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THE NON-SYMMETRIC STRONG MULTIPLICITY PROPERTY FOR SIGN PATTERNS*

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Abstract. We develop a non-symmetric strong multiplicity property for matrices that may or may not be symmetric. We say a sign pattern allows the non-symmetric strong multiplicity property if there is a matrix with the non-symmetric strong multiplicity property that has the given sign pattern. We show that this property of a matrix pattern preserves multiplicities of eigenvalues for superpatterns of the pattern. We also provide a bifurcation lemma, showing that a matrix pattern with the property also allows refinements of the multiplicity list of eigenvalues. We conclude by demonstrating how this property can help with the inverse eigenvalue problem of determining the number of distinct eigenvalues allowed by a sign pattern.

Key words. Strong property, Sign patterns, Inverse eigenvalue problem, Bifurcation, Eigenvalue multiplicity.

AMS subject classifications. 15B35, 15A18, 15A29, 05C50, 58C15.

1. Introduction. One type of inverse eigenvalue problem involves determining spectral information based on the combinatorial structure of the sign pattern of a matrix. The Perron–Frobenius theorem is an early example, using the nonnegativity of a matrix along with primitivity (which can be described in terms of the combinatorial cycle structure, see e.g. [17]) to deduce the existence of a dominant eigenvalue. Other developments include, but are not limited to, the exploration of the nonnegative inverse eigenvalue problem [10], sign-nonsingular sign pattern matrices [5], potentially nilpotent sign patterns [11], spectrally arbitrary sign patterns [9], and the minimum rank of matrices associated with a graph [13]. Recently, strong spectral properties have been developed to help with combinatorial inverse eigenvalue problems. The study of strong properties of matrices is rooted in the work of Colin de Verdière who developed the Strong Arnold property [7, 8]. This led to the development of strong properties for matrices associated with a graph in [2]. In particular, the strong spectral property and the strong multiplicity property were developed for symmetric matrices associated with a graph, as well as the strong spectral property for not necessarily symmetric matrices. We now introduce the strong multiplicity property for not necessarily symmetric matrices. We develop this tool using the approach in [12]. The approach involves finding a map that preserves the property of interest, demonstrating certain tangent spaces to a manifold have a transverse intersection, and applying a version of the inverse function theorem.

In Section 2, we describe some of the analytic tools to be used, then formally define the non-symmetric strong multiplicity property in Section 3. In Section 4, we develop our main results: (1) showing that the strong property of a sign pattern is preserved upon taking superpatterns; (2) providing a bifurcation result demonstrating that if a sign pattern with the strong multiplicity property allows a given multiplicity list of

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distinct eigenvalues, then the pattern also allows refinements of that list; (3) showing that a multiplicity list is preserved for superpatterns of patterns with the strong multiplicity property; and (4) demonstrating that the strong multiplicity property is inherited for block diagonal patterns (and reducible patterns) when the irreducible blocks do not require any shared eigenvalues. In Section 5, we give an example of a non-symmetric matrix having the strong multiplicity property but not the strong spectral property. We also demonstrate the usefulness of the strong multiplicity property for determining the number of distinct eigenvalues allowed by a sign pattern. We conclude in Section 6 with some open questions and include an Appendix with *Sage* code for checking if a non-symmetric matrix has the strong multiplicity property.

2. Analytic tools. In this section, we discuss some of the analytic tools used in [12]. We use these tools to generalize the strong multiplicity property, introduced in [2] for symmetric matrices, to non-symmetric matrices.

