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Random anti-commuting Hermitian matrices

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We consider pairs of anti-commuting $2p$-by-$2p$ Hermitian matrices that are chosen randomly with respect to a Gaussian measure. Generically such a pair decomposes into the direct sum of 2-by-2 blocks on which the first matrix has eigenvalues $\pm x_j$ and the second has eigenvalues $\pm y_j$. We call $\{(x_j, y_j)\}$ the skew spectrum of the pair. We derive a formula for the probability density of the skew spectrum, and show that the elements are repelling.

Keywords: Random matrix tuples; anti-commuting matrices.

1. Introduction

The study of random matrices goes back at least to the 1920's, but it came to prominence in physics with the work of of Wigner [13–15] and Dyson [5–7], who used results from random matrices to predict the eigenvalues of complicated Hamiltonians. See e.g. [10] for an account. What happens if we choose multiple Hamiltonians whose interaction forces them to satisfy certain algebraic relations? In [9], this question was studied when the Hamiltonians commute (see Sec. 2). The purpose of this note is to study the eigenvalue distribution of random pairs of anti-commuting Hermitian matrices.

First, let us define what we mean by a random d-tuple of matrices satisfying given algebraic relations. Let M_n denote the algebra of n-by-n complex matrices, and let Σ_n

denote the Hermitian matrices in M_n. Let $\mathrm{V}_n \subseteq \mathbb{M}_n^d$ be an algebraic set, by which we mean there are non-commutative polynomials $p_1,\dots,p_N$ in the $2d$ variables $\{x^1,(x^1)^*,\dots,x^d,(x^d)^*\}$ so that

$$\mathrm{V}_n = \{X \in \mathbb{M}_n^d : p_j(X) = 0 \ \forall\, 1 \le j \le N\}. \tag{1.1}$$

The set V_n can be thought of as a subset of $\mathbb{C}^{dn^2} = \mathbb{R}^{2dn^2}$, and as such there is a natural measure on it, consisting of Hausdorff measure of the real dimension of V_n. We will write this measure as dX. To convert this infinite measure to a probability measure, we multiply by something like a Gaussian weight.

For $X = (X^1,\dots,X^d)$ in $\mathbb{M}_n^d$ define its Frobenius (or Hilbert–Schmidt) norm by

$$\|X\|_F^2 := \sum_{r=1}^{d} \sum_{i,j=1}^{n} |X_{ij}^r|^2 .$$

Let w be a continuous non-negative function on $[0,\infty)$, which has enough moments that $w(\|X\|_F)dX$ is a finite measure on V_n. We will assume w is normalized so $w(\|X\|_F)dX$ is a probability measure.

Definition 1.1. *A random d-tuple in* V_n *is a random variable with values in* V_n *and with distribution* $w(\|X\|_F)dX$.

In particular, in this note we will study the set

$$\mathrm{A}_n := \{(X,Y) \in \Sigma_n^2 : XY + YX = 0\}.$$

We shall assume that $n = 2p$ is even. In Sec. 3, we will see that A_n has dimension $n^2 + \frac{n}{2}$, and that generically elements of A_n are unitarily equivalent to a pair of the form

$$X = \begin{pmatrix} \begin{matrix} x_1 & 0 \\ 0 & -x_1 \end{matrix} & & 0 \\ & \begin{matrix} x_2 & 0 \\ 0 & x_2 \end{matrix} & \\ 0 & & \ddots \end{pmatrix}, \quad Y = \begin{pmatrix} \begin{matrix} 0 & y_1 \\ y_1 & 0 \end{matrix} & & 0 \\ & \begin{matrix} 0 & y_2 \\ y_2 & 0 \end{matrix} & \\ 0 & & \ddots \end{pmatrix}, \tag{1.2}$$

where each x_j and y_j is positive. We shall call the pairs $\{(x_j,y_j) : 1 \le j \le p\}$ the *skew spectrum* of (X,Y).

Here is our main result.

Theorem 1.2. *Let $Z = (X,Y)$ be chosen randomly in A_n with distribution $w(\|Z\|_F)$. Then the probability distribution of the skew spectrum of (X,Y) on $\mathbb{R}_+^{2p}$ is given by*

$$\rho_n(x_1, y_1, \dots, x_p, y_p)$$
$$= C_n w(\|Z|_F^2) \prod_{1\le k\le p} x_k y_k \sqrt{x_k^2 + y_k^2} \prod_{1\le i<j\le p} [(x_i - x_j)^2 + (y_i - y_j)^2]$$
$$\times [(x_i + x_j)^2 + (y_i - y_j)^2][(x_i - x_j)^2 + (y_i + y_j)^2][(x_i + x_j)^2 + (y_i + y_j)^2]. \tag{1.3}$$

If x_i, x_j, y_i, y_j are bounded and bounded away from zero, the last factor in (1.3) is bounded above and below by

$$[(x_i - x_j)^2 + (y_i - y_j)^2]$$

(see Lemma 4.2). This quadratic vanishing is of the same order as in the Ginibre formula for commuting Hermitian pairs, showing that the repulsion between the elements of the skew spectrum is similar to, though more complicated than, the repulsion between the joint eigenvalues for a commuting Hermitian pair.

