Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

PROBABILISTIC ZERO FORCING WITH VERTEX REVERSION*

ZACHARY BRENNAN†

Abstract. Probabilistic zero forcing is a graph coloring process in which blue vertices "infect" (color blue) white vertices with a probability proportional to the number of neighboring blue vertices. This paper introduces reversion probabilistic zero forcing (RPZF), which shares the same infection dynamics but also allows for blue vertices to revert to being white in each round. A threshold number of blue vertices is produced such that the complete graph is entirely blue in the next round of RPZF with high probability. Utilizing Markov chain theory, a tool is formulated which, given a graph's RPZF Markov transition matrix, calculates the probability of whether the graph becomes all white or all blue as well as the time at which this is expected to occur.

Key words. Probabilistic zero forcing, Markov chains on graphs, Reversion, Discrete contact process, High probability.

AMS subject classifications. 15B51, 60J10, 60G50, 05C15, 05C81, 60J20, 60J22, 60C05.

1. Introduction.

Zero forcing is a graph coloring process in which blue vertices "infect" (color blue) neighboring white vertices. It was introduced independently as a condition for the control of quantum systems [8] and as a bound for the maximum nullity of a matrix in the study of the minimum rank problem [20]. Zero forcing has since been used extensively in the study of the minimum rank problem (see [21] and the references therein) and has been found to have further connections with graph search algorithms [30], power domination [4], and the Cops and Robbers game [2, 7]. These connections have led to the study of zero forcing in its own right, and variants of zero forcing have since emerged (see the workshop summary [16] for examples). This paper focuses on probabilistic variations of zero forcing, the first of which was introduced by Kang and Yi in [23]. Specifically, in this paper, we introduce reversion probabilistic zero forcing (RPZF), a process in which blue vertices can now revert back to being white. Probabilistic zero forcing (PZF) cannot move across graph components and is typically studied on connected graphs. Thus, unless otherwise stated, we assume G is a simple connected graph on n vertices.

The main results of this paper characterize the behavior of RPZF on complete and complete bipartite graphs with different densities of infected vertices. For the complete graph on n vertices, Theorem 3.8 and Theorem 3.9 show that $\sqrt{n \log n}$ is the threshold number of infected vertices for the graph to be fully infected in one time step. Theorem 3.13 gives evidence that this asymptotic behavior is similar to RPZF on the balanced complete bipartite graph when the infected vertices are evenly distributed. On the other hand, Theorem 3.15 shows that the star graph is more "difficult" to fully infect when compared to the complete and balanced complete bipartite graphs. This notion of difficulty is explored further with simulations in Section 5.

Section 2 defines RPZF, introduces RPZF parameters, and formulates RPZF as a Markov chain. This formulation is expanded in Section 4 to quantify the behavior of RPZF on any finite graph when provided its Markov transition matrix, and examples of such calculations are given in Section 5. Finally, Section 6 explores how RPZF is a discrete-time analog of the susceptible-infected-susceptible (SIS) contact process,

*Received by the editors on April 22, 2024. Accepted for publication on December 22, 2024. Handling Editor: Jephian Lin.
Corresponding Author: Zachary Brennan.
†Department of Mathematics, Iowa State University, Ames, Iowa 50011, USA (brennanz@iastate.edu).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

a basic continuous-time model for the spread of infection. In particular, RPZF is an example of a semi-heterogeneous model, where the infection rate is different for different vertices, but the recovery rate is uniform. The contact process is traditionally considered on the integer lattice due to its simplistic structure and the fact that clusters of vertices do not interact globally. This paper, on the other hand, develops contact process-like results on denser and more complex graph structures.

1.1. Basic notation. A *graph* is a pair $G = (V, E)$ where the set $E = E(G)$ of *edges* consists of 2-element subsets of $V = V(G)$, the finite set of *vertices*. Thus, all graphs discussed (except in Section 6) are simple, undirected, and finite. Two vertices $v, w \in V$ are *adjacent* if $\{v, w\} \in E$. The *open neighborhood of* v is the set of all vertices adjacent to v, denoted by $N(v) = \{w \in V : \{v, w\} \in E\}$. The *degree of* v is $\deg v = |N(v)|$, the number of vertices adjacent to v, and the *closed neighborhood of* v is $N[v] = N(v) \cup \{v\}$. We say G is *connected* if for every $v, w \in V$ there exists a *path* of vertices $v = v_0, v_1, \ldots, v_k = w$ such that $\{v_{i-1}, v_i\} \in E$ for all $i \in \{1, \ldots, k\}$. In this paper, we consider only connected graphs.

The identity matrix is denoted by I, and the matrix containing all zero entries is denoted by O. We will use $\mathbb{1} = [1\ 1\ \cdots\ 1]^T$ to denote the column vector containing all ones and $\mathbb{N} = \{0, 1, 2, \ldots\}$ to refer to the set of non-negative integers. Given a Markov chain (X_t) with state space $(S_0, \ldots, S_s)$, $\mathbf{P}_i[A] = \mathbf{P}[A \mid X_0 = S_i]$ denotes the probability of event A given the chain starts from state S_i, and $\mathbf{E}_i[Y] = \mathbf{E}_i[Y \mid X_0 = S_i]$ denotes the expected value of random variable Y given the chain starts from state S_i.

1.2. Zero forcing and Markov chains. Suppose G is colored so that every vertex is blue or white. The *(deterministic) zero forcing color change rule* describes how the vertices of G can change color: a blue vertex u will *force* (change) a white vertex w to be blue if w is the only white neighbor of u. This is denoted by $u \to w$. Probabilistic zero forcing is a variant of the deterministic model. Let $B \subseteq V(G)$ be a set of blue vertices. In one round of PZF, each blue vertex $u \in B$ attempts to force each of its white neighbors $w \in N(u) \setminus B$ independently with probability

$$\mathbf{P}(u \to w) = \frac{|N[u] \cap B|}{\deg u}.$$

The study of PZF with Markov chains was introduced in [23] and studied further in [10, 17]. Let $\mathcal{S} = (S_0, \ldots, S_s)$ be the *ordered state space* of the 2^n possible colorings of a graph G.[1] The PZF Markov transition matrix for G is $M = M(G; \mathcal{S})$ such that M_{ij} is the probability of transitioning from coloring S_i to S_j in one time step [23]. We use $|S_i|$ to denote the number of blue vertices in state S_i, and we say $\mathcal{S}$ is *properly ordered* if $|S_i| \le |S_{i+1}|$ for all $i \in \{0, \ldots, s-1\}$ [10]. It is often helpful to think of a state S_i as a set of blue vertices, with the remaining vertices $V(G) \setminus S_i$ being white. We will use these two notions as convenient.

2. Reversion probabilistic zero forcing. This section introduces reversion probabilistic zero forcing, a modification of the PZF process where blue vertices have the chance to revert back to being white at the end of each round. Two variations of this process are defined. Single absorption reversion probabilistic zero forcing (SARPZF) adds a second phase to each round of PZF.

Definition 2.1. Given a graph G and set B of currently blue vertices, in *phase 1* each blue vertex $u \in B$ attempts to force each of its white neighbors $w \in N(u) \setminus B$ independently and simultaneously with probability

[1] It is standard in probabilistic zero forcing to simplify the state space $\mathcal{S}$ by omitting unreachable states and to combine states that behave analogously into a single state.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

$$\mathbf{P}(u \to w) = \frac{|N[u] \cap B|}{\deg u},$$

as in PZF. We now have an updated set B' of blue vertices. In *phase 2*, each blue vertex $u \in B'$ *reverts* (changes to being white) independently with probability $p \in (0,1)$. Phases 1 and 2, done consecutively, define the *SARPZF color change rule.* A round of SARPZF is one application of the SARPZF color change rule.

Unlike in PZF, it is possible SARPZF may never reach the state S_s where all vertices in G are blue. Moreover, it is not hard to see that SARPZF will always eventually result in all vertices being white. We say SARPZF *dies out* when this occurs. However, SARPZF may lead to G being entirely blue any number of times before dying out. It is natural to ask if $V(G)$ will ever be entirely blue. If so, when is the first time we expect this to happen? To answer these questions, we introduce a stopping condition to SARPZF.

Definition 2.2. The *dual absorption reversion probabilistic zero forcing color change rule* (DARPZF color change rule) is defined by modifying the SARPZF color change rule as follows: after phase 1, if the set of currently blue vertices B' is the entire vertex set, then no vertices revert in phase 2. A round of DARPZF is one application of the DARPZF color change rule.

Collectively, SARPZF and DARPZF are referred to as *reversion probabilistic zero forcing*, or RPZF. We say that DARPZF *fully forces* G when every vertex is blue and *dies out* when every vertex is white. We say DARPZF is *absorbed* whenever it dies out or fully forces G. In SARPZF, *absorbed* refers only to SARPZF dying out, hence the terminology *single* and *dual* absorption.

Remark 2.3. We adopt the convention that $0 < p < 1$ in RPZF. When $p = 1$, SARPZF trivially dies out in one step. When $p = 0$, many of the results presented can be adapted to recover PZF results. However, most of these results are already known and so we refer the reader to the PZF literature. Moreover, some results, such as those involving the matrix Q_S introduced in (4.9), are not suitable for adaptation to PZF.

Let G be a graph with n vertices and let $\mathcal{S} = (S_0, \ldots, S_s)$ be a properly ordered state space with all 2^n colorings of $V(G)$. Then G, $\mathcal{S}$, and $p \in (0,1)$ determine the SARPZF and DARPZF Markov transition matrices $M_S(G;\mathcal{S},p)$ and $M_D(G;\mathcal{S},p)$, respectively. We suppress the dependencies on G, $\mathcal{S}$, and p when they are clear from context. The RPZF Markov transition matrices can be derived from the previously established PZF Markov transition matrices. Let $F = M(G;\mathcal{S})$ describe phase 1 of RPZF on G using the PZF color change rule. Now define R_S and R_D to be the matrices which describe vertex reversion in SARPZF and DARPZF, respectively. Explicitly, if we regard S_i as a set of blue vertices then

$$(R_S)_{ij} = \begin{cases} p^{|S_i|-|S_j|}(1-p)^{|S_j|}, & S_j \subseteq S_i \\ 0, & \text{otherwise} \end{cases}, \tag{2.1}$$

and

$$(R_D)_{ij} = \begin{cases} p^{|S_i|-|S_j|}(1-p)^{|S_j|}, & S_j \subseteq S_i \neq S_s \\ 1, & S_i = S_j = S_s \\ 0, & \text{otherwise} \end{cases}. \tag{2.2}$$

Namely, the (i,j)th entries of R_S and R_D give the probability of moving from state S_i to state S_j via vertex reversion. The RPZF transition matrices are now given as

$$M_S = FR_S \quad \text{and} \quad M_D = FR_D. \tag{2.3}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

The idea behind combining states of $\mathcal{S}$ to create a new state space $\mathcal{S}'$ is follows. For analogously behaving states $S_{j_1}, S_{j_2}, \ldots \in \mathcal{S}$ and any state $S_i \in \mathcal{S}$, the events $\{S_i \text{ to } S_{j_k}\}$ are all mutually exclusive. Hence, if $S_{j_1}, S_{j_2}, \ldots \in \mathcal{S}$ are combined into a single state $S'_j \in \mathcal{S}'$, the probability of moving from $S_i \in \mathcal{S}$ to $S'_j \in \mathcal{S}'$ is the sum of the probabilities to move from S_i to S_{j_k}. The probability distribution for *leaving* $S'_j \in \mathcal{S}'$ is the same as that of any one of the states $S_{j_1}, S_{j_2}, \ldots \in \mathcal{S}$ as a direct consequence of the component states behaving analogously.

Example 2.4 (The Complete Graph). Let K_n denote the complete graph on n vertices, let $\mathcal{S} = \{S_0, \ldots, S_n\}$ be the properly ordered state space where S_i is the state of K_n having i blue vertices, and let $p \in (0,1)$. From [10, Theorem 2.4], $F = M(K_n; \mathcal{S})$ is the $(n+1) \times (n+1)$ matrix given by

$$F_{ij} = \begin{cases} \binom{n-i}{j-i}\left(1-\left(1-\frac{i}{n-1}\right)^i\right)^{j-i}\left(\left(1-\frac{i}{n-1}\right)^i\right)^{n-j}, & 1 \le i \le j \le n \\ 1, & i = j = 0 \\ 0, & \text{otherwise.} \end{cases}$$

The RPZF reversion matrices for K_n are given by

$$(R_S)_{ij} = \begin{cases} \binom{i}{j} p^{i-j}(1-p)^j, & 0 \le j \le i \le n \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad (R_D)_{ij} = \begin{cases} \binom{i}{j} p^{i-j}(1-p)^j, & 0 \le j \le i < n \\ 1, & i = j = n \\ 0, & \text{otherwise.} \end{cases}$$

Then $M_S(K_n; \mathcal{S}, p) = FR_S$ and $M_D(K_n; \mathcal{S}, p) = FR_D$.

