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ORIGINAL ARTICLE

Slowly Vanishing Mean Oscillations: Non-uniqueness of Blow-ups in a Two-phase Free Boundary Problem

Matthew Badger[1] · **Max Engelstein**[2] · **Tatiana Toro**[3]

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Abstract
In Kenig and Toro's two-phase free boundary problem, one studies how the regularity of the Radon–Nikodym derivative $h = d\omega^-/d\omega^+$ of harmonic measures on complementary NTA domains controls the geometry of their common boundary. It is now known that $\log h \in C^{0,\alpha}(\partial\Omega)$ implies that pointwise the boundary has a unique blow-up, which is the zero set of a homogeneous harmonic polynomial. In this note, we give examples of domains with $\log h \in C(\partial\Omega)$ whose boundaries have points with non-unique blow-ups. Philosophically the examples arise from oscillating or rotating a blow-up limit by an infinite amount, but very slowly.

Keywords Two-phase free boundary problems · Harmonic measure · Uniqueness of blow-ups

Mathematics Subject Classification (2010) Primary 31B15 · 35R35

1 Introduction

In this note, we answer a question about uniqueness of blow-ups in non-variational two-phase free boundary problems for harmonic measure *in the negative*. Throughout, we let $\Omega^+ = \Omega \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ denote complementary unbounded domains with a common boundary $\partial\Omega = \partial\Omega^+ = \partial\Omega^-$. Furthermore, we require that $\Omega^\pm$ belong to the

Dedicado a Carlos Kenig, un gran maestro y amigo en conmemoración de sus 70 años.

✉ Max Engelstein
mengelst@umn.edu

Matthew Badger
matthew.badger@uconn.edu

Tatiana Toro
toro@uw.edu

1 Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA

2 Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

3 Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195-4350, USA

class of NTA domains in the sense of Jerison and Kenig [14]. Let $\omega^{\pm}$ denote harmonic measures on $\Omega^{\pm}$ with finite poles $X^{\pm}$ or with poles at infinity (see Kenig and Toro [18]). Finally, we assume $\omega^{+} \ll \omega^{-} \ll \omega^{+}$ and let

$$h = \frac{d\omega^{-}}{d\omega^{+}}$$

denote the Radon–Nikodym derivative of harmonic measure on one side of the boundary with respect to harmonic measure on the other side. We are interested in understanding how different regularity assumptions on h controls the geometry of $\partial\Omega$.

Following Kenig and Toro [20] and Badger [4], we know if $\log h \in \mathrm{VMO}(d\omega^{+})$ (vanishing mean oscillation) or $\log h \in C(\partial\Omega)$ (continuous), then the boundary admits a finite decomposition into pairwise disjoint sets,

$$\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_{d_0}, \tag{1.1}$$

where geometric blow-ups (tangent sets) of $\partial\Omega$ centered at any $Q \in \Gamma_d$ $(1 \le d \le d_0)$ are zero sets Σ_p of homogeneous harmonic polynomials (hhp) $p : \mathbb{R}^n \to \mathbb{R}$ of degree d. That is to say, given any boundary point $Q \in \Gamma_d$ and any sequence of scales $r_i > 0$ with $\lim_{i\to\infty} r_i = 0$, there exists a subsequence r_{i_j} and a hhp p of degree d such that

$$\lim_{j\to\infty} \max\left\{\mathrm{excess}\left(\frac{\partial\Omega - Q}{r_{i_j}} \cap B, \Sigma_p\right),\ \mathrm{excess}\left(\Sigma_p \cap B, \frac{\partial\Omega - Q}{r_{i_j}}\right)\right\} = 0 \tag{1.2}$$

for every ball B in $\mathbb{R}^n$. Here $\mathrm{excess}(S, T) = \sup_{s\in S} \inf_{t\in T} |s - t|$ when $S, T \subset \mathbb{R}^n$ are nonempty and $\mathrm{excess}(\emptyset, T) = 0$; see [10] for more information about this mode of convergence of closed sets (the Attouch–Wets topology). Following [6] and [8], we further know that the regular set Γ_1 is relatively open, Reifenberg flat with vanishing constant, and has Hausdorff and Minkowski dimensions $n - 1$, whereas the singular set $\partial\Omega \setminus \Gamma_1$ is closed and has Hausdorff and Minkowski dimension at most $n - 3$.

We remark that the maximum degree d_0 witnessed in the decomposition (1.1) can be bounded in terms of the ambient dimension and the NTA constants of $\Omega^{\pm}$. When $n = 2$, it is always the case that $\partial\Omega = \Gamma_1$. When $n = 3$, we have $\partial\Omega = \Gamma_1 \cup \Gamma_3 \cup \cdots \cup \Gamma_{2d_1+1}$ (odd degrees only) and for every odd $d \ge 1$, there exist two-sided domains with $\Gamma_d \neq \emptyset$. In dimensions $n \ge 4$, for every integer $d \ge 1$, even or odd, there exist two-sided domains with $\Gamma_d \neq \emptyset$. See [8] for details and [3, 23, 26] for additional results on the regularity of Γ_1.

One may ask: Are the blow-ups at each point in $\partial\Omega$ unique? In other words, is the zero set Σ_p in (1.2) independent of choice of the sequence of scales r_i? Under a stronger free boundary regularity hypothesis, the answer is *affirmative*. Following Engelstein [11] and [9], we know that if $\log h \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha > 0$ (Hölder continuous), then blow-ups are unique. Moreover, when $\log h \in C^{0,\alpha}(\partial\Omega)$, the regular set Γ_1 is actually a $C^{1,\alpha}$ embedded submanifold and the singular set $\partial\Omega \setminus \Gamma_1$ is $(n-3)$-rectifiable in the sense of geometric measure theory (see e.g. [21]). Below, we supply examples demonstrating that under the weaker regularity hypothesis $\log h \in C(\partial\Omega)$, there may exist points in the boundary that have non-unique blow-ups.

