

The Reciprocal Schur Inequality



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For Ilya Spitkovsky on his 70th birthday

Abstract Schur's inequality states that the sum of three special terms is always nonnegative. This note is a short review of inequalities for the sum of the reciprocals of these terms and of extensions of the latter inequalities to an arbitrary number of terms and thus to higher-order divided differences.

A favorite and central topic of Ilya Spitkovsky's work is several kinds of factorizations of scalar or matrix functions. A prominent representative of these factorizations is the Wiener-Hopf factorization introduced by Norbert Wiener and Eberhard Hopf in their paper [19] of 1931. Since then the name of Eberhard Hopf is well known in the communities of integral equations, mathematical physics, and operator theory; see, e.g., [7, 13]. The intention of this note is to highlight the result of Eberhard Hopf's dissertation [5], defended in 1926, in which he ultimately solved another important problem and which seems to be less known in our present days. This result concerns the positivity of higher-order divided differences and leads us back to an inequality that is commonly attributed to Issai Schur.

Schur's inequality says that

$$x^s(x-y)(x-z) + y^s(y-z)(y-x) + z^s(z-x)(z-y) \geq 0 \quad (1)$$

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for $x, y, z > 0$ and arbitrary $s \in \mathbb{R}$, with equality if and only if $x = y = z$. The reciprocal version of this inequality states that, for distinct $x, y, z > 0$,

$$\frac{1}{x^s(x-y)(x-z)} + \frac{1}{y^s(y-z)(y-x)} + \frac{1}{z^s(z-x)(z-y)} \quad (2)$$

is positive if $s > 0$ or $s < -1$, negative if $-1 < s < 0$, and zero if $s = 0$ or $s = -1$. It is well known that extension of Schur's inequality to arbitrarily many variables $x_1, \dots, x_n$ is impossible without imposing additional constraints on the variables; see, e.g., [8, 20]. However, the reciprocal Schur inequality may be extended to an arbitrary number of variables without any additional requirements. Here it is.

Theorem 1 *Let $x_1, \dots, x_n$ ($n \geq 2$) be distinct positive real numbers and let $s \in \mathbb{R}$. Then the sum*

$$\sum_{j=1}^n \frac{1}{x_j^s \prod_{k \neq j} (x_j - x_k)} \quad (3)$$

is zero if and only if $s \in \{0, -1, -2, \dots, -(n-2)\}$ and otherwise the sign of this sum equals the sign of $(-1)^{n+1}s(s+1) \cdots (s+n-2)$.

Thus, if, for example, $n = 4$, then the sum (3) is zero for $s \in \{0, -1, -2\}$, it is positive if $s \in (-\infty, -2) \cup (-1, 0)$, and it is negative if $s \in (-2, -1) \cup (0, \infty)$.

Theorem 1 is actually a special case of the following theorem.

Theorem 2 *Let $-\infty \leq \alpha < \beta \leq \infty$, let $n \geq 2$, and let $f : (\alpha, \beta) \rightarrow \mathbb{R}$ be $n-1$ times continuously differentiable in (α, β) . Then*

$$\sum_{j=1}^n \frac{f(x_j)}{\prod_{k \neq j} (x_j - x_k)} \quad (4)$$

is nonnegative (resp. nonpositive) for arbitrary distinct $x_1, \dots, x_n$ in (α, β) if and only if $f^{(n-1)}$ is nonnegative (resp. nonpositive) on (α, β) .

