

Lieb-Schultz-Mattis theorems for symmetry-protected topological phases

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ARTICLE INFO

Keywords:

Symmetry protected topological Phase
Lieb-Schultz-Mattis theorem
Magnetic Translation Symmetry
Topological insulator
Topological superconductor
Short Range Entangled Phase
Invertible Phase

ABSTRACT

The Lieb-Schultz-Mattis (LSM) theorem and its generalizations are a class of powerful no-go theorems that rule out any short-range-entangled (SRE) symmetric ground state irrespective of the specific Hamiltonian, based only on certain microscopic inputs, such as symmetries and particle filling numbers. In this work, we introduce and provide physical arguments for a new class of LSM-type theorems, where any symmetry-allowed SRE ground state must be a symmetry-protected topological (SPT) phase with robust gapless edge states, such as topological insulators and superconductors. The key ingredient is to replace the lattice translation symmetry in usual LSM theorems by the magnetic translation symmetry. These theorems provide new insights into realistic models and experimental realizations of SPT phases in interacting bosons and fermions.

1. Introduction

The Lieb-Schultz-Mattis (LSM) theorem [1] and its descendants [2–13] are powerful theorems that dictate long-distance low-energy (infrared) properties of a lattice-translation-invariant system from its microscopic (ultraviolet) input, such as a global $U(1)$ charge/spin conservation symmetry and the filling number per unit cell (u.c.). Irrespective of the microscopic Hamiltonian, remarkably, these generic ultraviolet inputs imply that a lattice-translation-invariant ground state at a non-integer filling is either gapless (e.g. in metals), or spontaneously breaks the $U(1)$ symmetry (e.g. in superconductors), or develops intrinsic topological orders [14] (e.g. in fractional quantum Hall states). In all cases, the system forbids a short-range-entangled (SRE) ground state [15] that preserves both global and lattice translation symmetries (e.g. a featureless Mott insulator without fractionalization). Since LSM theorems apply to a generic interacting system, they provide great insights in the study of quantum many-body systems beyond one spatial dimension (1d), which remains intractable in most analytic or numeric efforts.

While usual LSM theorems forbid a SRE symmetric ground state at fractional fillings, the interplay of symmetry and topology gives rise to a rich structure of SRE symmetric states, coined symmetry protected topological (SPT) phases [16,17]. Characterized by protected edge/surface states, topological insulators and superconductors [18,19] are the examples of SPT phases in non-interacting fermions. In spite of extensive theoretical studies, so far strongly-interacting SPT phases are still in lack of realizations beyond 1d spin chains [20,21].

In this work we intend to fill this gap by introducing and proving a new class of LSM theorems, summarized in Tables 1–2, whose SRE symmetric ground state must be a SPT phase. Focusing on two spatial dimensions (2d), we show that the key step is to replace the pure lattice translations in usual LSM theorem by magnetic translations [22], with a rational $\phi = 2\pi \frac{p}{q}$ flux piercing through each plaquette. We will always refer a unit cell (u.c.) as the original cell generated by Bravais lattice translations, in contrast to enlarged magnetic unit cell containing one flux quantum (or 2π flux). We show that a symmetric SRE ground state with proper degrees of

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<https://doi.org/10.1016/j.aop.2024.169806>

Received 18 May 2024; Accepted 13 September 2024

Available online 23 September 2024

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freedom (d.o.f.) per u.c. (typically a fractional filling $\bar{\rho} \notin \mathbb{Z}$) must be a SPT phase with protected edge modes. As will become clear later, a key idea is the charge-flux binding in SPT phases, which is closely related to the decorated-domain-wall picture of SPT phases [23].

2. LSM theorems for fermion SPT phases

Topological insulators (TIs) and topological superconductors (TSCs) are well-known examples of fermion SPT phases. In the Altland-Zirnbauer (AZ) 10-fold way of symmetry classes, nontrivial fermion SPT phases exist in 5 symmetry classes in each spatial dimension [24,25]. In two spatial dimensions (2d), below we demonstrate that a LSM-type theorem favoring a SPT ground state exists for 4 AZ symmetry classes: i.e. class D, DIII, A and AII as summarized in Table 1.

Symmetry class D describes superconductors with no symmetry, with a \mathbb{Z} classification in 2d. Characterized by an integer-valued topological index $\nu \in \mathbb{Z}$, they host chiral Majorana edge modes with a chiral central charge $c_- = \nu/2$ [26]. The simplest $\nu = 1$ SPT phase is the chiral $p_x + ip_y$ superconductor of spinless (or spin-polarized) electrons in 2d. One significant property of a $\nu = \text{odd}$ topological superconductor is an odd number of Majorana zero modes (MZMs) localized at each superconducting vortex core [27], robust against any perturbations. One MZM can be viewed as “half” of a fermion and it has been proven that a unique symmetric SRE ground state is not allowed in a translational invariant system with odd Majoranas per unit cell (u.c.) [28]. On the other hand, magnetic translations allow a SRE TSC ground state, as our theorem states:

Theorem 2.1. *For a generic interacting fermion system with an odd number of Majoranas per u.c., in the presence of magnetic translation symmetry [22]*

$$\tilde{T}_1 \tilde{T}_2 \tilde{T}_1^{-1} \tilde{T}_2^{-1} = e^{i\phi \hat{F}}, \quad \hat{F} = \text{total fermion number.} \quad (1)$$

with $\phi = \pi$ flux per u.c., if there is a unique symmetric and gapped ground state on torus, it must be a $\nu = \text{odd}$ TSC in class D with chiral Majorana edge states.

Now that all $\nu = \text{odd}$ chiral TSC necessarily breaks time reversal symmetry, there is a no-go theorem as a straightforward corollary of Theorem 2.1:

For a generic interacting fermion system with an odd number of Majoranas per u.c., in the presence of time reversal and magnetic translation symmetry (1) with $\phi = \pi$, it is impossible to have a symmetry-preserving unique gapped ground state on torus.

We demonstrate Theorem 2.1 by a square lattice π -flux model with 1 Majorana $\gamma_{\mathbf{r}}$ at each site $\mathbf{r} = (x, y)$, as illustrated in Fig. 1. Choosing a Landau gauge for magnetic translation algebra (1)

$$\tilde{T}_y = T_y, \quad \tilde{T}_x = T_x e^{i\phi \sum_{\mathbf{r}} y \hat{F}_{\mathbf{r}}} \quad (2)$$

where $\hat{F}_{\mathbf{r}}$ is the fermion number on site \mathbf{r} , the nearest neighbor (NN) $\phi = \pi$ -flux Hamiltonian writes

$$\hat{H}_0^D = \sum_{\mathbf{r}} i[t_x \gamma_{\mathbf{r}} \gamma_{\mathbf{r}+\hat{x}} + t_y (-1)^x \gamma_{\mathbf{r}} \gamma_{\mathbf{r}+\hat{y}}] + h.c. \quad (3)$$

It is straightforward to identify two zero-energy Majorana cones at $(k_x, k_y) = (0, 0)$ and $(0, \pi)$ related by magnetic translation \tilde{T}_x in (2), and there is only one mass term $m\hat{F}_0$ allowed by $\tilde{T}_{x,y}$ symmetries (details in Appendix A.1), realized by e.g. next nearest-neighbor (NNN) hoppings between Majoranas (see Fig. 1). In the usual band inversion story of TIs, opposite signs of mass m lead to a trivial insulator and a TI respectively. However for the Majorana cones in model (3), both signs of the mass term lead to a TSC in class D, whose topological index $\nu = \text{Sgn}(m)$ and chirality of edge modes depends on the mass sign. In Fig. 1 we show the chiral edge modes of such a symmetry-enforced TSC on square lattice.

Symmetry class DIII describes time-reversal-invariant (TRI) superconductors, with a \mathbb{Z}_2 classification in 2d. The TSC in class DIII is a triplet TRI p -wave superconductor [29], a 2d analog of $^3\text{He B}$ phase. One of its defining character is one Kramers pair of MZMs $\{\gamma_{\uparrow}, \gamma_{\downarrow}\}$ at each vortex core, stable against any time-reversal-invariant perturbations. One can also argue for a no-go theorem (see Appendix B.2) that rules out any symmetric SRE ground state in a translational invariant system with an odd number of Majorana Kramers pairs $\{\gamma_{a,\uparrow}, \gamma_{a,\downarrow} | 1 \leq a < 2N\}$ per u.c.. On the other hand, similar to class D, magnetic translation symmetry however allows a SRE TSC ground state:

Theorem 2.2. *For a generic interacting fermion system with an odd number of Majorana Kramers pairs per u.c., in the presence of time reversal symmetry $\hat{T}^2 = (-1)^{\hat{F}}$ and magnetic translation symmetry (1) with $\phi = \pi$ flux per u.c., any unique symmetric and gapped ground state on torus must be a TSC in class DIII with helical Majorana edge states.*

To demonstrate Theorem 2.2, we again consider a square lattice NN π -flux model with one Kramers pair $\{\gamma_{\mathbf{r},\uparrow}, \gamma_{\mathbf{r},\downarrow}\}$ of Majoranas per site \mathbf{r} :

$$\hat{H}_0^{\text{DIII}} = \sum_{\mathbf{r},\sigma} i\sigma[t_x \gamma_{\mathbf{r},\sigma} \gamma_{\mathbf{r}+\hat{x},\sigma} + t_y (-1)^x \gamma_{\mathbf{r},\sigma} \gamma_{\mathbf{r}+\hat{y},\sigma}] + h.c. \quad (4)$$

Similar to model (3) in class D, in the basis of $\phi_{\mathbf{k}} = \frac{1}{\sqrt{L_x L_y/2}} \sum_{(x,y)} e^{-i(k_x x + k_y y)} (\gamma_{(2x,y),\sigma}, \gamma_{(2x+1,y),\sigma})^T$, it is straightforward to show that NN model (4) leads to two Dirac points at $\mathbf{k} = (0, 0)$ and $(0, \pi)$, described by low-energy Dirac Hamiltonian

$$\hat{H}_0^{\text{DIII}} \rightarrow - \sum_{|\mathbf{q}| \ll 1} \Phi_{\mathbf{q}}^\dagger \left(\frac{t_x}{2} q_x \tau_x + t_y q_y \tau_z \mu_z \right) \sigma_z \Phi_{\mathbf{q}} + O(|\mathbf{q}|^2), \quad (5)$$

