle may also inspire ways that QL circuits could be constructed for more sophisticated processing operations.

A main conclusion from this example is that it is feasible to implement various QL gates on the network states. For instance, a Hadamard gate that ‘rotates’ $|a_1\rangle$ to the superposition $(|a_1\rangle + |a_2\rangle)/\sqrt{2}$ could be achieved by activating the coupling edges between subgraphs and activating the network corresponding to G_{a_2} . Then, the state with the frequency of the emergent state—recall that it is fixed by the valency of the k -regular graphs—becomes the required superposition. These gates are quantum *like* because the states and gates are composed by dynamic structures of the network, not by matrix operations. Nevertheless, the set of ways the

Gate	Alice's QL-bit	Phase map of QL State		State Bob sees	
		G_A (Alice)	coupling sign	G_B (Bob)	
Identity gate	$ \phi_A\rangle = 0\rangle_A + 1\rangle_A$		+		$ \phi_+\rangle = \frac{1}{\sqrt{2}} [0\rangle_A 0\rangle_B + 1\rangle_A 1\rangle_B]$
not gate	$ \phi_A\rangle = 1\rangle_A + 0\rangle_A$		+		$ \Psi_+\rangle = \frac{1}{\sqrt{2}} [0\rangle_A 1\rangle_B + 1\rangle_A 0\rangle_B]$
phase-flip gate	$ \phi_A\rangle = 0\rangle_A - 1\rangle_A$		-		$ \phi_+\rangle = \frac{1}{\sqrt{2}} [0\rangle_A 0\rangle_B - 1\rangle_A 1\rangle_B]$
not and phase-flip gates	$ \phi_A\rangle = 1\rangle_A - 0\rangle_A$		-		$ \Psi_-\rangle = \frac{1}{\sqrt{2}} [0\rangle_A 1\rangle_B - 1\rangle_A 0\rangle_B]$

Figure 7. A table that shows how Alice rotates the state of her QL-bit by a series of quantum gates and how that changes the QL state observed by Bob. The operations within the networks that produce the QL states are indicated schematically as a ‘phase map’.

network can be reconfigured offers a representation for the set of matrix operations that, in turn, represent maps on states in the relevant Hilbert space.

6. Discussion

In this study, we have asked: Is it worth looking for quantum effects in very complex, ‘messy’ systems? The standard answer is that quantum states are too fragile to exist in such systems, including biological settings. However, given the many relevant new ideas that have emerged over the past decades—including, for instance, more sophisticated understandings of networks, synchronization phenomena, and advances in solving quantum dynamics for large and complicated systems—we argue that the time is right to revisit this question. Furthermore, perhaps we do not need true ‘quantum states’, but instead sufficiently QL states will suffice. That inspires a shift from the common idea that we need to make chemical systems pristine in order to host quantum states in the form of qubits, to the idea that there are new opportunities to discover by identifying more abstract, but surprisingly robust states. These states can even be realized in electronic circuits. The main objective of this work is to propose concrete examples of very large and complex networks that host states resilient to disorder and displaying QL properties.

By considering suitably constructed graphs with expander properties, it turns out that we can find robust QL states that can inhabit huge, complex systems and that satisfy the axioms proposed in §3(a). A requirement for states of any large and complex quantum system, whose structure is, in a sense, ‘organic’, is that they must be especially stable with respect to various kinds of disorder. This is where the concept of emergent or collective phenomena is important because it allows the state of interest to be distinguished from the many other states in the spectrum. This emergence property thus protects the state. To abstract a suitable model, we need to account for an underlying structure that could be encoded by some kind of rule

(here, the rule is that the graph is k -regular). Then, we allow disorder in that structure or on the structure so that there is a large margin for flexibility in the precise structure. Frequency disorder at sites in a network is an example of disorder *on* the structure [23]. Disorder caused by randomly deleting edges from the network is an example of disorder *in* the structure [8]. We explicitly included this latter kind of disorder in the examples exhibited in the present work.

We identified k -regular graphs as examples of networks that can serve as the basis for QL states. However, any expander graph would serve this purpose. The best possible expanders are Ramanujan graphs [56–58], but ideal expansion properties do not appear to be crucial because in many interesting or useful ‘real-world’ scenarios, it is desirable for the graphs to be disordered—so that precise construction is not needed. The expander graphs also have intrinsically interesting properties that are desirable for networks involved in communication and information transfer and processing [38,59,60]. Expanders are not necessary for the constructions, just sufficient. Other graphs might be of interest, including chiral graphs that could host states analogous to electron spins or photon polarization.