Clearly, Theorem 1 follows from taking $(\alpha, \beta) = (0, \infty)$ and $f(t) = 1/t^s$, in which case

$$f^{(n-1)}(t) = (-1)^{n+1}s(s+1) \cdots (s+n-2) \frac{1}{t^{s+n-1}}.$$

The sum (4) is known as the n th divided difference of the function f and is usually denoted by $f[x_1, \dots, x_n]$. In [1] and [3], Theorem 2 is proved as follows. The Curry–Schoenberg B-spline associated with $x_1 < \dots < x_n$ is the function

$$F(t; x_1, \dots, x_n) = \frac{n-1}{2} \sum_{j=1}^n \frac{|x_j - t|(x_j - t)^{n-3}}{\prod_{k \neq j} (x_j - x_k)}.$$

This is actually a probability density supported on $[x_1, x_n]$. We have in particular, $F(t; x_1, \dots, x_n) \geq 0$ for $t \in [x_1, x_n]$. An identity known as Peano's formula says that if $f : [x_1, x_n] \rightarrow \mathbb{R}$ is in C^{n-1} , then

$$f[x_1, \dots, x_n] = \frac{1}{(n-1)!} \int_{x_1}^{x_n} f^{(n-1)}(t) F(t; x_1, \dots, x_n) dt. \quad (5)$$

See, for instance, [2, Theorem 3.7.1]. This formula implies the “if” portion of the theorem. As for the “only if” part, notice that if $f^{(n-1)}(t_0) < 0$ for some t_0 , then, by the already proved “if” part, $f[x_1, \dots, x_n] < 0$ for all $x_1, \dots, x_n$ sufficiently close to the point t_0 .

Requiring that f be $n-1$ times continuously differentiable is actually too much. The ultimate answer to the question about minimal conditions needed for a result like Theorem 2 was given by Eberhard Hopf in his dissertation [5], defended in 1926. Incidentally, the referees of the dissertation were Erhard Schmidt and Issai Schur.

Hopf first notes that Theorem 2 is true if only the existence of $f^{(n-1)}$ on (α, β) is required. He refers to a theorem by H. A. Schwarz [15], according to which for an $n-1$ times differentiable function f there is a $t \in (x_1, x_n)$ such that

$$f[x_1, \dots, x_n] = \frac{f^{(n-1)}(t)}{(n-1)!}. \quad (6)$$

See also equality (1.33) in [11]. Obviously, this formula shows that $f[x_1, \dots, x_n] \geq 0$ if $f^{(n-1)}(t) \geq 0$ on (α, β) . To tackle the case where $f^{(n-1)}(t_0) < 0$ for some t_0 and continuity of $f^{(n-1)}$ at t_0 is not guaranteed, Hopf has recourse to a result by T. J. Stieltjes [16], which says that $f[x_1, \dots, x_n]$ converges to $f^{(n-1)}(t_0)/n!$ whenever $x_1, \dots, x_n$ converge to t_0 with the additional condition that t_0 stays between the minimum and the maximum of $x_1, \dots, x_n$.

Here is Hopf's final theorem on the positivity of divided differences.

Theorem 3 *Let $-\infty \leq \alpha < \beta \leq \infty$, let $n \geq 3$, and let $f : (\alpha, \beta) \rightarrow \mathbb{R}$ be a function. Then*

$$\sum_{j=1}^n \frac{f(x_j)}{\prod_{k \neq j} (x_j - x_k)}$$

is nonnegative (resp. nonpositive) for arbitrary distinct $x_1, \dots, x_n$ in (α, β) if and only if f is $n-3$ times differentiable and $f^{(n-3)}$ is convex (resp. concave) on (α, β) .

This is Satz 1 on page 24 of [5]. Repeating Hopf's full proof here is beyond the scope of this note. We therefore confine ourselves to its basic steps.

For $n = 3$ and $x_1 < x_2 < x_3$, the inequality $f[x_1, x_2, x_3] \geq 0$ reads

$$f(x_2) \leq f(x_1) \frac{x_3 - x_2}{x_3 - x_1} + f(x_3) \frac{x_2 - x_1}{x_3 - x_1},$$

and this holds for all $x_1 < x_2 < x_3$ if and only if f is convex. So let $n \geq 4$ and suppose the theorem is true for $n - 1$. Without loss of generality assume that α and β are finite and that $x_1 < \dots < x_n$. Let $f[x_1, \dots, x_n] \geq 0$ for all $x_1, \dots, x_n$. The first goal is to prove that then f is differentiable. Take $0 < \delta < (\beta - \alpha)/2$ and fix $n - 1$ points $a_1, \dots, a_{n-1}$ in $(\alpha, \alpha + \delta)$ as well as $n - 1$ points $b_1, \dots, b_{n-1}$ in $(\beta - \delta, \beta)$. A simple identity for divided differences gives