Consider two finite dimensional normed vector spaces U and W over $\mathbb{R}$. A map F from an open subset $V \subseteq U$ to W is *differentiable* at $\mathbf{a} \in V$ if and only if there exists a unique linear mapping $\dot{F}_{\mathbf{a}} : U \to W$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{F(\mathbf{a}+\mathbf{h}) - F(\mathbf{a}) - \dot{F}_{\mathbf{a}}(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

The map $\dot{F}_{\mathbf{a}}$ is called the *derivative* of F at $\mathbf{a} \in U$. Similarly, we can define the *directional derivative* of F at $\mathbf{a}$ in the direction of vector $\mathbf{v}$ to be

$$D_{\mathbf{v}}F(\mathbf{a}) = \lim_{t\to 0} \frac{F(\mathbf{a}+t\mathbf{v}) - F(\mathbf{a})}{t}.$$

Notice that if F is differentiable at $\mathbf{a}$, then the directional derivative $D_{\mathbf{v}}F(\mathbf{a})$ exists for all choices of vector $\mathbf{v}$ and $D_{\mathbf{v}}F(\mathbf{a}) = \dot{F}_{\mathbf{a}}(\mathbf{v})$. This allows us to determine the range of $\dot{F}_{\mathbf{a}}$:

Observation 2.1. *Let $F : V \to W$ be differentiable at $\mathbf{a}$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a spanning set for* $\operatorname{span} V$. *Then* $\operatorname{range}(\dot{F}_{\mathbf{a}}) = \operatorname{span}\{D_{\mathbf{v}_1}F(\mathbf{a}), \ldots, D_{\mathbf{v}_k}F(\mathbf{a})\}$.

Using the derivative we can invoke the following version of the inverse function theorem.

Theorem 2.2 (Inverse Function Theorem). *Let U and W be finite-dimensional vector spaces over $\mathbb{R}$. Let F be a continuously differentiable function from an open subset of U to W with $F(\mathbf{u}_0) = \mathbf{w}_0$. If the derivative $\dot{F}_{\mathbf{u}_0}$ is surjective, then there is an open subset $W' \subseteq W$ containing $\mathbf{w}_0$ and a continuously differentiable function $T : W' \to U$ such that $T(\mathbf{w}_0) = \mathbf{u}_0$ and $F \circ T$ is the identity map on W', specifically F is surjective on $T(W')$.*

In this paper, we will equip all subspaces of $\mathbb{R}^{n\times n}$ with the *Frobenius inner product* $\langle A, B\rangle := \operatorname{tr}(B^T A)$ and the norm induced by this inner product $\|\cdot\|$. When working with subspaces of $\mathbb{R}^{n\times n}$, we use $\|\cdot\|$ to describe the open subset W' in Theorem 2.2; without loss of generality, there is an $\epsilon > 0$ such that $W' = \{\mathbf{w} \in W : \|\mathbf{w} - \mathbf{w}_0\| < \epsilon\}$.

3. Defining the non-symmetric strong multiplicity property. The non-symmetric strong spectral property was introduced in [2] along with the strong multiplicity and strong spectral properties for symmetric matrices associated with a graph. Here we introduce the non-symmetric strong multiplicity property for a matrix. The zero matrix is denoted O.

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DEFINITION 3.1. A matrix $A \in \mathbb{R}^{n\times n}$ has the *non-symmetric strong multiplicity property* (nSMP) provided $X = O$ is the only matrix such that $A \circ X = O$, $[A, X^T] = O$ and $\operatorname{tr}(X^T A^k) = 0$ for $k = 0, \dots, n-1$.

LEMMA 3.2. *If A has the nSMP and B is obtained from A via permutation similarity, diagonal similarity, transposition, or nonzero scalar multiplication, then B has the nSMP.*

Proof. Suppose $B = S^{-1}AS$ for some invertible matrix S. Then

$$\begin{aligned}[A, X^T] &= AX^T - X^T A = SBS^{-1}X^T - X^T SBS^{-1}\\ &= S(BS^{-1}X^T S - S^{-1}X^T SB)S^{-1}.\end{aligned}$$

Let $Y^T = S^{-1}X^T S$. Then

$$[A, X^T] = S(BY^T - Y^T B)S^{-1},$$

which is zero if and only if $[B, Y^T] = O$. Further,

$$\begin{aligned}\operatorname{tr}(X^T A^k) &= \operatorname{tr}(X^T SB^k S^{-1})\\ &= \operatorname{tr}(S^{-1}X^T SB^k)\\ &= \operatorname{tr}(Y^T B^k).\end{aligned}$$

Also, if S is either a permutation (or a diagonal) matrix, then the entries of A and X are simply rearranged (or scaled) in B and Y, respectively, and hence $A \circ X = O$ if and only if $B \circ Y = O$. It follows that A has the nSMP if and only if B has the nSMP.

