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Stochastic proof of the sharp symmetrized Talagrand inequality

Une preuve stochastique de l'inégalité de Talagrand symétrisée optimale

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Abstract. A new proof of the sharp symmetrized form of Talagrand's transport-entropy inequality is given. Compared to stochastic proofs of other Gaussian functional inequalities, the new idea here is a certain coupling induced by time-reversed martingale representations.

Résumé. Nous donnons une nouvelle preuve de la version symétrisée de l'inégalité de transport-entropie de Talagrand avec constante optimale. En comparaison avec d'autres preuves stochastiques d'inégalités fonctionnelles gaussiennes, l'élément nouveau ici est l'utilisation d'un couplage induit par un retournement du temps sur des représentations de martingales.

Keywords. Transport inequalities, Gaussian inequalities, Blaschke–Santaló inequality, Martingale representations.

Mots-clés. Inégalités de transport, inégalités Gaussiennes, inégalité de Blaschke–Santaló, représentations de martingales.

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1. Introduction

We aim to give a short stochastic proof of the following sharp symmetrized Talagrand inequality:

Theorem 1 ([8, Theorem 1.1]). *For Borel probability measures μ,ν on $\mathbb{R}^n$ with finite second moments and μ centered,*

$$W_2(\mu,\nu)^2 \le 2D(\mu\|\gamma) + 2D(\nu\|\gamma), \tag{1}$$

where W_2 is 2-Wasserstein distance, D is relative entropy, and γ is the standard Gaussian measure on $\mathbb{R}^n$.

By duality, (1) is formally equivalent to the functional Blaschke–Santaló inequality [13, Theorem 1.2], which states that if Borel functions $f,g:\mathbb{R}^n \to \mathbb{R}$ satisfy

$$\int_{\mathbb{R}^n} x\,\mathrm{e}^{-f(x)}\,\mathrm{d}x = 0 \text{ and } f(x)+g(y) \ge \langle x,y\rangle,\ \forall x,y\in\mathbb{R}^n,$$

then

$$\left(\int_{\mathbb{R}^n} \mathrm{e}^{-f(x)}\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^n} \mathrm{e}^{-g(x)}\,\mathrm{d}x\right) \le (2\pi)^n. \tag{2}$$

Equality holds for quadratic f, and $g = f^*$, its convex conjugate. Despite the equivalence, (1) may be regarded as a formal strengthening of (2) in the sense that (2) is recovered from Theorem 1 by weak duality: briefly, for f,g satisfying the hypotheses demanded by (2), take $\mathrm{d}\mu(x) \propto \mathrm{e}^{-f(x)}\,\mathrm{d}x$ and $\mathrm{d}\nu(x) \propto \mathrm{e}^{-g(x)}\,\mathrm{d}x$ in (1) and simplify to obtain (2). The reverse implication corresponds to strong duality, and is more difficult. See [10, Theorem 11] and [8] for details.

Inequality (2) is a functional generalization of the earlier Blaschke–Santaló inequality for the volume product of convex sets, earlier proofs of which were accomplished by calculus of variations [20] and symmetrization arguments [16, 19]. The functional form was proved in [2] (and earlier in K. Ball's Ph.D. thesis [3] in a restricted setting of even functions). The original proof relied on the usual Blaschke–Santaló inequality applied to level sets. Lehec later gave two alternative proofs; one using induction on the dimension [13], and the other [14] using the Prekópa–Leindler inequality and the Yao–Yao partition theorem. This last proof actually yields a more general statement, originally due to [9], but the present work shall be restricted to the classical setting. More recently, a new semigroup proof of the inequality for even functions was established in [18] using improved hypercontractive estimates for the heat flow, and then simplified in [6]. Let us also mention a recent generalization to several functions under a symmetry assumption, due to Kolesnikov and Werner [12].