$$\begin{aligned} & \frac{n-1}{n} \left\{ f[x_1, \dots, x_{n-1}] - f[a_1, \dots, a_{n-1}] \right\} \\ &= (x_1 - a_1) f[a_1, x_1, \dots, x_{n-1}] + \dots + (x_{n-1} - a_{n-1}) f[a_1, \dots, a_{n-1}, x_{n-1}] \end{aligned}$$

for every choice of $x_1, \dots, x_{n-1}$ in $(\alpha + \delta, \beta - \delta)$. As the right-hand side is nonnegative, it follows that $f[x_1, \dots, x_{n-1}] \geq f[a_1, \dots, a_{n-1}]$. It can be shown in an analogous fashion that $f[x_1, \dots, x_{n-1}] \leq f[b_1, \dots, b_{n-1}]$. Consequently, there is a constant M such that $|f[x_1, \dots, x_{n-1}]| \leq M$ for all $x_1, \dots, x_n$. A Hilfssatz proved on page 12 says that this implies that f is differentiable on $(\alpha + \delta, \beta - \delta)$, and as $\delta > 0$ can be made arbitrarily small, one gets differentiability on all of (α, β) .

Now the induction step. It is based on two remarkable theorems. The first of them is a generalization of formula (6). This generalization, Satz 1 on page 9, states that if f is differentiable, then there are $t_1, \dots, t_{n-1}$ with $t_j \in (x_j, x_{j+1})$ for all j such that

$$f[x_1, \dots, x_n] = \frac{1}{n} f'[t_1, \dots, t_{n-1}]. \quad (7)$$

The second of the two theorems, Satz 2 on page 11, says that if f is differentiable, then there is a $t \in (x_1, x_n)$ such that

$$\frac{1}{n} f'[x_1, \dots, x_{n-1}] = f[t, x_1, \dots, x_{n-1}]. \quad (8)$$

Using equalities (7) and (8) and the differentiability of f proved in the previous paragraph, one gets that $f[x_1, \dots, x_n] \geq 0$ for all $x_1, \dots, x_n$ if and only if f is differentiable and $f'[x_1, \dots, x_{n-1}] \geq 0$ for all $x_1, \dots, x_{n-1}$. By the induction hypothesis, the latter is equivalent to the requirement that the $(n - 4)$ th derivative of f' exists and is convex, which is the same as requiring that $f^{(n-3)}$ exists and is convex. This completes Hopf's proof.

One of the marvels in divided differences is that $f[x_1, \dots, x_n]$ is a complete homogeneous symmetric (CHS) polynomial if f is a monomial of sufficiently large

degree; see, e.g., [9, Lemma 4] or [11, Theorem 1.2.1]. To be precise, if q is a nonnegative integer and $f(t) = t^{q+n-1}$, then

$$f[x_1, \dots, x_n] = h_q(x_1, \dots, x_n) := \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_q \leq n} x_{j_1} x_{j_2} \cdots x_{j_q},$$

with the convention that $h_0(x_1, \dots, x_n) := 1$. This is in fact Jacobi's bialternant formula, which says that, for every nonnegative integer q ,

$$h_q(x_1, x_2, \dots, x_n) V(x_1, x_2, \dots, x_n) = \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{q+n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{q+n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{q+n-1} \end{pmatrix},$$

where $V(x_1, x_2, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$ is the Vandermonde determinant.