The *SARPZF* and *DARPZF Markov Chains* are the Markov chains (X_t^S) and (X_t^D) with transition matrices M_S and M_D, respectively. That is, for any $t \in \mathbb{N}$ and any $i, j \in \{0, \ldots, s\}$,

$$\mathbf{P}[X_{t+1}^S = S_j \mid X_t^S = S_i] = (M_S)_{ij} \quad \text{and} \quad \mathbf{P}[X_{t+1}^D = S_j \mid X_t^D = S_i] = (M_D)_{ij}.$$

Just as probabilistic propagation time and expected propagation time were introduced for PZF in [17], we introduce new parameters of interest for RPZF.

Definition 2.5. Let G be a graph with properly ordered state space $(S_0, \ldots, S_s)$, B a set of blue vertices in G, and (X_t^S) and (X_t^D) the SARPZF and DARPZF Markov chains on G with reversion probability $p \in (0,1)$. The *probabilistic time of absorption for B under SARPZF* (respectively *DARPZF*) is the first time at which every vertex turns the same color, denoted

$$\text{pta}_S(G; B, p) = \min\{t \ge 0 : X_t^S = S_0\} \quad \text{and} \quad \text{pta}_D(G; B, p) = \min\{t \ge 0 : X_t^D \in \{S_0, S_s\}\},$$

where we define $\min \varnothing = \infty$. The *expected time of absorption for B under SARPZF* (*DARPZF*) is then

$$\text{eta}_S(G; B, p) = \mathbf{E}[\text{pta}_S(G; B, p)] \quad \text{and} \quad \text{eta}_D(G; B, p) = \mathbf{E}[\text{pta}_D(G; B, p)].$$

Finally, the *expected time of absorption under SARPZF* (*DARPZF*) of a connected graph G is the minimum of the expected time of absorption for B under SARPZF (DARPZF) over all one-vertex sets B of $V(G)$ and is denoted by $\text{eta}_S(G; p)$ ($\text{eta}_D(G; p)$). The S and D subscripts and superscripts are omitted for general RPZF chains.

Definition 2.5 is concerned with *when* RPZF is absorbed. But given a starting state (coloring) S_i, for what probability p does G have an equal chance of dying out or being fully forced in DARPZF?

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

Definition 2.6. The *critical reversion probability*, denoted $p_D(G,S_i)$, is the reversion probability such that the DARPZF Markov chain starting from state S_i has equal probability of dying out or fully forcing.

We show how to calculate $p_D(G,S_i)$ in Section 4 and prove that it exists for all connected graphs G and all nonabsorbing states S_i.

3. Threshold results for RPZF. In this section, we consider the behavior of RPZF as the number of vertices tends to infinity. To start, we establish fundamental results regarding the expected number of blue vertices after one step of the SARPZF Markov chain, following the methods of Theorem 3.1 in [10]. Recall that a state S_i, and thus a random variable X_t of an RPZF chain, can be viewed as a set of blue vertices.

Proposition 3.1. *Let (X_t) be a SARPZF Markov chain on a connected graph G with reversion probability $p\in(0,1)$ and properly ordered state space $(S_0,S_1,\dots,S_s)$. Let F_t be the number of vertices forced during phase 1 of time t (before reversion). Then for all $i\in\{0,\dots,s\}$, $\mathbf{E}_i[|X_1|]=(1-p)(|S_i|+\mathbf{E}_i[F_1])$.*

Proof. Suppose $X_0=S_i$. Let $v_1,\dots,v_{|S_i|+F_1}$ be the vertices that are blue after phase 1 of the SARPZF color change rule at time $t=1$. Notice that $\mathbf{1}[v_j\notin X_1]$ are i.i.d. indicator random variables for the event of v_j reverting, $j=1,\dots,|S_i|+F_1$. Thus,

$$\mathbf{E}_i[|X_1|]=|S_i|+\mathbf{E}_iF_1-\mathbf{E}_i\left[\sum_{j=1}^{|S_i|+F_1}\mathbf{1}[v_j\notin X_1]\right]=|S_i|+\mathbf{E}_iF_1-\mathbf{E}_i[|S_i|+F_1]p=(1-p)(|S_i|+\mathbf{E}_iF_1),$$

by independence, linearity of expectation, and the fact that v_1 reverts with probability p. ◻

Proposition 3.2. *Let G be a connected graph with b blue vertices and let $p\in(0,1)$ be the reversion probability. The expected number of blue vertices in G after one step of the SARPZF color change rule is bounded above by $(1-p)(b+b^2)$.*

Proof. Let (X_t) be the SARPZF chain on G with starting state $X_0=S_i$ corresponding to the vertices $B=\{v_1,\dots,v_b\}$ colored blue and let $V(G)\setminus B=\{w_1,\dots,w_\ell\}$. Let F_t be the number of vertices forced during phase 1 of time t (before reversion). We start by bounding $\mathbf{E}_iF_1$. For each $j=1,\dots,b$, let B_j be the set of vertices forced by v_j in phase 1. Note that multiple blue vertices may force the same white vertex and so the B_j's may intersect. Then

$$\mathbf{E}_iF_1=\mathbf{E}_i[|B_1\cup\dots\cup B_b|]\le\mathbf{E}_i\left[\sum_{j=1}^{b}|B_j|\right]=\sum_{j=1}^{b}\mathbf{E}_i|B_j|.$$

Let $Y_j(k)=1$ if v_j forces w_k and 0 otherwise. Then $|B_j|=\sum_{k=1}^{\ell}Y_j(k)$ and so

$$\mathbf{E}_i|B_j|=\sum_{k=1}^{\ell}\mathbf{E}_iY_j(k)=\sum_{k=1}^{\ell}\mathbf{P}_i[v_j\to w_k]=|N(v_j)\setminus B|\frac{|N(v_j)\cap B|}{|N(v_j)|}\le|N(v_j)\cap B|.$$

It follows that $\mathbf{E}_iF_1\le\sum_{j=1}^{b}|N[v_j]\cap B|\le b|B|=b^2$, and we conclude that $\mathbf{E}_i[|X_1|]\le(1-p)(b+b^2)$ from Proposition 3.1 since $|S_i|=b$. ◻

Observe that for any connected graph with SARPZF and DARPZF Markov processes (X_t^S) and (X_t^D), $\mathbf{E}_i[|X_1^D|]\ge\mathbf{E}_i[|X_1^S|]$ for all states S_i. Indeed, in phase 1, SARPZF and DARPZF share the same distribution by (2.3), and in phase 2, all vertices revert with probability p *unless* DARPZF steps into absorbing state S_s.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

3.1. The complete graph. When utilizing the RPZF Markov chain (X_t) on the complete graph, it will be convenient to consider the random variables X_t as taking integer values representing the number of blue vertices at the end of time t. For notational convenience, we denote the probability that a particular vertex in K_n is forced by one of its b blue neighbors by

$$q(n,b) = 1 - \left(1 - \frac{b}{n-1}\right)^b. \tag{3.4}$$

It will be useful to have an explicit description of the one-step Markov transition probability $\mathbf{P}_b[X_1 = k]$. For SARPZF, this can be done in terms of known probability distributions. Indeed, let B be the set of currently blue vertices with $|B| = b$, let X be a random variable equal to the number of vertices from B which do not revert, and let Y be a random variable equal to the number of vertices from $V \setminus B$ which are forced blue and do not revert. Then X and Y are independent with $X \sim \text{Binomial}(b, 1-p)$ and $Y \sim \text{Binomial}(n-b, (1-p)q(n,b))$. Moreover, if $X_0^S = b$, then $X_1^S = X + Y$ follows a Poisson binomial distribution with $p_1 = \cdots = p_b = 1-p$ and $p_{b+1} = \cdots = p_n = (1-p)q(n,b)$. This is used in the following result.

Proposition 3.3. *Let (X_t^S) and (X_t^D) be the SARPZF and DARPZF Markov chains on K_n, $n \geq 2$, with reversion probability $p \in (0,1)$, and let $q(n,b)$ be defined as in (3.4). Then for any $1 \leq b \leq n$ and $0 \leq k \leq n$, $\mathbf{P}_b[X_1^S = k]$ is equal to both of the following:*

(1) $\displaystyle\sum_{i=\max\{b,k\}}^{n} \binom{n-b}{i-b}\binom{i}{k}(1-p)^k p^{i-k} q(n,b)^{i-b}(1-q(n,b))^{n-i}$

(2) $\displaystyle\sum_{i=0}^{\min\{b,k\}} \binom{n-b}{k-i}\binom{b}{i}(1-p)^i p^{b-i}[(1-p)q(n,b)]^{k-i}[1-(1-p)q(n,b)]^{n-b-(k-i)}.$

Additionally,

$$\mathbf{P}_b[X_1^D = k] = \mathbf{P}_b[X_1^S = k] - \binom{n}{k} p^{n-k}(1-p)^k q(n,b)^{n-b} + \delta_{nk}\,(q(n,b))^{n-b}, \tag{3.5}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Proof. Let $M_S = FR_S$ and $M_D = FR_D$ denote the Markov transition matrices for the SARPZF and DARPZF chains (X_t^S) and (X_t^D) on K_n. Observe that $F_{bi} = 0$ when $i < b$ and $(R_S)_{ik} = 0$ when $i < k$. Formula (1) now follows from $\mathbf{P}_b[X_1^S = k] = (M_S)_{bk}$ and matrix multiplication. If $X_0^S = b$, then X_1^S is distributed as a Poisson binomial random variable with $p_1 = \cdots = p_b = 1-p$ and $p_{b+1} = \cdots = p_n = (1-p)q(n,b)$, the probability mass function of which is formula (2). Considering $\mathbf{P}_b[X_1^D = k]$, observe that $(R_S)_{ik} = (R_D)_{ik}$ for $k \leq i \leq n-1$ and so $\mathbf{P}_b[X_1^D = k] = \mathbf{P}_b[X_1^S = k] - F_{bn}(R_S)_{nk} + F_{bn}(R_D)_{nk}$. This simplifies to (3.5). ∎

We now calculate how the number of blue vertices in K_n is expected to change after one step of an RPZF process.

Theorem 3.4. *Let (X_t^S) and (X_t^D) be the SARPZF and DARPZF Markov chains on K_n, $n \geq 2$, with reversion probability $p \in (0,1)$, and let $q(n,b)$ be defined as in (3.4). Then*

$$\mathbf{E}_b[X_1^S] = (1-p)(b + (n-b)q(n,b)) \quad \textit{and} \quad \mathbf{E}_b[X_1^D] = \mathbf{E}_b[X_1^S] + np\,q(n,b)^{n-b}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

Proof. The value of $\mathbf{E}_b[X_1^S]$ is the known mean of a Poisson binomial random variable. To calculate $\mathbf{E}_b[X_1^D]$, let F_1 be the number of vertices forced blue at time $t = 1$. Observe that by Proposition 3.1 and linearity, $\mathbf{E}_b[X_1^S] = (1-p)\mathbf{E}_b[F_1 + b]$. Hence, by the definition of expected value

$$\begin{aligned}\mathbf{E}_b[X_1^D] &= (1-p)\sum_{k=b}^{n-1} k\mathbf{P}_b[F_1 + b = k] + n\mathbf{P}_b[F_1 + b = n]\\ &= \mathbf{E}_b[X_1^S] - (1-p)n\mathbf{P}_b[F_1 + b = n] + n\mathbf{P}_b[F_1 + b = n].\end{aligned}$$

Substituting $\mathbf{P}_b[F_1 = n - b] = q(n,b)^{n-b}$ and simplifying finishes the proof. ∎

The remainder of this section is dedicated to threshold-like results for RPZF on K_n. These results concern the necessary number of blue vertices b_n for a particular event to occur in one step of the RPZF chain with high probability, where b_n is a function of n, the total number of vertices in the graph, and $n \to \infty$.