Equivalence between integral inequalities of the form (2) and transport inequalities of the form (1) via duality goes back to [4], where they studied Talagrand quadratic transport-entropy inequality [21] (which is (1) in the particular case $\mu = \gamma$). Duality for transport inequalities involving three measures, such as (1), has been considered in [10] and [11, Proposition 8.2].

Stochastic proofs of functional inequalities, in particular using Brownian motion and Girsanov's theorem, go back to Borell's stochastic proof of the Prekópa–Leindler inequality [5]. Our present work is motivated by Lehec's short stochastic proofs of various functional inequalities [15], including in particular Talagrand's transport-entropy inequality.

2. A Stochastic Proof

We'll work on the Wiener space $(\Omega,\mathscr{B},\mathbb{P})$, where Ω is the set of continuous paths $\omega:[0,1]\to\mathbb{R}^n$ starting at 0, $\mathscr{B}$ is the usual Borel σ-algebra, and $\mathbb{P}$ is the Wiener measure. Let $B_t(\omega) := \omega(t)$ be the coordinate process, so that $B = (B_t)_{0\le t\le 1}$ is a standard Brownian motion, and so is the time-reversed process $\widehat{B}_t := B_1 - B_{1-t}$. Let $\mathscr{F} = (\mathscr{F}_t)_{0\le t\le 1}$ and $\mathscr{F}^+ = (\mathscr{F}_t^+)_{0\le t\le 1}$ denote the filtrations generated by B and $\widehat{B}$, respectively. For each $t \in [0,1]$, $\mathscr{F}_t$ and $\mathscr{F}^+_{1-t}$ are complementary, in the sense that they are independent and $\mathscr{B} = \sigma(\mathscr{F}_t \cup \mathscr{F}^+_{1-t})$. Henceforth, $\|\cdot\|$ denotes the ℓ^2 norm, and I denotes the identity matrix.

Apart from standard facts in stochastic calculus, we'll need two lemmas. The first is a variational representation of entropy, obtained as a consequence of Girsanov's theorem; it has been applied to study rigidity and stability of various functional inequalities (see, e.g., [1, 7, 17]).

Lemma 2 ([7]). *For a centered probability measure μ on $\mathbb{R}^n$ with finite second moments, we have*

$$D(\mu\|\gamma) = \inf_F \frac{1}{2}\int_0^1 \frac{\mathbb{E}[\|F_t - \mathrm{I}\|^2]}{1-t}\,\mathrm{d}t, \tag{3}$$

where the infimum is over $\mathscr{F}$-adapted $\mathbb{R}^{n\times n}$-valued processes $F = (F_t)_{0\le t\le 1}$ such that $\int_0^1 F_t\,\mathrm{d}B_t \sim \mu$.

Remark 3. The same representation holds if we consider $\mathscr{F}^+$-adapted F with $\int_0^1 F_t\,\mathrm{d}\widehat{B}_t \sim \mu$.

Stochastic proofs of several other functional inequalities use representation formulas for the entropy and linear couplings of Brownian motions (cf. [7, 15]). Our proof will similarly rely on the representation formula (3) for the entropy[1], but makes use of a new coupling induced by time-reversed martingale representations. The next lemma is the crucial new ingredient; it relates martingale representations in terms of $(B_t)_{0\le t\le 1}$ and its time-reversal $(\widehat{B}_t)_{0\le t\le 1}$.

Lemma 4. *If $X \in L^2(\Omega,\mathscr{B},\mathbb{P})$ is a $\mathbb{R}^n$-valued random vector with martingale representations*

$$X = \int_0^1 F_t\,\mathrm{d}B_t = \int_0^1 G_t\,\mathrm{d}\widehat{B}_t,$$

then

$$\int_t^1 \mathbb{E}[\|F_s - \mathrm{I}\|^2]\,\mathrm{d}s \ge \int_0^{1-t} \mathbb{E}[\|G_s - \mathrm{I}\|^2]\,\mathrm{d}s,\ \forall 0\le t\le 1. \tag{4}$$