Theorem 1.1 *For each $d \in \{1, 3\}$, there exist complementary NTA domains $\Omega^{\pm} \subset \mathbb{R}^3$ such that $\log h \in C(\partial\Omega)$, but there exists a point in Γ_d at which geometric blow-ups of $\partial\Omega$ are not unique.*

Remark 1.2 In fact, the domains that we construct below have *locally finite perimeter* and *Ahlfors regular* boundaries: that is, there exists $C > 0$ (depending on Ω) such that

$$C^{-1}r^{n-1} \le \mathcal{H}^{n-1}(\partial\Omega \cap B(Q,r)) \le Cr^{n-1} \quad \text{for all } Q \in \partial\Omega \text{ and } r > 0,$$

where $\Omega \subset \mathbb{R}^n$ and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Even more, the boundaries of the domains are smooth surfaces outside of a single point.

The basic strategy is to start with a blow-up domain $\Omega_p^{\pm} = \{X \in \mathbb{R}^n : \pm p(X) > 0\}$ associated to a hhp p of degree d, which has $\log h \equiv 0$ and $0 \in \Gamma_d$. We then deform the domain near the origin by introducing rotations/oscillations at each scale $0 < r \le 1/100$ so that the magnitude of the oscillation at scale r vanishes as $r \to 0$. The tension in the proof becomes choosing the correct speed of vanishing. On the one hand, by choosing the speed to be sufficiently *quick*, we can guarantee by making estimates on elliptic measure that the deformed domain has $\log h \in C(\partial\Omega)$. On the other hand, by choosing the speed to be sufficiently *slow*, we can guarantee that the deformed domain has uncountably many blow-ups at the origin, each of which are rotations of the original domain.

Remark 1.3 By a suitable modification, the technique introduced in the case $d = 3$ can be used to show existence of domains with $\log h \in C(\partial\Omega)$ and non-unique blow-ups at an isolated point $Q \in \Gamma_d$ for any value of $d \ge 2$. When $d \ge 3$ is odd, the examples can be produced in $\mathbb{R}^3$. When $d \ge 2$ is even, the examples can be produced in $\mathbb{R}^4$.

In a related context, Allen and Kriventsov [1] use conformal maps to construct domains $\Omega^{\pm} = \{u^{\pm} > 0\} \subset \mathbb{R}^n$ $(n \ge 2)$ associated to non-negative subharmonic functions $u^{\pm}$ for which the Alt–Caffarelli–Friedman functional

$$\Phi(r, u^+, u^-) = \frac{1}{r^4}\int_{B_r(0)} \frac{|\nabla u^+|^2}{|X|^{n-2}} \int_{B_r(0)} \frac{|\nabla u^-|^2}{|X|^{n-2}}$$

has a positive limit as $r \to 0$, but whose interface $\partial\Omega = \partial\Omega^+ = \partial\Omega^-$ does not have a unique tangent plane at the origin. It would be interesting to know whether a suitable modification of their examples satisfy $\log h \in C(\partial\Omega)$. For more on the connection between the ACF functional and two-phase free boundary problems for harmonic measure (originally observed by Kenig, Preiss, and Toro [16]), see [2, §2.2] and the references within.

We handle the case $d = 3$ of Theorem 1.1 in Section 2 and the case $d = 1$ in Section 3.

2 The First Example: Non-Unique Singular Tangents

2.1 Description and Geometric Properties

We begin with Szulkin's example [24] of a degree 3 hhp,

$$s(x, y, z) = x^3 - 3xy^2 + z^3 - 1.5(x^2 + y^2)z,$$

with the interesting feature that its zero set Σ_s is homeomorphic to $\mathbb{R}^2$. See Fig. 1. Because Σ_s is a cone (s is homogeneous) and $\Sigma_s \cap S^2$ is a smooth curve[1], it follows that $\Omega_s^{\pm} = \{(x, y, z) \in \mathbb{R}^3 : \pm s(x, y, z) > 0\}$ are complementary NTA domains. Note that the positive z-axis belongs to Ω_s^+ and the negative z-axis belongs to Ω_s^-, since $s(0, 0, \pm 1) = \pm 1$.

[1] One can check that $\nabla s(x, y, z) = 0 \Leftrightarrow (x, y, z) = (0, 0, 0)$.

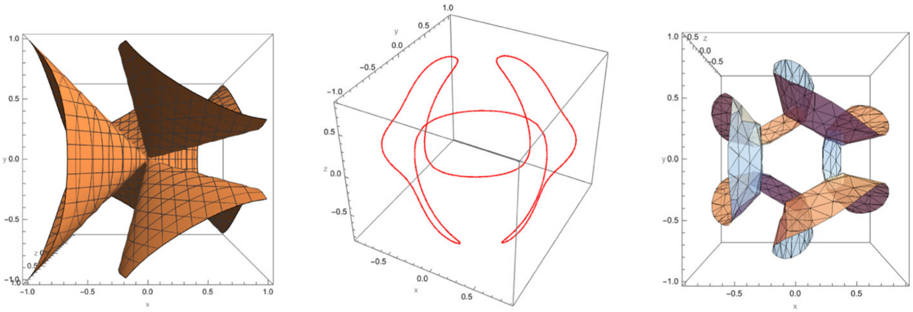

Fig. 1 Left: Szulkin Σ_s, viewed from the z-axis. Center: the curve formed by intersection of Szulkin Σ_s and $\mathbb{S}^2$, viewed from a different angle. Right: Szulkin Σ_s inside of the annulus $1/2 < r < 1$, viewed from the z-axis

