"Factorization Algebra" for Encyclopedia of Mathematical Physics

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Abstract

Factorization algebras are local-to-global objects living on manifolds, and they arise naturally in mathematics and physics. Their local structure encompasses examples like associative algebras and vertex algebras; in these examples, their global structure encompasses Hochschild homology and conformal blocks. In the setting of quantum field theory, factorization algebras articulate a minimal set of axioms satisfied by the observables of a theory, and they capture concepts like the operator product and correlation functions. This article gives the definitions and key examples, compares this approach with other approaches to mathematically formalizing field theory, describes key results, and explains how higher symmetries can be encoded in this framework.

Factorization algebras offer a flexible framework for describing the observables and symmetries of field theories in physics, but they also appear naturally in several areas of mathematics, notably in topology and representation theory. In this way there is an interesting transfer of insights between different subjects, and much of the research around them benefits from this transfer.

Factorization algebras are a *minimal* axiom system for the observables of a quantum field theory (QFT). The idea is to write down the weakest possible axioms that capture the concepts of *operators* (called observables in Costello and Gwilliam (2017, 2021)), *operator products*, and *correlation functions*.

For chiral conformal field theories (CFT), these aspects of a field theory are captured by a vertex algebra; similarly, for non-chiral CFTs, these ideas are described by the conformal bootstrap axioms. Factorization algebra aim to capture these concepts for a general quantum field theory, while imposing the weakest restrictions that can usefully describe this part of QFT. In contrast with algebraic quantum field theory, we do not demand that observables form an associative algebra. *A priori* it is not obvious why one can assign an associative algebra to a quantum field theory (and indeed one cannot in Euclidean signature). Instead, as discussed in section "AQFT and Factorization Algebras", associative algebras appear in Lorentzian signature as a *consequence* of the axioms.

Basics of Factorization Algebras and Their Physical Meaning

Let us now state the axioms of a prefactorization algebra, which are very simple.

Definition 1 *Let M be a topological space (which the reader can take to be Euclidean $\mathbb{R}^n$). A prefactorization algebra $\mathcal{F}$ on M with values in vector spaces is the following data*
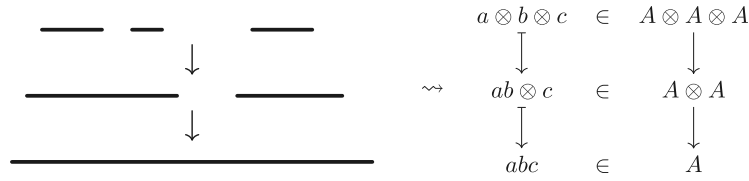
$$a \otimes b \otimes c \quad \in \quad A \otimes A \otimes A$$
$$\downarrow \qquad\qquad \downarrow$$
$$\rightsquigarrow \quad ab \otimes c \quad \in \quad A \otimes A$$
$$\downarrow \qquad\qquad \downarrow$$
$$abc \quad \in \quad A$$

Fig. 1 The prefactorization algebra A^{fact} of an associative algebra A.

- There is a (topological) vector space $\mathcal{F}(U)$ for each open set $U \subset M$
- There is a linear map $m_V^U : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for each inclusion $U \subset V$ of open sets
- There is a linear map $m_V^{U_1, \ldots, U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \rightarrow \mathcal{F}(V)$ for every finite collection of open sets where each $U_i \subset V$ and where the U_i are pairwise disjoint. The following picture represents the situation
- The maps are compatible in the obvious way, so that if $U_{i,1} \sqcup \cdots \sqcup U_{i,n_i} \subseteq V_i$ and $V_1 \sqcup \cdots \sqcup V_k \subseteq W$, the following diagram commutes

For an explicit example of the associativity, consider the following picture.

The case of $k = n_1 = 2$, $n_2 = 1$

Factorization algebras have an additional local-to-global axiom that we discuss in section "From Prefactorization Algebras to Factorization Algebras".

One should think of the vector space $\mathcal{F}(U)$ as the space of measurements one make in the region U. For instance, in a Lagrangian QFT, $\mathcal{F}(U)$ will be built from integrals of local operators against test functions supported in U, as well as similar multi-local expressions.

The product on prefactorization algebras is the operator product of QFT. It is only defined on disjoint opens because, in a Euclidean quantum field theory, the operator product expansion has singularities when operators have coincident position. In a Lorentzian QFT, the singularities can be on the light cone, which suggests a Lorentzian variant of the axioms which we will discuss shortly.

Factorization Algebras and Associative Algebras

A crucial example of a factorization algebra is given by an associative algebra. Every associative algebra A defines a prefactorization algebra A^{fact} on the real line $\mathbb{R}$, as follows. To each open interval (a, b), we set $A^{fact}((a, b)) = A$. To any open set $U = \coprod_j I_j$, where each I_j is an open interval, we set $A^{fact}(U) = \otimes_j A$. The structure maps simply arise from the multiplication map for A. **Fig. 1** displays the structure of A^{fact}. Notice the resemblance to the notion of an A_∞ algebra, or algebra over the little 1-disks operad.

Conversely, if a prefactorization algebra $\mathcal{F}$ on $\mathbb{R}$ has the property that the structure map $\mathcal{F}(I) \rightarrow \mathcal{F}(I')$ is an isomorphism for any inclusion of intervals $I \subset I'$, then $\mathcal{F}$ gives rise to an associative algebra. Moreover, if the prefactorization algebra is equivariant under translation along $\mathbb{R}$, we find an associative algebra with an infinitesimal automorphism, which one might call the Hamiltonian.

This class of examples lets us connect with quantum mechanics, viewed as a one-dimensional field theory. For example, let A be the algebra of operators for a quantum mechanical system. We view the associated prefactorization algebra as describing when measurements happen in time: during the period (a, b), we have $A^{fact}((a, b)) \cong A$ describing possible measurements.

Similarly, if $\mathfrak{g}$ is a Lie algebra of symmetries of the system, there is a map of Lie algebras $J : \mathfrak{g} \rightarrow A$ and hence a map of associative algebra $J : U\mathfrak{g} \rightarrow A$, where $U\mathfrak{g}$ denotes the universal enveloping algebra. This *quantum moment map* determines a map of prefactorization algebras $J^{fact} : U\mathfrak{g}^{fact} \rightarrow A^{fact}$. One views $U\mathfrak{g}^{fact}$ as a current algebra and J^{fact} as encoding how currents determine operators for the quantum system.

Correlation Functions From Prefactorization Algebras

Although the prefactorization algebra axioms are very simple, it is perhaps a little more surprising that this data is enough to capture concepts such as correlation functions and the operator product expansion. To see this, it is easiest to work on $\mathbb{R}^n$ and assume that our theory has Euclidean symmetry.

A *Euclidean-invariant* prefactorization algebra $\mathcal{F}$ has a compatible action of the Euclidean group $\text{Iso}(\mathbb{R}^n)$, giving isomorphisms between $\mathcal{F}(U)$ and $\mathcal{F}(g(U))$ for any $g \in \text{Iso}(\mathbb{R}^n)$ in a way respecting products.

A *state* is a linear map

$$\langle - \rangle : \mathcal{F}(\mathbb{R}^n) \rightarrow \mathbb{C}$$

that is invariant under the $\text{Iso}(\mathbb{R}^n)$ action.

A *local operator* at x is an element of the inverse limit

$$\mathcal{F}_x = \lim_r \mathcal{F}(D(x, r))$$

where $D(x,r)$ is the disc of radius r around x. Given a local operator $\mathcal{O}$ at 0, we can use the translation action to build a local operator $\mathcal{O}(x)$ at any point x.

Given a state, we define correlation function of local operators, as follows. Given n such operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$, the correlation function is then defined as

$$\langle m(\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n))\rangle \in \mathbb{C}$$

where

$$m : \mathcal{F}_{x_1} \otimes \ldots \otimes \mathcal{F}_{x_n} \to \mathcal{F}(\mathbb{R}^n)$$

is the factorization product. The limit involved in defining the space of local operators means that m is defined as long as the points x_i are disjoint.