By positing a QL bit structure based on coupled expander graphs and combining that with the concept of QL states, here an ‘existence proposal’ for huge quantum states was described. The work opens up many questions for future work. Below, we discuss two open questions: (i) Is there a ‘QL’ advantage for function? (ii) Could QL states and function be used by neural networks in the brain?

(a) Is there a QL advantage?

Synchronized networks of QL states could serve as a hardware platform for information processing. There are three big questions: (i) what is the QL advantage for certain functions? (ii) what platforms could be designed to produce and test QL states? and (iii) what experiments could test the QL functions of QL states?

The main point argued in the present work is that we defined construction of a QL bit $|A\rangle$ so that its emergent states ‘live in’ a Hilbert space $\mathcal{H}_A$, just like quantum states. A set of axioms is proposed that guide the properties the states need to have. We exhibited examples of these states and the way they are measured. It was also shown how QL gates can be implemented by controlling phases—specifically, signs of edges within each QL bit. Based on this framework, one can in principle adapt circuit designs from quantum processing systems. This is a key direction for future work that will provide answers to how QL states can have a QL advantage for function. However, two elements of the program need consideration. First, what are appropriate ways to scale the number of QL bits? Here, we proposed a simple mapping as a starting point, but in further work, we will report a one-to-one mapping between graph constructions and arbitrary tensor product states. Second, we need to evaluate quantitatively how the classical resources are required to make QL states scale.

Compelling demonstration of QL state function requires a hardware platform for producing those states. The obvious approach is to design circuits of coupled electronic oscillators [61]. These kinds of systems could be miniaturized, and may be well suited for tasks like pattern recognition [62]. QL states might also be implemented in the framework of spiking neural networks, thereby taking advantage of neuromorphic hardware systems [63]. It would also be interesting to explore connections with and applications using random circuit sampling [64].

Later work can adapt the concepts for more complex networks in optical, biological, chemical and other systems. Biological systems are certainly an intriguing target because they comprise a huge number and diversity of networks. Many times, these networks are so dense and complicated that they are referred to as ‘hairballs’ [65]. The QL-states proposed here provide a platform for rethinking quantum biology because it is much more likely to find QL-states than quantum states in such complex systems.

It is well-known that neurons are connected to form the complex networks of the brain. It is now generally accepted that the ways these networks process signals and information to produce responses and functions do not simply involve signal transmission. Rather, information can be encoded in the timing and spatial coherence of signal oscillations [22,66–72]. We are not attempting to review that field here, but simply to sketch a small part of what is known to give examples of how networks function in what is arguably nature's most sophisticated machine—the brain.

One key discovery is that information processing seems to be facilitated by oscillations of neuron activity. There are characteristic frequencies for these oscillations, grouped into bands denoted delta (<4 Hz), theta (4–8 Hz), alpha (8–12 Hz) and beta (12–30 Hz), in the low-frequency range, to the high frequencies in the gamma band (30–80 Hz) and above.

One example is shown in figure 8. Here, gamma band oscillations are studied in the visual cortex of a monkey as it was presented with an image [73]. In the experiment, neuronal activity was detected using an electrocorticographic grid that monitored the left brain hemisphere of a macaque monkey. A number of prior studies had shown that neuronal gamma-band synchronization in the visual cortex is associated with various kinds of functions. This study suggested that 50–80 Hz gamma-band activity plays a key role in the process of viewing. Figure 8 shows the main result of the paper, a unipolar local field potential trace taken as the monkey looks at two oranges.

A second example serves to illustrate that brain function involves the interaction of widely distributed cortical regions. Long-range synchronization has been implicated in enabling the coordination of these regions across the brain and to communicate between brain regions [69]. For example, how do we map sound to meaning [68]? In Hipp *et al.* [74], the authors study the perception process. They used an analysis of EEG recordings in humans that allowed correlation of brain oscillations to be mapped. The experiment involved the subject watching a screen where two bars come together, then move apart. There is an audio cue when the bars meet. The measured power response during the stimulus is collected in maps, depicting the theta (4–8 Hz), alpha (8–16 Hz), beta (16–32 Hz), low gamma (32–64 Hz) and high gamma (64–128 Hz) band activity across the brain. The main finding of the work was enhanced long-range beta-synchrony during stimulus processing, which was accompanied by local suppression of beta-band activity.