To build $\Omega^{\pm}$, we deform $\Omega_s^{\pm}$ by rotating spherical shells $\Sigma_s \cap \partial B_r(0)$ in the xy-plane. More precisely, we put $\Omega^{\pm} = \{\pm s_{\text{twist}} > 0\}$, where $s_{\text{twist}} \equiv s \circ \Phi_{-\theta}$ and $\Phi_{\pm\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ are homeomorphisms given by

$$\begin{aligned} \Phi_{\pm\theta}(x, y, z) &= (x\cos(\pm\theta) - y\sin(\pm\theta), x\sin(\pm\theta) + y\cos(\pm\theta), z), \\ \theta \equiv \theta(r) &:= \log(-\log(r)) \quad \text{for all } 0 < r := \sqrt{x^2+y^2+z^2} \le 1/100 \end{aligned} \tag{2.1}$$

and we smoothly interpolate to $\theta(r) := 0$ for all $r \ge 1$. See Fig. 2.

If $s_{\text{twist}}(x, y, z) = 0$, then $\Phi_{-\theta}(x, y, z) \in \Sigma_s$. Hence, the interface $\Sigma = \partial\Omega^{\pm} = \Phi_\theta(\Sigma_s)$. Similarly, $\Omega^{\pm} = \Phi_\theta(\Omega_s^{\pm})$.

Remark 2.1 Let us collect some simple, but useful observations about θ and Φ_θ.

(i) For any $\theta_0 \in [0, 2\pi)$, there exists a sequence $r_i \downarrow 0$ such that $\theta(r_i) = \theta_0 \pmod{2\pi}$, i.e. such that $\min_{k\in\mathbb{Z}} |\theta(r_i) - \theta_0 - 2\pi k| = 0$ for all $i \ge 1$.
(ii) For any sequence $r_i \downarrow 0$, there exists $\theta_0 \in [0, 2\pi)$ and a $r_{i_j} \downarrow 0$ such that $\theta(r_{i_j}) \to \theta_0 \pmod{2\pi}$, i.e. $\lim_{j\to\infty} \min_{k\in\mathbb{Z}} |\theta(r_{i_j}) - \theta_0 - 2\pi k| = 0$.
(iii) For all $0 < r \le 1/100$, we have $|\nabla\theta| = 1/(-r\log(r))$ and $|\partial_{ij}\theta| \le C/(-r^2\log(r))$ for all $1 \le i, j \le 3$.

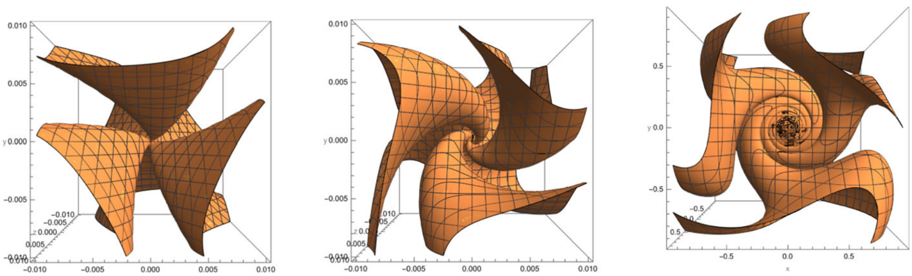

Fig. 2 Examples of twisted Szulkin domains $\Omega^{\pm}$ defined using various rotation functions $\theta(r)$. Left: $\theta(r) = \log(-\log(r))$; the domains $\Omega^{\pm}$ are NTA and $\log h \in C(\partial\Omega)$. Center: $\theta(r) = -\log(r)$; the domains $\Omega^{\pm}$ are NTA, but $\log h \notin \mathrm{VMO}(d\omega^+)$. Right: $\theta(r) = (-\log(r))^2$; the domains $\Omega^{\pm}$ are not NTA

(iv) For all (x, y, z) with $0 < r \leq 1/100$, we can write $D\Phi_\theta = R_\theta + E_\theta$, where

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a rotation matrix and the "error matrix" E_θ is such that $\|E_\theta\|_\infty \leq C/(-\log(r))$, where the norm is the sup norm on the entries of E_θ.

(v) The map $\Phi_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ is a quasiconformal homeomorphism, with $\Phi_\theta^{-1} = \Phi_{-\theta}$. Moreover, Φ_θ is asymptotically conformal at the origin.

Proof The first property holds since $\theta(r)$ is continuous in r and $\theta(r) \to \infty$ as $r \downarrow 0$. The second property is true by compactness of the torus $\mathbb{R}/2\pi$. The third property is a straightforward computation. By another straightforward (if tedious) computation, $D\Phi_\theta = R_\theta + E_\theta$, where R_θ is as above and E_θ is the rank 1 matrix given by

$$E_\theta = \begin{pmatrix} -x\sin(\theta) - y\cos(\theta) \\ x\cos(\theta) - y\sin(\theta) \\ 0 \end{pmatrix} \begin{pmatrix} \theta_x & \theta_y & \theta_z \end{pmatrix}.$$

Let's examine the (1,1) entry of E_θ. Since $\theta_x = \theta'(r)r_x = \theta'(r)x/r$ and $|x| \leq r$, we have

$$|x\theta_x \sin(-\theta) + y\theta_x \cos(-\theta)| \leq 2r|\theta'(r)| \leq 2/(-\log r).$$