2. Random Commuting Matrices

In this section, we give some results about random commuting Hermitian matrices. We shall not use these explicitly in the following sections, but they serve as a guide to what we would like to achieve in the anti-commuting case. When $d = 1$, Ginibre [8] proved that for the Gaussian Hermitian ensemble, the eigenvalues of a random Hermitian matrix in Σ_n have the distribution

$$\rho(\lambda_1, \dots, \lambda_n) = C_n e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j^2} \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^2 \tag{2.1}$$

on R^n. We use C_n to denote a constant that depends on n, and may vary from one occurrence to another. An analogous formula to (2.1) turns out to hold not just in the Gaussian case, but if the matrices are chosen with respect to any weight that depends only on $\|X\|_F$ — see e.g. [12] for an account. Wigner proved in [15], subject to all moments having bounds independent of n, that if X_n is chosen in Σ_n with distribution $w(X)$ that depends only on $\|X\|_F$, then the density of eigenvalues of $\frac{1}{\sqrt{n}} X_n$ converges almost surely to the semi-circular distribution

$$\frac{1}{2\pi}\sqrt{4 - x^2}\,dx$$

on $[-2,2]$.

Now let $d > 1$, and let C^d_n denote the set of commuting d-tuples in Σ^d_n. Let $w(X)$ be a weight on C^d_n that depends only on $\|X\|_F$ and is normalized to have

$\int_{\mathfrak{C}_n^d} w(X)dX$. An eigenvalue of X is now a d-tuple in $\mathbb{R}^d$ (since the matrices commute, they have common eigenvectors). If $(\lambda_1,\ldots,\lambda_n)$ are the eigenvalues of X, then

$$\sum_{j=1}^{n} |\lambda_j|^2 = \|X\|_F^2,$$

where $|\lambda| = \sqrt{\sum_{r=1}^{d} |\lambda^r|^2}$ is the Euclidean norm in $\mathbb{R}^d$. Therefore, there is a function $\tilde{w} : (\mathbb{R}^d)^n \to \mathbb{R}$ so that

$$w(X) = \tilde{w}(\lambda).$$

In [9] it was shown that the Ginibre formula still holds.

Theorem 2.1. *For X in $\mathbb{C}^d_n$ with distribution w as above, the eigenvalues of X have density*

$$\kappa_n(\lambda_1,\ldots,\lambda_n) = C_n\, \tilde{w}(\lambda) \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^2. \tag{2.2}$$

Any X in $\mathfrak{C}_n^d$ is unitarily equivalent to a d-tuple of diagonal matrices. The unitary implementing this is generically unique up to multiplication by a diagonal unitary. Let $\mathrm{U}(n)$ denote the unitary group in M_n, and let T^n be the subgroup of diagonal unitaries. Let ν be volume measure on the homogeneous space $\mathrm{U}(n)/\mathrm{T}^n$. Then (2.2) asserts that the measure $w(X)dX$ decomposes as

$$w(X)dX = C_n\, \tilde{w}(\lambda) \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^2\; d\lambda d\nu.$$

Let us now restrict to the Gaussian case $w(X) = C_n e^{-\gamma\|X\|_F^2}$. The equilibrium measure with respect to the logarithmic potential is the probability measure μ_d that minimizes the logarithmic energy

$$I(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \log\frac{1}{|x-y|} d\mu(x)d\mu(y) + \int_{\mathbb{R}^d} \gamma|x|^2 d\mu(x). \tag{2.3}$$

The equilibrium measure exists, is unique, and is compactly supported [1, Theorem 4.4.14]. The eigenvalue density, scaled by $\frac{1}{\sqrt{n}}$, converges to the equilibrium measure. We shall let $\mathbb{E}_n$ denote expectation at the nth level of the process.

Theorem 2.2 ([9]). *Let X_n be chosen in $\mathbb{C}^d_n$ with distribution $C_n e^{-\gamma\|X\|_F^2} dX$. Let φ be a continuous bounded function on $\mathbb{R}^d$. Then*

$$\lim_{n\to\infty} \mathbb{E}_n\left[\frac{1}{n}\mathrm{tr}\left(\phi\left(\frac{1}{\sqrt{n}}X_n\right)\right)\right] = \int_{\mathbb{R}^d} \phi(x)d\mu_d(x).$$

In this Gaussian case, the equilibrium measures have been calculated explicitly by Chafai, Saff and Womersley [2, 3]. We let $\sigma_{R_d}^{d-1}$ denote normalized surface area on the sphere of radius R_d in $\mathbb{R}^d$, and use 1_A to denote the indicator function of a set A.