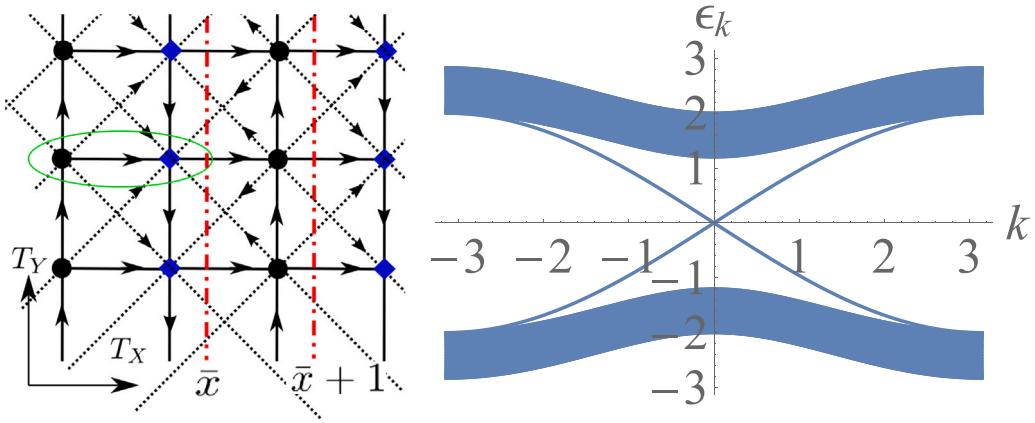


Fig. 1. Majorana hopping model (3) with $\phi = \pi$ flux per u.c. on square lattice (left) and its edge spectrum (right). Arrows represent the signs of Majorana hoppings, while the green oval stands for the doubled magnetic u.c.. The edge spectrum is obtained on a $L_y = 50$ open cylinder (periodic along \hat{x} direction), where NNN coupling is chosen as $t_2/t_1 = 0.3$ with NN couplings $t_x = t_y = t_1$. The two edge modes with opposite chirality are located on two open edges separately.

Table 1

Summary of LSM theorems for SPT phases (i.e. topological insulators/superconductors) of a generic interacting fermion system in two dimensions. The trivial insulators/superconductors have topological index $\nu, \sigma_{xy} = 0$.

Physical systems		Microscopic input			Output of LSM theorem		
AZ [30] class	Applications	Symmetry group	d.o.f. per unit cell	Flux per unit cell	Topological invariant	Edge states	Chiral central charge
D	Majorana vortex lattice [31]	$(-1)^{\hat{F}}$	Odd Majoranas $\{\gamma_1, \dots, \gamma_{2N-1}\}$	$\phi = \pi$	$\nu = \text{odd}$ $\in \mathbb{Z}$	Chiral Majorana	$c_- = \nu/2$
DIII	Kitaev-type spin liquid [32]	$\hat{\mathcal{T}}^2 = (-1)^{\hat{F}}$	Odd Majorana Kramers pairs	$\phi = \pi$	$\nu = 1$ $\in \mathbb{Z}_2 = \{0, 1\}$	Helical Majorana	$c_- = 0$
A	Integer QHE in Hofstadter models	$U(1)_{\text{charge}}$	Charge $e \cdot \bar{\rho}_f$	$\phi = 2\pi \frac{e}{q}$	$p\sigma_{xy} = q\bar{\rho}_f \pmod{q}$ $\sigma_{xy} \in \mathbb{Z}$	Chiral fermion	$c_- = \sigma_{xy} \pmod{8}$ [26]
AII	QSHE in π -flux model [33]	$U(1)_{\text{charge}}$ $\hat{\mathcal{T}}^2 = (-1)^{\hat{F}}$	Charge $e \cdot \bar{\rho}_f$ ($\bar{\rho}_f = \text{odd}$)	$\phi = \pi$	$\nu = 1$ $\in \mathbb{Z}_2 = \{0, 1\}$	Helical fermion	$c_- = 0$

where $\vec{\tau}$, $\vec{\mu}$ and $\vec{\sigma}$ are Pauli matrices for sublattice (in a doubled magnetic cell), valley and spin indices. With the following symmetry operations on Dirac spinor Φ_q :

$$\Phi_q \xrightarrow{T_y} \mu_z \Phi_q, \quad \Phi_q \xrightarrow{\hat{T}_x} \tau_x \mu_x \Phi_q, \quad \Phi_q \xrightarrow{\mathcal{T}} i \sigma_y \Phi_{-q}.$$

There is only one symmetric mass term for Dirac Hamiltonian (5) $\hat{H}_0 = \tau_y \mu_z \sigma_z$, realized by e.g. TRI NNN couplings shown in Fig. 1. Irrespective of the mass sign, the gapped Dirac Hamiltonian always leads to a TRI TSC with helical Majorana edge modes.

Symmetry class A describes fermion insulators (or metals) with $U(1)$ charge conservation symmetry, where lattice translation symmetry allows a well-defined filling number $\bar{\rho}_f$ per u.c.. The SPT phases in class A corresponds to Chern insulators characterized by integer-valued Hall conductance $\sigma_{xy} \in \mathbb{Z}$ (in unit of e^2/h) [34]. In the usual LSM theorem [1–3] with lattice translations, an insulating ground state without fractionalization is impossible at non-integer fillings $\bar{\rho}_f \notin \mathbb{Z}$. However with ϕ flux per u.c. and associated magnetic translation symmetry (1), an unfractonized SPT ground state becomes a possibility even at a fractional filling [35]. Specifically the Hall conductance σ_{xy} is constrained by the following LSM theorem [35]:

Theorem 2.3. Consider a generic interacting fermion system preserving $U(1)$ charge conservation and magnetic translation symmetry (1), with charge density $\bar{\rho}$ and flux density ϕ per u.c., if there is a unique symmetric and gapped ground state on torus, its Hall conductance σ_{xy} satisfy the following condition:

$$\sigma_{xy} \cdot \frac{\phi}{2\pi} = \bar{\rho} \pmod{1}. \quad (6)$$

In a special case, half-filled ($\bar{\rho}_f = 1/2$) fermions with $\phi = \pi$ must have an odd Hall conductance $\sigma_{xy} = 1 \pmod{2}$ for any unique gapped ground state, therefore forbidding a unique gapped TRI ground state. This is consistent with the well-known Dirac spectrum of NN π -flux model on square lattice [36].

Symmetry class AII describes half-integer-spin fermions preserving $U(1)$ charge conservation and time reversal symmetry \mathcal{T} with $\mathcal{T}^2 = (-1)^{\hat{F}}$. The SRE fermion phases are characterized by a \mathbb{Z}_2 -valued index $\nu = 0, 1$ where $\nu = 1$ corresponds to a QSH insulator with protected helical edge modes. With lattice translation symmetry, an unfractonized insulating ground state is only possible at

Table 2

Summary of LSM theorems for SPT phases of a generic interacting boson system in two dimensions. Hall conductance σ_{xy} is defined in unit of $1/h$, where the unit charge of microscopic bosons is set to 1, (p, q) are mutually-primed integers and $a \in \mathbb{Z}$.

Global symmetries and SPT classification			Microscopic input		Output of LSM theorem
Symmetry group G_s	Classification $H^3(G_s, U(1))$	Topological invariants	Density/d.o.f. per unit cell	Flux per unit cell	Topological index
$U(1)$	$2\mathbb{Z}$	$\sigma_{xy} = \text{even}$	$\bar{\rho} = \frac{2a}{q}$	$\phi = 2\pi \frac{p}{q}$	$p \cdot \sigma_{xy} = 2a \pmod{q}$ (BIQH states)
$U(1) \rtimes Z_2^T$	\mathbb{Z}_2	$\nu_f = 0, 1 \in \mathbb{Z}_2$	$\bar{\rho} \in \mathbb{Z} + \text{an odd number of Kramers doublets}$	$\phi = \pi$	$\nu_f = 1$ (BQSH state)
$Z_2 \times Z_2^T$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\nu, \nu_f = 0, 1 \in \mathbb{Z}_2$	An odd number of Kramers doublets	$\phi = \pi$	$\nu_f = 1$
$U(1)_A \times U(1)_B$	$(2\mathbb{Z})^2 \times \mathbb{Z}$	$\sigma_{xy}^A, \sigma_{xy}^B = \text{even}$ $\sigma_{xy}^{AB} = \sigma_{xy}^{BA} \in \mathbb{Z}$	$(\bar{\rho}_A, \bar{\rho}_B)$	(ϕ_A, ϕ_B)	$(\sigma_{xy}^A, \sigma_{xy}^B, \sigma_{xy}^{AB})$ satisfying (9)
$U(1)_A \times (Z_q)_B$	$2\mathbb{Z} \times (\mathbb{Z}_q)^2$	$\sigma_{xy}^A = \text{even}$ $\nu^B, \nu^{AB} \in \mathbb{Z}_q$	$\bar{\rho}_A = \frac{a}{q}$	$\phi_B = 2\pi \frac{p}{q}$ $\phi_A = 0, \pi$	$p \cdot \nu^{AB} = a \pmod{q}$

even fermion filling $\bar{\rho}_f \in 2\mathbb{Z}$ [7]. Time reversal symmetry is only compatible with $\phi = \pi$ flux per u.c., and the associated magnetic translation symmetry (1) brings in a new possibility at an odd filling [33]:

Theorem 2.4. Consider a generic interacting fermion system preserving $U(1)$ charge conservation, time reversal $\mathcal{T}^2 = (-1)^{\hat{F}}$ and magnetic translation symmetry (1), with fermion density $\bar{\rho}_f = 1 \pmod{2}$ and $\phi = \pi$ flux per u.c., if there is a unique symmetric and gapped ground state on torus, it must be a QSH insulator.

3. LSM theorems for boson SPT phases

While TIs and TSCs are realizable even in a system of non-interacting (free) fermions, in a boson system strong interactions are necessary to evade Bose–Einstein condensation and to achieve a gapped symmetric ground state. Among them, boson SPT phases with symmetry group G_s are SRE symmetric ground states with G_s -symmetry-protected edge/surface excitations. Below we present LSM theorems for various global symmetry G_s (see Table 2) containing $U(1) \subset G_s$ as a subgroup. We label the $U(1)$ charge density per u.c. as $\bar{\rho}$.

One minor (notational) difference from fermions is that magnetic translation algebra (1) will be written in a more generic context:

$$\tilde{T}_1 \tilde{T}_2 \tilde{T}_1^{-1} \tilde{T}_2^{-1} = e^{i\phi \hat{N}}, \quad \hat{N} = \text{total } U(1) \text{ charge.} \quad (7)$$

where ϕ is the flux per u.c. associated with the $U(1)$ symmetry. Similar to the TSC case, $U(1)$ charge conservation is not required to define the above magnetic translation symmetry. Even if the $U(1)$ group is broken down to a discrete Z_q subgroup generated by $\hat{R}_q \equiv e^{i\frac{2\pi}{q}\hat{N}}$, a flux of $\phi = \frac{2\pi p}{q}$ (with $p, q \in \mathbb{Z}$) is still well-defined in (7).