The other non-zero entries of E_θ obey the same estimate. This gives the fourth property. To prove that Φ_θ is quasiconformal (see e.g. [13]), it suffices to check that $\Phi_\theta \in W^{1,n}_{\mathrm{loc}}$ and there exists $1 \leq L < \infty$ such that the a.e. defined singular values $\lambda_1 \leq \lambda_2 \leq \lambda_3$ of $D\Phi_\theta$ satisfy $\lambda_3 \leq L\lambda_1$ a.e. These facts follow from property (iv) and the variational characterization of the minimum and maximum singular values. Furthermore, as $r \downarrow 0$, the maximum ratio of λ_3/λ_1 in B_r goes to 1. Therefore, Φ_θ is asymptotically conformal at the origin. □

The *Hausdorff distance* $\mathrm{HD}(A, B) = \max\{\mathrm{excess}(A, B), \mathrm{excess}(B, A)\}$ for all nonempty sets $A, B \subset \mathbb{R}^n$. Note that $\mathrm{HD}(\lambda A, \lambda B) = \lambda \mathrm{HD}(A, B)$ for any dilation factor $\lambda > 0$.

Lemma 2.2 (Twisted Szulkin vs. rotations of Szulkin) *If* $r, \epsilon, R > 0$ *and* $0 < Rr \leq 1/100$, *then* $\mathrm{HD}(\Sigma \cap B_{Rr}, R_{\theta(r)}\Sigma_s \cap B_{Rr}) \leq C \max(\epsilon r, \sup\{q|\theta(q) - \theta(r)| : \epsilon r \leq q \leq Rr\})$.

Proof For any $p \in B_{\epsilon r}$, we have $\mathrm{dist}(p, R_{\theta(r)}\Sigma_s \cap B_{Rr}) \leq 2\epsilon r$ and $\mathrm{dist}(p, \Sigma \cap B_{Rr}) \leq 2\epsilon r$, since $0 \in R_{\theta(r)}\Sigma_s$ and $0 \in \Sigma$. Thus, the main issue is to estimate distances inside $B_{Rr} \setminus B_{\epsilon r}$.

Let $p \in \Sigma \cap B_{Rr} \setminus B_{\epsilon r}$, say $p \in \Sigma \cap \partial B_q$ with $\epsilon r \leq q \leq Rr$. Then we may write $p = R_{\theta(q)}x$ for some $x \in \Sigma_s$. Let's estimate $\mathrm{dist}(p, R_{\theta(r)}\Sigma_s \cap B_{Rr})$ from above by the distance of p to the point $y = R_{\theta(r)}x \in R_{\theta(r)}\Sigma_s \cap \partial B_q$. Note that $y = R_{\theta(r)}x = R_{\theta(r)}R_{-\theta(q)}p = R_{\theta(r)-\theta(q)}p$ and $|y| = |p| = q$. Hence

$$\begin{aligned} |p - y| &\leq q|(1, 0, 0) - (\cos(\theta(q) - \theta(r)), \sin(\theta(q) - \theta(r)), 0)| \\ &= q(2 - 2\cos(\theta(q) - \theta(r)))^{1/2} \\ &\leq Cq|\theta(q) - \theta(r)|, \end{aligned}$$

where the first inequality holds by geometric considerations and the last inequality used the Taylor series expansion for cosine.

A similar inequality holds starting from any $p \in R_{\theta(r)}\Sigma_s \cap B_{Rr} \setminus B_{\epsilon r}$. □

Lemma 2.3 *With $\theta(r) = \log(-\log(r))$, the twisted Szulkin domains $\Omega^\pm$ as defined above are chord-arc domains (i.e. NTA domains with Ahlfors regular boundaries). The interface $\Sigma = \partial\Omega^\pm$ has a continuum of blow-ups at the origin, each of which is a rotation of Σ_s in the xy-plane.*

Proof The domains $\Omega^\pm = \Phi_\theta(\Omega_s^\pm)$ are NTA, because global quasiconformal maps send NTA domains to NTA domains. Every boundary of an NTA domain is lower Ahlfors regular (see e.g. [5, Lemma 2.3]). Thus, Σ is lower Ahlfors regular. To check upper Ahlfors regularity, first note that Σ_s is upper Ahlfors regular, since Σ_s can be covered by a finite number of Lipschitz graphs. Since $\|\det(D\Phi_\theta)\|_\infty < \infty$, it follows that $\Sigma = \Phi_\theta(\Sigma_s)$ is upper Ahlfors regular, as well.

Let's address the blow-ups of $\partial\Omega$ at the origin. Let $r_i \downarrow 0$ and suppose initially that $\theta(r_i) = \theta_0 \pmod{2\pi}$ for all i. Let $\epsilon(r)$ be a function of r to be specified below. Let $R \gg 1$ be a large radius. By Lemma 2.2, the homogeneity of the Hausdorff distance, and the mean value theorem, we have

$$\begin{aligned}
&\mathrm{HD}(r_i^{-1}\Sigma \cap B_R, R_{\theta_0}\Sigma_s \cap B_R)\\
&\quad\le C r_i^{-1} \max\left(\epsilon(r_i)r_i, \sup\{q|\theta(q) - \theta(r_i)| : \epsilon(r_i)r_i \le q \le Rr_i\}\right)\\
&\quad\le C \max\left(\epsilon(r_i), \sup\{t|\theta(tr_i) - \theta(r_i)| : \epsilon(r_i) \le t \le R\}\right)\\
&\quad\le C \max\left(\epsilon(r_i), R(R-1)r_i \sup\{|\theta'(tr_i)| : \epsilon(r_i) \le t \le R\}\right).
\end{aligned}$$