Operator Products Expansions and Prefactorization Algebras

Suppose we have a Euclidean-invariant prefactorization algebra on $\mathbb{R}^n$. Then $\mathcal{F}(D(0,r))$ is the space of *almost-local operators*; it is built from products of operators that have been smeared over a disc of radius r. The factorization product map

$$m : \mathcal{F}(D(0,r)) \otimes \mathcal{F}(D(x,r)) \to \mathcal{F}(D(0,R)),$$

where $2r < \|x\| < R + r$, has the physical interpretation of the *operator product*.

Wilson's operator product expansion arises as follows. First, we take the limit as $r \to 0$, replacing $\mathcal{F}(D(x,r))$ by $\mathcal{F}_x$ and $\mathcal{F}(D(0,r))$ by $\mathcal{F}_0$. Then the product is defined for $0 < \|x\| < R$. Given two operators $\mathcal{O}_1$ and $\mathcal{O}_2$, the product

$$m(\mathcal{O}_1(0),\mathcal{O}_2(x))$$

depends smoothly (and, in practice, real-analytically) on $x \in D(0,R)$. (Here we are using the fact that $\mathcal{F}(D(0,R))$ is really a topological vector space.) By construction, for $\|x\| < \varepsilon$, this product lives in $\mathcal{F}(D(0,\varepsilon))$.

If there exists an asymptotic expansion of $m(\mathcal{O}_1(0),\mathcal{O}_2(x))$ as $\|x\| \to 0$, then it automatically will live in the space $\mathcal{F}_0 = \lim_r \mathcal{F}(D(0,r))$ of local operators. Such an asymptotic expansion will be of the form

$$m(\mathcal{O}_1(0),\mathcal{O}_2(x)) \sim \sum F_i(x)\mathcal{O}^i(0)$$

where $\mathcal{O}^i(0) \in \mathcal{F}_0$ and $F_i(x)$ are real-analytic functions. The axioms of a prefactorization algebra do not guarantee the existence of such an asymptotic expansion, but if it does exist, it is unique, and in perturbative QFT this expansion does exist.

The associativity axiom of a prefactorization algebra is intended as a replacement for associativity of the OPE, which for non-conformal theories is very challenging to formulate. For chiral theories in dimension 2, one can show that this OPE satisfies the axioms of a vertex algebra, and the associativity axiom of a vertex algebra is a consequence of the associativity axiom of a prefactorization algebra (see Costello and Gwilliam (2021)).

A History of Factorization Algebras, and Their Relationship to Other Axioms

We connect these notions to other approaches and place them in historical context.

A Brief History

There are three different notions of factorization algebra in the literature, and while they share the same spirit, they differ in detail. The original version is due to Beilinson and Drinfeld (2004), from the early 1990s, and it arose as a recasting of vertex algebras (and related notions from chiral conformal field theory) in the language of algebraic geometry; it has primarily been developed in the setting of complex algebraic curves. Their theory has profound applications in geometric representation theory, notably the geometric Langlands program, and its higher-dimensional consequences are still being explored. The second version began with Francis, Gaitsgory, and Lurie around 2007 (although with roots in earlier work of McDuff (1975), Salvatore (2001), Segal (1973), and others). These authors recognized that a topological analogue should exist to the Beilinson-Drinfeld version, much as a vertex algebra is reminiscent of an algebra over the little disks operad (i.e., an E_2 algebra). It works in the setting of manifolds (not necessarily smooth), possibly equipped with data like an orientation or a G-bundle, but it is not sensitive to more geometric structure (like a metric). This theory is called *topological chiral homology* by Lurie (n.d.) or *factorization homology* by Ayala and Francis (2015) or *blob homology* by Morrison and Walker (2012), and it has beautiful applications in algebraic topology itself (notably a nonabelian generalization of Poincaré duality), but it also offers a class of computable topological field theories, a topic we discuss below. (See Ayala and Francis (2020) for a recent extensive survey. Morrison and Walker came to these ideas by a different route). The third version grew in response to the other two, as we recognized around 2008 that the observables produced by BV quantization seemed to form factorization algebras. In Costello and Gwilliam (2017, 2021) we developed axioms that fit well with differential geometry and field theory. All three versions have overlapping domains of applicability, but comparison is somewhat subtle and technical. We will focus here on the third version, and its offshoots.

Segal's Axioms and Prefactorization Algebras

The concept of factorization algebra has a close relationship with Segal's axioms for quantum field theory (see Kontsevich and Segal (2021) and Segal (1988, 2010, 2014)). Segal (following work with Atiyah on topological field theories, and ideas suggested by Witten) proposed that a quantum field theory in dimension d is a functor Z that assigns to a closed Riemannian manifold (N, g) of dimension $d - 1$, a (topological) vector space $Z(N, g)$, and that assigns to a Riemannian cobordism, a (continuous) linear map.

A quantum field theory in the sense of Segal gives rise to a prefactorization algebra on any Riemannian manifold M, as follows. Let Z be a Segal QFT. If $U \subset M$ is open and ∂U is its boundary, then we set

$$\mathcal{F}_Z(U) = Z(\partial U).$$

(To be pedantic, one needs to restrict to opens $U \subset M$ that have a nice boundary.) If $U_1, \ldots, U_n \subset V$ are disjoint opens, then the closure W of the manifold $V \backslash (U_1 \cup \ldots U_n)$ provides a cobordism between $\partial U_1 \cup \ldots \partial U_n$ and ∂V, and the factorization product

$$m : \bigotimes_i \mathcal{F}_Z(U_i) \to \mathcal{F}_Z(V)$$

is the map $Z(W)$.

Segal's axioms, however, are much stronger than those of a prefactorization algebra. In some ways this is a disadvantage: it is very hard to construct a solution to Segal's axioms, even in perturbation theory. By contrast, one can rigorously construct factorization algebras associated to field theories in perturbation theory.

AQFT and Factorization Algebras

Algebraic quantum field theory arises from the same impulse as factorization algebras: one should organize observables by their spacetime support (see Haag and Kastler (1964) and Haag (1996)). The context and details differ, however, as AQFT focuses on Lorentzian field theories and typically works with C^* or von Neumann algebras. Moreover, an AQFT only assigns an algebra to a specific class of subspaces or manifolds.

Relationships between the approaches are emerging, especially as modifications and generalizations of traditional AQFT are being actively explored:

- In Benini *et al.* (2020), Benini, Perin, and Schenkel carefully compare the two frameworks, and they introduce a useful variation — called *time-orderable prefactorization algebras* — that are well-suited to Lorentzian field theories. Adding the natural hypotheses of Cauchy constancy and additivity, they produce an equivalence between AQFTs and time-orderable prefactorization algebras.
- Gwilliam and Rejzner (2020) compare the BV quantization of free theories (such as a free scalar field or a free fermion) in the AQFT and factorization setting, showing how they are equivalent and explicating how to transfer information between the two descriptions of observables. In Gwilliam and Rejzner (n.d.) they show that the perturbative AQFT construction for interacting field theories can be extended to produce a factorization algebra of observables.
- In Benini *et al.* (n.d.) show, using a different framework for quantizing free Lorentzian theories, how to compare their AQFT and factorization algebras.

The works of these authors demonstrates one nice feature of factorization algebras: the associative algebras demanded in algebraic quantum field theory arise as a *consequence* of the apparently weaker definition of factorization algebra.

We have already seen this phenonemon in our discussion of quantum mechanics, where we saw that prefactorization algebras on $\mathbb{R}$ (satisfying an extra axiom) give rise to associative algebras, where the associative product comes from the factorization product.

As shown in Benini *et al.* (2020), something similar happens for prefactorization algebras on a Lorentzian manifold M. One must modify the definition of prefactorization algebras in two ways:

(i) First, the factorization product is only defined when the opens $U_1, \ldots, U_n \subset V$ are *causally* disjoint (and not just disjoint), which means that the opens U_i and U_j cannot be connected by a light ray for $i \neq j$. This condition is needed because the operator product in Lorentzian signature has poles on the light cone.
(ii) The *Cauchy constancy axiom* states that if $U \subset V$ are opens in M that are both globally hyberbolic and that share a common Cauchy hypersurface $X \subset U \cap V$, then the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism. This axiom is a form of causality: it says that anything that can be measured in a globally hyperbolic manifold V can be measured in a small neighborhood of a Cauchy hypersurface.