Boson integer quantum Hall (QIHE) states: In a simplest case we consider a boson system with $G_s = U(1)$. The associated boson SPT phases are BIQH states, characterized by an even Hall conductance $\sigma_{xy} \in 2\mathbb{Z}$ (in unit of $1/h$ where unit charge is set to 1). Analogous to symmetry class A of fermions, the usual LSM theorem forbids a unique gapped ground state at any non-integer filling $\bar{\rho} \notin \mathbb{Z}$ with lattice translation symmetry. In the presence of magnetic translation (7), a SPT ground state with $\sigma_{xy} \neq 0$ becomes possible even at a fractional filling. Specifically, the LSM theorem for interacting bosons with $G_s = U(1)$ symmetry has the same form as Theorem 2.3 for fermions, also yielding the constraint (6).

For any rational flux density $\phi = 2\pi \frac{p}{q}$ and commensurate charge density $\bar{\rho} = \frac{2a}{q}$, we have $p\sigma_{xy} = 2a \pmod{q}$ for any SRE symmetric ground state shown in Table 2. This necessarily leads to a nonzero Hall conductance, thus a BIQH state.

Bosonic quantum spin Hall (BQSH) states: With both $U(1)$ charge conservation and time reversal symmetry i.e. $G_s = U(1) \rtimes Z_2^T$, SRE symmetric boson states are classified by a \mathbb{Z}_2 -valued index $\nu = 0, 1$, where $\nu = 1$ corresponds to the nontrivial BQSH state [37]. In addition to protected edge states [37] there is another defining character for BQSH states: each π flux in the bulk is bound to a Kramers doublet transformed as $\mathcal{T}^2 = -1$. This is equivalent to the decorated-domain-wall picture of BQSH state [23], since a π flux can be viewed as the end/boundary of a 1d domain wall corresponding to symmetry element $e^{i\pi\hat{Q}}$, \hat{Q} being the total $U(1)$ charge. In the 2d BQSH state, the domain wall of $e^{i\pi\hat{Q}}$ symmetry is decorated by a 1d SPT phase with $G_s = U(1) \rtimes Z_2^T$ symmetry, and hence a π flux as its boundary must carry the edge mode of the 1d SPT phase, which is nothing but a Kramers doublet. A half-integer spin is an example of a Kramers doublet in contrast to an integer spin. If there is an odd number of Kramers doublets per u.c., usual LSM theorems with lattice translation symmetry forbids any SRE symmetric ground state [7]. In the presence of magnetic translation (7) with TRI $\phi = \pi$ flux per u.c., however, a BQSH ground state with an odd number of Kramers doublets per u.c. becomes possible:

Theorem 3.1. Consider a generic interacting boson system preserving $U(1)$ charge conservation, time reversal \mathcal{T} and magnetic translation symmetry (7), with charge density $\bar{\rho} \in \mathbb{Z}$, flux density $\phi = \pi$ and an odd number of Kramers doublets (e.g. spin-1/2's with $\mathcal{T}^2 = -1$) per u.c., if there is a unique symmetric and gapped ground state on torus, it must be a BQSH state.

The boson system in [Theorem 3.1](#) consists of two parts: charged bosons that transform as Kramers singlets ($\mathcal{T}^2 = +1$), and half-integer spins that transform as Kramers doublets ($\mathcal{T}^2 = -1$). Therefore the π flux per u.c. is only visible to charge d.o.f. but not to spins. In fact, the $U(1)$ symmetry above can be broken down to a discrete Z_2 subgroup, resulting in a symmetry group $G_s = Z_2 \times Z_2^T$. The associated SRE symmetric states have a $(\mathbb{Z}_2)^2$ classification [16,37], where one \mathbb{Z}_2 index (v_T in [Table 2](#)) comes from the binding of each π flux to a Kramers doublet [23]. Although the charge will no longer conserve with $G_s = Z_2 \times Z_2^T$, the π flux per u.c. is still well-defined and our LSM states that *any unique gapped ground state on torus with an odd number of Kramers doublets per u.c. must be a SPT state* where a π flux is bound to a Kramers doublet (see [Appendix B.5](#)).

Two-component BIQH states: For a two-component system with two species of conserved bosons, symmetry group $G_s = U(1)_A \times U(1)_B$ leads to $(2\mathbb{Z})^2 \times \mathbb{Z}$ classification of SRE symmetric states. They are characterized by a (real symmetric) 2×2 Hall conductance tensor $\sigma_{xy}^{\alpha\beta}$, $\alpha, \beta = A, B$. While the intra-species Hall conductance $\sigma_{xy}^A, \sigma_{xy}^B \in 2\mathbb{Z}$ must be even integers, the inter-species Hall conductance $\sigma_{xy}^{AB} = \sigma_{xy}^{BA} \in \mathbb{Z}$ can take any integer value. The magnetic translation symmetry here is defined by two flux ϕ_A and ϕ_B for the two components:

$$\tilde{T}_1 \tilde{T}_2 \tilde{T}_1^{-1} \tilde{T}_2^{-1} = e^{i(\phi_A \hat{N}_A + \phi_B \hat{N}_B)}. \quad (8)$$

Our LSM theorem states the following:

Theorem 3.2. *Consider a generic interacting boson system of two components A and B, separately conserved with a symmetry group $G_s = U(1)_A \times U(1)_B$. In the presence of magnetic translation symmetry (8), with charge density $(\bar{\rho}_A, \bar{\rho}_B)$ and flux density (ϕ_A, ϕ_B) per u.c., if there is a unique symmetric and gapped ground state on torus, its Hall conductance tensor must satisfy*

$$\frac{1}{2\pi} \begin{pmatrix} \sigma_{xy}^A & \sigma_{xy}^{AB} \\ \sigma_{xy}^{BA} & \sigma_{xy}^B \end{pmatrix} \begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix} = \begin{pmatrix} \bar{\rho}_A \\ \bar{\rho}_B \end{pmatrix} \mod 1 \quad (9)$$

This can be viewed as a generalization of [Theorem 2.3](#) with $G_s = U(1)$, and can be further generalized to a multi-component system. At any fractional filling i.e. $\rho_{A,B} \notin \mathbb{Z}$, the Hall conductance tensor cannot vanish identically, leading to a SPT ground state. Furthermore, one of the two $U(1)$ symmetries here can be broken down to a Z_q subgroup, as we discuss below.

Two-component magnets: In a two-component magnetic system where species A has a $U(1)$ conservation (e.g. of \hat{z} -component spin) and species B with only a discrete Z_q symmetry, the global symmetry $G_s = U(1)_A \times (Z_q)_B$ leads to a $2\mathbb{Z} \times (\mathbb{Z}_q)^2$ classification of 2d SPT phases [16]. Compared to previous $G_s = U(1)_A \times U(1)_B$ case, while $\sigma_{xy}^A \in 2\mathbb{Z}$ still serves as a valid topological index with $U(1)_A$ symmetry, $\sigma_{xy}^{AB} = \sigma_{xy}^{BA}$ and σ_{xy}^B are only well-defined modular q once $U(1)_B$ is broken down to $(Z_q)_B$, yield two \mathbb{Z}_q -valued indices:

$$v^{AB} \equiv \sigma_{xy}^{AB} = \sigma_{xy}^{BA} \mod q, \quad v^B \equiv \sigma_{xy}^B \mod 2q.$$

Our LSM theorem for this case reads the following:

Theorem 3.3. *Consider a generic interacting two-component (A and B) spin system with global symmetry $G_s = U(1)_A \times (Z_q)_B$, in the presence of magnetic translation symmetry (8), with conserved $U(1)_A$ density $\bar{\rho}_A$ and flux density $(\phi_A, \phi_B = 2\pi \frac{p}{q})$ per u.c., if there is a unique symmetric and gapped ground state on torus, its topological indices must satisfy*

$$\sigma_{xy}^A \frac{\phi_A}{2\pi} + v^{AB} \frac{\phi_B}{2\pi} = \sigma_{xy}^A \frac{\phi_A}{2\pi} + v^{AB} \frac{p}{q} = \bar{\rho}_A \mod 1. \quad (10)$$

Now that σ_{xy}^A is an even integer, by choosing flux $\phi_A = 0, \pi$ condition (10) immediately leads to $p \cdot v^{AB} = q\bar{\rho}_A \mod q$ as shown in [Table 2](#). This indicates any SRE symmetric ground state at fractional filling $\bar{\rho}_A \notin \mathbb{Z}$ must be a SPT state with $v^{AB} \neq 0 \mod q$.

4. Physical picture and sketch of arguments

What are the basic ideas behind these LSM theorems for SPT phases? Given the global and lattice translation symmetries, the usual LSM theorems dictate the “integer filling” (per u.c.) conditions on the Hilbert space that allow a symmetric SRE ground state: e.g. integer filling $\bar{\rho}$ per u.c. for conserved $U(1)$ charges [1–3], an even number of Kramers doublets per u.c. for TRI systems [7], and an even number of Majorana fermions per u.c. for superconductors [28]. When pure lattice translations are replaced by magnetic translations, there is an additional “background flux” in each u.c., in addition to “bare” symmetry charges (the “filling number”). A key feature of many SPT phases is the binding of a “fractionalized” symmetry charge to a symmetry flux [38]: e.g. charge-flux binding in quantum Hall states, the binding of a Kramers doublet to each π -flux in QSH states [39,40], and the binding of a MZM to each π flux in a chiral TSC [27]. For SPT states with magnetic translation symmetries, a “background polarization charge” comes together with the background flux, leading to a total symmetry charge different from the “bare” symmetry charges. This explains why a SPT ground state at fractional filling evades the usual LSM theorem requiring integer fillings, with the help of magnetic translations.

Take $G_s = U(1)$ as an example, with bare symmetry charge $\bar{\rho}$ and background polarization charge $-\sigma_{xy} \frac{\phi}{2\pi}$, the “total charge” per u.c. $\rho_{\text{total}} = \bar{\rho} - \sigma_{xy} \frac{\phi}{2\pi}$ must be an integer as dictated by the usual LSM theorem. This is exactly the condition (6) in [Theorem 2.3](#).