Our task is to choose $\epsilon(r_i)$ so that

$$\lim_{i\to\infty} \epsilon(r_i) = 0 \quad \text{and} \quad \lim_{i\to\infty} \sup\{r_i|\theta'(tr_i)| : \epsilon(r_i) \le t \le R\} = 0. \tag{2.2}$$

Since $|\theta'(r)| = 1/(-r\log r)$, we have $\sup\{r_i|\theta'(tr_i)| : \epsilon(r_i) \le t \le R\} \le 1/(-\epsilon(r_i)\log(Rr_i))$ for all sufficiently large i (i.e. for all sufficiently small r_i). Thus, (2.2) is satisfied (e.g.) by choosing $\epsilon(r) = |\log(r)|^{-1/2}$. It follows that $\lim_{i\to\infty} \mathrm{HD}(r_i^{-1}\Sigma \cap B_R, R_{\theta_0}\Sigma_s \cap B_R) = 0$ for all $R > 0$. This implies that Σ/r_i converge to $R_{\theta_0}\Sigma_s$ in the sense of (1.2).

In the general case, starting from any sequence $r_i \downarrow 0$, pass to a subsequence such that $\theta(r_i) \to \theta_0 \pmod{2\pi}$. One can readily check that $R_{\theta(r_i)}\Sigma_s$ converges to $R_{\theta_0}\Sigma_s$ in the Attouch–Wets topology. Therefore, Σ/r_i converges to $R_{\theta_0}\Sigma_s$ in the sense of (1.2) by the special case and the triangle inequality for excess. □

Remark 2.4 For all exponents $0 < p < 1$, the twisted Szulkin domains defined using the rotation function $\theta(r) = (-\log(r))^p$ also satisfy the conclusions of Lemma 2.3. However, there is phase transition at $p = 1$. When $\theta(r) = -\log(r)$, one can show that the blow-ups of Σ are no longer zero sets of hhp. The essential difference is that the "speed of rotation" vanishes as one zooms-in at the origin when $p < 1$, but the "speed of rotation" is constant when $p = 1$. When $p > 1$, the "speed of rotation" goes to infinity as one zooms-in at the origin and the associated twisted Szulkin domains $\Omega^\pm$ are not even NTA. See Fig. 2.

2.2 Potential Theory for the First Example

Let $r_i \downarrow 0$ be an arbitrary sequence of radii going to zero and let $K \gg 1$. Recall that $\Sigma \cap (B_{Kr_i} \backslash B_{r_i/K}) = \Phi_\theta(\Sigma_s \cap (B_{Kr_i} \backslash B_{r_i/K}))$. Set

$$\tilde{u}_i^\pm(x) = \frac{u^\pm \circ \Phi_{-\theta}^{-1}(r_i x) r_i}{\omega^\pm(B_{r_i})}, \tag{2.3}$$

where $u^{\pm}$ are the Green's functions with poles at infinity for $\Omega^{\pm}$. Then in $\Omega_s^{\pm} \cap B_K \backslash B_{1/K}$, we have that $\tilde{u}_i^{\pm}$ satisfies

$$-\mathrm{div}(B(r_i x)\nabla \tilde{u}_i^{\pm}) = 0, \qquad B = (\det D\Phi_\theta)^{-1}(D\Phi_\theta)(D\Phi_\theta)^T$$

and Φ_θ is as in (2.1).

To see that $B(r_i x)$ is Lipschitz regular, we note that Remark 2.1(iii) implies that $\|DB\| \leq \frac{C}{r\log(r)}$. Therefore, using the fundamental theorem of calculus along curves which stay in the annulus $B_K \backslash B_{1/K}$

$$\|B(r_i x) - B(r_i y)\| \leq Cr_i|x-y| \sup_{B_{Kr_i}\backslash B_{r_i/K}} \|DB\| \leq \frac{CK}{|\log(r_i)|}|x-y|, \quad \forall x, y \in B_K \backslash B_{1/K}, \tag{2.4}$$

where $C > 0$ is independent of i, K. This uniform Lipschitz continuity immediately implies the next result:

Lemma 2.5 *Let $\alpha \in (0,1)$, $K > 1$. The sequence $\tilde{u}_i^{\pm}$ is pre-compact in $C^{1,\alpha}(\Omega_s^{\pm} \cap B_K \backslash B_{1/K})$. Furthermore, there exists a subsequence along which $\tilde{u}_i^{\pm} \to \kappa s$, uniformly on compacta, where s is the Szulkin polynomial, for some $\kappa > 0$.*

Proof We see that $\tilde{u}_i^{\pm}$ solves an elliptic PDE with coefficients that are Lipschitz continuous and elliptic with coefficients independent of i. Furthermore,

$$\sup_{B_{4K}} |\tilde{u}_i^{\pm}| \leq C \quad \Leftrightarrow \quad \sup_{B_{4Kr_i}} |u^+| \leq C\frac{\omega^+(B_{r_i})}{r_i}.$$

The latter inequality holds (with a $C > 0$ that depends on K) by the Caffarelli–Fabes–Mortola–Salsa and doubling estimates on harmonic measure in NTA domains, see e.g. [14]. Then Schauder theory tells us that $\tilde{u}_i^{\pm}$ are uniformly in $C^{1,\alpha}(\overline{\Omega_s^{+}} \cap B_K \backslash B_{1/K})$ for any $\alpha \in (0,1)$; see [12, Theorem 8.3]. The precompactness follows.