Lemma 3.5. *Let (X_t^D) be the DARPZF Markov chain on K_n with reversion probability $p \in (0,1)$. If $b_n \geq \sqrt{n \log n^{2+\gamma}}$ with $\gamma > 0$, then*

$$\lim_{n\to\infty} |n - \mathbf{E}_{b_n}[X_1^D]| = 0.$$

Proof. Observe $\mathbf{E}_{n-1}[X_1^D] = \mathbf{E}_n[X_1^D] = n$, so we may assume $b_n \leq n-2$. Let $b_n \geq \sqrt{n\log n^{2+\gamma}}$ with $\gamma > 0$. Throughout this proof, let

$$g(n,b_n) = 1 - q(n,b_n) = \left(1 - \frac{b_n}{n-1}\right)^{b_n}.$$

Using Theorem 3.4, observe that

$$\mathbf{E}_{b_n}[X_1^D] = (1-p)\big(b_n + (n-b_n)\,(1-g(n,b_n))\,\big) + np\big(1 - g(n,b_n)\big)^{n-b_n},$$

which, after distributing $(n-b_n)$ and canceling the b_n terms, is equal to

$$(1-p)\big(n - (n-b_n)g(n,b_n)\big) + np\big(1-g(n,b_n)\big)^{n-b_n}.$$

Now distribute the $(1-p)$ and simplify to get

$$\begin{aligned}\mathbf{E}_{b_n}[X_1^D] &= (1-p)n - (1-p)(n-b_n)g(n,b_n) + np\big(1-g(n,b_n)\big)^{n-b_n}\\ &= n - (1-p)(n-b_n)g(n,b_n) - np\Big(1 - \big(1-g(n,b_n)\big)^{n-b_n}\Big).\end{aligned}$$

Additionally, $\mathbf{E}_{b_n}[X_1^D] \leq n$ because $X_1^D \leq n$ and so $|n - \mathbf{E}_{b_n}[X_1^D]| = n - \mathbf{E}_{b_n}[X_1^D]$, which in turn simplifies to

$$(1-p)(n-b_n)g(n,b_n) + np\Big[1 - \big(1-g(n,b_n)\big)^{n-b_n}\Big].$$

We show that each of these two terms converges to 0 as $n \to \infty$.

For the first term, it suffices to show that $ng(n,b_n) \to 0$ as $n \to \infty$ since $n - b_n$ and $g(n,b_n)$ are nonnegative. Recalling the Taylor expansion $\log(1-x) = -\sum_{k=1}^{\infty} x^k/k$ for $|x| < 1$, we have

$$\begin{aligned}g(n,b_n) = \exp\left\{b_n \log\left(1 - \frac{b_n}{n-1}\right)\right\} &= \exp\left\{b_n\left(-\frac{b_n}{n-1} - \sum_{k=2}^{\infty}\frac{b_n^k}{k(n-1)^k}\right)\right\}\\ &= \exp\left\{-\frac{b_n^2}{n-1}\right\}\exp\left\{-\sum_{k=2}^{\infty}\frac{b_n^{k+1}}{k(n-1)^k}\right\},\end{aligned}\tag{3.6}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

for all $b_n < n-1$. Then $g(n,b_n) \leq e^{-b_n^2/(n-1)}$ because $b_n > 0$. Using the inequality $b_n \geq \sqrt{n \log n^{2+\gamma}}$,

$$g(n,b_n) \leq \exp\left\{-\frac{b_n^2}{n-1}\right\} \leq \exp\left\{-\frac{n\log n^{2+\gamma}}{n-1}\right\} < \exp\left\{-\frac{n\log n^{2+\gamma}}{n}\right\} = n^{-(2+\gamma)}. \tag{3.7}$$

Hence,

$$0 \leq (1-p)\,(n-b_n)\,g(n,b_n) \leq \exp\left\{-\frac{b_n^2}{n-1}\right\} n < n^{-(1+\gamma)} \to 0,$$

as $n \to \infty$.

It is left to show

$$\lim_{n\to\infty} np\Big(1-\big(1-g(n,b_n)\big)^{n-b_n}\Big) = 0.$$

Define $H(n) = (n-b_n)\log\left(1-g(n,b_n)\right)$ so that $1-\big(1-g(n,b_n)\big)^{n-b_n} = 1-e^{H(n)}$. It thus suffices to show that $n(1-e^{H(n)}) \to 0$ as $n \to \infty$. Observe

$$H(n) = (n-b_n)\log(1-g(n,b_n)) \leq 0,$$

since $0 \leq g(n,b_n) < 1$. Hence, to prove $n(1-e^{H(n)}) \to 0$, it is enough to show

$$H(n) \geq \log\left(1-n^{-(1+\gamma)}\right),$$

for sufficiently large n, since then

$$0 \leq n(1-e^{H(n)}) \leq n(1-(1-n^{-(1+\gamma)})) = n^{-\gamma}.$$

To this end, notice that because $H(n) = (n-b_n)\log(1-g(n,b_n)) \leq 0$ and $b_n \leq n$,

$$H(n) \geq n\log(1-g(n,b_n)) = -n\sum_{k=1}^{\infty}\frac{g(n,b_n)^k}{k}.$$

We already showed in (3.7) that $g(n,b_n) \leq n^{-(2+\gamma)}$. Hence,

$$H(n) \geq -\sum_{k=1}^{\infty}\frac{n^{-k(2+\gamma)+1}}{k} \geq -\sum_{k=1}^{\infty}\frac{n^{-k(1+\gamma)}}{k} = \log(1-n^{-(1+\gamma)}),$$

and so $n(1-e^{H(n)}) \leq n^{-\gamma}$. It follows that

$$np\left[1-\left(1-\left(1-\frac{b_n}{n-1}\right)^{b_n}\right)^{n-b_n}\right] = np(1-e^{H(n)}) \to 0,$$

as $n \to \infty$. We have shown that each term of $|n - \mathbf{E}_{b_n}[X_1^D]|$ converges to 0 and so $|n - \mathbf{E}_{b_n}[X_1^D]| \to 0$. ∎

Considering the SARPZF chain (X_t^S), notice that

$$(1-p)n - \mathbf{E}_b[X_1^S] = (1-p)(n-b)\left(1-\frac{b}{n-1}\right)^b.$$

Notice also that $\mathbf{E}_b[X_1^S] \leq \mathbf{E}_n[X_1^S] = (1-p)n$ by Theorem 3.4. Following the proof of Lemma 3.5 through (3.7), one gets the following corollary that gives a threshold for the expected number of blue vertices in SARPZF to be close to $(1-p)n$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

Corollary 3.6. *Let (X_t^S) be the SARPZF Markov chain on K_n with reversion probability $p \in (0,1)$. If $b_n \geq \sqrt{n \log n^{1+\gamma}}$ with $\gamma > 0$, then*

$$\lim_{n\to\infty} |(1-p)n - \mathbf{E}_{b_n}[X_1^S]| = 0.$$

In other words, if K_n has $b_n = \Omega(\sqrt{n\log n})$ blue vertices, then with constant $C_1 > \sqrt{2}$ we expect n blue vertices after one application of the DARPZF color change rule, and with constant $C_2 > 1$ we expect $(1-p)n$ blue vertices after one application of the SARPZF color change rule. We refer the reader to Appendix A for a review of asymptotic notation and their common properties.

It turns out $\sqrt{n\log n}$ is the threshold for this behavior. Indeed, if $b_n = O(\sqrt{n \log n})$ with constant $C < 1$, then the SARPZF and DARPZF Markov chains on K_n converge to each other while getting arbitrarily far from n. This is made precise in the next result.

Proposition 3.7. *Let (X_t^S) and (X_t^D) denote the SARPZF and DARPZF Markov chains on K_n. If $b_n \leq \sqrt{n \log n^{1-\gamma}}$ with $\gamma > 0$, then $|\mathbf{E}_{b_n}[X_1^S] - \mathbf{E}_{b_n}[X_1^D]| \to 0$ as $n \to \infty$. Moreover, $|(1-p)n - \mathbf{E}_{b_n}[X_1^S]| \to \infty$ and $|n - \mathbf{E}_{b_n}[X_1^D]| \to \infty$ as $n \to \infty$.*

Proof. Let $b_n \leq \sqrt{n\log n^{1-\gamma}}$ with $\gamma > 0$ and let $g(n, b_n) = \left(1 - \frac{b_n}{n-1}\right)^{b_n}$. By Theorem 3.4, to prove $|\mathbf{E}_{b_n}[X_1^S] - \mathbf{E}_{b_n}[X_1^D]| \to 0$, it suffices to show that $np(1-g(n,b_n))^{n-b_n}$ converges to 0 as $n \to \infty$. Observe

$$g(n,b_n) = e^{b_n \log\left(1-\frac{b_n}{n-1}\right)} = e^{b_n\left(-\frac{b_n}{n-1} + O\left(\frac{b_n^2}{n^2}\right)\right)} = e^{b_n\left(-\frac{b_n}{n} - \frac{b_n}{n^2-n} + O\left(\frac{b_n^2}{n^2}\right)\right)} = e^{-\frac{b_n^2}{n}} e^{O\left(\frac{b_n^3}{n^2}\right)},$$

since $-\frac{b_n}{n^2-n} = O(b_n^2/n^2)$. Using the expansion $e^x = \sum_{i=0}^\infty x^i/i!$ this is equal to

$$e^{-\frac{b_n^2}{n}}\left[1 + O\left(\frac{b_n^3}{n^2}\right) + O\left(\frac{b_n^6}{n^4}\right) + \cdots\right] = e^{-\frac{b_n^2}{n}}\left[1 + O\left(\frac{b_n^3}{n^2}\right)\right].$$

Now apply the assumption $b_n \leq \sqrt{n\log n^{1-\gamma}}$ to get $g(n,b) \geq \frac{1}{n^{1-\gamma}}\left[1 + O\left(\frac{b_n^3}{n^2}\right)\right]$.

Turning to $(1-g(n,b_n))^{n-b_n}$, this can be written as

$$\exp\left\{-(n-b_n)\sum_{k\geq 1}\frac{g(n,b_n)^k}{k}\right\} \leq e^{-(n-b_n)g(n,b_n)},$$

and substituting $-g(n,b) \leq -\frac{1}{n^{1-\gamma}}\left[1 + O\left(\frac{b_n^3}{n^2}\right)\right]$ gives

$$(1-g(n,b_n))^{n-b_n} \leq \exp\left\{-(n-b_n)n^{-(1-\gamma)}\left[1 + O\left(\frac{b_n^3}{n^2}\right)\right]\right\}.$$

Finally, apply $b_n \leq \sqrt{n\log n^{1-\gamma}}$ and simplify to find

$$0 \leq (1-g(n,b_n))^{n-b_n} \leq \exp\left\{-n^\gamma\left(1 - \frac{\sqrt{\log n^{1-\gamma}}}{\sqrt{n}}\right)\left[1 + O\left(\frac{b_n^3}{n^2}\right)\right]\right\}.$$

Since $\sqrt{\log n^{1-\gamma}/n} \to 0$ and $O(b_n^3/n^2) \to 0$ as $n \to \infty$, we conclude that $(1-g(n,b_n))^{n-b_n} = O(e^{-n^\gamma})$ from which it follows that $np(1-g(n,b_n))^{n-b_n} \to 0$ as $n \to \infty$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

Additionally,

$$|n-\mathbf{E}_{b_n}[X_1^D]| \geq (1-p)(n-b_n)g(n,b_n) = |(1-p)n-\mathbf{E}_{b_n}[X_1^S]|,$$

and from $g(n,b) \geq \frac{1}{n^{1-\gamma}}\left[1+O\left(\frac{b_n^3}{n^2}\right)\right]$ it follows that

$$\begin{aligned}(n-b_n)g(n,b_n) \geq \left(n^\gamma - \frac{b_n}{n^{1-\gamma}}\right)\left[1+O\left(\tfrac{b_n^3}{n^2}\right)\right] &\geq \left(n^\gamma - \frac{\sqrt{n\log n^{1-\gamma}}}{n^{1-\gamma}}\right)\left[1+O\left(\tfrac{b_n^3}{n^2}\right)\right] \\ &= n^\gamma\left(1-\frac{\sqrt{n\log n^{1-\gamma}}}{n}\right)\left[1+O\left(\tfrac{b_n^3}{n^2}\right)\right],\end{aligned}$$

which tends to infinity as $n\to\infty$. ☐

This result, combined with Lemma 3.5, shows that $\sqrt{n\log n}$ is the threshold number of blue vertices for DARPZF on the complete graph to fully force in one step.

Theorem 3.8. *Let (X_t^D) be the DARPZF Markov chain on K_n with reversion probability $p\in(0,1)$, and let $\gamma>0$.*

- *If $b_n \leq \sqrt{n\log n^{1-\gamma}}$ then $|n-\mathbf{E}_{b_n}[X_1^D]|\to\infty$ as $n\to\infty$, and*
- *if $b_n \geq \sqrt{n\log n^{2+\gamma}}$ then $|n-\mathbf{E}_{b_n}[X_1^D]|\to 0$ as $n\to\infty$.*

That is, $\sqrt{n\log n}$ is the threshold function for expecting DARPZF to fully force in one step.

In fact, if $b_n \geq \sqrt{n\log n^{2+\gamma}}$, then since $n\geq X_1^D\geq 0$ we have $|n-\mathbf{E}_{b_n}[X_1^D]| = \mathbf{E}_{b_n}[|n-X_1^D|]\to 0$ as $n\to\infty$. Formally, one says that if $b_n\geq\sqrt{n\log n^{2+\gamma}}$, then X_1^D *converges in mean* to n.

The next result, as a consequence of previous, states that if K_n has asymptotically greater than $\sqrt{n\log n}$ blue vertices, then with high probability K_n will be entirely blue after one step of the DARPZF Markov chain. If the number of blue vertices is asymptotically below $\sqrt{n\log n}$, then with high probability the DARPZF chain will not have n blue vertices.