In this situation Benini *et al.* (2020), shows that $\mathcal{F}(U)$ is an associative algebra for any globally hyperbolic submanifold $U \subset M$. We feel that this construction of Benini *et al.* provides an important conceptual improvement to algebraic quantum field theory.

Example 1 *Let us outline their argument in the case of* $\mathbb{R}^2$. *Recall that the causal diamond centered on* (x_0, t_0) *of width* r *is*

$$\mathbb{D}(x_0, t_0, r) = \{x, t \,|\, |x - x_0| + |t - t_0| < r\}.$$

We will define the product on the diamond $\mathbb{D}(0, 0, 1)$.

Let $\mathcal{O}_1, \mathcal{O}_2$ be two elements of $\mathcal{F}(\mathbb{D}(0,0,1))$. Assuming a version of the local-to-global factorization algebra axiom, one can show that $\mathcal{O}_2$ lives in $\mathcal{F}(\mathbb{D}(0,\varepsilon,1-\varepsilon))$ for some small ε. This assumption is physically reasonable: any given observable is a measurement in some bounded subregion of the causal diamond.

By Cauchy constancy, we can then lift $\mathcal{O}_1$ to some small neighborhood of the Cauchy hypersurface $t = 0$, and also lift $\mathcal{O}_2$ to some small neighborhood of $t = \varepsilon$, which is a Cauchy hypersurface in $\mathcal{F}(\mathbb{D}(0,\varepsilon,1-\varepsilon))$. These neighborhoods are causally disjoint, so we can define the factorization product of $\mathcal{O}_1$ with $\mathcal{O}_2$. This product turns out to be associative, and it defines the algebra seen in AQFT.◇

Chiral Conformal Field Theory and Factorization Algebras

Conformal field theory (CFT) in two dimensions, particularly chiral CFT, has experienced extensive exploration by mathematicians as well as physicists, with deep consequences for representation theory, complex geometry, topology, and operator algebras. There are now many different mathematical formulations — including (but not limited to) vertex algebras, conformal nets, and Segal's functors — and comparisons in various directions. We list here a few results that entangle factorization algebras, in the style described here, with those other approaches.

- Chapter 5 of Costello and Gwilliam (2017) examines prefactorization algebras on $\mathbb{C}$ and identifies properties that allow one to extract a vertex algebra, leading to a functor from this class of prefactorization algebras to the category of vertex algebras. This functor produces many examples, such as the universal affine vertex algebra for an affine Lie algebra $\hat{L\mathfrak{g}}$, Virasoro vertex algebras (by Williams (2017)), and chiral differential operators (by Gwilliam et al. (2020)).
- Bruegmann (n.d.) gives a functor from $\mathbb{Z}$-graded vertex algebras to factorization algebras on $\mathbb{C}$. His construction provides a kind of inverse to the functor of Costello and Gwilliam (2017), and it builds upon work of Huang (1997).
- Henriques (2018) shows that a conformal net of finite index is a factorization algebra on the circle. He uses this result in a larger project to understand the 3-category of conformal nets and to construct important examples such as WZW theories and Chern-Simons theories.
- Hennion and Kapranov (2023) provide a functor from locally constant Beilinson-Drinfeld factorization algebras to locally constant Costello-Gwilliam factorization algebras. It would be interesting to extend it to encompass the non-locally constant algebras.

Topological Field Theory and Factorization Algebras

Topological field theories (TFTs) provide another area where mathematicians have explored deeply, along with physicists. On the one hand, there is a collection of examples that seem endlessly productive, such as the A- and B-models appearing in mirror symmetry, Chern-Simons and Rozansky-Witten theories related to knot and 3-manifold invariants, and four-dimensional gauge theories related to the Donaldson and Seiberg-Witten invariants. On the other hand, there is the functorial framework, introduced by Atiyah (1988) and Segal (2004), that has evolved into a powerful tool for analyzing TFTs, particularly nonperturbative aspects rooted in topology (see Freed et al. (n.d.), Gaiotto et al. (2015) and Sharpe (2015)). Joining the two hands is a work in progress. Factorization algebras connect with both directions.

- In Section "Observables as Prefactorization Algebras" of Lurie (2009), Lurie explains why factorization homology should yield a fully extended framed n-dimensional TFT from an E_n-algebra, and Scheimbauer gave a detailed development of this idea in her thesis (Scheimbauer, 2014). Further structural results by Ayala and Francis (2019), Ayala et al. (2017) and Morrison and Walker (2012) enrich this story.
- This approach has been deployed on concrete examples, notably by Ben-Zvi, Brochier, and Jordan, who computed the factorization homology on surfaces of the representations of a quantum group in Ben-Zvi et al. (2018a, 2018b). There are many outgrowths of this work, such as Brochier and Woike (2023), Brochier et al. (2021), Cooke (2023), Keller and Müller (2023) and Kirillov and Tham (2022)
- Elliott and Safronov (2019) show how a prefactorization algebra on $\mathbb{R}^n$ with an action of a super translation algebra (e.g., the observables of a supersymmetric field theory) determines an E_n algebra after choosing a "twist" in the super Lie algebra.
- There is extensive work on the rigorous construction of perturbative topological field theories, such as Chern-Simons theory or Poisson σ-models, going back to Axelrod and Singer (1994) and Kontsevich (1994). Due to the main theorem of Costello and Gwilliam (2021), such work yields factorization algebras, with the local behavior producing an E_n-algebra (see Elliott and Safronov (2019)) as well. For some examples, the values of factorization homology have been computed, such as for topological B-models by Li and Li (2016), Rozansky-Witten theories by Chan et al. (2017), and the Kapustin-Witten gauge theories by Elliott et al. (n.d.)

A recent expository source is Tanaka (2020).

From Prefactorization Algebras to Factorization Algebras

Now let us get a bit more formal and discuss the local-to-global axiom that distinguishes factorization algebras from pre-factorization algebras.

A factorization algebra is, in essence, a prefactorization algebra whose value on large open sets is determined by its behavior on small open sets. In other words, its global behavior is governed by its local structure. We will use terminology from sheaf theory to make the local-to-global condition precise, but similar intuition motivates ordinary homology in algebraic topology (which is covariant in open sets, just as a prefactorization algebra is). See Remark 2 for more on this comparison.

We have already seen how to define correlation functions using prefactorization algebras. Let us generalize the discussion and consider a prefactorization algebra on a manifold M instead of on $\mathbb{R}^n$. A state is given by a linear map

$$\langle \cdots \rangle : \mathcal{F}(M) \to \mathbb{C}$$

that allows us to define correlators of operators in disjoint opens $U_1 \ldots, U_n$ via the composition

$$\mathcal{F}(U_1) \otimes \ldots \otimes \mathcal{F}(U_n) \to \mathcal{F}(M) \to \mathbb{C}.$$

In particular, we have n-point correlation functions for any n-tuple of operators supported at points.

We would like to say that a state is the same data as its collection of correlation functions of operators. Equivalently, we would like to say that the vector space $\mathcal{F}(M)$ is the *universal recipient* of correlation functions. That is, there are correlation functions valued in $\mathcal{F}(M)$, and any set of correlation functions valued in some other vector space V is obtained via a linear map from $\mathcal{F}(M)$ to V.

The factorization algebra axiom aims to capture this idea. It was inspired by the work of Ayala and Francis (2015), Beilinson and Drinfeld (2004) and Lurie (n.d.). Before we give the definition, let us point out that this axiom is tailored to *local* operators. In general, it does not aim to capture extended defects.

The key definition that allows us to formulate the local-to-global axiom is the following.