This simple physical picture not only provides a generic construction of these LSM theorems, but also allows us to argue for their validity. Below we only sketch the physical arguments, leaving details to the Appendices. Consider a many-body system on a

$L_x \times L_y$ torus (or an infinitely-long cylinder, periodic along \hat{y} direction with a finite circumference L_y). In the presence of magnetic translation symmetry (1) or (7) with ϕ flux per u.c., we choose a circumference L_y satisfying $\phi L_y \neq 0 \pmod{2\pi}$: e.g. for $\phi = \pi$ flux we choose $L_y = \text{odd}$. In the Landau gauge (2) with pure lattice translation symmetry $\hat{T}_y = T_y$ along \hat{y} direction, clearly the boundary condition along \hat{y} direction changes with the coordinate x (hence no well-defined magnetic translation \hat{T}_x). Specifically the \hat{y} -direction boundary condition Θ_y of a many-body state $|\Psi(\Theta_y)\rangle$ with wavefunctions $\Psi(\mathbf{X}_1, \mathbf{X}_2, \dots)$ can be defined as [41] ($\mathbf{X}_i \equiv (x_i, y_i)$ are coordinates of i th particle)

$$\Psi(\mathbf{X}_1 + L_y \hat{y}, \mathbf{X}_2, \dots) = e^{i\Theta_y} \Psi(\mathbf{X}_1, \mathbf{X}_2, \dots). \quad (11)$$

Equivalently it can be written as $e^{iL_y \hat{K}_y} |\Psi(\Theta_y)\rangle = e^{i\Theta_y} |\Psi(\Theta_y)\rangle$, where $\hat{\mathbf{K}}^1$ is the momentum of one single particle. Although magnetic translation \hat{T}_x is absent on the torus/cylinder of chosen size L_y , the change of boundary condition Θ_y upon pure lattice translation $x \xrightarrow{\hat{T}_x} x + 1$ leads to an important condition:

$$\hat{T}_x |\Psi(\Theta_y)\rangle = |\Psi(\Theta_y + \phi L_y)\rangle. \quad (12)$$

which imposes a strong constraint on any gapped and symmetric ground state on this torus/cylinder.

Take $G_s = U(1)$ (Theorem 2.3) for example, a SRE insulator can be characterized by a well-defined polarization [42,43] $\hat{P}_x \equiv e^{\frac{2\pi}{L_x} \sum_{\mathbf{r}} x \hat{n}_{\mathbf{r}}}$ where $\hat{n}_{\mathbf{r}}$ labels the $U(1)$ charge on lattice site \mathbf{r} . Due to the non-commutative algebraic relation $\hat{T}_x \hat{P}_x \hat{T}_x^{-1} = e^{-iL_y \bar{\rho}} \hat{P}_x$, condition (12) implies the change of ground state polarization $\langle \hat{P}_x \rangle$ upon twisting boundary condition (11):

$$\langle \Psi(\Theta_y) | \hat{P}_x | \Psi(\Theta_y) \rangle = e^{iL_y \bar{\rho}} \langle \Psi(\Theta_y + \phi L_y) | \hat{P}_x | \Psi(\Theta_y + \phi L_y) \rangle$$

Since polarization is physically the “center of mass” of all charges, its change indicates pumping of a charge $L_y \bar{\rho}$ upon insertion of flux ϕL_y (and hence change of boundary condition). This is a direct evidence for nontrivial Hall conductance $\sigma_{xy} \neq 0$ in the gapped ground state. Similar arguments apply to Theorems 3.2–3.3 as well.

Generally with global symmetry G_s , the change of “generalized polarization” under translation \hat{T}_x must be compatible with the pumping of “generalized symmetry charges” upon flux insertion (i.e. boundary condition twist). This implies a symmetric SRE ground state must be a SPT phase with certain (generalized) flux-charge binding. For instance in fermion symmetry class D (Theorem 2.1), we consider “fermion parity polarization” which is nothing but the total fermion parity $(-1)^{\hat{F}}$ on $L_y = \text{odd}$ torus: it changes sign by either translation \hat{T}_x . The only compatible SRE ground state is a $\nu = \text{odd}$ chiral TSC in class D, such as spinless $p_x + i p_y$ superconductor [27], whose fermion parity changes upon switching between periodic and antiperiodic boundary conditions. In other examples with time reversal symmetry (Theorems 2.4–3.1) and an odd number of Kramers doublets in each u.c., one can observe the change of “time reversal polarization” [44] under translation \hat{T}_x : it is detected by the presence/absence of Kramers pairs in entanglement spectra [7,45] at different entanglement cuts related by \hat{T}_x (e.g. \bar{x} and $\bar{x} + 1$ in Fig. 1). This implies the pumping of one Kramers doublet by switching between periodic and antiperiodic boundary conditions ($\phi L_y = \pi \pmod{2\pi}$), only compatible with a QSH ground state [44,46]. All theorems in Tables 1–2 can be argued following this line of thoughts as we show in Appendix B.

5. Applications

Majorana vortex lattice: The simplest application of Theorem 2.1 (fermions in symmetry class D) is the vortex lattice of a 2d chiral p -wave TSC [27], or of a 3d TI-superconductor heterostructure [47]. In both cases there is a single MZM in each vortex core. Magnetic translation symmetry also naturally emerges in a vortex lattice.

As shown in Ref. [31], on a Majorana vortex lattice whose plaquette is a polygon of n vortices, there is a $(n\frac{\pi}{2} - \pi)$ flux per plaquette in a Majorana hopping model between the vortices. Therefore on both triangular ($n = 3$) and square ($n = 4$) lattice, there is only one Majorana fermion per u.c. whose tunneling amplitudes preserve magnetic translation (1) with a flux density of $\phi = \pi$. According to Theorem 2.1, any unique gapped ground state of the Majoranas must be a $\nu = \text{odd}$ TSC with chiral Majorana edge modes. This is precisely the case for a triangular vortex lattice, as shown in Fig. 2.

Quantum spin liquids [48] provide another platform to realize magnetic translation of fermionic spinons with an emergent gauge flux of $\phi = \pi$ per u.c. [36]. On square lattice, Lieb’s theorem [49] dictates a π -flux per square plaquette in lowest-energy spinon ground state, in the presence of particle-hole symmetry.

One well-known example is the $U(1)$ Dirac spin liquid in square-lattice large- N Heisenberg model [36], where N -flavor fermionic spinons at half filling ($\bar{\rho} = \frac{1}{2}$ for each flavor) hop in the background of $\phi = \pi$ flux per plaquette. According to Theorem 2.3, gapping out the Dirac spectrum of the $U(1)$ spin liquid without breaking translation symmetry will result in an odd Chern number $\sigma_{xy} = 1 \pmod{2}$, corresponding to a chiral spin liquid [50,51] in the context of a spin system.

Another example is in Kitaev-type Z_2 spin liquids [26], where fermionic spinons form a “superconductor” and they can see a background Z_2 gauge flux of $\phi = 0, \pi$. Due to particle-hole symmetry in superconductors, Lieb’s theorem [49] again applies and points to a π -flux ground state on square lattice. Ref. [32] introduces such a NN square-lattice model, where one Kramers pair of Majorana spinons $\{\gamma_{\mathbf{r},\uparrow}, \gamma_{\mathbf{r},\downarrow}\}$ per site \mathbf{r} hops under an emergent $\phi = \pi$ flux as described in (4). According to Theorem 2.2, gapping out the Dirac spinon spectrum of NN model (A.1) while preserving translation symmetry must lead to a TSC of Majorana spinons: as realized by 3-spin interactions involving NNNs [32].

BIQH states [37,52] are also important applications of LSM theorems in Table 2. In Hofstadter models [53] of bosons hopping in a magnetic field of ϕ flux per u.c., the Hall conductance of an unfractonized insulator ground state must satisfy relation (6) for

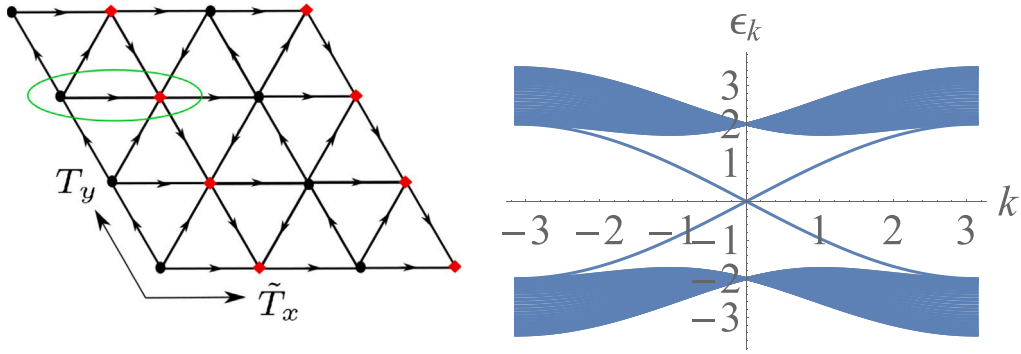


Fig. 2. The NN tunneling model of MZMs on a triangular vortex lattice (left) and its edge spectrum (right). The doubled magnetic unit cell is labeled by the green oval. The edge spectrum is obtained on a $L_y = 50$ cylinder periodic along \tilde{T}_x direction, where two counter-propagating edge modes are located separately on two opposite edges.

one-component bosons, or relation (9) for two-component bosons. Hence we can use LSM theorems to choose proper boson fillings that enforces a BIQH ground state.

One example is the correlated hopping model of half-filled bosons on honeycomb lattice introduced in Ref. [54]. With only hoppings within the same sublattice, there are two $U(1)$ conservation laws (one on each sublattice) with $\bar{\rho}_A = \bar{\rho}_B = \frac{1}{2}$, and in the presence of $\phi_A = \phi_B = \pi$ flux we must have $\sigma_{xy}^{AB} = 1 \bmod 2$ for a unique gapped ground state, as dictated by Theorem 3.2. Indeed this is the SPT state observed in numerical studies of Ref. [54].

6. Discussions

In this work we introduce and provide physical arguments for a new class of LSM theorems in 2d, which relies on magnetic translation symmetries in contrast to the usual LSM theorems with pure lattice translations. While no symmetric SRE ground states is allowed in usual LSM theorems, our theorems imply that a symmetric SRE ground state at fractional filling must belong to a SPT phase with protected edge modes. As summarized in Table 1–2, our LSM theorems apply to many different physical systems with various global symmetries. They will serve as useful guidance to construct realistic models of interacting (especially bosonic) SPT phases, and to future experimental realizations of SPT phases. While the current work focuses on 2d systems with magnetic translation symmetries, it will be interesting to generalize these ideas to other magnetic space group symmetries and to higher spatial dimensions such as 3d, which we leave for future work.

Although the magnetic translation symmetry plays a crucial role in establishing these LSM theorems for SPT phases, we emphasize that the SPT phases discussed in this work are all strong SPT phases, which by themselves do not require the protection of magnetic translation symmetry. They are not weak SPT phases which are protected by both on-site and translational symmetries [55–58].