Theorem 2.3 (Chafai, Saff, Womersley). *The equilibrium measure that minimizes* (2.3) *is supported on the ball of radius R_d, and is given by*

$$\frac{2}{\pi R_1^2}\sqrt{(R_1^2 - x^2)_+}dx, \qquad R_1 := \sqrt{\frac{2}{\gamma}}, \quad d = 1;$$

$$\frac{1}{\pi R_2^2} 1_{|x|<R_2} dx^1 dx^2, \qquad R_2 := \frac{1}{\sqrt{\gamma}}, \quad d = 2;$$

$$\frac{1}{\pi^2 R_3^2}\frac{1}{\sqrt{R_3^2 - |x|^2}} 1_{|x|<R_3} dr d\sigma_1^2, \qquad R_3 := \sqrt{\frac{2}{3\gamma}}, \quad d = 3;$$

$$\sigma_{R_d}^{d-1}, \qquad R_d := \frac{1}{\sqrt{2\gamma}}, \quad d \geq 4.$$

3. Generic Elements in A_n

MLet B_n denote $\{(X,Y) \in M_n^2 : XY + YX = 0\}$. The set of commuting pairs in $_n$ is an irreducible variety [11], but B_n is not. In [4, Proposition 4.10] Chen and Wang showed that for each triple (q,m,r) of non-negative integers that satisfy

$$2q + m + r = n$$

there is an irreducible variety $Z_{q,m,r}$ so that

$$B^n = \bigcup_{2q+m+r=n} \mathfrak{Z}_{q,m,r}.$$

These varieties can be described as follows. Let $U_{q,m,r}$ be the set of pairs (X,Y) in B_n that are jointly similar to a block-diagonal pair of the form

$$\begin{pmatrix} \begin{matrix} x_1 & 0 \\ 0 & -x_1 \end{matrix} & & & & \\ & \ddots & & & \\ & & \begin{matrix} x_q & 0 \\ 0 & -x_q \end{matrix} & & \\ & & & A_m & \\ & & & & 0_r \end{pmatrix} \qquad \begin{pmatrix} \begin{matrix} 0 & y_1 \\ z_1 & 0 \end{matrix} & & & & \\ & \ddots & & & \\ & & \begin{matrix} 0 & y_q \\ z_q & 0 \end{matrix} & & \\ & & & 0_m & \\ & & & & B_r \end{pmatrix}$$

$X =$, $Y =$,

where A_m and B_r are arbitrary diagonal matrices, of size m-by-m and r-by-r, respectively, X has rank $2q + m$, Y has rank $2q + r$, all the non-zero eigenvalues of X are distinct, and all the non-zero eigenvalues of Y are distinct. Then Chen and Wang showed that $\mathfrak{Z}_{q,m,r}$ equals the Zariski closure of $\mathfrak{U}_{q,m,r}$, and has complex dimension $n^2 + q$ [4, Propositions 3.2 and 3.4].

What does this tell us about $\mathcal{A}_n$? As $\mathcal{A}_n = \mathcal{B}_n \cap \Sigma^2_n$, we have

$$\mathcal{A}^n = \bigcup_{2q+m+r=n} \mathfrak{Z}_{q,m,r} \cap \Sigma^2_n.$$

Similar arguments to the ones given in [4] show that the real dimension of $\mathfrak{Z}_{q,m,r} \cap \Sigma^2_n$ is $n^2 + q^2$, so only the largest component, $\mathfrak{Z}_{p,0,0} \cap \Sigma_n$ will have positive measure with respect to $w(\|X\|_F)dX$. Moreover, since X is self-adjoint, we can always choose $x_j > 0$ by swapping coordinates if necessary. Since Y is self-adjoint, we have $\bar{z}_j = y_j$, and we can choose y_j to be positive by conjugating the jth block by an appropriate diagonal unitary. So we can restrict our attention to what we will call the generic elements in $\mathcal{A}_n$, namely the set of full measure consisting of pairs that are jointly unitarily equivalent to some (X,Y) as in (1.2) with $\{x_1,\ldots,x_p\}$ and $\{y_1,\ldots,y_p\}$ both consisting of p distinct positive numbers. The skew spectrum of such a pair will be the p points $\{(x_j, y_j)\}$ in $\mathbb{R}^2_+$, where $\mathbb{R}_+$ denotes the positive reals.