Further, $[A, X^T] = O$ if and only if $[A^T, X] = O$, $\operatorname{tr}(X^T A^k) = \operatorname{tr}(X {A^T}^k)$ and $A \circ X = O$ if and only if $A^T \circ X^T = O$. Thus, the nSMP is preserved under transpose. Finally, it is straightforward to check that the nSMP is preserved under nonzero scalar multiplication. ∎

A *sign pattern* is an $n \times n$ matrix with entries in $\{+, -, 0\}$. A real matrix A has sign pattern S if $A_{ij} > 0$ when $S_{ij} = +$, $A_{ij} < 0$ when $S_{ij} = -$ and $A_{ij} = 0$ when $S_{ij} = 0$. The *qualitative class* of a sign pattern S, denoted $\mathcal{Q}(S)$, is the set of all matrices that have sign pattern S. A sign pattern S *allows the nSMP* if there is a matrix $A \in \mathcal{Q}(S)$ that has the nSMP. If every matrix $A \in \mathcal{Q}(S)$ has the nSMP, then S *requires the nSMP*.

Based on Lemma 3.2, we say that a pattern P is *equivalent* to a sign pattern S if P can be obtained from S via permutation similarity, signature similarity, transpose, and/or negation. (A pattern S is *signature similar* to pattern P if there is a diagonal matrix H with signed unit entries on the diagonal, such that $HSH = P$.) The *digraph* of a sign pattern S, denoted $D(S)$, is the directed graph on n vertices, with an arc from vertex i to vertex j if $S_{ij} \neq 0$. A *cycle* is a digraph whose vertices can be labeled $\{1, \dots, n\}$ so that there is an arc from vertex i to vertex j if and only if $j = i + 1 \mod n$. The next example gives a class of sign patterns that require the nSMP.

EXAMPLE 3.3. Let S be an $n \times n$ sign pattern and assume that $D(S)$ is a cycle with $n > 2$. By equivalence, we may assume S is a pattern with positive entries on the superdiagonal and $S_{n,1} \in \{+, -\}$. Let $A \in \mathcal{Q}(S)$. By diagonal similarity, we may assume that A has unit entries on the superdiagonal. Let X^T be a matrix in the commutator of A. Then, since A is nonderogatory, X^T is a polynomial in A (see e.g. [17, Theorem 3.2.4.2]). In particular, $X^T = \sum_{i=0}^{n-1} c_i A^i$ for some coefficients $c_0, \dots, c_{n-1}$. Note that unless k is a multiple of n, A^k has a zero diagonal and $A^n = A_{n,1} I$. Hence, for $0 < k < n$, $\operatorname{tr}(X^T A^k) = \operatorname{tr}\left(\sum_{i=0}^{n-1} c_i A^{i+k}\right) = nc_{n-k}A_{n,1}$, and $\operatorname{tr}(X^T) = nc_0$. Thus, $\operatorname{tr}(X^T A^k) = 0$ for $0 \le k < n$ implies that each coefficient is zero and therefore $X = 0$. Thus, S requires the nSMP.

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4. Main results. In this section, we develop our results about superpatterns of patterns that allow the nSMP, as well as our bifurcation result, our multiplicity list preservation theorem, and our results on reducible patterns.