Proof. By the Pythagorean theorem, convexity, and independence of $\mathscr{F}_t$ and $\mathscr{F}^+_{1-t}$, we have

$$\mathbb{E}[\|X\|^2] - \mathbb{E}[\|\mathbb{E}[X|\mathscr{F}_t]\|^2] = \mathbb{E}[\|X - \mathbb{E}[X|\mathscr{F}_t]\|^2] \ge \mathbb{E}[\|\mathbb{E}[X - \mathbb{E}[X|\mathscr{F}_t]|\mathscr{F}^+_{1-t}]\|^2] = \mathbb{E}[\|\mathbb{E}[X|\mathscr{F}^+_{1-t}]\|^2].$$

Since $\mathbb{E}[X|\mathscr{F}_t] = \int_0^t F_s\,\mathrm{d}B_s$ and $\mathbb{E}[X|\mathscr{F}^+_{1-t}] = \int_0^{1-t} G_s\,\mathrm{d}\widehat{B}_s$, three applications of Itô's isometry give

$$\int_t^1 \mathbb{E}[\|F_s\|^2]\,\mathrm{d}s = \mathbb{E}[\|X\|^2] - \mathbb{E}[\|\mathbb{E}[X|\mathscr{F}_t]\|^2] \ge \mathbb{E}[\|\mathbb{E}[X|\mathscr{F}^+_{1-t}]\|^2] = \int_0^{1-t} \mathbb{E}[\|G_s\|^2]\,\mathrm{d}s,\ \forall 0\le t\le 1.$$

Next, by applying Itô's isometry to each martingale representation of X, we find

$$\int_t^1 \mathbb{E}[\mathrm{Tr}(F_s)]\,\mathrm{d}s = \mathbb{E}[\langle X, B_1 - B_t\rangle] = \mathbb{E}[\langle X, \widehat{B}_{1-t}\rangle] = \int_0^{1-t} \mathbb{E}[\mathrm{Tr}(G_s)]\,\mathrm{d}s,\ \forall 0\le t\le 1.$$

Combining the previous two observations gives (4). □

Proof of Theorem 1. The inequality is invariant with respect to translations of ν, so we may also assume ν is centered. Let $F = (F_t)_{0\le t\le 1}$ be any $\mathscr{F}$-adapted process such that $\int_0^1 F_t\,\mathrm{d}B_t \sim \mu$, and let $H = (H_t)_{0\le t\le 1}$ be any $\mathscr{F}^+$-adapted process such that $\int_0^1 H_t\,\mathrm{d}\widehat{B}_t \sim \nu$. Let $G = (G_t)_{0\le t\le 1}$ be the martingale representation of $\int_0^1 F_t\,\mathrm{d}B_t$ in terms of the time-reversed Brownian motion $\widehat{B}$; i.e., G is $\mathscr{F}^+$-adapted, satisfying $\int_0^1 G_t\,\mathrm{d}\widehat{B}_t = \int_0^1 F_t\,\mathrm{d}B_t \sim \mu$. By the Tonelli theorem and Lemma 4, we have

$$\begin{aligned}\int_0^1 \frac{\mathbb{E}[\|G_s - \mathrm{I}\|^2]}{s}\,\mathrm{d}s &= \int_0^1 \mathbb{E}[\|G_s - \mathrm{I}\|^2]\,\mathrm{d}s + \int_0^1 \frac{1}{(1-t)^2}\left(\int_0^{1-t} \mathbb{E}[\|G_s - \mathrm{I}\|^2]\,\mathrm{d}s\right)\mathrm{d}t\\ &\le \int_0^1 \mathbb{E}[\|F_s - \mathrm{I}\|^2]\,\mathrm{d}s + \int_0^1 \frac{1}{(1-t)^2}\left(\int_t^1 \mathbb{E}[\|F_s - \mathrm{I}\|^2]\,\mathrm{d}s\right)\mathrm{d}t = \int_0^1 \frac{\mathbb{E}[\|F_s - \mathrm{I}\|^2]}{1-s}\,\mathrm{d}s.\end{aligned}$$

[1] The representation (3) is not the same as that used in [15]. However, it is derived from [15, Theorem 4] by combination with the martingale representation theorem.