In the case of even $q = 2p$, we have

$$(t^{2p+n-1})^{(n-1)} = (2p+n-1)(2p+n-2) \cdots (2p+1)t^{2p} \geq 0$$

on $(\alpha, \beta) = (-\infty, \infty)$, and hence Theorem 2 implies that

$$\sum_{j=1}^n \frac{x_j^{2p+n-1}}{\prod_{k \neq j} (x_j - x_k)} = h_{2p}(x_1, \dots, x_n) \quad (9)$$

is nonnegative for arbitrary distinct real numbers $x_1, \dots, x_n$. In the discussion of [17] it is shown that

$$h_q(x_1, x_2, \dots, x_n) = \frac{1}{q!} \mathbb{E}((x_1 Z_1 + x_2 Z_2 + \cdots + x_n Z_n)^q) \quad (10)$$

for arbitrary real numbers $x_1, x_2, \dots, x_n$, where $Z_1, Z_2, \dots, Z_n$ are independent and identically exponentially distributed random variables with parameter 1. Clearly, this is another way to see that $h_q(x_1, x_2, \dots, x_n) \geq 0$ if $q = 2p$ is even. We want to note that (10) is the case $g(t) = t^q/q!$ of the more general identity

$$\mathbb{E}(g(x_1 Z_1 + x_2 Z_2 + \cdots + x_n Z_n)) = H[x_1, x_2, \dots, x_n]$$

where $H(s) = s^{n-2}G(1/s)$ and $G(s)$ is the Laplace transform of $g(t)$. More about the latter identity will be said elsewhere.

Actually a sharp lower bound for CHS polynomials of even degree is known. Namely, a famous theorem by Hunter [6] says that

$$h_{2p}(x_1, \dots, x_n) \geq \frac{1}{2^p p!} (x_1^2 + \dots + x_n^2)^p \quad (11)$$

for arbitrary $(x_1, \dots, x_n) \in \mathbb{R}^n$. Combining (9) and (11) we see that Hunter's inequality is actually the following reciprocal Schur inequality.

Theorem 4 *Let $x_1, \dots, x_n$ ($n \geq 2$) be distinct real numbers and let p be a nonnegative integer. Then*

$$\sum_{j=1}^n \frac{x_j^{2p+n-1}}{\prod_{k \neq j} (x_j - x_k)} \geq \frac{1}{2^p p!} (x_1^2 + \dots + x_n^2)^p.$$

The following theorem of Farwig and Zwick [3] provides us with another lower bound for $f[x_1, \dots, x_n]$. It is applicable if $f^{(n-1)}$ exists and is convex (and thus, in particular, continuous).

Theorem 5 *Let $x_1, \dots, x_n$ ($n \geq 2$) be distinct real numbers and let $f : (\alpha, \beta) \rightarrow \mathbb{R}$ be a function whose $(n-1)$ th derivative exists and is convex. Then, for distinct $x_1, \dots, x_n$ in (α, β) ,*

$$\sum_{j=1}^n \frac{f(x_j)}{\prod_{k \neq j} (x_j - x_k)} \geq \frac{1}{(n-1)!} f^{(n-1)}\left(\frac{x_1 + \dots + x_n}{n}\right).$$

For example, if $f(t) = t^6$, then Theorem 4 (with $p = 2$ and $n = 3$) and Theorem 5 deliver the lower bounds

$$\frac{(x_1^2 + x_2^2 + x_3^2)^2}{8} \quad \text{and} \quad \frac{5}{27} (x_1 + x_2 + x_3)^4,$$

respectively. If $x_1 + x_2 + x_3 = 0$, then the first of the bounds is better than the second, but if x_1, x_2, x_3 are very close to one another, then the second of the two bounds is larger than the first.

The proof of Theorem 5 given in [3] starts with the fact that

$$\int_{x_1}^{x_n} t F(t; x_1, \dots, x_n) dt = \frac{x_1 + \dots + x_n}{n} =: c$$

for $x_1 < \dots < x_n$. Since $f^{(n-1)}$ is convex, we may invoke Jensen's inequality to obtain

$$\begin{aligned} f^{(n-1)}(c) &= f^{(n-1)}\left(\int_{x_1}^{x_n} t F(t; x_1, \dots, x_n) dt\right) \\ &\leq \int_{x_1}^{x_n} f^{(n-1)}(t) F(t; x_1, \dots, x_n) dt, \end{aligned}$$

and Peano's formula tells us that the right-hand side of this inequality is nothing but $(n-1)! f[x_1, \dots, x_n]$. This completes the proof.