A sign pattern S is a proper *superpattern* of a sign pattern P if S can be obtained from P by changing at least one zero entry to a signed nonzero entry. In the next theorem, we will observe that if a matrix A has the nSMP, then superpatterns of the sign pattern of A will also allow the nSMP. In fact, this invariance to superpatterns is a common feature that has been observed of strong properties more generally (such as for spectrally arbitrary patterns [15], the strong Arnold property [7], the strong spectral property [2], the strong multiplicity property [2], the non-symmetric strong spectral property [2], and the strong inner product property for orthogonal matrices [6]). We state the next definition in order to formalize this common feature.

Definition 4.1. For the $n \times n$ matrices A, a strong system is a family of homogeneous systems of linear equations in n^2 variables $x_{i,j}, 1 \leq i, j \leq n$, whose coefficients are fixed multivariate polynomials in the entries of A, such that the system includes the equation $A \circ X = O$ with $X = (x_{i,j})$. A particular matrix is said to have the *strong property* if the only solution to the strong system is $X = O$.

Given $A \in \mathbb{R}^{m \times n}$, $\alpha \subseteq \{1, \dots, m\}$, and $\beta \subseteq \{1, \dots, n\}$, the notation $A[\alpha, \beta]$ represents the submatrix of A with rows indexed by α and columns indexed by β. The notation $A[:, \beta]$ is a shorthand for $A[\{1, \dots, m\}, \beta]$.

Lemma 4.2. *Let $\Phi : \mathbb{R}^{n \times n} \to \mathbb{R}^{m \times k}$ be a continuous transformation. If $\Phi(A)[:, \beta]$ has linearly independent columns for some $\beta \subseteq \{1, \dots, k\}$, then for ϵ sufficiently small and $M \in \mathbb{R}^{n \times n}$ with $||M - A|| < \epsilon$, the columns of $\Phi(M)[:, \beta]$ are linearly independent.*

Proof. Suppose $\Phi(A)[:, \beta]$ has linearly independent columns for some $\beta \subseteq \{1, \dots, k\}$. Then there exists $\alpha \subseteq \{1, \dots, m\}$ such that $|\alpha| = |\beta|$ and $\det(\Phi(A)[\alpha, \beta]) \neq 0$. By the continuity of Φ and continuity of the determinant, there exists an $\epsilon > 0$ such that for all $M \in \mathbb{R}^{n \times n}$ with $||M - A|| < \epsilon$, then $\det(\Phi(M)[\alpha, \beta]) \neq 0$. For such M, the columns of $\Phi(M)[:, \beta]$ are linearly independent. ∎

The next theorem formally states that superpatterns can preserve strong properties (see also [1, Cor. 2.3]).

Theorem 4.3. *Let A be an $n \times n$ matrix that has a strong property. If S is a superpattern of the sign pattern of A, then S allows a matrix that has the strong property. In particular, if $M \in \mathcal{Q}(S)$ is sufficiently close to A, then M will have the strong property.*

Proof. Since A has a strong property, there is a corresponding strong homogeneous system of linear equations whose variables are the entries of X including $A \circ X = O$. Let $\Phi : \mathbb{R}^{n \times n} \to \mathbb{R}^{m \times n^2}$ be the map that takes A to the coefficient matrix of this system, where m is the number of equations of the strong system. Let $z(A)$ be the set of indices of the variables in X corresponding to the positions (i, j) with $A_{ij} = 0$. Note that $\Phi(A)[:, z(A)]$ has linearly independent columns if and only if A has the strong property. Let S be a superpattern of the sign pattern of A. By Lemma 4.2, we can select $M \in \mathcal{Q}(S)$ with $||M - A||$ sufficiently close to zero such that $\Phi(M)[:, z(A)]$ has linearly independent columns and $z(M) \subseteq z(A)$, which ensures the columns of $\Phi(M)[:, z(M)]$ are also linearly independent. Thus, M has the strong property and therefore S allows the strong property. ∎

The spectrum of $A \in \mathbb{R}^{n \times n}$ is denoted by $\text{spec}(A) = \{\lambda_1^{(m_1)}, \dots, \lambda_k^{(m_k)}\}$, where m_i denotes the algebraic multiplicity of λ_i for $i = 1, \dots, k$. Let $\mathbf{m}(A)$ be the list of algebraic multiplicities of the eigenvalues of A.