By definition of W_2, Itô's isometry, convexity of $\|\cdot\|^2$, and the previous estimate,

$$W_2(\mu,\nu)^2 \le \mathbb{E}\left\|\int_0^1 (G_t - H_t)\,\mathrm{d}\widehat{B}_t\right\|^2 = \int_0^1 \mathbb{E}[\|G_t - H_t\|^2]\,\mathrm{d}t \le \int_0^1 \frac{\mathbb{E}[\|G_t - \mathrm{I}\|^2]}{t}\,\mathrm{d}t + \int_0^1 \frac{\mathbb{E}[\|H_t - \mathrm{I}\|^2]}{1-t}\,\mathrm{d}t$$
$$\le \int_0^1 \frac{\mathbb{E}[\|F_t - \mathrm{I}\|^2]}{1-t}\,\mathrm{d}t + \int_0^1 \frac{\mathbb{E}[\|H_t - \mathrm{I}\|^2]}{1-t}\,\mathrm{d}t.$$

With the help of Lemma 2, optimizing over F and H completes the proof. □

3. Remarks on the Approach

3.1. *Equality cases*

The equality cases for (1) are also evident from the given proof. Indeed, if $D(\mu\|\gamma) < \infty$, then the infimum in (3) is a.s.-uniquely achieved by an $\mathscr{F}$-adapted process $F = (F_t)_{0\le t\le 1}$. Defining $X := \int_0^1 F_t\,\mathrm{d}B_t \sim \mu$ for this particular F, equality in (1) implies equality in (4) for a.e. $t \in [0,1]$, which requires that $X - \mathbb{E}[X|\mathscr{F}_t]$ is $\mathscr{F}^+_{1-t}$-measurable for a.e. $t \in [0,1]$. By Proposition 5 below, this ensures $X \sim \mu$ is Gaussian. By symmetry, any extremal ν is also Gaussian, and explicit computation shows that μ,ν are extremizers in (1) if and only if $\mu = N(0,C)$ and $\nu = N(\theta, C^{-1})$ for some $\theta \in \mathbb{R}^n$ and positive definite $C \in \mathbb{R}^{n\times n}$.

Proposition 5. *Let $X \in L^2(\Omega,\mathscr{B},\mathbb{P})$ admit martingale representation $X = \int_0^1 F_t\,\mathrm{d}B_t$. If $X - \mathbb{E}[X|\mathscr{F}_t]$ is $\mathscr{F}^+_{1-t}$-measurable for a.e. $t \in [0,1]$, then X is Gaussian.*

Proof. Define $M_t := \int_0^t F_s\,\mathrm{d}B_s$. The hypothesis is equivalent to requiring that $(M_1 - M_t)$ is $\mathscr{F}^+_{1-t}$-measurable for each $t \in \mathscr{D}$, where $\mathscr{D}$ is dense in $[0,1]$. Fix any $s,t \in \mathscr{D}$, with $s \le t$. Since $(M_1 - M_t)$ is $\mathscr{F}^+_{1-t}$-measurable by hypothesis, and $(M_t - M_s)$ is $\mathscr{F}_t$-measurable by definition, complementarity ensures $(M_1 - M_t)$ and $(M_t - M_s)$ are independent. Iterating this procedure on the $(M_1 - M_t)$ term allows us to conclude that $(M_t)_{0\le t\le 1}$ has independent increments, provided the endpoints of the increments are in $\mathscr{D}$. Since $X \in L^2(\Omega,\mathscr{B},\mathbb{P})$, a version of $(M_t)_{0\le t\le 1}$ admits continuous sample paths, and we conclude by density of $\mathscr{D}$ that $(M_t)_{0\le t\le 1}$ has (square-integrable) independent increments generally, and is thus a Gaussian process. □