Passing to a subsequence, we get that the sequences converges to functions $\tilde{u}_\infty^{\pm}$, which solves $-\mathrm{div}(B_\infty \nabla \tilde{u}_\infty^{\pm}) = 0$ in $\Omega_s^{\pm} \cap B_K \backslash B_{1/K}$. From (2.4) we see that $B_\infty = \mathrm{Id}$ and so, invoking a diagonal argument, $\tilde{u}_i^{\pm} \to \tilde{u}_\infty^{\pm}$, uniformly on compacta in $\mathbb{R}^3$. Furthermore, $\tilde{u}_\infty^{\pm}$ are positive harmonic functions in $\Omega_s^{\pm}$ that vanish on $(\Omega_s^{\pm})^c$.

Since $(\Omega_s^{\pm})^c$ are (global) NTA domains, the boundary Harnack inequality implies that there are scalars $\kappa_{\pm} > 0$ such that $\tilde{u}_\infty^{\pm} = \kappa_{\pm} s$ (see [18, Lemma 3.7 and Corollary 3.2]).

To wrap up, let us again note that the points $(0,0,\pm 1) \in \Omega_s^{\pm}$ are invariant under Φ_θ. Furthermore, by symmetry $u^+(0,0,1) = u^-(0,0,-1)$ and $\omega^+(B_r) = \omega^-(B_r)$ for all r. Thus, $u_\infty^+(0,0,1) = u_\infty^-(0,0,-1)$ and this number determines the constant of proportionality with s. □

Finally, the proof of the continuity of $\log h$ follows immediately:

Proof of $\log h \in C(\partial\Omega)$ We note that away from the origin, $\partial\Omega$ is smooth so continuity of the Radon–Nikodym derivative follows from classical potential theory. Furthermore, arguing by symmetry (that is, $-\Omega^+ = \Omega^-$) we have that $\omega^+(B(0,r)) = \omega^-(B(0,r))$ for all $r > 0$. Thus, recalling that $u^{\pm}$ are the Green's function for $\Omega^{\pm}$ respectively, we are done if we can show that

$$\lim_{\partial\Omega \ni Q \to 0} \frac{|\nabla u^+|(Q)}{|\nabla u^-|(Q)} = 1.$$

(Recall that where $\partial\Omega$ is smooth, $C^{1,\alpha}$ is sufficient, the Radon–Nikodym derivative is given by the ratio of the derivatives of the Green functions [15]).

Let $Q_i \in \partial\Omega$ with $Q_i \to 0$ and let $|Q_i| = r_i \downarrow 0$. Let $\tilde{u}_i^{\pm}$ be given by (2.3). Then

$$\frac{\omega^{\pm}(B_{r_i})}{r_i^2} D\Phi_\theta(r_i x)\nabla\tilde{u}_i^{\pm}(x) = \nabla u^{\pm}(\Phi_{-\theta}^{-1}(r_i x)).$$

Let $\tilde{Q}_i = \Phi_\theta(Q_i)/r_i \in \Sigma_s \cap \partial B_1$. We have shown that

$$\frac{|\nabla u^+|(Q_i)}{|\nabla u^-|(Q_i)} = \frac{|D\Phi_\theta(r_i\tilde{Q}_i)\nabla\tilde{u}_i^+(\tilde{Q}_i)|}{|D\Phi_\theta(r_i\tilde{Q}_i)\nabla\tilde{u}_i^-(\tilde{Q}_i)|}.$$

Continuity of $\log h$ follows from Lemma 2.5 (the lemma implies that $\tilde{u}^{\pm} \to \kappa s$ in $C^{1,\alpha}(\overline{\Omega_s} \cap B_2 \backslash B_{1/2})$) and the fact that along some subsequence $D\Phi_\theta(r_i x) \to R_{\theta_0}$ for some θ_0 (depending on the subsequence). □

3 The Second Example: Non-Unique Flat Tangents

3.1 Description and Geometric Properties

To show non-uniqueness at "flat points" we adapt an example from [25]. We set $\Omega^{\pm} = \{(x, y, z) \in \mathbb{R}^3 : \pm(z - v(x, y)) > 0\}$, where $v : \mathbb{R}^2 \to \mathbb{R}$ is defined by setting $v(0, 0) = 0$,

$$v(x, y) = x \log|\log(r)| \sin(\log|\log(r)|) \quad \text{when } 0 < r = (x^2 + y^2)^{1/2} \le 1/100,$$

and smoothly (e.g. $C^{1,\alpha}$) interpolating to $v(x, y) = 1$ when $r \ge 1$.

Lemma 3.1 (see [25, Example 2]) *The graph domains $\Omega^{\pm}$ are chord-arc domains. The interface $\Sigma = \partial\Omega^{\pm}$ has a continuum of blow-ups at the origin, each of which is a plane $z = mx$ with "slope" $-\infty \le m \le \infty$* (Fig. 3).