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Note that $\mathbf{m}(A) = \mathbf{m}(B)$ if the multiset of the list $\mathbf{m}(A)$ is equal to the multiset of the list of $\mathbf{m}(B)$. We write $\mathrm{gm}_A(\lambda)$ for the geometric multiplicity of an eigenvalue λ.

In order to develop the theory behind the nSMP, we make use of certain polynomial functions near the identity map. In particular, for each $\mathbf{c} = (c_0, \dots, c_{n-1}) \in \mathbb{R}^n$, we define the polynomial $p_\mathbf{c}(x) = x + \sum_{i=0}^{n-1} c_i x^i$ and choose $\|\mathbf{c}\|$ to be small.

LEMMA 4.4. *Let $A \in \mathbb{R}^{n\times n}$ have spectrum $\mathrm{spec}(A) = \{\lambda_1^{(m_1)}, \dots, \lambda_t^{(m_t)}\}$. Then there exists an $\epsilon > 0$ such that for all $\mathbf{c} \in \mathbb{R}^n$ with $\|\mathbf{c}\| < \epsilon$ and for all $L \in \mathbb{R}^{n\times n}$ with $\|L\| < 1$, the matrix $M = (I+L)^{-1} p_\mathbf{c}(A)(I+L)$ has spectrum $\mathrm{spec}(M) = \{\eta_1^{(m_1)}, \dots, \eta_t^{(m_t)}\}$ that satisfies $|\eta_i - \lambda_i| < \frac{1}{2}|\lambda_j - \lambda_\ell|$ for all i, j and ℓ with $j \neq \ell$. In particular, $\mathbf{m}(A) = \mathbf{m}(M)$. Moreover, $\mathrm{gm}_A(\lambda_i) = \mathrm{gm}_M(\eta_i)$ for $i = 1, \dots, t$.*

Proof. Begin by writing $A = W^{-1}JW$, where $J = J_1 \oplus \cdots \oplus J_s$ is the Jordan canonical form of A. Let $\epsilon > 0$ be chosen such that for every $\mathbf{c} = (c_0, \dots, c_{n-1}) \in \mathbb{R}^n$ satisfying $\|\mathbf{c}\| < \epsilon$, $|\sum_{k=0}^{n-1} c_k \lambda_i^k| < \frac{1}{2}|\lambda_j - \lambda_\ell|$ for all i, j and ℓ with $j \neq \ell$. Let $\mathbf{c} = (c_0, \dots, c_{n-1}) \in \mathbb{R}^n$ and $L \in \mathbb{R}^{n\times n}$ satisfy $\|\mathbf{c}\| < \epsilon$ and $\|L\| < 1$, respectively. Then $I + L$ is nonsingular and

$$\begin{aligned}(I+L)^{-1} p_\mathbf{c}(A)(I+L) &= (I+L)^{-1} W^{-1} p_\mathbf{c}(J) W (I+L) \\ &= (I+L)^{-1} W^{-1} \big(p_\mathbf{c}(J_1) \oplus \cdots \oplus p_\mathbf{c}(J_s)\big) W (I+L).\end{aligned}$$

Observe that for each $r = 1, \dots, s$ the matrix $p_\mathbf{c}(J_r) = J_r + \sum_{k=0}^{n-1} c_k J_r^k$ is upper triangular with diagonal entries of the form $\eta_i = \lambda_i + \gamma_i$, where $\gamma_i = \sum_{k=0}^{n-1} c_k \lambda_i^k$. Since $\|\mathbf{c}\| < \epsilon$, $|\eta_i - \lambda_i| < \frac{1}{2}|\lambda_j - \lambda_\ell|$ for all $j \neq \ell$, as required.