It is insightful to comment on the relation between the LSM theorems for SPT phases in this work and the usual LSM theorems [1] which forbid SRE ground states. Both the usual LSM theorems and the new LSM theorems discussed in this work can be understood in the language of UV-IR anomaly matching [59]. In the usual LSM theorems with e.g. the usual lattice translation symmetry, the UV data describing the degree of freedom (d.o.f.) in each unit cell must match the IR data of the ground state properties, if the ground state is compatible with the microscopic d.o.f. while preserving the translational symmetry. For example, a fractional charge per unit cell is not compatible with any symmetric insulator without intrinsic topological order, which has an integer charge per unit cell. In our LSM-SPT theorems with magnetic translation symmetry, since the magnetic translation associates each unit cell with a global (on-site) symmetry flux, the anomaly matching condition changes accordingly. For the ground state to be a SRE phase without intrinsic topological order, the global symmetry flux in the SRE ground state (IR data) must match the microscopic d.o.f. in each unit cell (UV data). For example, in a π -flux system of symmetry class DIII (see Theorem 2.2), if there is a Majorana Kramers pair in each unit cell, it must match the π -flux in each unit cell in the ground state, and therefore such a SRE ground state must be a topological superconductor in class DIII.

After this work was posted on arXiv, a number of related works of LSM theorems for SPT phases have been published, such as a concurrent work for bosonic SPT phases in two dimensions [60], and follow-up works on 3d SPT phases [61–63].

CRediT authorship contribution statement

Yuan-Ming Lu: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Acknowledgments

I thank Ying Ran and Masaki Oshikawa for related collaborations [35], Yin-Chen He for discussions on Ref. [54], and Michael Levin for useful discussions and especially for pointing out Ref. [31]. I thank KITP for hospitality during “topoquant16” program where part of this work was performed. This work is supported by National Science Foundation under award number NSF DMR-1653769 and NSF DMR-2011876, and in part by National Science Foundation under Grant No. NSF PHY11-25915.

Appendix A. Proofs for LSM theorems for free fermions

A.1. Symmetry class D

A.1.1. Majorana hopping model (3) on square lattice

With only nearest neighbor hoppings on square lattice, the π -flux model with one Majorana $\gamma_{\mathbf{r}}$ per site \mathbf{r} writes

$$\begin{aligned}\hat{H}_0^D &= \sum_{\mathbf{r}} i[t_x \gamma_{\mathbf{r}} \gamma_{\mathbf{r}+\hat{x}} + t_y (-1)^x \gamma_{\mathbf{r}} \gamma_{\mathbf{r}+\hat{y}}] + h.c. \\ &= - \sum_{\mathbf{k}} \phi_{-\mathbf{k}}^T [2t_y \sin k_y \tau_z + t_x (\sin k_x \tau_x + (1 - \cos k_x) \tau_y)] \phi_{\mathbf{k}}\end{aligned}\quad (\text{A.1})$$

where we define 2-component spinor

$$\phi_{\mathbf{k}} \equiv \frac{1}{\sqrt{L_x L_y / 2}} \sum_{x,y} e^{-i(k_x x + k_y y)} \begin{pmatrix} \gamma_{2x,y} \\ \gamma_{2x+1,y} \end{pmatrix} \quad (\text{A.2})$$

and $\bar{\tau}$ are Pauli matrices for the sublattice index in a doubled magnetic unit cell (u.c.). Clearly the dispersion vanishes at two Majorana cones (two “valleys”) at $(k_x, k_y) = (0, 0)$ and $(0, \pi)$. A mode expansion around these Majorana cones leads to 4-component spinor

$$\begin{aligned}\hat{H}_0^D &= - \sum_{\mathbf{q}} \Phi_{-\mathbf{q}}^T [t_x q_x \tau_x + 2t_y q_y \tau_z \mu_z] \Phi_{\mathbf{q}} + O(|\mathbf{q}|^2), \\ \Phi_{\mathbf{q}}^T &\equiv (\phi_{\mathbf{q}}^T, \phi_{(0,\pi)+\mathbf{q}}^T).\end{aligned}\quad (\text{A.3})$$

where $\bar{\mu}$ are Pauli matrices for the valley index. Under magnetic translations the Majoranas transform as

$$\begin{aligned}\gamma_{(x,y)} &\xrightarrow{T_y} \gamma_{(x,y+1)}, \\ \gamma_{(x,y)} &\xrightarrow{\tilde{T}_x} (-1)^y \gamma_{(x+1,y)}.\end{aligned}\quad (\text{A.4})$$

Therefore the 4-component low-energy spinor transforms as

$$\Phi_{\mathbf{q}} \xrightarrow{T_y} \mu_z \Phi_{\mathbf{q}}, \quad (\text{A.5})$$

$$\Phi_{\mathbf{q}} \xrightarrow{\tilde{T}_x} \tau_x \mu_x \Phi_{\mathbf{q}}. \quad (\text{A.6})$$

It is straightforward to see that among all possible mass terms $\tau_y \mu_{0,x,z}$ and $\tau_z \mu_y$ to Dirac Hamiltonian (A.3), the only mass that preserves magnetic translation $\{\tilde{T}_x, T_y\}$ is

$$\hat{M} = m \cdot \hat{\Gamma}_0, \quad \hat{\Gamma}_0 = \tau_y \mu_z \quad (\text{A.7})$$

This symmetric mass drives the system into a spinless $p_x \pm i p_y$ TSC, whose chirality $v = \text{Sgn}(m)$ depends on the sign of the mass term. This mass term can be realized by next nearest neighbor (diagonal) hoppings that preserve magnetic translations.

A.1.2. Proof of Theorem 2.1 for free fermions

For a generic free-fermion system with an odd number of Majoranas per u.c., one can go beyond perturbing around nearest neighbor model (A.1), and prove Theorem 2.1 non-perturbatively. In particular, the momentum-space spinor $\phi_{\mathbf{k}}$ transforms under magnetic translations as

$$\phi_{\mathbf{k}} \xrightarrow{T_y} e^{i k_y} \phi_{\mathbf{k}}, \quad (\text{A.8})$$

$$\phi_{(k_x, k_y)} \xrightarrow{\tilde{T}_x} \begin{pmatrix} 0 & 1 \\ e^{i k_x} & 0 \end{pmatrix} \phi_{(k_x, k_y + \pi)} \quad (\text{A.9})$$

Generically a gapped 2d superconductor in class D can be diagonalized as

$$\hat{H}^D = \sum_{\mathbf{k}} \phi_{-\mathbf{k}}^T \hat{h}_{\mathbf{k}} \phi_{\mathbf{k}} = \sum_{\mathbf{k}} \Gamma_{-\mathbf{k}}^T \hat{A}_{\mathbf{k}} \Gamma_{\mathbf{k}}, \quad (\text{A.10})$$

$$A_{\mathbf{k}} = i \oplus_{\mathbf{k}} \begin{pmatrix} 0 & E_{\mathbf{k}} \\ -E_{\mathbf{k}} & 0 \end{pmatrix}, \quad E_{\mathbf{k}} > 0,$$

$$\Gamma_{\mathbf{k}} = W_{\mathbf{k}} \phi_{\mathbf{k}}, \quad W_{-\mathbf{k}} = W_{\mathbf{k}}^*. \quad (\text{A.11})$$

where $W_{\mathbf{k}}$ is a unitary matrix representing the Bloch wavefunction at momentum \mathbf{k} . As shown in [27,64], the parity of topological index ν for a 2d superconductor in class D is given by

$$(-1)^\nu = \text{Sgn}(\text{Pf}(\hat{h}_{k_x=0}) \cdot \text{Pf}(\hat{h}_{k_x=\pi})) = \text{Sgn}\left(\prod_{\mathbf{Q}=-\mathbf{Q}} \det W_{\mathbf{Q}}\right) \quad (\text{A.12})$$

where $\text{Pf}(\hat{h})$ denotes the Pfaffian of antisymmetric matrix \hat{h} . Here $\mathbf{Q} = -\mathbf{Q}$ represents the 4 time reversal invariant momenta (TRIM) i.e. $(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π) . Meanwhile magnetic translation symmetry (A.9) dictates that

$$W_{(k_x, k_y+\pi)} = W_{(k_x, k_y)} \cdot \begin{pmatrix} 0 & 1 \\ e^{ik_x} & 0 \end{pmatrix} \quad (\text{A.13})$$

and hence we have (notice that $\det W_{\mathbf{Q}} = \pm 1$ for any TRIM \mathbf{Q})

$$\begin{aligned} \det W_{(0,0)} \cdot \det W_{(0,\pi)} &= \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1, \\ \det W_{(\pi,0)} \cdot \det W_{(\pi,\pi)} &= \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -1. \end{aligned}$$

Therefore we have proven that $\nu=\text{odd}$ for any gapped superconductor i.e. Theorem 2.1 for any free-fermion system.

A.2. Symmetry class DIII

For simplicity, we consider a π -flux model on square lattice, with an $N_f = \text{odd}$ number of Kramers pairs $\{\gamma_{\mathbf{r},\uparrow}^a, \gamma_{\mathbf{r},\downarrow}^a | 1 \leq a \leq N_f\}$ per u.c. \mathbf{r} . We can therefore define an odd number of complex fermions per u.c.:

$$f_{\mathbf{r},a} \equiv \frac{\gamma_{\mathbf{r},\uparrow}^a + i\gamma_{\mathbf{r},\downarrow}^a}{2}. \quad (\text{A.14})$$

Under time reversal symmetry (TRS) each complex fermion transforms as a Kramers doublet

$$f_{\mathbf{r},a} \xrightarrow{\mathcal{T}} -i f_{\mathbf{r},a}^\dagger, \quad \mathcal{T}^2 = (-1)^{\hat{F}} = \prod_{\mathbf{r},a} (i\gamma_{\mathbf{r},\uparrow}^a \gamma_{\mathbf{r},\downarrow}^a). \quad (\text{A.15})$$

where τ are Pauli matrices for the Nambu indices. Meanwhile, clearly there is also a particle-hole symmetry (PHS) for the Nambu spinor $\psi_{\mathbf{r}} \equiv (f_{\mathbf{r}}, f_{\mathbf{r}}^\dagger)^T$

$$\psi_{\mathbf{r}} = \tau_x \psi_{\mathbf{r}}^*. \quad (\text{A.16})$$

In this Nambu basis, the BdG Hamiltonian for superconductors is mapped to a Bloch Hamiltonian for band insulators, where the above PHS is essentially a half-filling condition (N_f particles per u.c.) for the “band insulator”. In the meantime, TRS (A.15) is implemented as

$$\psi_{\mathbf{r}} \xrightarrow{\mathcal{T}} \tau_y \psi_{\mathbf{r}} \quad (\text{A.17})$$