Definition 2 *Given a manifold M and an open set $U \subset M$, a Weiss cover of U is a collection of open subsets $\mathfrak{B} = \{V_i\}_{i \in I}$ with the property that any finite set of points $\{x_1, \ldots, x_n\}$ in U is contained in some V_i from $\mathfrak{B}$.*

This notion is stronger than the usual notion of cover, which simply requires that any single point in U is contained in some open set in the cover. A typical Weiss cover of U is given by the collection of opens in U that are finite disjoint unions of open balls of radius less than δ, for some small δ. It is a Weiss cover because any finite subset is inside such a union of open balls. Typically, Weiss covers contain infinitely many elements, because for a manifold, we can always find a finite set not contained in some finite collection of proper open subsets.

Definition 3 *A factorization algebra on M is a prefactorization algebra satisfying*

(a) *If U_1 and U_2 are disjoint opens in M, then the map $\mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \to \mathcal{F}(U_1 \coprod U_2)$ is an isomorphism, and*

(b) *$\mathcal{F}$ satisfies the cosheaf condition (dual to the sheaf condition) for Weiss covers, which means that if the $\{V_i\}$ form a Weiss cover of U, then the sequence*

$$\bigoplus_{i \neq j} \mathcal{F}(V_i \cap V_j) \to \bigoplus_i \mathcal{F}(V_i) \to \mathcal{F}(U) \to 0 \tag{1}$$

is exact on the right and in the middle.

Remark 1 *In our book (Costello and Gwilliam, 2017) we use the derived version of this condition. In that case, each $\mathcal{F}(U)$ is a cochain complex, and we ask that $\mathcal{F}(U)$ is quasi-isomorphic to theČech double complex arising from a Weiss cover. In order to make this survey as broadly accessible as possible, we have decided to present the un-derived version of the definition.⋄*

Let us unpack this definition in the case that we use the Weiss cover built from disjoint unions of open balls of radius less than some small δ. Given points $x_1, \ldots, x_n \in U$ and radii $\varepsilon_1, \ldots, \varepsilon_n < \delta$ such that the balls $D(x_i, \varepsilon_i)$ are pairwise disjoint, we have a map

$$\bigotimes_i \mathcal{F}(D(x_i, \varepsilon_i)) \to \mathcal{F}(U).$$

For $\mathcal{F}$ a factorization algebra, this map satisfies two conditions. First, the span of the images of these maps, as x_i and ε_i varies, is all of $\mathcal{F}(U)$. That is, any element of $\mathcal{F}(U)$ is built from a sum of products of almost-local operators. Therefore, any linear functional on $\mathcal{F}(U)$ is indeed determined by its correlation functions on operators in $\mathcal{F}(D(x, \varepsilon))$ for $\varepsilon < \delta$ and δ arbitrarily small.

The second condition deals with a redundancy in this construction. Pick a linear function on $\mathcal{F}(U)$, giving correlation functions for almost-local operators. Pick some operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ in $\mathcal{F}(D(x_i, \varepsilon_i))$. Suppose $D(y_i, \delta_i)$ are open discs obtained by moving the discs $D(x_i, \varepsilon_i)$ a little, and suppose that $\mathcal{O}_i$ come from the intersection $\mathcal{F}(D(y_i, \delta_i) \cap D(x_i, \varepsilon_i))$. Then the correlators of the operators $\mathcal{O}_i$, viewed as elements of $\mathcal{F}(D(x_i, \varepsilon_i))$ and as elements of $\mathcal{F}(D(y_i, \delta_i))$, should be the same. Asking that the sequence (1) is exact in the middle ensures that this is so (and imposes other similar constraints).

Given a factorization algebra $\mathcal{F}$ on M, we often call its global sections $\mathcal{F}(M)$ (and especially the derived version) the *factorization homology* of $\mathcal{F}$ on M. A common notation is $\int_M \mathcal{F}$.

An example might help clarify what information is contained in factorization homology.

Example 2 *Let A be an associative algebra. It determines a factorization algebra A^{fact} on the circle with values in vector spaces, where*

- *For any proper open set $U = I_1 \sqcup \cdots \sqcup I_k$ consisting of finitely many open intervals, $A^{fact}(U) = A^{\otimes k}$ and*
- *$A^{fact}(S^1) = A/[A,A]$, the quotient of A by the vector subspace spanned by commutators $ab - ba$.*

This quotient space is the source of all trace maps: any linear map $\tau : A \to R$ satisfying $\tau(ab) = \tau(ba)$ for all $a, b \in A$ corresponds to a linear map $A/[A,A] \to R$. (Compare with the use of density matrices in quantum mechanics.)⋄

This example generalizes substantially. We will state a version that is easy to interpret.

Theorem 1 *Let A be a dg algebra. It determines a factorization algebra on S^1 assigning A to each open interval inside S^1 and whose factorization homology is quasi-isomorphic to the Hochschild complex $C_*^{Hoch}(A, A)$.*

One can thus view factorization homology as providing a systematic generalization of Hochschild homology, with $\int_M \mathcal{F}$ reflecting the shape of the given manifold M.

Remark 2 *A factorization algebra $\mathcal{F}$ is locally constant if the structure map $\mathcal{F}(D) \to \mathcal{F}(D')$ is a quasi-isomorphism whenever we have an inclusion of disks $D \subset D'$. In this setting, Ayala and Francis show that the factorization homology satisfy axioms akin to the Eilenberg-Steenrod axioms for homology theories, although the "coefficients" (the value on a disk) are now much richer than an abelian group. (For framed n-dimensional manifolds, the coefficients are E_n-algebras, i.e., algebras over the little n-dimensional disks operad.) Their characterization offers a powerful toolkit in the setting of algebraic topology, notably some alternative computational methods. For a physicist, the observables of a topological field theory provide examples of locally constant prefactorization algebras.*⋄

Factorization Algebras From Deformation Quantization

Axiom schemes for quantum field theory are not so useful unless one can produce solutions to these axioms. In Costello and Gwilliam (2017, 2021) we show that essentially any perturbative field theory in Riemannian signature has a factorization algebra of observables (see Section "Key Results for Perturbative Theories" for more). These books also they explore how many standard notions and constructions can be articulated in terms of factorization algebras. Using these methods, one obtains a direct path between concrete Lagrangian formulations and the structural points of view. Recent work indicates that analogous results hold for perturbative theories in Lorentzian signature (see Benini *et al.* (2020) and Gwilliam and Rejzner (n.d.)).

In this section we will outline why prefactorization algebras appear in classical field theory (even nonperturbatively) and why perturbative quantization deforms this classical structure. We discuss nonperturbative issues in Section "Noether's Theorem Beyond the Perturbative".

Observables as Prefactorization Algebras

So far in our discussion of factorization algebras, we have focused on factorization algebras valued in vector spaces. For a proper discussion of field theory, we need to use factorization algebras valued in cochain complexes. This is because we work in the BV-BRST formalism, which is needed to quantize gauge theories (and is useful even without gauge symmetry). The reader uncomfortable with this homological language can generally ignore this extra technology and continue to take the factorization algebras to be valued in vector spaces.

A classical field theory $\mathcal{T}$, specified by a Lagrangian density or action functional, yields a system of partial differential equations that we will call the equations of motion for $\mathcal{T}$. In a setting like a gauge theory, we may want to consider solutions to these equations up to an equivalence relation (e.g., gauge transformation) that respects locality. (It is better, in mathematical practice, to work with a stack of solutions that encodes how different solutions are identified.) In brief, a classical field theory produces a sheaf $\mathrm{Sol}_\mathcal{T}$ on spacetime M that assigns to each open set U, a space $\mathrm{Sol}_\mathcal{T}(U)$ of solutions to the equations of motion on U. (This space might be a manifold, possibly infinite-dimensional, or a stack or a derived stack, depending on the context.) It is a sheaf because PDE impose constraints that are local in spacetime, and a solution is determined by its behavior on a cover.