The pair (X^2,Y) will be in $\mathcal{C}^2_n$, the set of commuting pairs of Hermitian matrices. Its spectrum will consist of the joint eigenvalues $\{(x_j,\pm y_j) : 1 \le j \le p\} \subset \mathbb{R}_+ \times \mathbb{R}$.

4. Distribution of the Skew Spectrum

Let $n = 2p$. Let $(\mathbb{R}^2_+)^p_{\text{gen}}$ be the set of p-tuples $\{(x_j, y_j) : 1 \le j \le p\}$ in $\mathbb{R}^2_+$ such that all the x_j's are distinct, and all the y_j's are distinct. Let $\mathcal{A}_{n,\text{gen}}$ denote the generic elements of $\mathcal{A}_n$, as described in Sec. 3. The map from $\mathcal{U}(n) \times (\mathbb{R}^2_+)^p_{\text{gen}}$ to $\mathcal{A}_{n,\text{gen}}$ that sends $(U,\{(x_j,y_j)\})$ to

$$\left(U \left[\bigoplus_{j=1}^{p} \begin{pmatrix} x_j & 0 \\ 0 & -x_j \end{pmatrix} \right] U^*, U \left[\bigoplus_{j=1}^{p} \begin{pmatrix} 0 & y_j \\ y_j & 0 \end{pmatrix} \right] U^* \right)$$

will not be injective, as any unitary U that is the direct sum of 2-by-2's of the form $\begin{pmatrix} e^{i\theta_j} & 0 \\ 0 & e^{i\theta_j} \end{pmatrix}$ will leave the block-diagonal matrices invariant. To rectify this, we shall consider

$$G : \mathcal{U}(n)/\mathbb{T}^p \times (\mathbb{R}^2_+)^p_{\text{gen}} \to \mathfrak{A}_{n,\text{gen}}$$

$$U, \{(x_j, y_j)\} \mapsto \left(U \left[\bigoplus_{j=1}^{p} \begin{pmatrix} x_j & 0 \\ 0 & -x_j \end{pmatrix} \right] U^*, U \left[\bigoplus_{j=1}^{p} \begin{pmatrix} 0 & y_j \\ y_j & 0 \end{pmatrix} \right] U^* \right). \tag{4.1}$$

Then G is a bijection, and it follows from the proof of Theorem 4.1 that it is a diffeomorphism as the Jacobian does not vanish. Let ν be volume measure on the homogeneous space $\mathrm{U}(n)/\mathrm{T}^p$.

To reduce the use of superscripts, we shall let Z be an element of A_n, and write its two components as

$$Z = (Z^1, Z^2) = (X, Y).$$

If $x = \{x_1, \ldots, x_p\} \in \mathrm{C}^p$, let

$$A_x = \bigoplus_{j=1}^{p} \begin{pmatrix} x_j & 0 \\ 0 & -x_j \end{pmatrix}, \quad B_x = \bigoplus_{j=1}^{p} \begin{pmatrix} 0 & x_j \\ x_j & 0 \end{pmatrix}.$$

Theorem 4.1. *Let $Z = (X, Y)$ be chosen randomly in A_n with distribution $w(\|Z\|_F)$. Then the probability distribution of the skew spectrum of (X, Y) on $\mathbb{R}^{2p}_+$ is given by*

$$\begin{aligned} &\rho_n(x_1, y_1, \ldots, x_p, y_p) \\ &\quad = C_n \, w(\|Z\|_F) \prod_{1 \le k \le p} x_k y_k \sqrt{x_k^2 + y_k^2} \prod_{1 \le i < j \le p} [(x_i - x_j)^2 + (y_i - y_j)^2] \\ &\qquad \times [(x_i + x_j)^2 + (y_i - y_j)^2][(x_i - x_j)^2 + (y_i + y_j)^2] \\ &\qquad \times [(x_i + x_j)^2 + (y_i + y_j)^2]. \end{aligned} \tag{4.2}$$

Proof. Fix some point $Z = (X, Y) \in \mathrm{A}_{n,\text{gen}}$. Without loss of generality, we can choose a basis so that U in (4.1) is the identity, and (X, Y) has the form (1.2) with skew spectrum in $(\mathbb{R}^2_+)^p_{\text{gen}}$. The derivative of G is a map between the tangent spaces.

$$dG : (T_{[I/\mathrm{T}_p]} \mathrm{U}(n)/\mathrm{T}_p) \times \mathrm{R}^{2p} \to T_{(X,Y)} \mathrm{A}_n.$$

If we view dG as a real linear map, then the Jacobian will be $\mathcal{J} = \sqrt{\det(dG^* dG)}$. So we will have

$$w(\|Z\|_F)dZ = w(\|Z\|_F)\mathcal{J}\, d\nu dx_1 dy_1 \dots dx_p dy_p. \tag{4.3}$$

Integrating with respect to v, we get that ρ_n equals the Jacobian times $w(\|Z\|_F)$; we must prove that this has the form (4.2).