In the presence of magnetic translational symmetry with π -flux per u.c., a magnetic u.c. consists of 2 u.c. and the Bloch spinor in momentum space is defined as

$$\Psi_{\mathbf{k}=(k_x, k_y)} \equiv \frac{1}{\sqrt{N/2}} \sum_{x,y} e^{-i(xk_x + yk_y)} \begin{pmatrix} \psi_{(2x,y)} \\ \psi_{(2x+1,y)} \end{pmatrix}. \quad (\text{A.18})$$

Under PHS and TRS it transforms as

$$\Psi_{\mathbf{k}} \xrightarrow{\text{PHS}} \tau_x \Psi_{-\mathbf{k}}^*, \quad \Psi_{\mathbf{k}} \xrightarrow{\mathcal{T}} \tau_y \Psi_{-\mathbf{k}} \quad (\text{A.19})$$

Meanwhile under magnetic translations in the Landau gauge

$$\tilde{T}_x = T_x \cdot (-1)^{\sum_{\mathbf{r}} y \sum_a f_{\mathbf{r},a}^\dagger f_{\mathbf{r},a}} \quad (\text{A.20})$$

the “Bloch” spinor $\Psi_{\mathbf{k}}$ transforms as

$$\Psi_{\mathbf{k}} \xrightarrow{\tilde{T}_x} \begin{pmatrix} 0 & 1 \\ e^{ik_x} & 0 \end{pmatrix}_{\tilde{\mu}} \Psi_{(k_x, k_y+\pi)}, \quad \Psi_{\mathbf{k}} \xrightarrow{T_y} e^{ik_y} \Psi_{\mathbf{k}}. \quad (\text{A.21})$$

Clearly the magnetic translation \tilde{T}_x shifts the k_y component by π . We use Pauli matrices $\vec{\mu}$ for sublattice index, and $\vec{\tau}$ for the Nambu index.

As shown in Ref. [44,65], the \mathbb{Z}_2 -valued bulk invariants of topological superconductors in symmetry class DIII is quite similar to the QSHE in symmetry class AII [44], given by the “time reversal polarization”

$$\nu = (-1)^{P_\Theta(k_y=0) - P_\Theta(k_y=\pi)} \in \pm 1 \quad (\text{A.22})$$

In particular, the time reversal polarization is given by [44]

$$(-1)^{P_\Theta(k_y)} = e^{\frac{i}{2} (\int_0^\pi dk_x - \int_{-\pi}^0 dk_x) A_{\mathbf{k}}^x} \frac{\text{Pf}[w(k_x = \pi, k_y)]}{\text{Pf}[w(k_x = 0, k_y)]} \quad (\text{A.23})$$

We define the Berry connection for all filled bands

$$\mathbf{A}_{\mathbf{k}} = (A_{\mathbf{k}}^x, A_{\mathbf{k}}^y) \equiv i \sum_{\alpha=\text{filled}} \langle \mathbf{k}, \alpha | \vec{\nabla}_{\mathbf{k}} | \mathbf{k}, \alpha \rangle \quad (\text{A.24})$$

and the anti-symmetric “time reversal” matrix

$$w_{\alpha,\beta}(\mathbf{k}) \equiv \langle -\mathbf{k}, \alpha | \mathcal{T} | \mathbf{k}, \beta \rangle, \quad \alpha, \beta = \text{filled}. \quad (\text{A.25})$$

In our case with magnetic translational symmetry (A.21), it is straightforward to show that

$$\begin{aligned} A_{(k_x, k_y)}^x - A_{(k_x, k_y + \pi)}^x &= - \sum_{\alpha=\text{filled}} \langle \mathbf{k}, \alpha | \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\vec{\mu}} | \mathbf{k}, \alpha \rangle \\ &= - \sum_{\alpha=\text{filled}} \langle (k_x, k_y + \pi), \alpha | \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\vec{\mu}} | (k_x, k_y + \pi), \alpha \rangle \end{aligned} \quad (\text{A.26})$$

Note that one can always choose a (smooth) gauge so that time reversal symmetry (A.19) leads to

$$w_{\mathbf{Q}} \equiv (i \tau_y) \otimes \hat{1}_{N_f \times N_f} \quad (\text{A.27})$$

at the four TRIM $\mathbf{Q} = -\mathbf{Q}$. Therefore the Pfaffians in (A.23) cancels out each other, and the \mathbb{Z}_2 -valued invariant is simply given by

$$\begin{aligned} \nu &= (-1)^{P_\Theta(k_y=0) - P_\Theta(k_y=\pi)} \\ &= e^{-\frac{i}{2} \int_0^\pi dk_x \sum_{\alpha=\text{filled}} \langle (k_x, 0), \alpha | (k_x, 0), \alpha \rangle} \\ &= e^{-i\pi \frac{2N_f}{2}} = (-1)^{N_f} \end{aligned} \quad (\text{A.28})$$

As a result, we have shown that a gapped superconducting ground state can only be a $\nu = -1$ topological superconductor in symmetry class DIII, if we have $N_f = \text{odd}$ Kramers pairs of Majorana fermions per u.c..

A.3. Symmetry class AII and A

Symmetry class AII corresponds to topological insulators, with a \mathbb{Z}_2 classification in 2d associated with quantum spin Hall effects. Its bulk topological invariant is also given by time reversal polarization [44] in (A.22)–(A.23). In the presence of magnetic translation symmetry (A.21) with π -flux per u.c., the bulk invariant can be computed in complete parallel to previous case of class DIII. One can similarly show that $\nu = (-1)^{\bar{\rho}_f}$ when there is $\bar{\rho}_f$ spin-1/2 fermions per u.c.. Therefore a gapped ground state can only be a quantum spin Hall insulator with $\nu = -1$, at half-filling with $\bar{\rho}_f = \text{odd}$.

In the case insulators (symmetry class A) with ϕ flux and $\bar{\rho}_f$ fermions per u.c., the theorem

$$\sigma_{xy} \frac{\phi}{2\pi} = \bar{\rho}_f \pmod{1}. \quad (\text{A.29})$$

was proved in the context of free fermion band theory [66]. In the special case of $\phi = 2\pi/q$ and $\bar{\rho}_f = p/q$, it reduces to the formula

$$\sigma_{xy} = p \pmod{q} \quad (\text{A.30})$$

Appendix B. Physical arguments for LSM theorems for interacting systems

Without loss of generality, we always consider a square lattice for simplicity. In the case of square lattice, each u.c. consists of just one site, therefore we also refer it to a site in proper context. We will always choose Landau gauge for simplicity, where translation along \hat{y} direction is the pure crystal translation T_y , while along \hat{x} direction there is a magnetic translation \tilde{T}_x . Our arguments however do not depend on the lattice geometry, as long as magnetic translation symmetries are preserved.

B.1. Fermion: Symmetry class D

Consider a generic interacting system with $N_y = \text{odd}$ Majorana fermions $\{\gamma_{\mathbf{r},a} | 1 \leq a \leq N_y\}$ per site \mathbf{r} on a $L_x \times L_y$ torus, where $L_y = \text{odd}$ and $L_x = \text{even}$. On such a $L_y = \text{odd}$ torus, although translation T_y is still intact, the magnetic translation \hat{T}_x in (A.4) is in fact broken. As shown in Fig. 1, in contrast to periodic boundary condition along \hat{x} direction, the boundary condition along \hat{y} direction will switch between periodic and antiperiodic in different columns. If we translate the torus along \hat{x} direction by one u.c., the boundary condition along \hat{y} direction will be twisted by a phase factor of $e^{i\pi} = -1$. In other words, denoting a ground state $|\Psi\rangle$ with boundary condition $e^{i\Theta_y}$:

$$\Psi(\mathbf{X}_1 + L_y \hat{y}, \mathbf{X}_2, \dots) = e^{i\Theta_y} \Psi(\mathbf{X}_1, \mathbf{X}_2, \dots). \quad (\text{B.1})$$

as $|\Psi(\Theta_y)\rangle$, we have

$$\hat{T}_x |\Psi(\Theta_y)\rangle = |\Psi(\Theta_y + \phi L_y)\rangle. \quad (\text{B.2})$$

where $\phi L_y = \pi \pmod{2\pi}$ here.

Note that with $L_y N_y = \text{odd}$ Majorana fermions per column of the torus, the crystal translation T_x plays the role of a supersymmetry [28] which changes fermion parity $(-1)^{\hat{F}}$:

$$T_x (-1)^{\hat{F}} = (-1)^{\hat{F}} T_x \cdot (-1)^{L_y N_y} = -(-1)^{\hat{F}} T_x \quad (\text{B.3})$$

Therefore the two ground states in (B.2) related by switching periodic/antiperiodic boundary conditions will have opposite fermion parities if $N_y = \text{odd}$:

$$\begin{aligned} \langle \Psi(\Theta_y) | (-1)^{\hat{F}} | \Psi(\Theta_y) \rangle &= \\ (-1)^{L_y N_y} \langle \Psi(\Theta_y + \pi L_y) | (-1)^{\hat{F}} | \Psi(\Theta_y + \pi L_y) \rangle \end{aligned} \quad (\text{B.4})$$

In symmetry class D, all superconductors are classified by an integer index $\nu \in \mathbb{Z}$. Among them, $\nu = \text{odd}$ topological superconductors (e.g. spinless $p_x + ip_y$ superconductor has $\nu = 1$) are distinguished from $\nu = \text{even}$ ones with 3 sharp features [26,27,67]: (i) an odd number of chiral Majorana modes on an open boundary, with half-integer-valued chiral central charge $c_- = \nu/2$; (ii) one robust Majorana zero mode in each vortex (i.e. π flux) core; (iii) change of fermion parity if the boundary condition along one direction (\hat{y} -direction in our case) is switched from periodic to anti-periodic.

The last feature (iii) i.e. change of fermion parity upon twisting boundary condition along one (say \hat{y}) direction can be intuitively understood as the following, by making a connection to the well-known feature (ii) i.e. a single Majorana zero mode trapped at each π flux. Twisting boundary condition along \hat{y} direction is equivalent as dragging a π flux across the whole system along \hat{x} direction. Now that a Majorana bound state is localized around each π flux in a $\nu = \text{odd}$ topological superconductor, bringing this single Majorana fermion across the system will necessarily change the fermion parity [28,64].

Therefore in our case with magnetic translation symmetry, as dictated by condition (B.4) with $L_y = \text{odd}$, the only unique gapped ground state compatible with feature (iii) must be a $\nu = \text{odd}$ topological superconductor. This argument justifies the LSM theorem for class D.