Now consider the observables of this theory, which should describe functions on solutions. Let $\mathrm{Obs}_\mathcal{T}$ assign to each open set U, the commutative algebra of functions on $\mathrm{Sol}_\mathcal{T}(U)$. If $\mathrm{Sol}_\mathcal{T}(U)$ is a (possibly derived) stack, then this algebra $\mathcal{O}(\mathrm{Sol}_\mathcal{T}(U))$ will be a dg commutative algebra. Let $\mathcal{O}$ denote the functor that takes a space X (e.g., a derived stack for some theories) and produces its algebra of functions $\mathcal{O}(X)$ (e.g., a dg commutative algebra for those theories). This functor $\mathrm{Obs}_\mathcal{T} = \mathcal{O} \circ \mathrm{Sol}_\mathcal{T} : \mathrm{Open}(M) \to \mathrm{CAlg}$ is covariant, because $\mathcal{O}$ is contravariant and $\mathrm{Sol}_\mathcal{T}$ is also contravariant in open sets.

Note that this functor $\mathrm{Obs}_{\mathcal{T}}$ provides a prefactorization algebra because

$$\mathrm{Sol}_{\mathcal{T}}(U \sqcup V) \cong \mathrm{Sol}_{\mathcal{T}}(U) \times \mathrm{Sol}_{\mathcal{T}}(V)$$

for disjoint open sets U, V, and so there is a canonical map

$$\mathcal{O}(\mathrm{Sol}_{\mathcal{T}}(U)) \otimes \mathcal{O}(\mathrm{Sol}_{\mathcal{T}}(V)) \to \mathcal{O}(\mathrm{Sol}_{\mathcal{T}}(U) \times \mathrm{Sol}_{\mathcal{T}}(V))$$

that gives the structure map

$$\mathrm{Obs}_{\mathcal{T}}(U) \otimes \mathrm{Obs}_{\mathcal{T}}(V) \to \mathrm{Obs}_{\mathcal{T}}(U \sqcup V).$$

This map just says that observables supported on disjoint regions can be multiplied. All the other structure maps can be built from this one and the maps $\mathrm{Obs}_{\mathcal{T}}(U) \to \mathrm{Obs}_{\mathcal{T}}(U')$ arising from an inclusion $U \subset U'$.

This argument is robust, and it does not depend on the signature of the metric or other details of the theory, just the locality of classical Lagrangian field theory. It applies nonperturbatively.

Two separate questions now arise. First, under what conditions is $\mathrm{Obs}_{\mathcal{T}}$ a factorization algebra? And second, what happens when we quantize $\mathcal{T}$?

The second question depends on what notion of quantization is used. So far, research has focused on BV quantization, as it produces a deformation of the differential on the observables and hence is well-suited to this setting. Other methods of quantization might also apply, but they have not yet been explored in this context, so far as we know. Hence we will discuss only BV quantization here.

Even at the classical level, one does not expect the observables of an arbitrary classical field theory to form a factorization algebra. Physically, it is because the local-to-global axiom is linked to local operators, and classical gauge theories can have extended defects such as Wilson lines. Mathematically, it is because the functor $\mathcal{O}$ that takes a space X and produces its algebra of functions $\mathcal{O}(X)$ can lose information; for example, functions on a quotient stack like $BG = */G$ are trivial but the space is not. It is thus unlikely that the composite $\mathrm{Obs}_{\mathcal{T}} = \mathcal{O} \circ \mathrm{Sol}_{\mathcal{T}}$ is a factorization algebra. (The observables of a Dijkgraaf-Witten theory are typically not a factorization algebra.)

Remark 3 *One takeaway is that prefactorization algebras appear naturally, but we do not yet know what local-to-global axiom is appropriate in nonperturbative settings. For TFTs, variants are know (Morrison and Walker, 2012): introduced blob homology, and (Ayala et al., 2018) have enhanced factorization homology, in steps toward a potential proof of the cobordism hypothesis described by Ayala and Francis (n.d.).* ◇

In the perturbative setting, however, the situation improves considerably. By a perturbative classical field theory, we mean that we fix a global solution $\phi \in \mathrm{Sol}_{\mathcal{T}}(M)$ and we study formal deformations of this solution. In this case, there is a robust mathematical framework, known as derived deformation theory and compatible with the BV-BRST formalism, and a key results is the for a formal moduli space, its algebra of functions knows everything (if one remembers a canonical filtration). Since the perturbative space $\mathrm{Sol}_{\mathcal{T},\phi}$ of solutions is a sheaf and hence satisfies a local-to-global condition, the observables then also satisfy a local-to-global condition. In fact, these classical observables form a factorization algebra.

If a BV quantization exists (i.e., the anomaly vanishes), it produces, in essence, a deformation of the differential that produces a deformation of $\mathrm{Obs}_{\mathcal{T}}$ as a factorization algebra. Such a BV quantization involves issues of renormalization and homological algebra, and it is these issues that are the focus of Costello and Gwilliam (2021), Fredenhagen and Rejzner (2013), Gwilliam and Rejzner (n.d.) and Benini et al. (2020).

Key Results for Perturbative Theories

In this section, whenever we speak of a perturbative classical or quantum field theory, we will mean the definitions articulated in Costello and Gwilliam (2017, 2021), Costello (2011), which encompass the Euclidean versions of many field theories studied in physics and mathematics. The main result is the following.

Theorem 2 *The observables of a perturbative classical field theory on M form a factorization algebra Obs^{cl} that assigns to every open set, a dg commutative algebra. A Batalin-Vilkovisky quantization of this theory, if it exists, yields a factorization algebra Obs^{q} that is a flat deformation of Obs^{cl} over $\mathbb{R}[[\hbar]]$.*

This theorem provides an elegant interpretation of Batalin-Vilkovisky formalism as a kind of deformation quantization: the classical world is commutative and the quantum world deforms it in a noncommutative direction. An important refinement, which we will not examine here, is that the classical observables have a 1-shifted Poisson structure (the BV anti-bracket), which is the analogue of the usual, unshifted Poisson bracket in traditional deformation quantization.

Remark 4 *One might hope that a similar statement holds for Lorentzian theories as well, and recent work (discussed in Section "AQFT and Factorization Algebras") gives cause for optimism.* ◇

Symmetries play a key role in analyzing and organizing field theories, and they can be encoded using factorization algebras. We saw in section "Factorization Algebras and Associative Algebras" how a Lie algebra $\mathfrak{g}$ acting on an associative algebra A by inner derivations (i.e., given a Lie algebra map $\mathfrak{g} \to A$) determines an associative algebra map $U\mathfrak{g} \to A$ and hence a map of factorization algebras $U\mathfrak{g}^{fact} \to A^{fact}$. This pattern continues. Any Lie algebra $\mathfrak{g}$ (or L_∞ algebra) has an *enveloping factorization algebra* $\mathbb{U}\mathfrak{g}$ on a manifold M (see Knudsen (2017)) for a thorough treatment.) More generally, any reasonable sheaf $\mathcal{G}$ of Lie algebras on a manifold M has an *enveloping factorization algebra* $\mathbb{U}\mathcal{G}$ (see chapter 3, section 6 of Costello and Gwilliam (2017), where $\mathcal{G}$ is a *local L_∞ algebra*). A rich source of examples are the current algebras of physics, and these allow for a systematic extension of constructions from mechanics (like quantum moment maps) and conformal field theory (like the Kac-Moody or Virasoro vertex algebras).

An interesting feature of this construction is that $\mathbb{U}\mathcal{G}$ has a natural filtration and its associated graded is a 1-shifted Poisson algebra. In other words, $\mathbb{U}\mathcal{G}$ is itself a kind of BV quantization, much as the universal enveloping algebra $U\mathfrak{g}$ is a deformation quantization of $\mathrm{Sym}(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^*)$. This feature leads to the following refinement of Noether's (first) theorem in this setting for field theory. See the final chapters of Costello and Gwilliam (2021) for a full discussion and many examples.

Theorem 3 *Let a local L_∞ algebra $\mathcal{G}$ act as symmetries of a classical field theory on M; in particular, it acts by Hamiltonian vector fields for shifted Poisson structure on Obs^{cl}. Then there is a map of factorization algebras*

$$J^{cl} : \mathrm{Sym}(\mathcal{G}_c[1]) \to \mathrm{Obs}^{cl}$$

that respects shifted Poisson structures.