The tangent space at the identity of U(n) is the space of skew-symmetric matrices. The tangent space of $U(n)/\mathbb{T}^p$ at $[I\mathbb{T}^p]$ is the skew-symmetric matrices whose diagonals are of the form $(\pm\iota\theta_j)$. (We shall write ι for $\sqrt{-1}$ to distinguish from i used as an index). We have

$$dG|_{(I\mathbb{T}^p,x,y)}(S,a,b) = \frac{d}{dt}\left(e^{tS}A_{x+ta}e^{-tS}, e^{tS}B_{y+tb}e^{-tS}\right)$$
$$= (SA_x - A_xS + A_a, SB_y - B_yS + B_b). \tag{4.4}$$

We want to pick a basis for the tangent space that facilitates computation.

For any $k,\ell \in \{1,\dots,n\}$ let $E_{k,l}$ denote the elementary matrix with 1 in the (k,ℓ) place and 0 elsewhere. We shall let i,j range between 1 and p, and α and β range over $\mathbb{Z}_2$ (where 1 + 1 = 0). Define a basis as follows. For each $1 \le k \le p$, we have 3 matrices:

$$R_k = \frac{1}{\sqrt{2}}[E_{2k-1,2k} - E_{2k,2k-1}], \quad S_k = \frac{\iota}{\sqrt{2}}[E_{2k-1,2k} + E_{2k,2k-1}],$$
$$T_k = \frac{\iota}{\sqrt{2}}[E_{2k-1,2k-1} - E_{2k,2k}].$$

For each pair (i,j) in $\{1,\dots,p\}$ with $i < j$ and each $\alpha,\beta \in \mathbb{Z}_2$, we have two matrices

$$R_{ij,\alpha\beta} = \frac{1}{\sqrt{2}}[E_{2i-\alpha,2j-\beta} - E_{2j-\beta,2i-\alpha}], \quad S_{ij,\alpha\beta} = \frac{\iota}{\sqrt{2}}[E_{2i-\alpha,2j-\beta} - E_{2j-\beta,2i-\alpha}].$$

Finally, for a basis of $\mathbb{R}^{2p}$, thought of as the tangent space to $(\mathbb{R}^2_+)^p_{\text{gen}}$, we let $\{e^1_k : 1 \le k \le p\}$ be the standard basis for the first slot, and $\{e^2_k : 1 \le k \le p\}$ be the standard basis in the second slot.

Straightforward calculations show

$$[R_k, A_x] = 2\iota x_k S_k, \qquad [R_k, B_y] = -2\iota y_k T_k,$$
$$[S_k, A_x] = -2\iota x_k R_k, \qquad [S_k, B_y] = 0,$$
$$[T_k, A_x] = 0, \qquad [T_k, B_y] = 2\iota y_k R_k$$

and

$$[R_{ij,\alpha\beta}, A_x] = \iota((-1)^\beta x_j - (-1)^\alpha x_i) S_{ij,\alpha\beta},$$

$$[R_{ij,\alpha\beta},B_y] = \iota(y_i S_{ij,\alpha+1,\beta} - y_j S_{ij,\alpha,\beta+1}),$$

$$[S_{ij,\alpha\beta},A_x] = \iota((-1)_\alpha x_i - (-1)_\beta x_j) R_{ij,\alpha\beta},$$

$$[S_{ij,\alpha\beta},B_y] = \iota(y_j R_{ij,\alpha,\beta+1} - y_i R_{ij,\alpha+1,\beta}).$$

The matrix for dG has a block form. It maps the 5 (real) dimensional space spanned by $\{R_k,S_k,T_k,e^1{}_k,e^2{}_k\}$ into the six-dimensional space that is spanned by $\{R_k,S_k,T_k\}$ in both the first and second slots, and it maps the eight-dimensional space spanned by $\{R_{ij,\alpha\beta},S_{ij,\alpha\beta} : \alpha,\beta \in \mathrm{Z}_2\}$ into two copies of the same space. Because of the block structure, the Jacobian of the whole map will be the product of the Jacobians for each block.