Additionally, ϵ may be chosen small enough so that the superdiagonal of each J_r is arbitrarily close to that of $p_\mathbf{c}(J_r)$. For such an ϵ, $\mathrm{rank}(p_\mathbf{c}(J_r) - \eta_i I) = \mathrm{rank}(J_r - \lambda_i I)$ whenever λ_i is an eigenvalue corresponding to the Jordan block J_r. Thus, $\mathrm{gm}_A(\lambda_i) = \mathrm{gm}_M(\eta_i)$ for all $i = 1, \dots, t$. ∎

Let $S = [s_{ij}]$ be an $n \times n$ sign pattern and let $\mathcal{Q}^{\circledast}(S)$ be the set of real matrices $M = [m_{ij}]$ such that $m_{ij} = 0$ if $s_{ij} = 0$. Note that $\mathcal{Q}^{\circledast}(S)$ is a vector space. Let $\mathcal{M}^{n\times n}$ be the set of $n\times n$ real matrices M such that $\|M\| < 0.5$. Consider the functions $G : \mathcal{Q}^{\circledast}(S) \times \mathcal{M}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $H : \mathcal{Q}^{\circledast}(S) \times \mathcal{M}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ given by

$$G(B, L, \mathbf{c}) = (I+L)^{-1} p_\mathbf{c}(A+B)(I+L), \tag{4.1}$$

and

$$H(B, L, \mathbf{c}) = (I+L)^{-1} p_\mathbf{c}(A)(I+L) + B, \tag{4.2}$$

where $\mathbf{c} = (c_0, \dots, c_{n-1})$ and $p_\mathbf{c}(x) = x + \sum_{i=0}^{n-1} c_i x^i$.

For brevity, we often write $\dot{G}$ and $\dot{H}$ as shorthand for $\dot{G}_{(\mathrm{O},\mathrm{O},\mathbf{0})}$ and $\dot{H}_{(\mathrm{O},\mathrm{O},\mathbf{0})}$. Also, $V^\perp$ denotes the orthogonal complement of V.

PROPOSITION 4.5. *Let $A \in \mathbb{R}^{n\times n}$ have sign pattern S. Let G and H be given by (4.1) and (4.2) respectively, and $\mathcal{T}_A := \{AL - LA : L \in \mathbb{R}^{n\times n}\} + \mathrm{span}\{A^k : k = 0, \dots, n-1\}$. Then the following are equivalent:*

- *(i)* *A has the nSMP;*
- *(ii)* *$\mathcal{Q}^{\circledast}(S)^\perp \cap \mathcal{T}_A^\perp = \{\mathrm{O}\}$;*
- *(iii)* *$\mathrm{range}(\dot{G}_{(\mathrm{O},\mathrm{O},\mathbf{0})}) = \mathcal{Q}^{\circledast}(S) + \mathcal{T}_A = \mathbb{R}^{n\times n}$;*
- *(iv)* *$\mathrm{range}(\dot{H}_{(\mathrm{O},\mathrm{O},\mathbf{0})}) = \mathcal{Q}^{\circledast}(S) + \mathcal{T}_A = \mathbb{R}^{n\times n}$.*

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Proof. The directional derivatives of G at $(\mathrm{O},\mathrm{O},\mathbf{0})$ along $(B,\mathrm{O},\mathbf{0})$, $(\mathrm{O},L,\mathbf{0})$, and $(\mathrm{O},\mathrm{O},\mathbf{c})$, respectively, are

$$D_{(B,\mathrm{O},\mathbf{0})}G(\mathrm{O},\mathrm{O},\mathbf{0}) = \lim_{t\to 0}\frac{A+tB-A}{t} = B,$$

$$D_{(\mathrm{O},L,\mathbf{0})}G(\mathrm{O},\mathrm{O},\mathbf{0}) = \lim_{t\to 0}\frac{(I+tL)^{-1}A(I+tL)-A}{t} = AL-LA,$$

and

$$D_{(\mathrm{O},\mathrm{O},\mathbf{c})}G(\mathrm{O},\mathrm{O},\mathbf{0}) = \lim_{t\to 0}\frac{p_{t\mathbf{c}}(A)-A}{t} = \sum_{k=0}^{n-1} c_k A^k.$$