If the classical theory admits a BV quantization, then there may be a cohomological obstruction α to deforming J^{cl} to a map of factorization algebras $\mathbb{U}\mathcal{G} \to \mathrm{Obs}^q$. This cocycle determines a central extension of $\mathcal{G}$ as an L_∞-algebra, and hence there is a map of factorization algebras

$$J^q : \mathbb{U}_\alpha \mathcal{G} \to \mathrm{Obs}^q$$

where $\mathbb{U}_\alpha \mathcal{G}$ is the α-twisted enveloping factorization algebra of $\mathcal{G}$.

Because we use sheaves of L_∞-algebras, our formulation of Noether's theorem incorporates the infinitesimal higher symmetries of Gaiotto *et al.* (2015) as well as ordinary symmetries.

The central extension is called a *'t Hooft anomaly* in physics literature. (The term "anomaly" is a bit overloaded in physics. It means two quite distinct things: a gauge anomaly is an obstruction to quantizing a system, and a 't Hooft anomaly is a central extension of a symmetry algebra. The mathematical terminology, of obstruction and central extension, is typically clearer.)

This formalism, combining BV quantization, factorization algebras, and Noether's theorem, has been deployed in a number of nontrivial applications:

- Grady *et al.* (2017) give a direct connection between Fedosov quantization and the factorization algebra of a 1-dimensional field theory, obtaining a new proof of the algebraic index theorem of Nest and Tsygan (1995a, 1995b) (a cousin of the Atiyah-Singer index theorem). See Gui *et al.* (2021) for further progress.
- Gwilliam *et al.* (2020) constructed rigorously a 2-dimensional QFT depending on a Calabi-Yau manifold X, known as the curved $\beta\gamma$ system, and showed the factorization algebra recovers the chiral differential operators of X, verifying a conjecture of Witten and Nekrasov. Subsequently Gui and Li (n.d.) used this theory to produce a chiral algebraic index theorem.
- Li and Li (2016) constructed the topological B-model and show the factorization algebra recovers the expected topological vertex algebra (see Li (2023)) for a deeper treatment). This provides a direct connection to mirror symmetry, a topic of active interest in mathematics for the last three decades.
- Rabinovich (2019) showed how the local index theorem of Getzler and the axial anomaly are simultaneously captured by quantizing free fermions. Subsequently, he recovered the Quillen determinant line in Rabinovich (2020).
- Costello (n.d.) introduced a 4-dimensional gauge theory whose factorization algebra recovers the Yangian, a quantum group introduced by Drinfeld, and he subsequently gave extensive applications to quantum integrable systems with Witten and Yamazaki in Costello *et al.* (2018), Costello *et al.* (2019) and Costello and Yamazaki (n.d.).
- Costello, Gaiotto, Li, Paquette, and others have explored the AdS/CFT correspondence and its interaction with topological string theory. See Budzik and Gaiotto (n.d.), Costello and Gaiotto (n.d.), Costello and Paquette (2021) and Costello and Li (2020) as starting places.

Much of the work by the community that uses this formalism is focused on holomorphic or topological field theories, primarily because of the researchers' backgrounds, not because the techniques only apply to such theories.

Defects, Boundaries, and Compactifications

Prefactorization algebras apply to many different geometric situations, so that it is easy to work on manifolds with boundaries or corners and to incorporate defects. In this section we will briefly explain how other concepts from field theory are captured in the language of prefactorization algebras.

Basic Constructions

In the one-dimensional case, it is straightforward to include a module for the algebra by including a boundary.

Example 3 *Let A be an associative algebra and V a right A-module. This data determines a prefactorization algebra $\mathcal{F}_{A,V}$ on the half-line $\mathbb{R}_{\geq 0} = [0,\infty)$, as follows. To any open set U in the interior $(0,\infty)$, we set $\mathcal{F}_{A,V}(U) = A^{fact}(U)$, and we use the multiplication map of A to combine elements from disjoint intervals, as in Example ??. To an open set $U = [0,a)$, we set $\mathcal{F}_{A,V}(U) = V$. The module structure determines the structure maps involving open sets that contain the boundary. For instance, given open sets $U = [0,a)$ and $U' = (b,c)$, with $a < b$, and a larger open interval $U'' = [0,d)$, with $c < d$, we have*

$$\mathcal{F}_{A,V}(U) \otimes \mathcal{F}_{A,V}(U') \cong V \otimes A \to V \cong \mathcal{F}_{A,V}(U'').$$

One can similarly work with a closed interval $[0,T]$ and describe incoming and outgoing states. See Chiaffrino et al. (n.d.) for a detailed analysis of this situation using the BV formalism. ⋄

More generally, a QFT with a boundary condition gives rise to a factorization algebra on a manifold with boundary (in the case above, on $[0,\infty)$). A similar construction can be used for a "domain wall" between two systems: for $t < 0$, one has an algebra A_- and for $t > 0$, one has an algebra A_+, and for an open interval containing the origin, one has a bimodule.

Maps between spaces let one produce new factorization algebras.

Definition 4 *Let $p : M \to N$ be a continuous map. Let $\mathcal{F}$ be a factorization algebra on M. Its pushforward $p_*\mathcal{F}$ is the factorization algebra on N defined by*

$$p_*\mathcal{F}(U) = \mathcal{F}(p^{-1}U)$$

on open sets and likewise for structure maps

Example 4 *Consider a prefactorization algebra $\mathcal{F}$ for a theory on a manifold of the form $\mathbb{R} \times N$, where we view $\mathbb{R}$ as a time-like parameter and N as space-like. Let us assume that N is compat. If we push forward along the projection $p : \mathbb{R} \times N \to \mathbb{R}$, we get a prefactorization algebra $p_*\mathcal{F}$ on $\mathbb{R}$ that can be seen as the observables of a quantum mechanical system. In the case of a free scalar field, this push-forward operation gives the familiar Weyl algebra produced by canonical quantization.* ⋄

Thus the pushforward also offers a language for *compactification*.
This natural operation gives a view on the state-operator correspondence and on canonical quantization.

Example 5 *We return to Euclidean field theory on $\mathbb{R}^2$. Consider the map $p : \mathbb{R}^2 \to [0,\infty)$ sending a point (x,y) to its radius $r = \sqrt{x^2 + y^2}$. The interval $[0,a)$ has a disk centered at the origin as its preimage, and any open interval $(a,b) \subset [0,\infty)$ has an annulus as its preimage. The pushforward p_*Obs thus determines a prefactorization algebra whose value on an open interval (a,b) behaves like an algebra of operators (those from radial quantization) and whose value on $[0,a)$ behaves like the state space of that algebra. This state space is, however, the algebra of local operators in the two-dimensional theory. In this sense we have a state-operator correspondence.* ⋄

An Overview of Related Work

In the topological setting Ayala *et al.* (2017) and Morrison and Walker (2012) have given a powerful formalism that characterizes the local algebraic structure of topological defects, generalizing the relationship between E_n algebras and locally constant factorization algebras. They explain, for instance, how to obtain knot invariants by this process. This kind of relationship also offers a way to characterize defects using Koszul duality (see Paquette and Williams (2022)) for a nice discussion).

In Costello (2014) and Costello (n.d.), Costello explained why line defects, in a gauge theory dubbed *4-dimensional Chern-Simons theory*, satisfy the quantum Yang-Baxter equation and how the associated factorization algebra encodes the Yangian, a quantum group introduced by Drinfeld. Subsequently, this framework has been expanded to encompass many further features of quantum integrable systems (Costello *et al.*, n.d., 2018, 2019; Costello and Yagi, 2020; Costello and Yamazaki, n.d.).

In his thesis Rabinovich (n.d.) proved an analog of **Theorem 2** for manifolds with boundary: on a manifold with boundary, if a perturbative theory is topological in a direction normal to the boundary and a local boundary condition is imposed, then the observables form a factorization algebra whose observables in the bulk agree with those of Costello and Gwilliam (2021). This setting encompasses many examples (e.g., the Chern-Simons/WZW correspondence) but it also allows one to implement a standard ansatz for constructing defects (see Contreras *et al.* (2023), following Kapustin (2006) for the physical ansatz). Rabinovich's approach should extend to situation where the theory is scale-equivariant (or conformal) in a direction normal to the boundary.