B.2. Fermion: Symmetry class DIII

B.2.1. A no-go theorem for translational symmetric system

Before arguing for our LSM theorem with magnetic translation symmetries, we first justify a related theorem for a system with the usual crystal translation symmetry. The no-go theorem states the following:

Theorem B.1. *For a generic interacting fermion system with a $N_f = \text{odd}$ number of Kramers pairs of Majoranas $\{\gamma_{\mathbf{r},a,\sigma} | \sigma = \uparrow / \downarrow, 1 \leq a \leq N_f\}$ per u.c. \mathbf{r} , there is no unique gapped ground state that preserves both translations and time reversal symmetry.*

The argument of the theorem is simple. With crystal translational symmetry, we are allowed to put the many-body system on any periodic lattice [2], and we choose a $L_x \times L_y$ torus where both lengths are odd:

$$L_x, L_y = 1 \pmod{2}. \quad (\text{B.5})$$

Notice that under time reversal symmetry, the Majorana fermions transform as Kramers doublets

$$\begin{pmatrix} \gamma_{\mathbf{r},a,\uparrow} \\ \gamma_{\mathbf{r},a,\downarrow} \end{pmatrix} \xrightarrow{\mathcal{T}} \begin{pmatrix} \gamma_{\mathbf{r},a,\downarrow} \\ -\gamma_{\mathbf{r},a,\uparrow} \end{pmatrix} \quad (\text{B.6})$$

and therefore

$$\mathcal{T} (i\gamma_{\mathbf{r},a,\uparrow} \gamma_{\mathbf{r},a,\downarrow}) \mathcal{T}^{-1} = -i\gamma_{\mathbf{r},a,\uparrow} \gamma_{\mathbf{r},a,\downarrow} \quad (\text{B.7})$$

Note that the total fermion parity is given by

$$(-1)^{\hat{F}} = \prod_{\mathbf{r},a} (i\gamma_{\mathbf{r},a,\uparrow} \gamma_{\mathbf{r},a,\downarrow}) \quad (\text{B.8})$$

Therefore the fermion parity and time reversal symmetry satisfy the following algebra

$$\mathcal{T}(-1)^{\hat{F}}\mathcal{T}^{-1}(-1)^{\hat{F}} = (-1)^{N_f L_x L_y} \quad (\text{B.9})$$

This means on a odd by odd torus, time reversal symmetry \mathcal{T} serves as a supersymmetry [29] that changes the fermion parity. Since time reversal and fermion parity anticommutes with each other, they cannot be both preserved in a unique symmetric ground state. Therefore we have justified the no-go theorem.

B.2.2. Arguments for LSM Theorem 2.2 for class DIII

Now let us turn to the case with magnetic translation symmetry (A.20). This time we consider an infinite cylinder which is finite along \hat{y} direction, but infinite along \hat{x} direction. Again we choose the circumference length L_y along \hat{y} direction to be odd. Quite similar to the class D case, the boundary condition along \hat{y} direction switches between periodic and antiperiodic in different columns of the cylinder. When we translate the physical system by one u.c. along \hat{x} direction, we twist the \hat{y} -direction boundary condition by a phase of $e^{i\pi} = -1$.

To argue for the LSM theorem, we first assume a unique gapped ground state that preserves magnetic translation and time reversal symmetries. Using symmetry properties of the entanglement spectrum of a SRE state [7,68], we are able to show that this SRE ground state must be a topological superconductor in class DIII.

Next we consider the Schmidt decomposition of unique SRE ground state $|\Psi(\Theta_y)\rangle$ with boundary condition (B.1) across an entanglement cut along \hat{y} direction located at $x_0 - 1 < \bar{x} < x_0$ (see Fig. 1):

$$|\Psi(\Theta_y)\rangle = \sum_{\alpha} \lambda_{\bar{x},\alpha}^{\Theta_y} |\alpha, \Theta_y\rangle_{\bar{x},L} |\alpha, \Theta_y\rangle_{\bar{x},R} \quad (\text{B.10})$$

where $\lambda_{\bar{x},\alpha}$ are Schmidt weights. Note that in a generic Hamiltonian of Majorana fermions, the fermion number is not conserved and the Schmidt eigenstates do not generally have a fixed particle number. In the presence of time reversal symmetry \mathcal{T} , although fermion parity $(-1)^{\hat{F}}$ can fluctuate for each Schmidt state, $\{|\alpha\rangle_{\bar{x},L/R}\}$ must form a representation of the following algebra

$$\mathcal{T}(-1)^{\hat{F}}\mathcal{T}^{-1}(-1)^{\hat{F}} |\alpha, \Theta_y\rangle_{\bar{x},L} = e^{i\Phi_{\bar{x}}} |\alpha, \Theta_y\rangle_{\bar{x},L} \quad (\text{B.11})$$

where $e^{i\Phi_{\bar{x}}} = \pm 1$ is a phase factor depending on the entanglement cut (at \bar{x}), but independent of Schmidt eigenstate $|\alpha\rangle_{\bar{x},L}$. Similar to the class D case, the many-body symmetry (B.2) for ground state boundary condition exists in class DIII as well. According to relation (B.2), a Schmidt decomposition (B.10) of $|\Psi(\Theta_y)\rangle$ at entanglement cut \bar{x} leads to the same entanglement spectrum as that of $|\Psi(\Theta_y + \pi L_y)\rangle$ at entanglement cut $\bar{x} + 1$ (see Fig. 1)

$$|\Psi(\Theta_y + \pi L_y)\rangle = \sum_{\beta} \lambda_{\bar{x}+1,\alpha}^{\Theta_y + \pi L_y} |\alpha, \Theta_y + \pi L_y\rangle_{\bar{x},L} |\alpha, \Theta_y + \pi L_y\rangle_{\bar{x},R} \quad (\text{B.12})$$

with

$$\lambda_{\bar{x}+1,\alpha}^{\Theta_y + \pi L_y} = \lambda_{\bar{x},\alpha}^{\Theta_y}, \quad (\text{B.13})$$

$$|\alpha, \Theta_y + \pi L_y\rangle_{\bar{x}+1,L/R} = \hat{T}_{\bar{x}} |\alpha, \Theta_y\rangle_{\bar{x},L}. \quad (\text{B.14})$$

In comparison, the original ground state $|\Psi(\Theta_y)\rangle$ has the following Schmidt decomposition at entanglement cut $\bar{x} + 1$:

$$|\Psi(\Theta_y)\rangle = \sum_{\beta} \lambda_{\bar{x}+1,\beta}^{\Theta_y} |\beta, \Theta_y\rangle_{\bar{x}+1,L} |\beta, \Theta_y\rangle_{\bar{x}+1,R} \quad (\text{B.15})$$

In the following, we will compare the Schmidt decompositions (B.15) and (B.12) for two ground states with two boundary conditions differed by $\pi L_y = \pi \bmod 2\pi$, at the same entanglement cut $\bar{x} + 1$. First according to (B.11) and (B.14), clearly Schmidt eigenstates of $|\Psi(\Theta_y + \pi L_y)\rangle$ has a symmetry character $e^{i\Phi_{\bar{x}}}$:

$$\mathcal{T}(-1)^{\hat{F}}\mathcal{T}^{-1}(-1)^{\hat{F}} |\alpha, \Theta_y + \pi L_y\rangle_{\bar{x}+1,L} = e^{i\Phi_{\bar{x}}} |\alpha, \Theta_y + \pi L_y\rangle_{\bar{x}+1,L}, \quad \forall \alpha. \quad (\text{B.16})$$

What about Schmidt eigenstates $|\beta, \Theta_y\rangle_{\bar{x}+1,L}$ of ground state $|\Psi(\Theta_y)\rangle$? Note that the Schmidt eigenstates of $|\Psi(\Theta_y)\rangle$ at the two different cuts (B.10) and (B.15) are related by

$$|\beta, \Theta_y\rangle_{\bar{x}+1,L} = \sum_{p,\alpha} M_{\beta,\alpha}^p |p\rangle_{x_0} \otimes |\alpha, \Theta_y\rangle_{\bar{x},L} \quad (\text{B.17})$$

where $\{|p\rangle_{x_0}\}$ is a set of orthonormal basis for Hilbert space on column x_0 . Note that we have a $L_y = \text{odd}$ number of Majorana Kramers pairs on column x_0 , and therefore

$$\mathcal{T}(-1)^{\hat{F}}\mathcal{T}^{-1}(-1)^{\hat{F}} |p\rangle_{x_0} = (-1) \cdot |p\rangle_{x_0}. \quad (\text{B.18})$$

As a result, the symmetry character of Schmidt eigenstates (B.15) at cut $\bar{x} + 1$ have a symmetry character

$$\mathcal{T}(-1)^{\hat{F}}\mathcal{T}^{-1}(-1)^{\hat{F}} |\beta, \Theta_y\rangle_{\bar{x}+1,L} = e^{i\Phi_{\bar{x}+1}} |\beta, \Theta_y\rangle_{\bar{x}+1,L} = e^{i(\Phi_{\bar{x}} + \pi)} |\beta, \Theta_y\rangle_{\bar{x}+1,L} \quad (\text{B.19})$$

Comparing (B.19) and (B.16), we can see that after twisting boundary condition along \hat{y} direction, for the same entanglement cut at $\bar{x} + 1$, the entanglement spectrum of a ground state $|\Psi(\Theta_y + \pi L_y)\rangle$ with twisted boundary condition has different symmetry

character $\mathcal{T}(-1)^{\hat{F}} \mathcal{T}^{-1}(-1)^{\hat{F}} = e^{i\Phi_{\hat{x}}}$, as compared to the original ground state $|\Psi(\Theta_y)\rangle$ whose entanglement spectrum has symmetry character $\mathcal{T}(-1)^{\hat{F}} \mathcal{T}^{-1}(-1)^{\hat{F}} = e^{i\Phi_{\hat{x}+1}} = -e^{i\Phi_{\hat{x}}}$.