Theorem 3.9. *Let (X_t^D) be the DARPZF Markov chain on K_n with reversion probability $p\in(0,1)$, and let $\gamma>0$.*

- *If $b_n\leq\sqrt{n\log n^{1-\gamma}}$, then $\mathbf{P}_{b_n}[X_1^D=n]\to 0$ as $n\to\infty$, and*
- *if $b_n\geq\sqrt{n\log n^{1+\gamma}}$, then $\mathbf{P}_{b_n}[X_1^D=n]\to 1$ as $n\to\infty$.*

Proof. Let b_n be such that $0\leq b_n\leq n$ for all n. We may assume $b_n\leq n-2$ since $\mathbf{P}_{n-1}[X_1^D=n]=1$. Notice $\mathbf{P}_{b_n}[X_1^D=n]=q(n,b_n)^{n-b_n}=(1-g(n,b_n))^{n-b_n}$. If $b_n\leq\sqrt{n\log n^{1-\gamma}}$, then in Proposition 3.7, we showed $(1-g(n,b_n))^{n-b_n}=O(e^{-n^\gamma})$. Thus, $\mathbf{P}_{b_n}[X_1^D=n]\to 0$ as $n\to\infty$.

Suppose now $b_n\geq\sqrt{n\log n^{1+\gamma}}$. Define $H(n)=(n-b_n)\log(1-g(n,b_n))$ so that

$$1-\big(1-g(n,b_n)\big)^{n-b_n} = 1-e^{H(n)}.$$

It suffices to show that $1-e^{H(n)}\to 0$ as $n\to\infty$. Equivalently, we show $H(n)\geq\log(1-n^{-\gamma})$ for sufficiently large n. The proof of this is almost identical to that in Lemma 3.5. Indeed, notice that because $H(n)=(n-b_n)\log(1-g(n,b_n))<0$,

$$H(n) > n\log(1-g(n,b_n)) = -n\sum_{k=1}^{\infty}\frac{g(n,b_n)^k}{k}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

We showed in (3.7) that if $b_n \geq \sqrt{n \log n^{1+\gamma}}$, then $g(n, b_n) \leq n^{-(1+\gamma)}$. Hence,

$$H(n) > -\sum_{k=1}^{\infty} \frac{n^{-k(1+\gamma)+1}}{k} \geq -\sum_{k=1}^{\infty} \frac{n^{-k\gamma}}{k} = \log(1 - n^{-\gamma}),$$

and so $1 - (1 - g(n, b_n))^{n-b_n} \to 0$, which implies $\mathbf{P}_{b_n}[X_1^D = n] = (1 - g(n, b_n))^{n-b_n} \to 1$ as $n \to \infty$. ∎

Since $n \geq X_1^D$, this is equivalent to saying that X_1^D *converges in probability* to n when $b_n \geq \sqrt{n \log n^{1+\gamma}}$. Formally, $\mathbf{P}_{b_n}[|n - X_1^D| > \varepsilon] \to 0$ for all $\varepsilon > 0$ which is equivalent to $\mathbf{P}_{b_n}[|n - X_1^D| = 0] \to 1$, and so $X_1^D = n$ with high probability. As a complimentary result, we can directly calculate the limit of $\mathbf{P}_b[X_1 = 0]$ for RPZF on K_n as $n \to \infty$.

THEOREM 3.10. *Suppose K_n has $1 \leq b \leq n-2$ vertices colored blue and RPZF chain (X_t) with reversion probability $p \in (0, 1)$. Then the probability K_n dies out in one step of the RPZF chain converges to $p^b e^{b^2(p-1)}$ as $n \to \infty$.*

Proof. By formula (2) of Proposition 3.3,

$$\mathbf{P}_b[X_1^S = 0] = p^b[1 - (1-p)q(n, b)]^{n-b} = p^b \left[p + (1-p)\Big(1 - \frac{b}{n-1}\Big)^b\right]^{n-b}.$$

Similarly,

$$\mathbf{P}_b[X_1^D = 0] = p^b \left[p + (1-p)\Big(1 - \frac{b}{n-1}\Big)^b\right]^{n-b} - p^n \left(1 - \Big(1 - \frac{b}{n-1}\Big)^b\right)^{n-b}.$$

To calculate the limits of these values, notice first that $0 \leq \left(1 - (1 - \frac{b}{n-1})^b\right)^{n-b} \leq 1$ for all $0 \leq b \leq n-2$ and hence

$$0 \leq p^n \left(1 - \Big(1 - \frac{b}{n-1}\Big)^b\right)^{n-b} \leq p^n \to 0,$$

as $n \to \infty$ since $p \in (0, 1)$. We are left to consider $p^b \left(p + (1-p)(1 - \frac{b}{n-1})^b\right)^{n-b}$. Taking $n \to \infty$, this is equal to

$$p^b \exp\left\{\lim_{n\to\infty} \frac{1}{(n-b)^{-1}} \log\left[p + (1-p)\Big(1 - \frac{b}{n-1}\Big)^b\right]\right\},$$

and applying L'Hôpital's rule this simplifies to $p^b e^{b^2(p-1)}$. ∎

Finally, we establish when the upper bound presented in Proposition 3.2 is asymptotically tight.

PROPOSITION 3.11. *Let (X_t) be an RPZF Markov chain on K_n with reversion probability p. If $b_n \leq \frac{\sqrt{n}}{\log n}$, then for any $\varepsilon > 0$,*

$$(1-p)(b_n + (1-\varepsilon){b_n}^2) \leq \mathbf{E}_{b_n}[X_1] \leq (1-p)(b_n + {b_n}^2)$$

for n sufficiently large. In particular, if $b \in \mathbb{N}$ is fixed, then $\mathbf{E}_b[X_1] \to (1-p)(b + b^2)$ as $n \to \infty$.

Proof. Since $b_n \leq \frac{\sqrt{n}}{\log n} \leq \sqrt{n \log n}$, by Proposition 3.7, we need consider only the SARPZF chain. By Proposition 3.2, $\mathbf{E}_{b_n}[X_1] \leq (1-p)(b_n + b_n^2)$. Let F_1 denote the number of vertices forced at time $t = 1$ during phase 1. Then by Proposition 3.1, $\mathbf{E}_{b_n}[X_1] = (1-p)(b_n + \mathbf{E}_{b_n}[F_1])$. The authors of [10] showed in the proof of Theorem 3.1 that $\mathbf{E}_{b_n}[F_1] = b_n^2 - o(b_n^2)$ when $b_n \leq \frac{\sqrt{n}}{\log n}$. Consequently, since $b_n^2 = o(n)$, for any $\varepsilon > 0$, $\mathbf{E}_{b_n}[F_1] > (1-\varepsilon)b_n^2$ for sufficiently large n. Thus, $\mathbf{E}_{b_n}[X_1] > (1-p)(b_n + (1-\varepsilon)b_n^2)$ and for fixed $b \in \mathbb{N}$, taking $\varepsilon \to 0$ gives that $\mathbf{E}_b[X_1] \to (1-p)(b + b^2)$ as $n \to \infty$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

3.2. The balanced complete bipartite graph. The *complete bipartite graph* $K_{m,n}$ is the graph of order $m+n$ whose vertices can be partitioned into two parts $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ such that the edges of the graph are $u_i v_j$ for all $1 \le i \le m$ and all $1 \le j \le n$. If $m = n$, then $K_{n,n}$ is the *balanced* complete bipartite graph. We will see here and in Section 5 that $K_{n,n}$ behaves very similarly to K_{2n} in DARPZF.

LEMMA 3.12. *Let $K_{m,n}$ have vertex partitions U and V. Suppose b_U vertices in U are blue and b_V vertices in V are blue. The probability that U forces V entirely blue in one step is*

$$\mathbf{P}[U \to V] = \left(1 - \left(1 - \frac{b_V + 1}{|V|}\right)^{b_U}\right)^{|V| - b_V}.$$

Proof. Let B be the set of blue vertices in $K_{m,n}$. Let $u \in U \cap B$ be blue and $v \in V \setminus (B \cap V)$ be white. Then

$$\mathbf{P}[u \to v] = \frac{|N[u] \cap B|}{\deg u} = \frac{b_V + 1}{|V|},$$

and so the probability that v is forced by some vertex in U is

$$\mathbf{P}[U \to v] = 1 - \mathbf{P}[U \not\to v] = 1 - \prod_{u \in U \cap B} \mathbf{P}[u \not\to v] = 1 - \left(1 - \frac{b_V + 1}{|V|}\right)^{b_U}.$$

Thus,

$$\mathbf{P}[U \to V] = \prod_{v \in V \setminus (B \cap V)} \mathbf{P}[U \to v] = \left(1 - \left(1 - \frac{b_V + 1}{|V|}\right)^{b_U}\right)^{|V| - b_V}. \qquad \square$$

We now give an upper bound for the threshold number of blue vertices to fully force the balanced complete bipartite in one step with high probability, starting from one of two cases. The first case is when the vertex parts U and V of $K_{n,n}$ have the same number of blue vertices, and the second is when, without loss of generality, U is entirely blue and V is minimally blue.

THEOREM 3.13. *Let $K_{n,n}$ have vertex parts U and V with at least b_n^U and b_n^V blue vertices in U and V, respectively.*

- *If $b_n^U \ge \sqrt{n \log n^{1+\gamma_1}}$ and $b_n^V \ge \sqrt{n \log n^{1+\gamma_2}}$ for any $\gamma_1, \gamma_2 > 0$, then with high probability $K_{n,n}$ is blue after one application of the DARPZF color change rule as $n \to \infty$.*
- *If $b_n^U = n$ and $b_n^V \ge \log(n^{1+\gamma})$ for any $\gamma > 0$, then with high probability $K_{n,n}$ is blue after one application of the DARPZF color change rule as $n \to \infty$.*

Proof. Suppose first $b_n^U \ge \sqrt{n \log n^{1+\gamma_1}}$ and $b_n^V \ge \sqrt{n \log n^{1+\gamma_2}}$ for some $\gamma_1, \gamma_2 > 0$. Let $\gamma = \min\{\gamma_1, \gamma_2\}$. We may assume U and V each have $b_n \ge \sqrt{n \log n^{1+\gamma}}$ blue vertices because the probability of fully forcing in one step monotonically increases in both b_n^V and b_n^U. Since the events $\{U \to V\}$ and $\{V \to U\}$ are independent at time t, the probability that $K_{n,n}$ is blue after one step of DARPZF is

$$\mathbf{P}[U \to V]\mathbf{P}[V \to U] = \left(\left(1 - \left(1 - \frac{b_n + 1}{n}\right)^{b_n}\right)^{n - b_n}\right)^2 = \left(1 - \left(1 - \frac{b_n + 1}{n}\right)^{b_n}\right)^{2n - 2b_n},$$

by Lemma 3.12. Call this probability $P(b_n)$. We wish to show $P(b_n) \to 1$ as $n \to \infty$. Note that $P(b_n) = 1$ if $b_n = n$ so assume $b_n \le n - 1$. Define $f(n, b_n) = \left(1 - \frac{b_n+1}{n}\right)^{b_n}$ and let $H(n) = (2n - 2b_n)\log(1 - f(n, b_n))$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

Then $P(b_n) = e^{H(n)}$ and so $P(b_n) \to 1$ if $H(n) \to 0$. Observe that $H(n) < 0$ and hence

$$H(n) > 2n \log(1 - f(n, b_n)) = -2n \sum_{k=1}^{\infty} \frac{f(n, b_n)^k}{k}.$$

Now in the style of Theorem 3.9, if $f(n, b_n) \leq n^{-(1+\gamma)}$ then

$$H(n) > -2 \sum_{k=1}^{\infty} \frac{n^{-k(1+\gamma)+1}}{k} \geq -2 \sum_{k=1}^{\infty} \frac{n^{-k\gamma}}{k} = 2 \log(1 - n^{-\gamma}) \to 0,$$

as $n \to \infty$. To see that $f(n, b_n) \leq n^{-(1+\gamma)}$, observe

$$f(n, b_n) = \exp\left\{ b_n \log\left(1 - \frac{b_n + 1}{n}\right)\right\} = \exp\left\{ -b_n \sum_{k=1}^{\infty} \frac{(b_n+1)^k}{kn^k}\right\}$$
$$= \exp\left\{ -b_n \frac{b_n + 1}{n}\right\} \exp\left\{ -b_n \sum_{k=2}^{\infty} \frac{(b_n+1)^k}{kn^k}\right\},$$

which is bounded above by $e^{-b_n^2/n}$ since $b_n > 0$. Then because $b_n \geq \sqrt{n \log n^{1+\gamma}}$,

$$f(n, b_n) \leq e^{-b_n^2/n} \leq n^{-(1+\gamma)}.$$

Hence, $H(n) < 0$ implies $H(n) \to 0$ and thus $P(b_n) = e^{H(n)} \to 1$ as $n \to \infty$.