Remark 3.2 Moreover, $\Omega^{\pm}$ are vanishing chord-arc domains in the sense of [19]. This can be seen as follows. First, every pseudo blow-up (an Attouch–Wets limit Γ of $(\Sigma - Q_i)/r_i$ with $Q_i \to Q$ and $r_i \downarrow 0$) is a plane. Indeed, on the one hand, if $\limsup_{i\to\infty} |Q_i - Q|/r_i = \infty$, then Γ is a plane, because $\Sigma \setminus \{0\}$ is smooth. On the other hand, if $|Q_i|/r_i \le C$ for all i, then Γ is a translate of a blow-up at Q (see [10, Lemma 3.7]), and thus, Γ is a plane by Lemma 3.1. Because every pseudo blow-up is a plane, Σ is locally Reifenberg vanishing. Now, $v \in W^{2,2}(\mathbb{R}^2)$ (see [25]). Hence, by Sobolev embedding, the normal vector of the interface $\hat{n} \in \mathrm{BMO}(\partial\Omega)$ with small BMO norm. Therefore, $\Omega^{\pm}$ are vanishing chord-arc domains; see e.g. [7, 17].

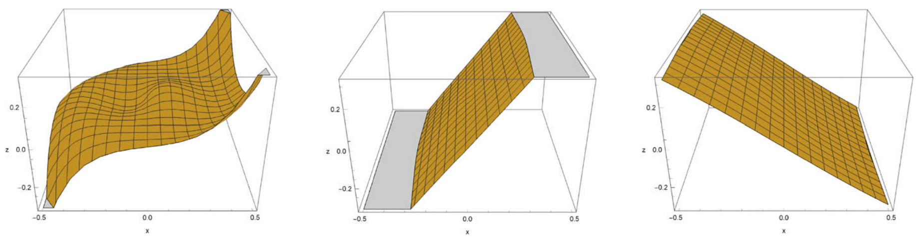

Fig. 3 Blow-ups Σ/r of the interface $\Sigma = \partial\Omega^{\pm}$ of the graph domains. Left: $r = 1$. Center: $r = 10^{-6}$. Right: $r = 10^{-12}$

3.2 Potential Theory for the Second Example

Following the approach of Section 2.2, we now prove that $\log h \in C(\partial\Omega)$.[2] As before, because $\partial\Omega$ is smooth outside of any neighborhood of the origin, $\log h \in C^\infty$ on $\partial\Omega \setminus B_r(0)$ for any $r > 0$. Thus, the key point is to show that $\log h$ is continuous at the origin.

Let $H^\pm = \{\pm z > 0\}$ denote the open upper and lower half-spaces. Let $r_i \downarrow 0$ be arbitrary, $K \gg 1$ and write

$$\{z = v(x, y)\} \cap \left(B_{Kr_i} \backslash B_{r_i/K}\right) = \Phi^{-1}\left(\{z = 0\} \cap (B_{Kr_i} \backslash B_{r_i/K})\right),$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is the homeomorphism given by

$$\Phi(x, y, z) \equiv (x, y, z - v(x, y)).$$

Set $\tilde{u}_i^\pm(p) = \frac{u^\pm \circ \Phi^{-1}(r_i p) r_i}{\omega^\pm(B_{r_i}(0))}$, where $u^\pm$ are the Green's functions with poles at infinity for $\Omega^\pm$, and the $\omega^\pm$ are the corresponding harmonic measures. In $H^\pm \cap B_K \backslash B_{1/K}$, $\tilde{u}_i^\pm$ satisfies

$$-\mathrm{div}\left(B(r_i p)\nabla \tilde{u}_i^\pm(p)\right) = 0, \qquad B = (\det D\Phi)^{-1}(D\Phi)(D\Phi)^T.$$

Lemma 3.3 *Let $\alpha \in (0, 1)$, $K > 1$. The sequence $\tilde{u}_i^\pm$ is pre-compact in $C^{1,\alpha}(\overline{H^\pm} \cap B_K \backslash B_{1/K})$. Furthermore, there exists a subsequence along which $\tilde{u}_i^\pm \to \kappa z_\pm$ for some $\kappa > 0$ uniformly on compact subsets of $\mathbb{R}^3$.*

Proof We claim that $\tilde{u}_i^\pm$ solves an elliptic PDE with Lipschitz continuous coefficients in $B_K \backslash B_{1/K} \cap H^\pm$. Indeed,

$$\begin{aligned} |B(r_i p) - B(r_i q)| &\leq C r_i |p - q| \|DB\|_{L^\infty(B_{Kr_i} \backslash B_{r_i/K})} \\ &\overset{[25]}{\leq} C K r_i \frac{\log|\log(r_i)|}{r_i |\log(r_i)|} |p - q| \leq CK|p - q|, \end{aligned} \tag{3.1}$$

by the fundamental theorem of calculus.

Arguing as in Lemma 2.5 above, $\tilde{u}_i^\pm$ are uniformly in $C^{1,\alpha}(\overline{H^+} \cap B_K \backslash B_{1/K})$ for any $\alpha \in (0, 1)$ and thus have the desired pre-compactness. Passing to a subsequence and invoking a diagonal argument $\tilde{u}_i^\pm \to \tilde{u}_\infty^\pm$ uniformly on compacta. Furthermore, $\tilde{u}_\infty^\pm > 0$ and solves $-\mathrm{div}(B_\infty \nabla \tilde{u}_\infty^\pm) = 0$ in $H^\pm$ and has $\tilde{u}_\infty^\pm(x, y, 0) = 0$. We see in (3.1) that B_∞ is constant (as $\log|\log(r_i)|/\log(r_i) \downarrow 0$) and so $-\mathrm{div}(B_\infty \nabla z) = 0$. Again, up to scalar multiplication there is a unique signed solution of $-\mathrm{div}(B_\infty \nabla -) = 0$ in $H^\pm$ which vanishes on $\{z = 0\}$ and that has subexponential growth at infinity. Continuing to follow the argument for Lemma 2.5, we conclude that $\tilde{u}_\infty^\pm = \kappa_\pm z_\pm$, with $\kappa_+ = \kappa_-$. (Remember that $-\{z > v(x, y)\} = \{z < v(x, y)\}$, because v is odd.) □