By Observation 2.1, $\mathrm{range}(\dot{G}) = \mathcal{Q}^{\circledast}(S) + \mathcal{T}_A$, where

$$\mathcal{T}_A = \{AL-LA : L\in\mathbb{R}^{n\times n}\} + \mathrm{span}\{A^k : k=0,\ldots,n-1\},$$

and so $\mathrm{range}(\dot{G}) = \mathbb{R}^{n\times n}$ if and only if $\mathcal{Q}^{\circledast}(S)^\perp \cap \mathcal{T}_A^\perp = \{\mathrm{O}\}$.

We now compute $\mathcal{T}_A^\perp$. Let $X\in\mathbb{R}^{n\times n}$ and suppose $0 = \langle AL-LA, X\rangle$ for every $L\in\mathbb{R}^{n\times n}$ and $0 = \left\langle A^k, X\right\rangle$ for $k=0,\ldots,n-1$. Then for every $L\in\mathbb{R}^{n\times n}$

$$0 = \mathrm{tr}(X^TAL - X^TLA) = \mathrm{tr}(LX^TA - LAX^T) = \left\langle X^TA - AX^T, L^T\right\rangle,$$

and $\mathrm{tr}(X^TA^k) = 0$ for $k=0,\ldots,n-1$. It follows that $X^TA - AX^T = \mathrm{O}$. Thus,

$$\mathcal{T}_A^\perp \subseteq \{X\in\mathbb{R}^{n\times n} : [A,X^T] = \mathrm{O}\}.$$

Now, let $X\in\mathbb{R}^{n\times n}$ satisfy $[A,X^T]=\mathrm{O}$. Then

$$\langle AL-LA, X\rangle = \left\langle X^TA - AX^T, L^T\right\rangle = \mathrm{O}.$$

Hence, $\mathcal{T}_A^\perp = \{X\in\mathbb{R}^{n\times n} : [A,X^T]=\mathrm{O} \text{ and } \mathrm{tr}(X^TA^k)=0 \text{ for } 0\le k<n\}$.

It is readily verified that $\mathcal{Q}^{\circledast}(S)^\perp = \{X\in\mathbb{R}^{n\times n} : A\circ X = \mathrm{O}\}$. Thus,

$$\mathcal{Q}^{\circledast}(S)^\perp \cap \mathcal{T}_A^\perp = \{X\in\mathbb{R}^{n\times n} : A\circ X = \mathrm{O}, [A,X^T]=\mathrm{O} \text{ and } \mathrm{tr}(X^TA^k)=0 \text{ for } 0\le k<n\},$$

and so A has the nSMP if and only if $\mathcal{Q}^{\circledast}(S)^\perp \cap \mathcal{T}_A^\perp = \{\mathrm{O}\}$.

Finally notice that the directional derivatives of H at $(\mathrm{O},\mathrm{O},\mathbf{0})$ along $(B,\mathrm{O},\mathbf{0})$, $(\mathrm{O},L,\mathbf{0})$, and $(\mathrm{O},\mathrm{O},\mathbf{c})$ are the same as those for G. Thus, we have $\mathrm{range}(\dot{G}) = \mathrm{range}(\dot{H})$. ◻

Lemma 4.6. *Let A be an $n\times n$ matrix, with sign pattern S, that has the nSMP. There is an $\epsilon>0$ such that for any M with $\|M-A\|<\epsilon$, there is a matrix $A'\in\mathcal{Q}(S)$ with $\mathbf{m}(A') = \mathbf{m}(M)$ and A' has the nSMP. Further, corresponding eigenvalues of A' and M will have the same geometric multiplicity.*