Now assume $b_n^U = n$ and $b_n^V \geq \log(n^{1+\gamma})$ for some $\gamma > 0$. Then the probability that $K_{n,n}$ is entirely blue after one step of DARPZF is

$$\mathbf{P}[U \to V] = \left(1 - \left(1 - \frac{b_n^V + 1}{n}\right)^n\right)^{n - b_n^V}.$$

Let $f(n, b_n^V) = \left(1 - \frac{b_n^V+1}{n}\right)^n$ and $H(n) = (n - b_n^V) \log(1 - f(n, b_n^V))$. Notice that f differs from before in the exponential. Like before, it suffices to show that $f(n, b_n^V) \leq n^{-(1+\gamma)}$ because then

$$H(n) > n \log(1 - f(n, b_n^V)) = -n \sum_{k=1}^{\infty} \frac{f(n, b_n^V)}{k} \geq -\sum_{k=1}^{\infty} \frac{n^{-k\gamma}}{k} = \log(1 - n^{-\gamma}).$$

Now,

$$f(n, b_n^V) = \exp\left\{ n \log\left(1 - \frac{b_n^V + 1}{n}\right)\right\} = \exp\left\{ -n \sum_{k=1}^{\infty} \frac{(b_n^V+1)^k}{kn^k}\right\}$$
$$= \exp\left\{ -n \frac{b_n^V + 1}{n}\right\} \exp\left\{ -n \sum_{k=2}^{\infty} \frac{(b_n^V+1)^k}{kn^k}\right\},$$

which is bounded above by $e^{-b_n^V}$ since $b_n^V > 0$. Then because $b_n^V \geq \log n^{1+\gamma}$ it follows that $f(n, b_n^V) \leq n^{-(1+\gamma)}$. Thus, $0 > H(n) > \log(1 - n^{-\gamma}) \to 0$ and so $\mathbf{P}[U \to V] = e^{H(n)} \to 1$ as $n \to \infty$. □

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

3.3. The star graph. The star graph on n vertices is $K_{1,n-1}$, and the singleton vertex is called the *universal vertex* because it is adjacent to all other vertices. In this section, we show that the star graph exhibits a large threshold value for one-step forcing. Let $K_{1,n-1}$ have universal vertex v and set of currently blue vertices B. Notice that if $|B| = b \leq n-2$, then v must be blue for $K_{1,n-1}$ to be fully forced in one step. Hence, when calculating the one-step threshold for $K_{1,n-1}$, we need consider only when $v \in B$. In that case, $K_{1,n-1}$ is fully forced in one step with probability

$$\mathbf{P}[v \to V(K_{1,n-1}) \setminus B] = \left(\frac{b}{n-1}\right)^{n-b}.$$

LEMMA 3.14. *Let (X_t^S) and (X_t^D) be the SARPZF and DARPZF Markov chains on $K_{1,n-1}$ with $n \geq 2$ and universal vertex v. If v blue at time $t = 0$, then*

$$\mathbf{E}_b[X_1^S] = (1-p)\left(b + (n-b)\left(\frac{b}{n-1}\right)\right),$$

and

$$\mathbf{E}_b[X_1^D] = \mathbf{E}_b[X_1^S] + np\left(\frac{b}{n-1}\right)^{n-b}.$$

Proof. Let $w_1, \ldots, w_{n-b}$ denote the white vertices. Then by Proposition 3.1,

$$\mathbf{E}_b[X_1^S] = (1-p)\big(b + (n-b)\mathbf{P}_b[v \to w_i]\big) = (1-p)\left(b + (n-b)\left(\frac{b}{n-1}\right)\right).$$

Using the same approach as that in Theorem 3.4, one also calculates

$$\mathbf{E}_b[X_1^D] = \mathbf{E}_b[X_1^S] + np\left(\frac{b}{n-1}\right)^{n-b}. \qquad \square$$

Observe that if $K_{1,n-1}$ has $b_n = n-1-C$ vertices blue, then

$$\mathbf{P}[v \to V(K_{1,n-1}) \setminus B] = \left(1 - \frac{C}{n-1}\right)^{1+C} \to 1,$$

as n grows to infinity. This turns out to be the threshold for fully forcing in DARPZF: if the distance between n and b_n is unbounded, then with high probability $K_{1,n-1}$ is not fully forced in the next step.

THEOREM 3.15. *Let (X_t^D) be the DARPZF Markov chain on $K_{1,n-1}$ with universal vertex v and reversion probability $p \in (0,1)$. When v is blue, we have the following:*

- *if $b_n = n-1-C$ for some constant $C \in \mathbb{N}$, then $|n - \mathbf{E}_{b_n}[X_1^D]| \to pC(C+1)$ as $n \to \infty$, and*
- *if $b_n = n-1-\omega(1)$, then $|n - \mathbf{E}_{b_n}[X_1^D]| \to \infty$ as $n \to \infty$.*

Proof. Let (X_t^D) be the DARPZF Markov chain on $K_{1,n-1}$ with reversion probability $p \in (0,1)$, let $0 \leq b_n \leq n-1$, and let v denote the universal vertex. Assume v is blue. If we replace $g(n, b_n) = \left(1 - \frac{b_n}{n-1}\right)^{b_n}$ with $1 - \frac{b_n}{n-1}$ in the proof of Lemma 3.5, then we may simplify $n - \mathbf{E}_{b_n}[X_1^D]$ as

$$n - \mathbf{E}_{b_n}[X_1^D] = (1-p)(n-b_n)\left(1 - \frac{b_n}{n-1}\right) + np\left[1 - \left(\frac{b_n}{n-1}\right)^{n-b_n}\right]. \tag{3.8}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

Suppose $b_n = n-1-C$ for some constant $C \geq 0$. Then

$$\begin{aligned} n - \mathbf{E}_{b_n}[X_1^D] &= (1-p)(n-(n-1-C))\left(1-\frac{n-1-C}{n-1}\right) + np\left(1-\left(\frac{n-1-C}{n-1}\right)^{n-(n-1-c)}\right) \\ &= (1-p)(1+C)\frac{C}{n-1} + p\left(n - n\left(1-\frac{C}{n-1}\right)^{C+1}\right). \end{aligned}$$

It is immediate that $(1-p)(1+C)\frac{C}{n-1} \to 0$ as $n \to \infty$. To see that $p\left(n-n(1-\frac{C}{n-1})^{C+1}\right) \to pC(C+1)$, observe

$$\begin{aligned} n - n\left(1-\frac{C}{n-1}\right)^{C+1} &= n - n\sum_{k=0}^{C+1}(-1)^k\binom{C+1}{k}\left(\frac{C}{n-1}\right)^k \\ &= n - n + n(C+1)\frac{C}{n-1} - n\sum_{k=2}^{C+1}(-1)^k\binom{C+1}{k}\left(\frac{C}{n-1}\right)^k \\ &= C(C+1)\frac{n}{n-1} + O(n^{-1}). \end{aligned}$$

Taking $n \to \infty$ gives $n - \mathbf{E}_{b_n}[X_1^D] \to pC(C+1)$.

On the other hand, suppose $b_n = n-1-f_n$ with $f_n = \omega(1)$. We want to show

$$n - \mathbf{E}_{b_n}[X_1^D] = (1-p)(n-b_n)\left(1-\frac{b_n}{n-1}\right) + np\left[1-\left(\frac{b_n}{n-1}\right)^{n-b_n}\right] \to \infty.$$

If $f_n = \omega(\sqrt{n})$, then

$$(1-p)(n-b_n)\left(1-\frac{b_n}{n-1}\right) = (1-p)(1+f_n)\frac{f_n}{n-1} = (1-p)\frac{f_n+f_n^2}{n-1} \to \infty,$$

so assume $f_n = O(\sqrt{n})$. Notice

$$np\left[1-\left(\frac{b_n}{n-1}\right)^{n-b_n}\right] = np\left[1-\left(1-\frac{f_n}{n-1}\right)^{1+f_n}\right],$$

and so define $h_n(x) = x^{1+f_n}$. Now $h_n'(x) = (1+f_n)x^{f_n}$ and then applying Taylor's theorem to $h_n(x)$ around $x_0 = 1$,

$$h_n(x) = h_n(1) + h_n'(\xi)(x-1),$$

for some $\xi \in (x,1)$. Thus,

$$h_n\left(1-\frac{f_n}{n-1}\right) = 1-(1+f_n)\xi_n^{f_n}\frac{f_n}{n-1},$$

for some $\xi_n \in \left(1-\frac{f_n}{n-1}, 1\right)$ and it follows that

$$np\left[1-\left(\frac{b_n}{n-1}\right)^{n-b_n}\right] = np\left[1-h_n\left(1-\frac{f_n}{n-1}\right)\right] = np\left[(1+f_n)\xi_n^{f_n}\frac{f_n}{n-1}\right],$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

which simplifies to $\frac{np}{n-1}\xi_n^{f_n}(f_n+f_n^2)$. By definition, $f_n+f_n^2 \to \infty$ when $f_n=\omega(1)$. However, $\xi_n^{f_n}$ must be accounted for; if we can show $\xi_n^{f_n} \geq M$ for some constant $M>0$ not dependent on n, then $\frac{np}{n-1}\xi_n^{f_n}(f_n+f_n^2) \geq \frac{np}{n-1}M(f_n+f_n^2) \to \infty$ as desired. Recall $\xi_n \in \left(1-\frac{f_n}{n-1},1\right)$ and so $\xi_n^{f_n} > \left(1-\frac{f_n}{n-1}\right)^{f_n}$. Observe

$$\left(1-\frac{f_n}{n-1}\right)^{f_n} = \exp\left\{f_n \log\left(1-\frac{f_n}{n-1}\right)\right\},$$

and using the Taylor expansion $\log(1-x) = -\sum_{k=1}^{\infty} x^k/x$ for $|x|<1$,

$$f_n \log\left(1-\frac{f_n}{n-1}\right) = -f_n \sum_{k\geq 1}\frac{1}{k}\left(\frac{f_n}{n-1}\right)^k = -\sum_{k\geq 1}\frac{f_n^{k+1}}{k(n-1)^k}.$$

It follows from $f_n = O(\sqrt{n})$ that

$$f_n \log\left(1-\frac{f_n}{n-1}\right) = O(1),$$

and so there exists an $M>0$ independent of n such that for sufficiently large n, $\left|f_n \log\left(1-\frac{f_n}{n-1}\right)\right| < M$. Equivalently, $\left(1-\frac{f_n}{n-1}\right)^{f_n} > e^{-M}$. Thus, $\xi_n \in \left(1-\frac{f_n}{n-1},1\right)$ implies $\xi_n^{f_n} > \left(1-\frac{f_n}{n-1}\right)^{f_n} > e^{-M} > 0$ and so

$$np\left[1-\left(\frac{b_n}{n-1}\right)^{n-b_n}\right] = \frac{2np}{n-1}\xi_n^{f_n}(f_n+f_n^2) \to \infty,$$

since $f_n = \omega(1)$.

It is left to consider when $f_n \neq O(\sqrt{n})$ and $f_n \neq \omega(\sqrt{n})$. In this case, define $g_n = \sup_{m\geq n} f_m$. Then $g_n \geq f_n$ and thus $n-1-f_n \geq n-1-g_n$. Since $\mathbf{E}_{b_n}[X_1^D]$ monotonically increases in b_n (keeping v blue), it suffices to show that $n-\mathbf{E}_{n-1-g_n}[X_1^D] \to \infty$. We claim that $g_n = \omega(\sqrt{n})$. Indeed, let $M>0$ and $N\in\mathbb{N}$ be arbitrary. Because $f_n \neq O(\sqrt{n})$, there exists an $n_0 > N$ such that $f_{n_0} > M\sqrt{n_0}$. Then

$$g_N = \sup_{m\geq N} f_m \geq f_{n_0} > M\sqrt{n_0} > M\sqrt{N},$$

and so $g_n = \omega(\sqrt{n})$. It follows from before that $n-\mathbf{E}_{n-1-g_n}[X_1^D] \to \infty$ and so $n-\mathbf{E}_{n-1-f_n}[X_1^D] \to \infty$. ☐

Observe that $|n-\mathbf{E}_{b_n}[X_1^D]| = \mathbf{E}_{b_n}[|n-X_1^D|]$ since $n \geq \mathbf{E}_{b_n}[X_1^D]$ and so Theorem 3.15 says X_1^D converges in mean to n exactly when $b_n = n-1$ for sufficiently large n. As a consequence, X_1^D converges to n in probability and hence $X_1^D = n$ with high probability.

4. Calculating RPZF parameters. This section presents standard Markov chain results for RPZF chains and parameters on any graph. These results are utilized in Section 5 to calculate the RPZF parameters of specific graph families. We provide brief explanations to support readers less familiar with probability. For an introduction to and proofs of the standard Markov chain results discussed here, the reader is directed to [19, Ch. 11]. More advanced results and details may be found in, e.g., [12].