There is an immense body of work exploring brain function. A key question is how different sensory inputs are linked to appropriate processing circuits to draw conclusions. It has been proposed that 'binding' of these units is facilitated by phase synchronization [66]. It has also been proposed that the oscillation frequencies allow these disparate computations to associate [70]. Further ideas hypothesize that a complex interplay of oscillations is important for perception [72].

In sum, on one hand, the brain functions using a complex network of networks that become synchronized during function. Here is where there is compelling evidence that coherent oscillations on complex networks are functionally operative in the brain, so a connection with the ideas hypothesized in the present paper—that QL function is possible—is feasible. However, it remains an enticing idea that requires further work. In particular, the way the dynamic networks interplay in the brain, and even how coherence is exploited are questions that have not yet been elucidated in detail. The present work at least provides a framework for devising testable hypotheses for additional, likely surprising, mechanisms that might be at play in the sophisticated networks of the brain. Other workers have investigated the idea that true quantum phenomena are at play in the brain. We leave this topic for discussion in future work, but a helpful recent summary can be found here [75].

In prior work, Khrennikov [76] has asked whether we can define the concept of a geometry for the mind, a 'mental space', that captures the hierarchical structure of information and

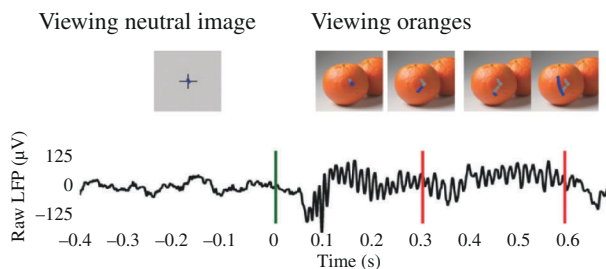


Figure 8. Gamma band oscillations observed in the visual cortex of a monkey as it was presented with an image. The monkey's focus is tracked by the line on the image. Reproduced with permission from monkey1.

connections. The graphs we have described in the present work may aid this quest. Treelike structures have been associated with information processing in the brain and other biological systems, but notably, the graphs proposed in the present work are not explicitly trees. However, our graphs do contain a great many trees as implicit subgraphs, and it might be worth exploring the roles of such embedded structure in future work.

7. Conclusion

We proposed a QL bit structure based on coupled expander graphs. Combining that with the concept of QL states, we suggest a way that huge, complex systems can host states that can process information in a QL fashion.

Specifically, it was suggested that building blocks suited for QL states are networks, possibly very complex, that we defined using graphs. These networks can be functionally sophisticated and encode a lot of information that is distilled into the emergent states we can exploit for QL like processing. The emergent states of networks are classical. They arise from feedback. We developed a way to build from these classical states, new states that have properties analogous to quantum states. Specifically, we based this construction on a QL bit that 'lives in' a Hilbert space $\mathcal{H}_A$. We demonstrate a candidate QL bit by partitioning an edge-disordered k -regular random graph G_A into two blocks, giving the subgraphs G_{a_1} and G_{a_2} , then connect those blocks to yield superposition states of the emergent states of each subgraph. The intrinsic k -regular random structure ensures that the emergent states are robust.

The work opens up many questions for future work. We explicitly discussed two major open questions: (i) Is there a 'QL' advantage for function? Perhaps this could be leveraged in hardware for QL-computing, which would be greatly simplified from that needed for true quantum computing, but may offer an attractive compromise between classical and quantum architectures. (ii) Could QL states and function be used by neural networks in the brain? We conclude that QL states can exist in arbitrarily complex systems and that it is worthwhile investigating further examples and working out where they can have a functional advantage. The primary advance of the paper was to overcome the main objection for various proposals of quantum states in biology, the brain and other complex systems—that is that quantum states are too fragile to serve a functional role. This was done by proposing, first that we should think about 'quantum like' states [7], not quantum states. Second, that we can devise a blueprint for suitable states, satisfying the axioms proposed above, by exploiting expander properties of graphs so that complex networks yield a single key state in their spectrum.

Data accessibility. The data are all calculated and can be reproduced using the methods described in the paper.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. G.D.S.: conceptualization, formal analysis, funding acquisition, methodology, writing—original draft, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. I declare I have no competing interests.

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