Interestingly, Pečarić and Zwick [10] proved that existence and convexity of $f^{(n-1)}$ implies that $f[x_1, \dots, x_n]$ is Schur convex. We first want to notice that if $f^{(n-1)}$ is continuous on (α, β) , then $f[x_1, \dots, x_n]$ may be continuously extended to a symmetric function on all of $(\alpha, \beta)^n$. Schur convexity means that $f[x_1, \dots, x_n] \leq f[y_1, \dots, y_n]$ whenever $x \prec y$, where $x \prec y$ in turn means that $x_1 \geq \dots \geq x_n$, $y_1 \geq \dots \geq y_n$, $\sum_{j=1}^k x_j \leq \sum_{j=1}^k y_j$ for $1 \leq k \leq n-1$, and $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$. For example, if $x_1 \geq \dots \geq x_n$, then

$$\left(\frac{x_1 + \dots + x_n}{n}, \dots, \frac{x_1 + \dots + x_n}{n}\right) \prec (x_1, \dots, x_n). \quad (12)$$

Taking into account that $f[c, \dots, c] = f^{(n-1)}(c)/(n-1)!$, we conclude from (12) and the Schur convexity of f that $f^{(n-1)}(c)/(n-1)! \leq f[x_1, \dots, x_n]$ with c equal to $(x_1 + \dots + x_n)/n$ for arbitrary points $x_1, \dots, x_n$. This is the proof of Theorem 5 presented in [10].

Our interest in the topic of this note was aroused by the question on whether the positivity of one of the exotic expressions in our recent paper [1] is related to Schur's inequality (1) of math contest fame. We could not prove extensions or generalizations of Schur's original inequality and rather became aware that we entered the business of inequalities for divided differences. The interpretation of these inequalities as reciprocal Schur inequalities seemed us worth exposing them in a note.

We remark that Schur's original inequality (1) is known to hold with x^s, y^s, z^s replaced by $f(x), f(y), f(z)$ with a function $f : (\alpha, \beta) \rightarrow \mathbb{R}$ and for $\alpha < x \leq y \leq z < \beta$ if and only if f belongs to the so-called class Q on (α, β) . The class Q was introduced in [4] and is defined as the set of all functions $f : (\alpha, \beta) \rightarrow \mathbb{R}$ satisfying

$$f((1-\tau)x + \tau z) \leq \frac{1}{1-\tau} f(x) + \frac{1}{\tau} f(z)$$

whenever $\alpha < x \leq y \leq z < \beta$ and $0 < \tau < 1$. Obviously, for nonnegative functions f , this condition is weaker than the convexity condition

$$f((1-\tau)x + \tau z) \leq (1-\tau)f(x) + \tau f(z).$$

It is readily seen that, besides the nonnegative convex functions, the class Q also contains all nonnegative monotone functions and all nonnegative functions f for which $\sup_t f(t) < 4 \inf_t f(t)$.

Two more occurrences of reciprocal Schur inequalities we want to mention are [14, 18]. Roventă and Temereancă [14] proved that $f[x_1, \dots, x_n] \geq 0$ if f is a polynomial such that $f^{(n-1)} > 0$, referring to [12] in this connection. Paper [18] by G. N. Watson contains a reciprocal Schur inequality in exactly our understanding. There it is proved that if $a > 0$ and $x_1, \dots, x_n$ are distinct real numbers, then $\sum_{j=1}^n \frac{a^{x_j}}{\prod_{k \neq j} (x_j - x_k)}$ is positive for $a > 1$, of the same sign as $(-1)^{n+1}$ if $0 < a < 1$, and zero for $a = 1$. Watson's proof is via representing the sum as determinant and then proceeding by induction on n . Obviously, the result follows from Theorem 2 with $f(t) = a^t$, in which case $f^{(n-1)}(t) = a^t (\log a)^{n-1}$.

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