The first set of blocks look like this.

$$\begin{array}{cccccc} & R_k & S_k & T_k & e_{1k} & e_{2k} \\ R_{kk} & 0_k & -2x_k & 0 & 0 & 0 \\ RTS_k & 2x00 & 000 & 200y & -\sqrt{}00\ 2 & 000 \\ T_{kkk} & -2y_k & 0 & 0_k & 0 & 0 \\ S & 0 & 0 & 0 & 0 & -\sqrt{2} \end{array} \times \iota. \qquad (4.5)$$

The second set looks like

$$\begin{array}{ccccccccc} & R_{ij}00 & R_{ij}10 & R_{ij}01 & R_{ij}11 & S_{ij}00 & S_{ij}10 & S_{ij}01 & S_{ij}11 \\ R_{ij}00 & 0 & 0 & 0 & 0 & x_i - x_j & 0 & 0 & 0 \\ R_{ij10} & 0 & 0 & 0 & 0 & 0 & -x_i - x_j & 0 & 0 \\ R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x + x_j \\ S & x - x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

S	0	$x + x$	0	0	0	0	0	0	
S	0		$0 -x -x$	0	0	0	0	0	
S	0	0	$00 -x + x$	00	0	0	0		$\times \iota.$

$R_{ijijijijijijijijijijijijijijij}$1011110110010000010010111101

$_j 0_{ij\, i}$ $_j 0_{ij\, i}$ $_j0_{ij}$ $_i$ $_j0_{ij}$ $_i$ 0_{ji} 0_{ji} $x_i +_{ji} x_j$ $_i0$

R	0	0		0		$-y$	y	0
R	0	0	0	0	$-y$	0	0	y_j
R	0	0	0	0	y	0	0	$-y_i$
R	0	0	0	0	0	y	$-y$	0
S	0	y	$-y$	0	0	0	0	0
S	y	0	0	$-y$	0	0	0	0
S	$-y$	0	0	y	0	0	0	0
S	0	$-y$	y	0	0	0	0	0

(4.6)

When (4.5) is premultiplied by its adjoint, one gets a diagonal matrix whose determinant is

$$4^4(x_k^2 + y_k^2)x_k^2 y_k^2.$$

When (4.6) is premultiplied by its adjoint, the resulting matrix has determinant

$$[((x_j - x_i)^2 + y_j^2 + y_i^2)^2 - 4y_j^2 y_i^2]^2[((x_j + x_i)^2 + y_j^2 + y_i^2)^2 - 4y_j^2 y_i^2]^2,$$ which equals

the square of

$$[(x_i - x_j)^2 + (y_i - y_j)^2][(x_i + x_j)^2 + (y_i - y_j)^2][(x_i - x_j)^2 + (y_i + y_j)^2]$$

$$\times[(x_i + x_j)^2 + (y_i + y_j)^2].$$

Multiplying all these together, we get that the Jacobian times $w(\|Z\|_F)$, when integrated with respect to ν, is (4.2). □

Lemma 4.2.

$$f(x_i,x_j,y_i,y_j) = [(x_i - x_j)^2 + (y_i - y_j)^2][(x_i + x_j)^2 + (y_i - y_j)^2]$$

$$\times[(x_i - x_j)^2 + (y_i + y_j)^2][(x_i + x_j)^2 + (y_i + y_j)^2]. \tag{4.7}$$

Let $d^2 = (x_i - x_j)^2 + (y_i - y_j)^2$. Let ε,M be positive constants, and assume that x_1,x_2,y_1 and y_2 are all between ε and M. Then we have

$$128\varepsilon^6 d^2 \leq f(x_i,x_j,y_i,y_j) \leq 200M^6 d^2. \tag{4.8}$$

Proof. The first factor of f is d^2. The other three factors are bounded below by $(2\varepsilon)^2(2\varepsilon)^2(8\varepsilon^2)$ and bounded above by $(5M^2)(5M^2)(8M^2)$. □

It follows from Lemma 4.2, that in compact subsets of $(0,\infty)^2$ the repulsion between elements of the skew spectrum, as given by (4.2), is of the order of the square of their Euclidean distance.

5. Fekete Points and the Limiting Distribution

In this section, we shall just consider the Gaussian case $w(Z) = e^{-\frac{1}{2}\|Z\|_F^2}$. For any fixed $n = 2p$, the skew spectrum is most likely to occur where the density ρ_n from (4.2) is highest. Using gradient descent, we numerically calculated what distribution of points maximized (ρ_n). See Fig. 1. They seem to be approximately equally spaced within the quarter-disk of radius $2\sqrt{n}$.

By way of comparison, we plot the joint eigenvalues that maximize the density κ_n from (2.2) for a pair of commuting self-adjoint matrices, with the same Gaussian weight. In this case, they are approximately equally spaced within the disk of radius $\sqrt{2n}$. See Fig. 2.

Write $z = (x,y)$ for a point in $\mathbb{R}^2_+$.

