The Matrix Equation $X^3 + Y^3 = 2Z^3$

The diophantine equation

$$x^3 + y^3 = 2z^3 \tag{1}$$

arises when one considers cubes in arithmetic progression: if $x^3 < z^3 < y^3$ are in arithmetic progression, then $y^3 - z^3 = z^3 - x^3$, which yields (1). Euler [**1**, Art. 247], Mordell [**2**, Ch. 15, Thm. 3], and Sierpiński [**3**, Ch. 2, p.79] showed that (1) has no solutions in integers except for the *trivial* solutions $x = -y$ and $z = 0$; or $x = y = z$. In particular, one cannot find three positive cubes in arithmetic progression.

We consider the matrix analogue $X^3 + Y^3 = 2Z^3$ of (1), where X, Y, Z are $n \times n$ matrices with integer entries. Whereas the corresponding scalar diophantine equation has no nontrivial solutions, we construct two families of nontrivial solutions to the matrix analogue.

We look for solutions of the form $X = P + Q$, $Y = P - Q$, and $Z = P$. This yields the matrix equation $\mathcal{T}(P) = Q^2P + QPQ + PQ^2 = 0$.

We first take $Q = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, the companion matrix of the polynomial x^2, so that $Q^2 = 0$. A calculation shows that $\mathcal{T}(P) = 0$ if and only if P is upper triangular.

We next take $Q = \left[\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right]$, the companion matrix of $x^2 + x + 1$, which has eigenvalues $\lambda = e^{2\pi i/3}$ and $\mu = e^{4\pi i/3}$. If $Q\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y}^*Q = \mu\mathbf{y}^*$ with $\mathbf{y} \neq \mathbf{0}$, then $\mathbf{x}\mathbf{y}^* \neq 0$ and $\mathcal{T}(\mathbf{x}\mathbf{y}^*) = (\lambda^2 + \lambda\mu + \mu^2)\mathbf{x}\mathbf{y}^* = 0$. Then, since Q has integer entries, there must be a nonzero matrix P with integer entries such that $\mathcal{T}(P) = 0$. A direct calculation shows that $\mathcal{T}(P) = 0$ if and only if P is of the form $\left[\begin{smallmatrix} a & b \\ a+b & -a \end{smallmatrix}\right]$.

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REFERENCES

[1] Euler L. Vollständige Anleitung zur niedern und höhern Algebra. Vol. 2. Cambridge: Cambridge University Press; 2009. Reprint of the 1797 original.
[2] Mordell LJ. Diophantine equations. London-New York: Academic Press; 1969. (Pure and Applied Mathematics; 30).
[3] Sierpiński W. Elementary theory of numbers. 2nd ed. Amsterdam: North-Holland Publishing Co.; 1988. (North-Holland Mathematical Library; 31).

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