Finally, the proof of the continuity of $\log h$ in this context follows exactly as in Section 2.2 except that we must be more careful estimating $|D\Phi(r_i \tilde{Q}_i)\nabla \tilde{u}^\pm(\tilde{Q}_i)|$. (We do not know that $D\Phi(r_i p)$ converges to a rotation as $r_i \downarrow 0$.) However, observe that $\tilde{u}^\pm \equiv 0$ on $\{z = 0\}$, so we know that $\nabla \tilde{u}^\pm(\tilde{Q}_i)$ is parallel to e_3. Thus, an elementary computation shows that

$$\frac{|D\Phi(r_i \tilde{Q}_i)\nabla \tilde{u}^+(\tilde{Q}_i)|}{|D\Phi(r_i \tilde{Q}_i)\nabla \tilde{u}^-(\tilde{Q}_i)|} = \frac{|\nabla \tilde{u}^+(\tilde{Q}_i)||D\Phi(r_i \tilde{Q}_i)e_3|}{|\nabla \tilde{u}^-(\tilde{Q}_i)||D\Phi(r_i \tilde{Q}_i)e_3|} = \frac{|\nabla \tilde{u}^+(\tilde{Q}_i)|}{|\nabla \tilde{u}^-(\tilde{Q}_i)|}.$$

The quantity on the right hand side converges to 1 by Lemma 3.3. As in Section 2.2, it follows that $\log h \in C(\partial\Omega)$.

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[2] One could prove the weaker result that $\log h \in \mathrm{VMO}(d\omega^+)$ using Remark 3.2 and standard properties of A_∞ weights.

4 Open Questions and Further Directions

We end by presenting some natural open questions. Our first question concerns the size of the set of non-uniqueness:

Question 4.1 Let $\Omega^{\pm} \subset \mathbb{R}^n$ be complementary NTA domains with $\log h \in C(\partial\Omega)$. Is it possible for

$$NU(\Omega) := \{Q \in \partial\Omega : \text{there is no unique (geometric) blow-up at } Q\}$$

to have Hausdorff dimension $n-1$?

We note that a local version of [26, Theorem 1.1] implies that the set Γ_1 of flat points in $\partial\Omega$ is uniformly rectifiable. Thus $\omega^{\pm}(NU) = 0 = \mathcal{H}^{n-1}(NU \cap \Gamma_1)$. Furthermore, by the main result of [8], $\dim \partial\Omega \backslash \Gamma_1 \leq n-3$. Thus, $\mathcal{H}^{n-1}(NU) = 0$. On the other hand, the example of [1] suggests that $\mathcal{H}^{n-2}(NU \cap \Gamma_1) > 0$ may be possible.

The example in Section 2 (twisted Szulkin) shows that it is possible for all singular points to have non-unique blowups and for the set of singular points with non-unique blowups to have positive $\mathcal{H}^{n-3}$-measure. (When $n \geq 4$, simply take $\Omega^{\pm} \times \mathbb{R}^{n-3}$.) This is sharp by [8]. Thus, the natural analogue of Question 4.1 is answered in the affirmative.

Our second question asks what are the possible tangent cones at points of non-unique blow-up:

Question 4.2 Let $C \subset G(n, n-1)$ be a compact, connected subset of the Grassmannian. Does there exist a pair of complementary NTA domains $\Omega^{\pm}$ with $\log h \in C(\partial\Omega)$ and a point $Q \in \partial\Omega$ at which $\mathrm{Tan}(\partial\Omega, Q) = C$?

In Section 3, we showed that the set $\mathrm{Tan}(\partial\Omega, 0)$ of blow-ups of the interface of the graph domains at the origin consists of all planes $z = mx$ with "slope" $-\infty \leq m \leq +\infty$. For any closed interval $I \subset \mathbb{R}$, it is not hard to adapt the example so that the blowups at the origin are exactly the planes $z = mx$ with $m \in I$. It is known that for any closed set $\Sigma \subset \mathbb{R}^n$ and $Q \in \Sigma$, the set $\mathrm{Tan}(\Sigma, Q)$ of all tangent sets of Σ at Q is closed and connected in the Attouch–Wets topology [10]; the statement and proof of this fact was originally motivated by similar statement for tangent measures [16, 22].

We may also ask a version of Question 4.2 at points where the blow-ups are homogeneous of higher degree:

Question 4.3 Let $\mathscr{H}_{n,d}$ be the set of degree d homogeneous harmonic polynomials p in $\mathbb{R}^n$ such that $\Omega_p^{\pm} = \{\pm p > 0\}$ are NTA domains. For each $n \geq 3$ and $d \geq 2$ and $C \subset \mathscr{H}_{n,d}$, which is compact and connected, does there exist complementary NTA domains $\Omega^{\pm}$ with $\log h \in C(\partial\Omega)$ and a point $Q \in \partial\Omega$ at which $\mathrm{Tan}(\partial\Omega, Q) = \{\Sigma_p : p \in C\}$?

The condition that $\mathbb{R}^n \backslash \Sigma_p$ is a union of two NTA domains is necessary for Σ_p to arise as a blow-up of the interface of complementary NTA domains. The first step to answering Question 4.3 may be to study the "moduli space" of $\mathscr{H}_{n,d}$ when $d \geq 2$. For example:

Question 4.4 If p and q lie in the same connected component of $\mathscr{H}_{n,d}$, is it true that Σ_q is bi-Lipschitz equivalent to Σ_p?

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