What kind of symmetric SRE ground state is compatible with the change of Schmidt eigenstate symmetry character upon twisting boundary condition? As discussed in the case of class D, twisting the boundary condition along \hat{y} direction can be viewed as dragging a π flux across the cylinder along \hat{x} direction. In symmetry class DIII, there are two classes ($\nu \in \mathbb{Z}_2$ classification) of time-reversal-symmetric (TRS) SRE superconductors: the trivial one with $\nu = 0$ and the topological superconductor with $\nu = 1$. While the π flux in a trivial superconductor has no stable low-energy bound states, the π flux in a 2d topological superconductor features a zero-energy Majorana Kramers pair [29] $\{\gamma_1, \gamma_1\}$. Therefore, dragging a π -flux across the cylinder along \hat{x} direction will also move this Kramers pair of Majoranas across the entanglement cut (at $\bar{x}+1$). Now that $\mathcal{T}(-1)^{\hat{F}} \mathcal{T}^{-1}(-1)^{\hat{F}} = -1$ when acting on an odd number of Majorana Kramers pairs, the symmetry character of entanglement spectrum will be switched by twisting boundary condition in such a topological superconductor. Therefore in the presence of magnetic translation with π flux per u.c., the only TRS SRE compatible with the above entanglement spectrum symmetry character is the $\nu = 1$ topological superconductor. Indeed, viewing the 2d $\nu = 1$ topological superconductor on a cylinder as a 1d system along x direction, upon twisting the boundary condition along y direction, the 1d system must change from a trivial superconductor to a topological one in class DIII, which has a \mathbb{Z}_2 classification in 1d. This justifies the LSM theorem for symmetry class DIII.

B.3. Symmetry group $G_s = U(1)_A \times U(1)_B$

In the presence of a global $U(1)$ symmetry, an insulating ground state $|\Psi\rangle$ on a $L_x \times L_y$ torus can be characterized by its polarization $\hat{P}_x \equiv e^{\frac{2\pi}{L_x} \sum_r x \hat{n}_r}$ (and \hat{P}_y can be defined similarly), where \hat{n}_r is the $U(1)$ charge on lattice site r [42,43,69]. Physically the polarization describes the center of mass of all $U(1)$ charges. In contrast to metals with no well-defined polarization i.e. $\langle \Psi | \hat{P}_\alpha | \Psi \rangle = 0$, insulators generally have a non-vanishing complex expectation value of each polarization component \hat{P}_x and \hat{P}_y .

Similar to previous cases, the boundary condition along \hat{y} direction generally changes on different columns of the torus. As a result, pure lattice translation \hat{T}_x can twist the \hat{y} -direction boundary condition as shown in (B.2). Meanwhile if there is a unique symmetric ground state separated from excitation states by a finite energy gap, one can adiabatically insert flux through the hole along \hat{x} direction without closing the gap, and \hat{y} -direction boundary condition can be adiabatically twisted in this flux insertion process:

$$\hat{F}_y(\phi L_y) |\Psi(\Theta_y)\rangle = e^{i\phi_0} |\Psi(\Theta_y + \phi L_y)\rangle. \quad (\text{B.20})$$

where ϕ_0 is an unimportant phase factor, and $F_y(\phi L_y)$ is the adiabatic ϕL_y -flux insertion operator.

Therefore according to (B.2) and (B.20), the system on a $L_x \times L_y$ torus has an emergent symmetry in the unique gapped ground state:

$$T'_x = F_y^{-1}(\phi L_y) \cdot T_x \quad (\text{B.21})$$

if we choose Landau gauge (i.e. preserving lattice translation $\hat{T}_y \equiv T_y$) for the magnetic translation algebra (7) and (8). Now that the insulator ground state has a non-vanishing expectation value for polarization \hat{P}_x , the emergent symmetry operation (B.21) must preserve the polarization operator \hat{P}_x [35].

In the presence of two $U(1)$ charge conservation symmetries $G_s = U(1)_A \times U(1)_B$, we have a more complicated magnetic translation algebra (8). There are also two polarization operators $\hat{P}_x^A = e^{\frac{2\pi}{L_x} \sum_r x \hat{n}_r^A}$ and $\hat{P}_x^B = e^{\frac{2\pi}{L_x} \sum_r x \hat{n}_r^B}$, and the emergent symmetry (B.21) on $L_x \times L_y$ torus becomes the following:

$$T'_x = [F_y^A(\phi_A L_y) F_y^B(\phi_B L_y)]^{-1} \cdot T_x \quad (\text{B.22})$$

Therefore both polarizations P_x^A and P_x^B must be preserved by the above emergent symmetry operation. Making use of the following commutation relations:

$$T_x P_x^\alpha T_x^{-1} = e^{-2\pi i \bar{\rho}_\alpha L_y} P_x^\alpha, \quad \alpha, \beta = A, B; \quad (\text{B.23})$$

$$F_y^\alpha(\phi_\alpha L_y) P_x^\beta [F_y^\alpha(\phi_\alpha L_y)]^{-1} = e^{i\sigma_{xy}^\beta \phi_\alpha L_y} P_x^\beta. \quad (\text{B.24})$$

and by requiring $[P_x^\alpha, T'_x] = 0$ we can immediately obtain relation (9). Therefore we have established Theorem 3.2.

Clearly, $G_s = U(1)$ is a special case of the above discussions, and one can easily justify relation (6) and Theorem 2.3 by requiring T'_x in (B.21) commutes with polarization P_x . This is studied in detail in Ref. [35]. Notice that in all discussions, we have not involve the statistics of microscopic particles, therefore the conclusions apply to interacting bosons and/or fermions.

B.4. Symmetry group $G_s = U(1)_A \times (Z_q)_B$

As mentioned in main text, breaking $U(1)_A \times U(1)_B$ down to its subgroup $G_s = U(1)_A \times (Z_q)_B$ leads to a $2\mathbb{Z} \times (\mathbb{Z}_q)^2$ classification of 2d SPT phases. They are characterized by Hall conductance σ_{xy}^A of conserved $U(1)_A$ charges, a $\nu^B \in \mathbb{Z}_q$ invariant associated with 2d $(Z_q)_B$ -SPT phases, plus another $\nu^{AB} \in \mathbb{Z}_q$ invariant describing the binding of $U(1)_A$ charges to $(Z_q)_B$ flux. Specifically, each 2π flux of $(Z_q)_B$ symmetry will trap $\nu^{AB} \in \mathbb{Z}_q$ units of $U(1)_A$ charges.

Again on $L_x \times L_y$ torus under Landau gauge (preserving T_y translation), the ground state $|\Psi(\theta_y^A, \theta_y^B)\rangle$ satisfy

$$\hat{T}_x |\Psi(\theta_y^A, \theta_y^B)\rangle = |\Psi(\theta_y^A + \phi_A L_y, \theta_y^B + \phi_B L_y)\rangle. \quad (\text{B.25})$$

The polarization P_x^A has the following dependence on \hat{y} -direction boundary conditions:

$$e^{-i L_y \bar{p}_A} \cdot \langle \Psi(\theta_y^A, \theta_y^B) | \hat{P}_x^A | \Psi(\theta_y^A, \theta_y^B) \rangle = \langle \Psi(\theta_y^A + \phi_A L_y, \theta_y^B + \phi_B L_y) | \hat{P}_x^A | \Psi(\theta_y^A + \phi_A L_y, \theta_y^B + \phi_B L_y) \rangle \quad (\text{B.26})$$

Now that twisting \hat{y} -direction boundary condition can be achieved by flux insertions through the hole along \hat{x} -direction, we immediately reach the relation (10). Therefore we have justified Theorem 3.3.

B.5. Symmetry group $G_s = U(1) \rtimes Z_2^T$ and $G_s = Z_2 \times Z_2^T$

We first consider a boson system with symmetry group $G_s = U(1) \rtimes Z_2^T$, consisting of two parts: (i) integer-spin $U(1)$ -charged bosons transformed as Kramers singlet ($\mathcal{T}^2 = +1$), and (ii) charge-neutral half-integer spin moments ($\mathcal{T}^2 = -1$). As stated in Theorem 3.1, we consider an odd number of half-integer spins together with $\phi = \pi$ flux of $U(1)$ symmetry in each unit cell. On a circumference- L_y cylinder which is infinitely long along \hat{x} direction and wrapped around along \hat{y} -direction, we consider a unique gapped ground state $|\Psi(\theta_y)\rangle$ where $\theta_y \in [0, 2\pi)$ denotes the boundary condition (11) along \hat{y} direction. Again the relation (B.2) holds on this infinite cylinder, imposing strong constraints on the ground state properties.

Similar to the arguments for Theorem 2.2, we again consider the Schmidt decompositions of two states $|\Psi(\theta_y)\rangle$ and $|\Psi(\theta_y + \pi L_y)\rangle$ at the same entanglement cut $\bar{x}+1$. General relations (B.10) and (B.12)–(B.15) still holds in this case, while the symmetry characters of the Schmidt eigenstates in this case becomes:

$$\mathcal{T}^2 |\alpha, \theta_y\rangle_{\bar{x}, L} = e^{i\Phi_{\bar{x}}} |\alpha, \theta_y\rangle_{\bar{x}, L}, \quad e^{i\Phi_{\bar{x}}} = \pm 1, \quad \forall \alpha, \quad (\text{B.27})$$

$$\mathcal{T}^2 |\alpha, \theta_y + \pi L_y\rangle_{\bar{x}+1, L} = e^{i\Phi_{\bar{x}}} |\alpha, \theta_y + \pi L_y\rangle_{\bar{x}+1, L}, \quad e^{i\Phi_{\bar{x}}} = \pm 1, \quad \forall \alpha, \quad (\text{B.28})$$

$$\mathcal{T}^2 |\beta, \theta_y\rangle_{\bar{x}+1, L} = e^{i\Phi_{\bar{x}+1}} |\beta, \theta_y + \pi L_y\rangle_{\bar{x}+1, L}, \quad e^{i\Phi_{\bar{x}+1}} = \pm 1, \quad \forall \beta. \quad (\text{B.29})$$

Again due to relation (B.17) and the fact

$$\mathcal{T}^2 |p\rangle_{x_0} = (-1)^{L_y} |p\rangle_{x_0}, \quad \forall p. \quad (\text{B.30})$$

we can easily show that

$$e^{i\Phi_{\bar{x}+1}} = e^{i(\Phi_{\bar{x}} + \pi)}, \quad \text{if } L_y = 1 \pmod{2}. \quad (\text{B.31})$$

Therefore on a $L_y = \text{odd}$ cylinder, the $\mathcal{T}^2 = \pm 1$ symmetry character of Schmidt eigenstates of ground state $|\Psi(\theta_y)\rangle$ at entanglement cut at $\bar{x}+1$ changes sign when boundary condition θ_y is changed by π . This indicates the pumping of one Kramers doublet across the cylinder as induced by the π flux insertion, which is only compatible with a BQSH state where π flux is bound to a Kramers doublet. Therefore we have justified Theorem 3.1.

Clearly the above argument goes through even if $U(1)$ symmetry is broken down to a discrete subgroup $H \in U(1)$, as long as H contains Z_2 as a subgroup. Therefore the LSM theorem for $G_s = Z_2 \times Z_2^T$ is also established.

The argument of Theorem 2.4 is very similar to the above discussions, also making use of Schmidt decomposition and symmetry characters of the Schmidt eigenstates. It is discussed in details by Ref. [33].

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