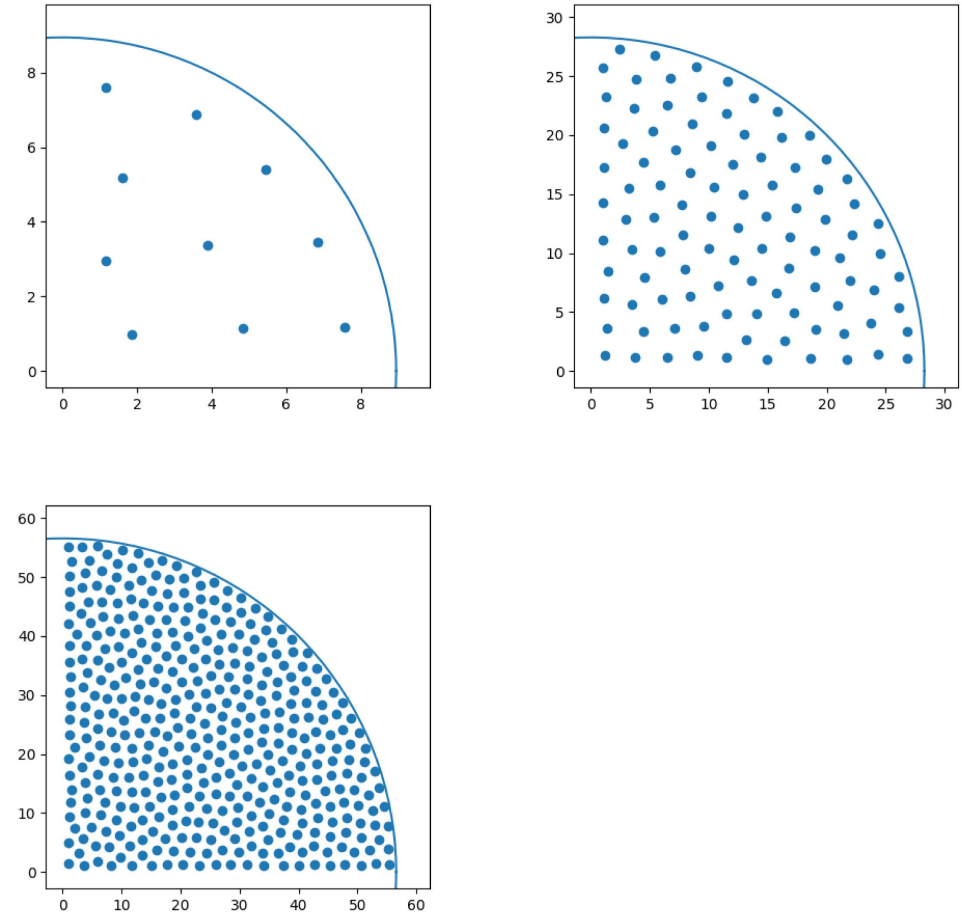

Fig. 1. Numerical simulation of points that maximize ρ_n for a pair of anti-commuting self-adjoint matrices, with sizes $n = 20,200,800$. Circle has radius $2\sqrt{n}$.

Definition 5.1. *Let $n = 2p$. We shall say a set $S = \{z_1, \dots, z_p\} \subseteq \mathbb{R}^2_+$ is a maximal likelihood set if ρ_n attains its maximum on S.*

It is not immediately obvious that maximal likelihood sets exist, since the domain is not bounded, but we shall prove that they do. It is more convenient to work with $\tau := -\log\rho_n$, a function from $\mathbb{R}^{2p}_+$ to $(-\infty, \infty]$ which we want to minimize. Let f be as in Lemma 4.2. Then

$$\tau(z_1, \dots, z_p) = \frac{1}{2}\sum_{k=1}^{p} |z_k|^2 - \log|x_k y_k z_k| - \frac{1}{2}\sum_{k\neq\ell} \log f(z_k, z_\ell).$$

Lemma 5.2. *For each $n \geq 1$, there exists a set S so that $\tau(S) \leq n^2$.*

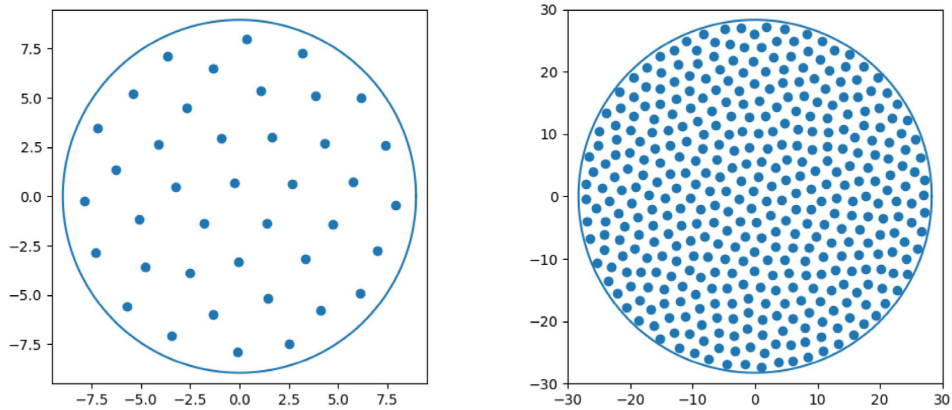

Fig. 2. Numerical simulation of points that maximize κ_n for a pair of commuting self-adjoint matrices, with sizes n = 40,400. Circle has radius $\sqrt{2n}$.