Noether's Theorem Beyond the Perturbative

In this section we discuss how higher symmetries act on a QFT using p re factorization algebras (c.f. Gaiotto *et al.*, 2015; Sharpe, 2015), inspired by our formulation of Noether currents using factorization algebras. It can be seen as a conjectural formulation of a non-perturbative Noether theorem. The language here is a little more mathematically sophisticated, as it is important to use derived stacks. Our formulation should hold in either Euclidean or Lorentzian signature. Everything we discuss here has been formulated and proved for perturbative theories in Costello and Gwilliam (2021).

From a mathematical perspective, there is an equivalence between an object with a symmetry and a family of objects. For example, let G be a discrete group. To give a G-action on $\mathbb{C}^n$ is then the same as giving a vector bundle $\mathcal{V}$ of rank n on the classifying space BG, together with a framing of the fiber of $\mathcal{V}$ at the base point in BG.

More generally, given some derived stack X with a base point $x \in X$, one can say that a vector space V with *X-symmetry* is a vector bundle $\mathcal{V}$ on X with an isomorphism $\mathcal{V}_x \simeq V$.

Similarly, for a quantum field theory, one should treat parameters in the theory (i.e., coupling constants) and symmetries on the same footing. We took this perspective in Costello and Gwilliam (2017, 2021); a somewhat similar perspective is taken in Córdova *et al.* (2020)

Since a quantum field theory on a manifold M is local on M, a symmetry should be given not just by a fixed space X but by a sheaf of spaces X on M. Here are some examples to bear in mind.

Example 6 *Let $\mathcal{L}$ be a sheaf of (homotopy) Lie algebras on a space M. There is a sheaf of formal derived stacks $B\mathcal{L}(U)$, the classifying space of the formal group associated to $\mathcal{L}(U)$. A theory with $\mathcal{L}$-symmetry is the same as a family of theories over $B\mathcal{L}(U)$. This is the context relevant to our perturbative Noether theorem, described above.*◇

Example 7 *Let $\mathrm{Conn}_G(U)$ be the stack of principal G-bundles on U with a connection, up to gauge equivalence. As U varies this construction forms a sheaf of stacks we call Conn_G. A theory T that can be coupled to a background G-gauge field is the same as a family of theories T^{Conn_G} over the sheaf Conn_G, which restricts to the original theory T at the trivial connection.*◇

Remark 5 *Physicists often conflate theories that have a G-symmetry and theories that live in a family over Conn_G. In general, these are not the same, although in physically relevant examples they tend to be.*◇

We can now formulate a general notion.

Definition 5 *Let X be a sheaf of (possibly infinite-dimensional) derived stacks over a manifold M, with a global section x. Let $\mathcal{T}$ be a quantum field theory on M. Giving X-symmetry to $\mathcal{T}$ means we give a family of theories $\mathcal{T}^X$ over X that restricts to $\mathcal{T}$ at the section x.*

Remark 6 *We will not attempt to be careful here about what infinite dimensional derived stack means, but see Alfonsi and Young (n.d.), Carchedi (n.d.) and Steffens (n.d.).* ◇

In the language of factorization algebras of Costello and Gwilliam (2021), this definition means that the observables (or operators) on an open U will be a kind of vector bundle over $X(U)$, and that the factorization product will be compatible with the sheaf structure on X. Since X is a sheaf of stacks, this notion encompasses both varying the coupling constants as well as compatibility with symmetry.

One can formulate a rather general Noether "theorem" once one has this language. More accurately, it is a proposal for capturing symmetries of the form just described, and it is conjectural until an adequate framework for nonperturbative field theory emerges.

There are a few preliminary steps. We let $X_c(U)$ denote the space of sections of X on U that are equal to our base section x outside a compact subset of U. It is the space of sections of X with compact support.

Note that if U_1, U_2 are disjoint open subsets of V, there is a natural map

$$X_c(U_1) \times X_c(U_2) \to X_c(V) \tag{2}$$

as follows. If s_1, s_2 denote elements of $X_c(U_1)$, $X_c(U_2)$, respectively, then s_i is equal to x outside a compact subset K_i of U_i. Hence s_1 extends to a section of X on the open $U_2 \backslash K_2$ by setting it to be x away from K_1. Similarly, s_2 extends to a section on the open $U_1 \backslash K_1$. These sections agree on their overlap and thus define a section on all of V. In short, we have shown that X_c defines a *prefactorization space*.

Example 8 *Consider the case that the space-time manifold is $\mathbb{R}$. Let G be a discrete group and let X be the sheaf of sections for the trivial bundle $\mathbb{R} \times BG \to \mathbb{R}$. For any open interval I, we find*

$$X_c(I) = G,$$

because compactly supported maps in BG is equivalent to the based loop space $\Omega BG \simeq G$. Alternatively, a compactly supported map into BG determines a principal G-bundle on I that is trivialized outside a compact set, and hence the bundle is determined by its parallel transport from

0 to 1. Furthermore, if I_1, I_2 are disjoint intervals inside I_3, the factorization product

$$X_c(I_1) \times X_c(I_2) \to X_c(I_3)$$

is the group multiplication $G \times G \to G$.

More generally, one can replace the real line with any manifold M, and this example shows how to encode 0-form symmetries. If $N \subset M$ is a compact and connected submanifold of codimension one, then a tubular neighborhood $U \supset N$ is diffeomorphic to $N \times \mathbb{R}$. Thus,

$$\mathrm{Map}_c(U, BG) \cong \mathrm{Map}_c(N \times \mathbb{R}, BG) \cong \mathrm{Map}(N, \mathrm{Map}_c(\mathbb{R}, BG)) \simeq \mathrm{Map}(N, G),$$

and as G is discrete, we have $\mathrm{Map}(N, G) = G$. In other words, we are labeling the submanifold N by group elements.

By replacing BG with $B^{k+1}A$, for A an abelian group, one can similarly encode k-form symmetries on codimension $k+1$-submanifolds.◇

To build a prefactorization algebra, we need to apply some linearization functor that takes spaces to vector spaces. This functor must be covariant. A natural functor to use is *distributions*, the (correct) linear dual to functions; since functions are contravariant, distributions are covariant. Let $\mathrm{Dist}(X_c(U))$ denote the space of distributions on $X_c(U)$, and let Dist_{X_c} denote the prefactorization algebra that assigns $\mathrm{Dist}(X_c(U))$ to each open set U.

Example 9 (Example 8 continued) *For G discrete, we care about BG as a homotopy type, and likewise for $X_c(U)$. The distributions on a homotopy type are encoded by the singular chain complex $C_*(X_c(U))$. (Functions are encoded by the singular cochains.) For the spacetime being $\mathbb{R}$, we find that*

$$\mathrm{Dist}(\mathcal{BG}_c)(I) \cong C_*(G) \simeq \mathbb{C}G,$$

so the we recover the group ring as the linearized symmetries.◇

Roughly speaking, the Noether theorem should say that when a theory has X-symmetry, the prefactorization algebra of observables receives a map from Dist_{X_c}. There is one subtlety, though: we need to take into account 't Hooft anomalies, which in a perturbative treatment lead to a central extension to the algebra.

To understand the 't Hooft anomaly, let us return to case that the space-time manifold is $\mathbb{R}$ and X is the constant sheaf BG, the classifying space of a discrete group. Here, we expect the 't Hooft anomaly to yield a central extension of G, namely a group $\hat{G}$ living in an exact sequence

$$1 \to \mathbb{C}^\times \to \hat{G} \to G \to 1.$$

The central extension $\hat{G}$ is a principal $\mathbb{C}^\times$-bundle over G, i.e., a line bundle $\mathcal{L}$. The fact that $\hat{G}$ is itself a group is reflected in the fact that the line bundle $\mathcal{L}$ is *multiplicative*: under the group multiplication map $m : G \times G \to G$, there is an isomorphism

$$m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}.$$

(This isomorphism must be associative in the natural sense). Since the product on G comes from the prefactorization structure on X_c, we introduce the following definition in general.