Let (X_t) be an RPZF Markov chain on a graph G with reversion probability $p\in(0,1)$, properly ordered state space $(S_0,\ldots,S_s)$, and Markov transition matrix M. Let $T(j) = \min\{t\geq 1 : X_t = S_j\}$ be the *time of first arrival* to S_j (sometimes called the *first return time*). Notice that the starting state X_0 is not explicit in the notation. Define

$$\rho_{ij} = \mathbf{P}_i[T(j)<\infty] = \mathbf{P}[X_t = S_j \text{ for some } t\geq 1 \mid X_0 = S_i],$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

to be the probability that the RPZF chain enters state S_j after starting from state S_i. Then the critical reversion probability $p_D(G, S_i)$ is the DARPZF reversion probability such that $\rho_{i0} = \rho_{is} = 1/2$. We can also classify states of the Markov chain. If $\rho_{ii} = 1$, then state S_i is said to be *recurrent*, if $\rho_{ii} < 1$, then state S_i is said to be *transient*, and if $\mathbf{P}_i[X_1 = S_i] = 1$, then state S_i is said to be *absorbing*.

PROPOSITION 4.1. *Let G be a connected graph and $(S_0, \ldots, S_s)$ be a properly ordered state space of G for an RPZF chain with reversion probability $p \in (0, 1)$. Then the state S_0 is absorbing, the state S_i is transient for all $i \in \{1, \ldots, s-1\}$, and the state S_s is absorbing in DARPZF and transient in SARPZF.*

Proof. Let (X_t) be an RPZF chain on a graph G with n vertices. It is immediate from the Markov matrices (see (4.9)) that S_0 is an absorbing state for SARPZF and DARPZF. Next, consider the state S_s where all vertices are blue. In the DARPZF chain it is immediate from (4.9) that S_s is absorbing. On the other hand, in the SARPZF chain $\mathbf{P}_s[T(s) = \infty] \geq \mathbf{P}_s[X_1 = S_0] = p^n > 0$ and so S_s is transient in SARPZF. Finally, let $i \in \{1, \ldots, s-1\}$ and consider $\rho_{ii} = \mathbf{P}_i[T(i) < \infty] = 1 - \mathbf{P}_i[T(i) = \infty]$. To prove S_i is transient, it suffices to show $\mathbf{P}_i[T(i) = \infty] > 0$. This follows from observing that an absorbing state is always reachable in one step of the chain. ∎

Given that every state in RPZF is either transient or absorbing, it is immediate that for any $B \subseteq V(G)$, $\mathbf{P}[\text{pta}(G, B) < \infty] = 1$. The division of states into absorbing and transient also gives a natural partition of the RPZF Markov transition matrices. In particular,

$$M_S = \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline \mathbf{r} & Q_S \end{array}\right] \quad \text{and} \quad M_D = \left[\begin{array}{c|c|c} 1 & 0 \cdots 0 & 0 \\ \hline \mathbf{a}_1 & Q_D & \mathbf{a}_2 \\ \hline 0 & 0 \cdots 0 & 1 \end{array}\right], \tag{4.9}$$

where the i^{th} row and column correspond to the state S_i. The column vectors $\mathbf{r}$ and $\mathbf{a}_1$ correspond to the absorbing state S_0 where all vertices are white and hence no vertex can be forced blue. Symmetrically, $\mathbf{a}_2$ corresponds to the absorbing state S_s in DARPZF where every vertex is blue and so by definition of DARPZF no reversions can occur. Additionally, for any $t \in \mathbb{N}$,

$$(M_S)^t = \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline * & (Q_S)^t \end{array}\right] \quad \text{and} \quad (M_D)^t = \left[\begin{array}{c|c|c} 1 & 0 \cdots 0 & 0 \\ \hline * & (Q_D)^t & * \\ \hline 0 & 0 \cdots 0 & 1 \end{array}\right]. \tag{4.10}$$

REMARK 4.2. The RPZF transition matrices are indexed from 0, and submatrices preserve the indexing of their parent matrix. So, for example, Q_S and Q_D are indexed from 1.

Notice that, as a consequence of Proposition 4.1, every state is either transient or absorbing and the only recurrent states are the absorbing states. Additionally, because $p > 0$, the submatrices Q_S and Q_D from (4.9) correspond exactly to transient states. From this it follows that $Q^t \to O$ as $t \to \infty$ for $Q \in \{Q_S, Q_D\}$.

We are next interested in how long an RPZF chain is expected to stay in transient states. For a connected graph G with properly ordered state space $(S_0, \ldots, S_s)$, let N_S and N_D be matrices such that

$$(N_S)_{ij} = \mathbf{E}_i[|\{t \geq 0 : X_t^S = S_j\}|] \quad \text{and} \quad (N_D)_{ij} = \mathbf{E}_i[|\{t \geq 0 : X_t^D = S_j\}|]. \tag{4.11}$$

It is a standard Markov chain result [19, Theorem 11.4] that $N \in \{N_S, N_D\}$ exists and that moreover $N = \sum_{k=0}^{\infty} Q^k = (I - Q)^{-1}$ where $Q \in \{Q_S, Q_D\}$ corresponds to the choice of N. Observe that summing across the ith row of N gives the expected number of rounds the chain spends in transient states, having started from state S_i. Formally, let $\mathbf{t} = N\mathbb{1}$ where $\mathbb{1}$ is the vector containing all ones. Then $\text{eta}(G; S_i, p) = \mathbf{t}_i$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

In the case of DARPZF, there are two potential absorbing states for the chain to enter. Which is more likely? Consider the DARPZF Markov chain on a connected graph G with properly ordered state space $(S_0, \ldots, S_s)$ and transition matrix M_D. Let $C = [c_{ij}]$ be the $(s-1) \times 2$ matrix such that, starting the chain from transient state S_i, c_{i1} is the probability that the graph dies out and c_{i2} is the probability that the graph is fully forced. Then

$$C = N_D[\mathbf{a}_1\ \mathbf{a}_2] = (I - Q_D)^{-1}[\mathbf{a}_1\ \mathbf{a}_2], \tag{4.12}$$

where the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ come from (4.9). Again, see [19, Theorem 11.6] for details.

PROPOSITION 4.3. *For any connected graph G with properly ordered state space $(S_0, \ldots, S_s)$ and for any transient state S_i, the critical reversion probability $p_D(G, S_i)$ exists. That is, there exists a reversion probability such that $\mathbf{P}_i[T(0) < \infty] = \mathbf{P}_i[T(s) < \infty] = 1/2$, where $T(j) = \inf\{t \geq 1 : X_t^D = S_j\}$.*

Proof. Let $C = [c_{ij}]$ be as in (4.12). The entries of Q_D are continuous functions in p and $I-Q$ is invertible for $p \in (0, 1)$. Hence, the entries of C are continuous functions in $p \in (0, 1)$. By taking p sufficiently small, $c_{i1} = c_{i1}(p) < 1/2$. Similarly, by taking p sufficiently large, $c_{i1} = c_{i1}(p) > 1/2$. Thus, $p_D(G, S_i)$ exists so that $c_{i1} = c_{i1}(p_D(G, S_i)) = 1/2$. Since $c_{i1} = \mathbf{P}_i[T(0) < \infty]$, $c_{i2} = \mathbf{P}_i[T(s) < \infty]$, and $c_{i2} = 1 - c_{i1}$, the result follows. ∎

5. DARPZF simulations and approximations. Consider the DARPZF chain on K_n with reversion probability $p \in (0, 1)$. Using (4.12) we calculate $p_D(K_n, S_1)$, the reversion probability such that K_n has equal probability to either die out or fully force when starting from a single blue vertex, for small n.

TABLE 1
Exact calculations of the critical reversion probability for the complete graph

n	$p_D(K_n, S_1)$	n	$p_D(K_n, S_1)$	n	$p_D(K_n, S_1)$	n	$p_D(K_n, S_1)$
3	.6	8	0.427761	13	0.433535	18	0.435628
4	0.466548	9	0.429115	14	0.434157	19	0.43585
5	0.437779	10	0.43052	15	0.434648	20	0.43604
6	0.428853	11	0.431747	16	0.435042	21	0.436203
7	0.427101	12	0.432745	17	0.435363	22	0.436346

These critical reversion probabilities, given in Table 1, have been converted from fractional to decimal form. We can further estimate $p_D(K_n, S_1)$ for larger n to arbitrary levels of precision, as provided in Table 2.

TABLE 2
Numerically approximated critical reversion probability for the complete graph.

n	$p_D(K_n, S_1)$	n	$p_D(K_n, S_1)$
12	0.43274 ± 0.00001	96	0.43805 ± 0.00005
16	0.43505 ± 0.00005	128	0.43815 ± 0.00005
32	0.43715 ± 0.00005	156	0.43818 ± 0.00005
64	0.4379 ± 0.00005	192	0.4382 ± 0.00005

Next, Figs. 1 and 2 approximate the expected time of absorption for various graphs using Monte Carlo simulations, starting from a single blue vertex. Let $V = \{v_1, \ldots, v_n\}$ be a set of vertices. The *path graph* P_n is the graph with edges $v_i v_{i+1}$ for $1 \leq i \leq n-1$. The endpoints are v_1 and v_n, and the midpoint is $v_{\lceil n/2 \rceil}$. The *cycle graph* is a path with the additional edge $v_n v_1$. When simulating DARPZF on the path, starting from an endpoint and starting from a midpoint has a noticeable effect on the expected time to absorption.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

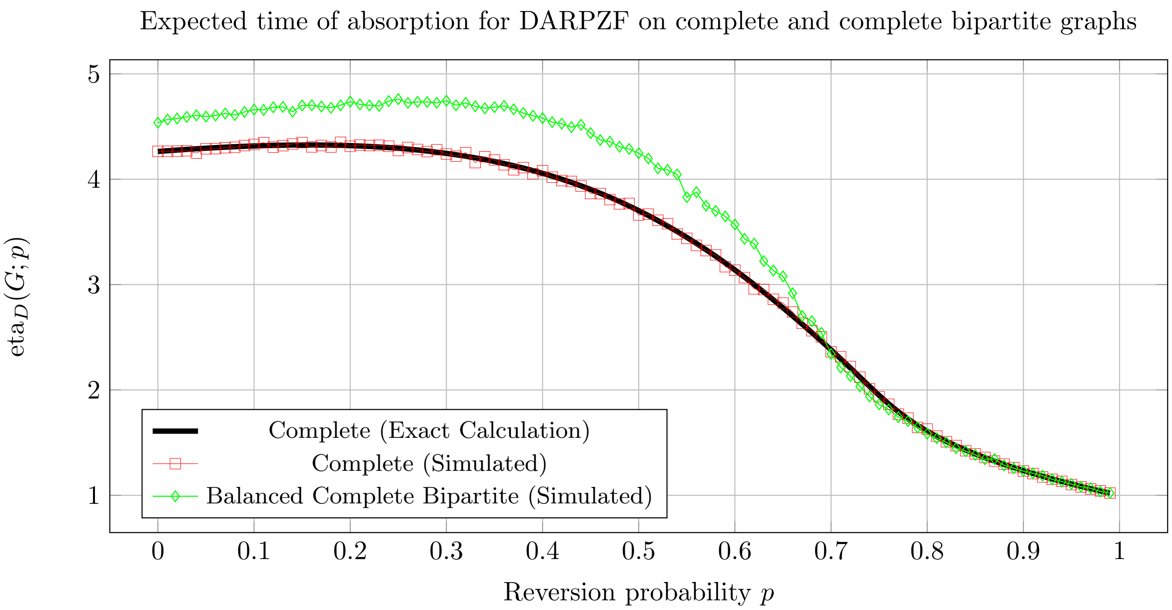

FIGURE 1. *Monte Carlo simulation of DARPZF on K_{32} and $K_{16,16}$ compared to the exact calculation for K_{32}, starting from a single vertex.*