Proof. Case: $p = q^2$ for $q \in \mathbb{N}$. Let $S_q = \{1,2,\ldots,q\}\times\{1,2,\ldots,q\}$. By Lemma 4.2, for any $k \neq \ell$ we have $f(z_k, z_\ell) \geq 1$. So

$$\begin{aligned}\tau(S_q) &\leq \frac{1}{2}\sum_{i,j=1}^{q}(i^2+j^2)\\ &= \frac{q^2(q+1)(2q+1)}{6}\\ &\leq q^4 = \frac{1}{4}n^2.\end{aligned}$$

General case: choose q so that $(q-1)^2 < p \leq q^2$. Let S be any p elements of S_q. Then

$$\begin{aligned}\tau(S) &\leq q^4\\ &\leq 4((q-1)^2+1)^2 \leq 4p^2 = n^2.\end{aligned}$$ □

Lemma 5.3. *Let $n = 2p$ be a positive even integer. Choose $K \geq 3p$ so that*

$$\frac{1}{2}K^2 - (3p+4p^2)\log(K) - \frac{p^2}{2}\log(400) - 4p^2 > 0. \tag{5.1}$$

Let $S = \{z_1, \ldots, z_p\} \in (\mathbb{R}^2_+)^p$ be a set so that $\tau(S) \leq 4p^2$. Then the maximum length of an element of S is at most K.

Proof. Let $M = \max\{|z_k| : 1 \leq k \leq p\}$. By Lemma 4.2, for $k \neq \ell$ we have

$$f(z_k, z_\ell) \leq 400M^8.$$

So

$$\begin{aligned} 4p^2 &\geq \tau(S) \\ &= \frac{1}{2}\sum_{k=1}^{p}(|z_k|^2 - \log|x_k y_k z_k|) - \frac{1}{2}\sum_{k\neq \ell}\log f(z_k, z_\ell) \\ &\geq \frac{1}{2}M^2 - p\log M^3 - \frac{p(p-1)}{2}\log(400M^8) \\ &\geq \frac{1}{2}M^2 - (3p+4p^2)\log M - \frac{p^2}{2}\log 400. \end{aligned}$$

The last inequality

$$4p^2 \geq \frac{1}{2}M^2 - (3p+4p^2)\log M - \frac{p^2}{2}\log 400 \tag{5.2}$$

fails at $M = K$ (by choice of K). Moreover, the right-hand side of (4.2) is increasing for $M \geq 3p$, so we must have $M < K$. □

Theorem 5.4. *Maximal likelihood sets exist.*

Proof. Let K be as in Lemma 5.3. Consider $\tau : [0,K]^{2p} \to (-\infty,\infty]$. This is a continuous function on a compact set, so attains its infimum. By Lemmas 5.2 and 5.3 this is a global infimum for τ on $\mathbb{R}^{2p}_{+}$. □

Definition 5.5. *A set $S \subseteq (\mathbb{R}^2_+)^p$ is a Fekete set of size p if the set $\sqrt{p}S$ is a maximal likelihood set of ρ_n.*

A measure is a Fekete measure of size p if it consists of p atoms of weight $\frac{1}{p}$ at each point of a Fekete set of size p.

Based on the numerical simulations shown in Fig. 1 and analogy with the commuting case, we are led to ask the following questions.

Question 5.6. Let μ_p be a sequence of Fekete measures of size p.

(1) Is there a compact set that contains the support of every μ_p?
(2) Does the sequence μ_p converge weakly (when integrated against bounded continuous functions) to a unique compactly supported measure μ?
(3) Define a random probability measure ν_p by choosing random anti-commuting self-adjoint pairs of size $2p$-by-$2p$ with the Gaussian measure $C_p e^{-\frac{1}{2}\|Z\|^2}dZ$, and letting ν_p have mass $\frac{1}{p}$ at each point of the skew spectrum of Z. Let E_p denote expectation with respect to this process. Is there a measure μ so that

$$\lim_{p\to\infty}\mathbb{E}_p\int_{\mathbb{R}^2_+}\phi(z)d\nu_p(z) = \int_{\mathbb{R}^2_+}\phi(z)d\mu(z) \quad \forall\,\phi \in C_b(\mathbb{R}^2_+)?$$

(4) If the answer to Questions 2 and 3 is yes, is μ equal to normalized area measure on the quarter disk in the first quadrant of radius $\sqrt{8}$?

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