3.2. *Importance of the coupling induced by time-reversal*

With the proof of Theorem 1 in hand and the equality cases characterized, we highlight the importance of the coupling based on time-reversal. Following previous stochastic proofs of functional inequalities, one could appeal to martingale representations $\int_0^1 F_t\,\mathrm{d}B^1_t \sim \mu$ and $\int_0^1 G_t\,\mathrm{d}B^2_t \sim \nu$ with linearly coupled Brownian motions B^1 and B^2 (equivalently, Brownian motions B^1 and B^2 adapted to a common filtration) to couple μ and ν. This approach cannot work, as we now explain.

Working in dimension $n = 1$ for simplicity, recall that when $\mu = N(0,\alpha)$ with $\alpha > 0$, the minimizer F in (3) has an explicit expression (e.g., [7, Sec. 2]). In particular,

$$\int F_t\,\mathrm{d}B^1_t \sim \mu \text{ and } D(\mu\|\gamma) = \frac{1}{2}\int_0^1 \frac{\mathbb{E}[\|F_t - \mathrm{I}\|^2]}{1-t}\,\mathrm{d}t \longrightarrow F_t = \frac{\alpha}{1-t+\alpha t}.$$

Likewise, for $\nu = N(0,\alpha^{-1})$, the "optimal" representation of ν with respect to B^2 satisfies

$$\int_0^1 G_t\,\mathrm{d}B^2_t \sim \nu \text{ and } D(\nu\|\gamma) = \frac{1}{2}\int_0^1 \frac{\mathbb{E}[\|G_t - \mathrm{I}\|^2]}{1-t}\,\mathrm{d}t \longrightarrow G_t = \frac{1}{\alpha(1-t)+t}.$$

Since B^1 and B^2 are linearly coupled standard Brownian motions, we can write

$$\begin{bmatrix} B^1_t \\ B^2_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma \\ \sigma & 1 \end{bmatrix}^{1/2} B_t,\ 0 \le t \le 1, \tag{5}$$

for some $|\sigma| \le 1$, where $(B_t)_{0\le t\le 1}$ is a 2-dimensional standard Brownian motion. This construction induces a coupling π_σ of $X := \int_0^1 F_t\,\mathrm{d}B_t^1 \sim \mu$ and $Y := \int_0^1 G_t\,\mathrm{d}B_t^2 \sim \nu$ satisfying

$$\mathbb{E}_{\pi_\sigma}\|X-Y\|^2 = \int_0^1 \left(F_t^2 + G_t^2 - 2\sigma F_t G_t\right)\mathrm{d}t = \begin{cases} \left(\alpha+\frac{1}{\alpha}\right) - \frac{4\sigma\log\alpha}{\left(\alpha-\frac{1}{\alpha}\right)} & \text{if } \alpha \neq 1 \\ 2(1-\sigma) & \text{if } \alpha = 1, \end{cases}$$

where we made use of Itô's isometry and (5). A simple calculation reveals that

$$\min_{\sigma:|\sigma|\le 1} \mathbb{E}_{\pi_\sigma}\|X-Y\|^2 \ge \alpha + \frac{1}{\alpha} - 2 = W_2(\mu,\nu)^2 = 2D(\mu\|\gamma) + 2D(\nu\|\gamma),$$

with equality if and only if $\alpha = 1$. So, with the exception of the trivial case $\mu = \nu = \gamma$, the established stochastic approach to proving functional inequalities using linearly coupled Brownian motions fails to produce the requisite optimal coupling between μ and ν in all extremal cases (at least, in this implementation). This suggests that coupling through time-reversal lends a useful new degree of freedom to the stochastic program for proving functional inequalities.

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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