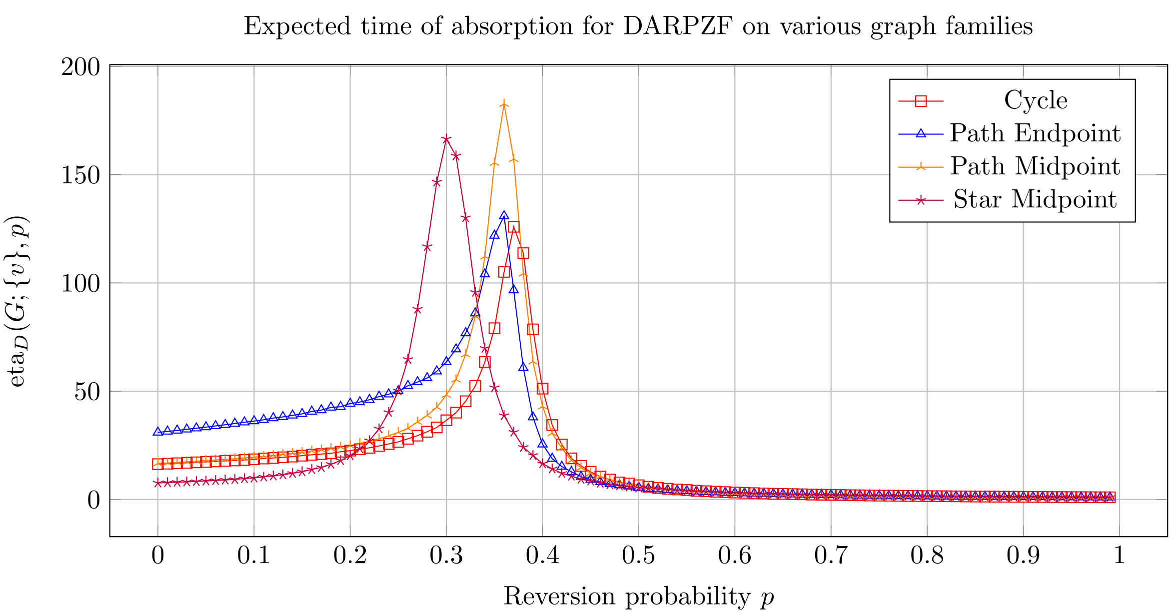

FIGURE 2. *Monte Carlo simulations of DARPZF on the cycle, path, and star on 32 vertices, starting from a single vertex.*

Finally, Fig. 3 presents Monte Carlo simulations of DARPZF on different 32 vertex graphs starting from one blue vertex. Notice that when the reversion probability $p > 0.4$, the cycle, path, and star die out with probability nearly one. On the other hand, the highly connected complete graph and balanced complete bipartite graph do not reach the same level die out until $p > 0.75$ and $p > 0.7$, respectively.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

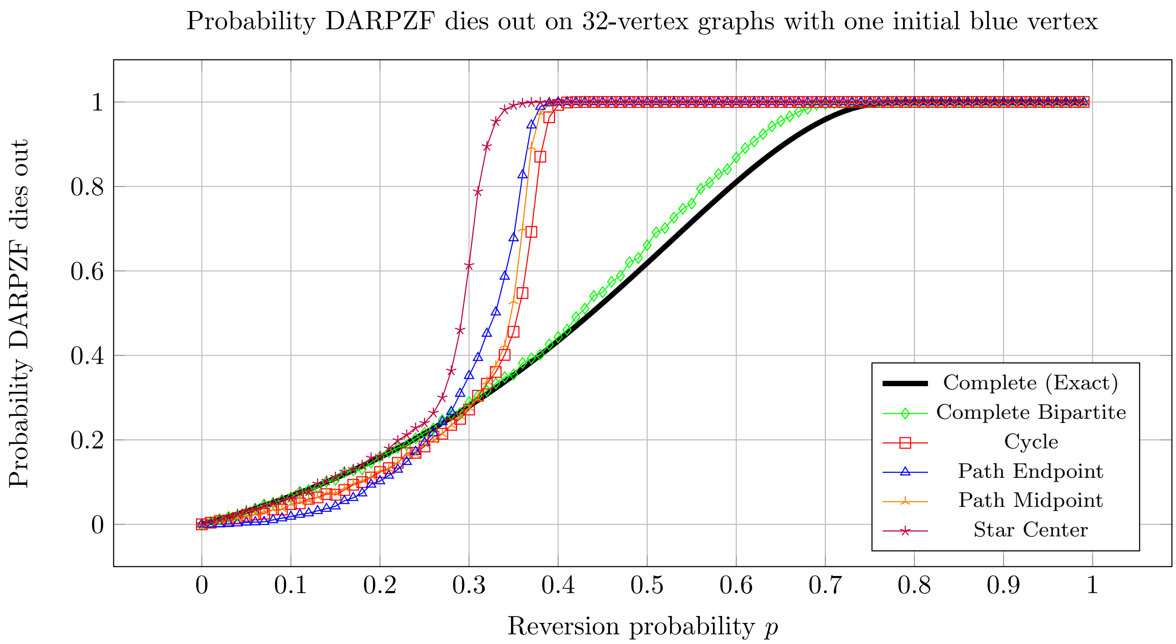

FIGURE 3. *Monte Carlo simulations of DARPZF on the cycle, path, star, and balanced complete bipartite graphs on 32 vertices, starting from a single blue vertex.*

The critical reversion probability starting from a single blue vertex can also be estimated from Fig. 3. Observe that for the cycle, path, and star graphs, the p which maximizes $\text{eta}_D(G;p)$ in Fig. 2 closely aligns with the approximate $p_D(G,\{v\})$ from Fig. 3. On the other hand, the maximums of $\text{eta}_D(K_{32};p)$ and $\text{eta}_D(K_{16,16};p)$ occur at a p noticeably smaller than their respective critical reversion probabilities. In the case of the complete graph, we know from Theorem 3.8 that only a small percentage of vertices are needed to have a high chance of forcing the entire graph blue. Compare this with the cycle or path graph, where the threshold for one-step fully forcing is of order n. Indeed, at least one-third of the vertices must be blue to even have a nonzero probability to fully force the cycle or path in one step.

Contrasting RPZF behavior with traditional PZF, the cycle and path are expected to take $\Theta(n)$ steps to force in PZF [17], and the complete graph has an expected propagation time of $\Theta(\log\log n)$ steps [10]. The cycle and path thus have more time to be stymied by the reversion of blue vertices in RPZF, which may result in longer-lasting oscillations of white and blue vertices. The complete graph, on the other hand, forces significantly more quickly and so does not as easily fall into these oscillation.

In RPZF, both the one-step threshold and expected propagation time quantify how "hard" a graph is to fully force, unline expected propagation time in traditional PZF. Consider the path or cycle on n vertices when starting from a single blue vertex. These graphs require only a single force to become deterministic in nature, but will always take at least $n/2$ rounds to fully force.

When starting from a single vertex, Figs. 1 and 3 indicate that the balanced complete bipartite graph behaves much more like the complete graph than the star graph (a severely "unbalanced" complete bipartite graph). This observation is supported by Theorem 3.13, which gives the one-step forcing threshold for $K_{n/2,n/2}$ as order $O\big(\sqrt{n\log(n/2)}\big)$, which is comparable to the K_n threshold of $\Theta\big(\sqrt{n\log n}\big)$ and significantly smaller than the $K_{1,n-1}$ threshold of $n-1-o(1)$ from Theorem 3.15.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

6. The contact process. RPZF can be viewed as a discrete-time analog of what is known in the probability literature as the contact process, a basic model for population growth and the spread of infection. Analogous to discrete-time Markov chains, we say that (η_t), $t \geq 0$, is a continuous-time Markov process on the state space $\mathcal{S}$ if for any $0 \leq u_0 < u_1 < \cdots < u_k$ and any states $S_j, S_i, S_{i_{k-1}}, \ldots, S_{i_0} \in \mathcal{S}$,

$$\mathbf{P}[\eta_{u_k+t} = S_j \mid \eta_{u_k} = S_i, \eta_{u_{k-1}} = S_{i_{k-1}}, \ldots, \eta_{u_0} = S_{i_0}] = \mathbf{P}[\eta_t = S_j \mid \eta_0 = S_i].$$

The *susceptible-infected-susceptible contact process* (SIS contact process) on a (possibly infinite) graph $G = (V, E)$ with infection parameter $\lambda \geq 0$ is a continuous-time Markov process (η_t) with state space $\{0,1\}^V$, where a state $\eta \in \{0,1\}^V$ is a configuration of zeros and ones on the graph. At any time $t \geq 0$, each vertex has status either 0 ("healthy") or 1 ("infected"). The state of the entire system at time t is then described by $\eta_t : V \to \{0,1\}$ where $\eta_t(v)$ is the status of vertex v at time t. Finally, for a vertex v and configuration η, we say that the contact process (η_t) evolves according to the local transition rates

$$\begin{aligned} 0 \to 1 &\quad \text{at rate} \quad \lambda \sum_{w \in N(v)} \eta(w), \\ 1 \to 0 &\quad \text{at rate} \quad 1. \end{aligned}$$

Specifically, these are the rates for exponential random variables whose value corresponds to the waiting time until vertex v changes status. Thus, infected vertices recover after some (exponential distribution) time with mean 1 independent of its neighbors, and healthy vertices become infected at a rate linearly proportional to its number of infected neighbors. G is often taken to be the d-dimensional infinite lattice, where vertex interactions are local in nature; that is, a vertex is adjacent only to those $2d$ vertices at Euclidean distance 1. This, combined with being vertex transitive, means that the contact process can be studied at the local level, from which global behaviors are deduced (with varying levels of simplifying assumptions) [6, 24]. That being said, recent work has also been done on other large graph structures such as scale-free and power-law graphs [18, 22].

A common topic in study of contact processes is the infection parameter λ and its relation to the process surviving or dying out. The contact process is said to die out if

$$\mathbf{P}[\eta_t \not\equiv 0 \ \forall t \geq 0] = 0,$$

where $\eta_t \not\equiv 0$ means there exists a $v \in V$ such that $\eta_t(v) \neq 0$. Otherwise, the process is said to survive. It is well known that on finite graphs, no matter the initial configuration or infection parameter, the process dies out [26]. The same is true for SARPZF. For a thorough introduction to the contact process and its fundamental results, the reader is directed to [25].

Let G now refer to the d-dimensional infinite lattice. An *additive process* is a generalization of the contact process where the infection rate λ and death rate δ (previously $\delta = 1$) at v are now a function of the finite subsets of $V(G)$ [13, Ch. 3]. The contact process is recovered when $\lambda(A) = \lambda$ for $A = \{x\}$ with $x \in N(v)$, $\delta(\varnothing) = 1$, and are 0 otherwise. SARPZF is formulated in the continuous-time regime as an additive process where, at a vertex v, the infection rate $\lambda(A)$ is a function of $A \subseteq \{w : d(v,w) \leq 2\}$ where $d(v,w)$ is equal to the length of the shortest path from v to w. A process is said to be *irreducible* if for all $t > 0$ and all $v \in V(G)$, $\mathbf{P}[\eta_t(v) = 1] > 0$ when starting from one blue vertex. Observe that both the contact process and continuous-time SARPZF are irreducible on connected graphs. It can be shown that if an irreducible additive process does not die out, then at large times it looks like the process starting from all vertices infected [5, 13].

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

6.1. The discrete SIS contact process. The discretized SIS contact process can be formally described as follows. Let G be a (possibly infinite) graph. At each time $t \in \mathbb{N}$, every infected vertex $v \in V(G)$ infects each of its neighbors independently with probability β. Simultaneously, every infected vertex v at time t recovers with probability p. The exact meaning of "simultaneously" differs depending on the particular discretization being considered. For instance, some models allow a vertex that recovers to be reinfected during the same time step, whereas others assert that a recovered vertex must remain recovered. SARPZF takes a hybrid approach wherein infected vertices can immediately recover, but recovered vertices remain recovered for that time step.

Over time, various formulations have been proposed to model the discrete-time SIS contact process. We describe a few of them and then contrast them to SARPZF. We consider only SARPZF because contact processes do not have the additional stopping condition that DARPZF has. For any vertex $v \in V(G)$ and time $t \in \mathbb{N}$, define $\mathfrak{p}_v(t)$ to be the probability that v is infected (blue) at time t and define

$$\mathfrak{q}_v(t+1) = \prod_{x \in N(v)} (1 - \beta \mathfrak{p}_x(t)),$$

to be the probability that v does not receive infection (is not forced) at time t. Finally, let p be the probability that a vertex recovers (reverts to white). Then, Wang et al. proposed the model [29]

$$1 - \mathfrak{p}_v(t+1) = (1 - \mathfrak{p}_v(t))\mathfrak{q}_v(t+1) + p\mathfrak{p}_v(t)\mathfrak{q}_v(t+1) + \frac{1}{2}p\mathfrak{p}_v(t)(1 - \mathfrak{q}_v(t+1). \tag{6.13}$$