Definition 6 *Given the prefactorization space X_c as above, a factorizing line bundle is a line bundle $\mathcal{L}(U)$ on every $X_c(U)$, with coherent isomorphisms*

$$m^*\mathcal{L}(V) \simeq \mathcal{L}(U_1) \boxtimes \mathcal{L}(U_2)$$

associated to the factorization products (2)

Equivalently, we note that sending every U to $B\mathbb{C}^\times$ defines a prefactorization space on M, where the factorization product is defined using the fact that $B\mathbb{C}^\times$ is an abelian group in stacks. A factorizing line bundle is a map of prefactorization spaces from X_c to $B\mathbb{C}^\times$.

Finally, we can formulate the non-perturbative analog of the Noether theorem:

Conjecture 1 *Let $\mathcal{T}$ be a quantum field theory on a manifold M, and let X be a sheaf of derived stacks on M.*

If $\mathcal{T}$ has X-symmetry, then there exists a factorizing line bundle $\mathcal{L}$ on the prefactorization space X_c and a map of prefactorization algebras

$$J : \mathrm{Dist}_{X_c}^{\mathcal{L}} \to \mathrm{Obs}_{\mathcal{T}}.$$

from distributions on X, twisted by $\mathcal{L}$, to $\mathrm{Obs}_{\mathcal{T}}$

This relationship should hold in both Lorentzian and Euclidean signature.

In the case that $\mathcal{T}$ is a perturbative theory and we work in Euclidean signature, and X is a sheaf of formal derived stacks (completed along the section x), then this conjecture is the Noether theorem proved in Costello and Gwilliam (2021).

Remark 7 *In Lorentzian signature, it is important to note that $\mathrm{Dist}_{X_c}^{\mathcal{L}}$ does not normally satisfy the Cauchy constancy axiom of Benini et al. (2020). This axiom fails even in the basic case when $X = \mathrm{Conn}_G$ is the stack of G-connections, roughly because a G-connection does not satisfy any PDE whatsoever and so is not determined by initial data. Thus, this Noether construction cannot be entirely formulated in AQFT in its standard form.*◇

Let us give some examples of these nonperturbative Noether currents.

Example 10 *Consider a quantum mechanical system with G-symmetry, where G is a discrete group. As we have seen, for any interval,* $\text{Conn}_{G,c}(I) \simeq G$. *The non-perturbative Noether theorem tells us that there is central extension of G and a ring homomorphism from the twisted group algebra* $\mathbb{C}_L[G]$ *to the algebra of operators of the system.*

In the case G is a Lie group, a similar statement applies, although the twisted group algebra should be interpreted as the algebra of distributions on G under convolution. Distributions supported at the identity in G give the (twisted) universal enveloping algebra of $\mathfrak{g}$, *recovering the perturbative statement of Noether's theorem.*◇

Example 11 *Let* $T = (\mathbb{C}^{\times})^n$ *be an Abelian complex algebraic group. Suppose that T acts as symmetries on a chiral conformal field theory on a Riemann surface M. For* $U \subset M$, *let* $\text{Bun}_T(U)$ *be the stack of holomorphic T-bundles on U. This has a basepoint given by the trivial bundle.*

Let $\mathcal{F}$ *be the factorization algebra associated to a chiral CFT with T-symmetry. Such a CFT can be coupled to a background holomorphic T-bundle and so has* Bun_T-*symmetry.*

Now observe that for U a very small disk — in fact, a formal disk $\hat{\mathcal{D}}$ *— the space* $\text{Bun}_{T,c}(\hat{\mathcal{D}})$ *is the affine (or loop) Grassmannian* Gr_T. *It has components labeled by* $\pi_1(T) \cong \mathbb{Z}^n$, *a lattice* (Beilinson and Drinfeld, 2004). *described, in their algebraic context, this factorization space, now known as the Beilinson-Drinfeld Grassmannian.*

Noether's theorem then says there will be a map from $\text{Dist}(\text{Bun}_{T,c}, L)$ *to observable of the chiral CFT, for a line bundle on L. There is a factorizing line bundle on* $\text{Bun}_{T,c}$ *associated to a pairing on the lattice* $\pi_1(T)$. *Beilinson and Drinfeld have shown that* $\text{Dist}(\text{Bun}_{G,c}, L)$ *encodes the lattice vertex algebra associated to the lattice* $\pi_1(T)$.

The map $J : \text{Dist}(\text{Bun}_{T,c}, L) \to \mathcal{F}$ *then encodes how this WZW model acts on the chiral CFT. Note that in this example, the perturbative Noether theorem is much weaker: it only yields a map from the Heisenberg subalgebra of the lattice vertex algebra.*◇

Example 12 *An important special case of the previous example is given when* $T = \mathbb{C}^{\times}$, *and* $\mathcal{F}$ *is the factorization algebra associated to a pair of chiral free fermion fields* ψ, ψ' *with Lagrangian* $\int \psi \bar{\partial} \psi'$. *(Physicists would call this system a chiral complex fermion.) In this case, the map*

$$J : \text{Dist}(\text{Bun}_{T,c}, L) \to \mathcal{F}$$

is an isomorphism, and it encodes the boson-fermion correspondence. It is essential that we use the non-perturbative Noether theorem for this statement, as the fermion fields ψ *and* ψ' *come from components of the affine Grassmanian disjoint from the identity component.*◇

Example 13 *Consider any quantum field theory on a manifold M of dimension n that can be coupled to a background* $U(1)$ *gauge field, i.e., has* $\text{Conn}_{U(1)}$-*symmetry. For an open subset* $U \subset M$, *the space* $\text{Conn}_{U(1),c}(U)$ *has components labeled by* $H_c^2(U)$, *which, by Poincar'e duality, is isomorphic to* $H_{n-2}(U)$. *The non-perturbative Noether theorem thus supplements the ordinary local Noether currents by non-local defects living on codimension 2 subspaces. This observation is behind several of the higher-form symmetries discussed in* Gaiotto et al. (2015). ◇

There are further examples of the non-perturbative Noether's theorem associated to homotopy types, as we discussed in Example 8 and which we can quickly generalize now.

Let (X, x) be a pointed topological space. The prefactorization space X_c on a manifold M assigns compactly supported maps into X; the natural linearization is given by singular chains, giving a prefactorization algebra $C_*(X_c)$. This prefactorization algebra Chains$(\mathcal{X}_c) = C_*(\text{Map}_c(U, X))$ with values in chain complexes is the *higher X-symmetry algebra*. (It is already well-understood in topology via *nonabelian Poincaré duality* of Ayala and Francis (2015), Lurie (n.d.), McDuff (1975), Segal (1973) and Salvatore (2001) Symmetries associated to such homotopy types encode generalized global symmetries. For $X = BG$, as we have already discussed, one encodes 0-form symmetry by a Noether map J.

Example 14 *Let G be a compact semisimple Lie group G, and let Z be the center of G. Using* $X = B^2 Z$, *one can encode the center symmetry of a G-gauge theory by such a map. Indeed, in this case, the relevant prefactorization space has the feature that*

$$\pi_0(\text{Map}_c(U, B^2 Z)) = H_c^2(U, Z) = H_{n-2}(U, Z).$$

We thus find symmetries associated to neighborhoods of codimension 2 manifolds.

Any gauge theory with G-symmetry can be coupled to a background field, which is a map to $B^2 Z$. *It is perhaps easiest to see this using the* Čech *picture: given a* Čech *2-cocycle valued in Z, one can modify the notion of principal G-bundle with connection by asking that the failure of the transition functions to satisfy the cocycle condition is given by the chosen Z-valued 2-cocycle. Taking such twisted principal G-bundles with connection as fields modifies the gauge theory.*

The Noether theorem then gives us a homomorphism from the prefactorization algebra Chains$(\text{Maps}_c(B^2 Z))$ *to observables of the theory. These observables are the standard higher symmetries associated to the center of the group, as explained in* Gaiotto et al. (2015) ◇

There is a great deal of further work to be done to flesh out this conjectural Noether theorem. In particular, we have been very vague about what one means by distributions on the infinite-dimensional derived stack that appear in the formulation of the conjecture, and we have explored few of the symmetry prefactorization algebras that appear in the theorem.

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