The first term is the probability of vertex v entering time $t+1$ healthy and then not being infected, the second term is the probability of v entering time $t+1$ infected, recovering, then not being reinfected, and the final term assumes that half of the time a vertex will undergo a "curing event" after being reinfected. Notice that this interpretation of "simultaneous" has recovery occurring before infection. This model is explored in more detail in [9]. Later models do away with the 1/2 probability "curing event" assumption, as well as stating the dynamics in terms of the probability of being infected, $\mathfrak{p}_v$, instead of the probability of being healthy, $1 - \mathfrak{p}_v$. For instance, Gómez et al. introduced the model [18]

$$\mathfrak{p}_v(t+1) = (1-p)\mathfrak{p}_v(t) + (1 - \mathfrak{q}_v(t+1))(1 - \mathfrak{p}_v(t)) + p(1 - \mathfrak{q}_v(t+1))\mathfrak{p}_v(t), \tag{6.14}$$

accounting for the cases of an infected vertex failing to recover, a susceptible vertex being infected, and an infected vertex recovering then becoming reinfected. This formulation also makes the assumption that a vertex which recovers at time t can immediately be reinfected at time t. Contrast (6.14) with the model presented by Ahn and Hassibi in [1] where

$$\mathfrak{p}_v(t+1) = (1-p)\mathfrak{p}_v(t) + (1 - \mathfrak{p}_v(t))(1 - \mathfrak{q}_v(t+1)), \tag{6.15}$$

doing away with the $p(1 - \mathfrak{q}_v(t+1))\mathfrak{p}_v(t)$ term and so asserting that a vertex that recovers cannot be reinfected in the same time step. This formulation seems most true to the notion of "simultaneous" vertex infection and recovery since vertices can only undergo one status change each time step. Equation (6.15) can be further simplified by truncating the terms of $\mathfrak{q}_v(t+1) = \prod_{x \in N(v)}(1 - \beta \mathfrak{p}_x(t))$ with powers of β greater than 1, giving

$$\mathfrak{p}_v(t+1) = (1-p)\mathfrak{p}_v(t) + (1 - \mathfrak{p}_v(t))\beta \sum_{x \in N(v)} \mathfrak{p}_x(t). \tag{6.16}$$

Note that this approximation is better for smaller values of β. Paré et al. demonstrate in [27] how (6.16) directly matches the model derived from applying Euler's method to the continuous-time mean field approx-

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

imation for the SIS contact process, as well as providing analysis on the accuracy of (6.15) and (6.16). For a graph G on n vertices, these models can all be used to solve for $\mathfrak{p}_v$ numerically from which tests of accuracy are typically derived. A commonly considered parameter is the graph's expected infection density ρ_t at time t. Given an infection rate β and recovery rate p, this is computed as

$$\rho_t = \frac{1}{n} \sum_{v \in V(G)} \mathfrak{p}_v(t).$$

So far, the models described have all been homogeneous, meaning that the infection rate β and recovery rate p are constant for all vertices. SARPZF, on the other hand, is more akin to a heterogenous model, wherein the infection rate and recovery rate depend on the vertex. In fact, SARPZF is somewhere in the middle, with a constant recovery rate but variable infection rate. Looking to model SARPZF in the same way as the above models, we find

$$\mathfrak{p}_v(t+1) = (1-p)\mathfrak{p}_v(t) + (1-p)(1-\mathfrak{p}_v(t))(1-\mathfrak{q}_v(t+1)). \tag{6.17}$$

The first term is the case of v being infected after time t and not reverting during time $t+1$, and the second term is the case of vertex v being health after time t, infected at time $t+1$, and not then reverting at time $t+1$. Notice that (6.17) is most similar to (6.15) but differs in two ways. First, the function $\mathfrak{q}_v$ is different. Second, the $(1-\mathfrak{p}_v(t))(1-\mathfrak{q}_v(t+1))$ is multiplied by $1-p$ due to the fact that in SARPZF, a vertex has the chance to revert in the same time step it is infected. This interpretation of "simultaneous" has recovery occurring after infection. One of the key differences between the models in the literature and SARPZF is the infection rate β and, subsequently, the probability that vertex v is not infected by a neighbor $\mathfrak{q}_v$. Let G be a graph on n vertices and let (X_t) be the SARPZF Markov chain on the properly ordered state space $\mathcal{S} = (S_0, \ldots, S_s)$ with reversion probability p. Notice that in SARPZF, if B is our set of infected vertices, then vertex v is "infected" (forced) with probability

$$\mathbf{P}[B \to v] = 1 - \mathbf{P}[B \not\to v] = 1 - \prod_{x \in B \cap N(v)} \mathbf{P}[x \not\to v] = 1 - \prod_{x \in B \cap N(v)} \left(1 - \frac{|B \cap N[x]|}{\deg x}\right).$$

Thus, if X_t is the set of blue vertices at time t and $X_0 = S_i$, then $\mathfrak{q}_v(t+1) = \mathbf{P}[X_t \not\to v]$. Applying the law of total probability over the state space $\mathcal{S}$,

$$\begin{aligned}\mathfrak{q}_v(t+1) &= \sum_{j=0}^{s} \mathbf{P}[X_t \not\to v \mid X_t = S_j]\mathbf{P}[X_t = S_j] \\ &= \sum_{j=0}^{s} \prod_{x \in S_j \cap N(v)} \left(1 - \frac{|S_j \cap N[x]|}{\deg x}\right) \prod_{x \in S_j} \mathfrak{p}_x(t) \prod_{x \in V(G) \setminus S_j} (1 - \mathfrak{p}_x(t)).\end{aligned}$$

Notice that in this case, the rate of infection is exponentially proportional to the number of infected neighbors and, moreover, is also dependent on the number of infected vertices at distance 2 from v. Intuitively, this means that an infected vertex is *more infectious* when it has more infected neighbors.

7. Conclusion and future directions. The focus of this paper was to develop tools and results in the analysis of RPZF on both general and densely connected graph structures. This included asymptotic thresholds for infection on the complete and complete bipartite graphs as well as calculations of the critical reversion probability for various graph families. The probabilistic literature, on the other hand, is often

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

concerned with behavior on simpler graph structures like the integer lattice which, unlike the complete graph, does not exhibit global vertex interaction. These results are typically realized either in the deduction of stationary distributions or mean field analyses, sometimes with conflicting results due to the nature of their simplifying assumptions (see, for instance, [3]). Both these types of results would be interesting to derive for RPZF. For instance, basic contact process results from, e.g., [13, Ch. 3] should be translatable to the RPZF domain. Another direction is to consider the behavior of RPZF on the Erdős-Rényi random graph model. Traditional PZF has already been considered on the random graph [15], and in the contact process literature the random graph model is used to capture the dynamic evolution of certain networks of interaction [28]. See [14] for a more thorough introduction to the contact process on random graphs.

Appendix A. Asymptotic notation. Let $f(n)$ and $g(n)$ be functions from the nonnegative integers to the real numbers, where g is strictly positive for sufficiently large input. Write $f = O(g)$ if there exists constants $C, N > 0$ such that for all $n > N$, $|f(n)| \leq Cg(n)$ and write $f = o(g)$ if for all $C > 0$ there exists an $N > 0$ such that for all $n > N$, $|f(n)| < Cg(n)$. Symmetrically, $f = \Omega(g)$ if there exists constants C, N such that for all $n > N$, $f(n) \geq Cg(n)$ and $f = \omega(g)$ if for all $C > 0$ there exists an $N > 0$ such that for all $n > N$, $f(n) > Cg(n)$. That is, $f = O(g)$ if and only if $g = \Omega(f)$ and $f = o(g)$ if and only if $g = \omega(f)$. One can also define this notions in terms of limit behavior. Namely, $f = o(g)$ when $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ and $f = \omega(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$. We refer the reader to , e.g., [11] for a more thorough introduction to asymptotic notation.

We make use of the following standard facts of asymptotic notation. Let f and g be real-valued functions.

Fact A.1. *If $f = o(g)$ then $f = O(g)$, and if $f = \omega(g)$ then $f = \Omega(g)$. Moreover, $f = O(f)$ and $f = \Omega(f)$.*

Fact A.2. *$O(f)O(g) = O(fg)$ and $\Omega(f)\Omega(g) = \Omega(fg)$. Moreover, $o(f)O(g) = o(fg)$ and $\omega(f)\Omega(g) = \omega(fg)$.*

Fact A.3. *If $f = O(g)$, then $O(f) + O(g) = O(g)$.*

Acknowledgment. This research was partially supported by NSF grant DMS-1839918, and the author thanks the National Science Foundation for this support. The author also thanks the reviewers for their feedback that has greatly improved the readability of the paper.

REFERENCES

[1] H.J. Ahn and B. Hassibi. Global dynamics of epidemic spread over complex networks. In *52nd IEEE Conference on Decision and Control*. IEEE, New York, NY, USA, 4579–4585, 2013.

[2] F. Barioli, W. Barrett, S.M. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. *J. Graph Theory*, 72(2):146–177, 2013.

[3] L. Belhadji, N. Lanchier, and M. Mercer. The contact process with an asymptomatic state. *Stochastic Process. Appl.*, 176:Paper No. 104417, 16, 2024.

[4] K.F. Benson, D. Ferrero, M. Flagg, V. Furst, L. Hogben, V. Vasilevska, and B. Wissman. Zero forcing and power domination for graph products. *Australas. J. Combin.*, 70:221–235, 2018.

[5] C. Bezuidenhout and L. Gray. Critical attractive spin systems. *Ann. Probab.*, 22(3):1160–1194, 1994.

[6] C. Bezuidenhout and G. Grimmett. The critical contact process dies out. *Ann. Probab.*, 18(4):1462–1482, 1990.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 49-73, January 2025.

[7] J. Breen, B. Brimkov, J. Carlson, L. Hogben, K.E. Perry, and C. Reinhart. Throttling for the game of Cops and Robbers on graphs. *Discrete Math.*, 341(9):2418–2430, 2018.

[8] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Phys. Rev. Lett.*, 99(10):100501, 2007.

[9] D. Chakrabarti, Y. Wang, C. Wang, J. Leskovec, and C. Faloutsos. Epidemic thresholds in real networks. *ACM Trans. Inf. Syst. Secur.*, 10(4):1–26, 2008.

[10] Y. Chan, E. Curl, J. Geneson, L. Hogben, K. Liu, I. Odegard, and M. Ross. Using Markov chains to determine expected propagation time for probabilistic zero forcing. *Electron. J. Linear Algebra*, 36:318–333, 2020.

[11] N.G. de Bruijn. *Asymptotic Methods in Analysis*, third edition. Dover Publications, Inc., New York, 1981.

[12] R. Durrett. *Essentials of Stochastic Processes*, third edition. Springer Texts in Statistics. Springer, Cham, 2016.

[13] R. Durrett. Ten lectures on particle systems. In: *Lectures on Probability Theory (Saint-Flour, 1993)*, vol. 1608. Lecture Notes in Math.. Springer, Berlin, 97–201, 1995.

[14] R. Durrett. Some features of the spread of epidemics and information on a random graph. *Proc. Natl. Acad. Sci. USA*, 107(10):4491–4498, 2010.

[15] S. English, C. MacRury, and P.ł Prałat. Probabilistic zero forcing on random graphs. *Eur. J. Comb.*, 91:103207, 2021.

[16] S. Fallat, S. Severini, and M. Young. Zero forcing and its applications. In *American Institute of Mathematics Workshop*, 2017. Summary at https://aimath.org/pastworkshops/zeroforcingrep.pdf.

[17] J. Geneson and L. Hogben. Expected propagation time for probabilistic zero forcing. *Australas. J. Combin.*, 83:397–417, 2022.

[18] S. Gómez, Alexandre Arenas, J. Borge-Holthoefer, S. Meloni, and Y. Moreno. Discrete-time Markov Chain approach to contact-based disease spreading in complex networks. *EPL*, 89(3):38009, 2010.

[19] C.M. Grinstead and J.L. Snell. *Introduction to Probability*, second edition. American Mathematical Society, Providence, RI, 1997.

[20] AIM Minimum Rank-Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra Appl.*, 428(7):1628–1648, 2008.

[21] L. Hogben, J.C.-H. Lin, and B.L. Shader. *Inverse Problems and Zero Forcing for Graphs*, vol. 270. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2022.

[22] E. Jacob and P. Mörters. The contact process on scale-free networks evolving by vertex updating. *R. Soc. Open Sci.*, 4(5):170081, 14, 2017.

[23] C.X. Kang and E. Yi. Probabilistic zero forcing in graphs. *Bull. Inst. Combin. Appl.*, 67:9–16, 2013.

[24] T.M. Liggett. *Interacting Particle Systems*, vol. 276. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985.

[25] T.M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, vol. 324. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.

[26] T. Mountford, D. Valesin, and Q. Yao. Metastable densities for the contact process on power law random graphs. *Electron. J. Probab.*, 18(103):1–36, 2013.

[27] P.E. Paré, J. Liu, C.L. Beck, B.E. Kirwan, and T. Başar. Analysis, estimation, and validation of discrete-time epidemic processes. *IEEE Trans. Control Syst. Technol.*, 28(1):79–93, 2020.

[28] M.D. Ryser, K. McGoff, D.P. Herzog, D.J. Sivakoff, and E.R. Myers. Impact of coverage-dependent marginal costs on optimal HPV vaccination strategies. *Epidemics*, 11:32–47, 2015.

[29] Y. Wang, D. Chakrabarti, C. Wang, and C. Faloutsos. Epidemic spreading in real networks: An eigenvalue viewpoint. In *22nd International Symposium on Reliable Distributed Systems, 2003. Proceedings*. IEEE Computer Society, Los Alamitos, CA, USA, 25–34, 2003.

[30] B. Yang. Fast-mixed searching and related problems on graphs. *Theoret. Comput. Sci.*, 507:100–113, 2013.