Proof. Let G be the function defined in Equation (4.1) and $\dot{G}$ be its derivative at 0. Since A has the nSMP, by Proposition 4.5, we know that $\mathrm{range}(\dot{G})$ is $\mathbb{R}^{n\times n}$. By Theorem 2.2, it follows that there is an open subset W' of $\mathbb{R}^{n\times n}$ containing A and a continuously differentiable function $T:\mathbb{R}^{n\times n}\to\mathcal{Q}^{\circledast}(S)\times\mathcal{M}^{n\times n}\times\mathbb{R}^n$ such that $T(A) = (\mathrm{O},\mathrm{O},\mathbf{0})$ and $G\circ T$ is the identity map on W' and hence G is surjective on $T(W')$. For any $\delta>0$, we can choose $\epsilon>0$ so that $\|M-A\|<\epsilon$ implies $\|T(M)\|<\delta$ for any $M\in W'$. This implies that for any M with $\|M-A\|<\epsilon$ there exists a point $(B',L',\mathbf{c}') = T(M)$ such that

$$G(B',L',\mathbf{c}') = (I+L')^{-1}p_{\mathbf{c}'}(A+B')(I+L') = M.$$

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Let $A' = A + B'$. Since $\|\mathbf{c}'\| < \|T(M)\|$, by choosing δ sufficiently small, Lemma 4.4 implies $\mathbf{m}(M) = \mathbf{m}(A')$ and corresponding eigenvalues have the same geometric multiplicity.

Since $\|T(M)\| \geq \|B'\| = \|A' - A\|$, by choosing δ sufficiently small, Lemma 4.3 implies that A' has the nSMP. ◻

Definition 4.7. The *filtered multiplicity list* for a real matrix A is a list consisting of the algebraic multiplicities of the real eigenvalues of A and the algebraic multiplicities for the conjugate pairs of eigenvalues of A. We write $(a_1, \dots, a_r; a_{r+1}, \dots, a_k)$ for the filtered multiplicity list of an $n \times n$ matrix A that has r real eigenvalues with multiplicities $a_1, \dots, a_r$ and $k - r$ complex pairs of eigenvalues with multiplicities $a_{r+1}, \dots, a_k$, and $(a_1 + \cdots + a_r) + 2(a_{r+1} + \cdots + a_k) = n$.

Definition 4.8. A *bifurcation* in a multiplicity list is a replacement of an integer $k > 1$ in the list with two positive integers that sum to k. A proper *refinement* of a multiplicity list $\mathbf{m}$ is a list obtained from $\mathbf{m}$ by one or more bifurcations. A *refinement* of the filtered multiplicity list $(a_1, a_2, \dots, a_r; a_{r+1}, \dots, a_k)$ is a filtered multiplicity list that consists of a refinement of the sequence $a_1, a_2, \dots, a_r$ and a refinement of the sequence $a_{r+1}, \dots, a_k$.

Example 4.9. Suppose that A is an 8×8 with spectrum $\{3, 3, 5 \pm 2i, 5 \pm 2i, 1 \pm i\}$. The filtered multiplicity list of A is $(2; 1, 2)$. There are three refinements of this filtered multiplicity list, namely $(1, 1; 1, 2)$, $(2; 1, 1, 1)$, and $(1, 1; 1, 1, 1)$.

The following is a bifurcation theorem for nSMP that mimics a corresponding bifurcation result for SMP [12, Corollary 3.4].

Theorem 4.10. *Let A be an $n \times n$ matrix, with sign pattern S, that has the nSMP and filtered multiplicity list $\mathbf{m}$. For any refinement $\mathbf{m}'$ of $\mathbf{m}$ there is a matrix $A' \in \mathcal{Q}(S)$ with the nSMP and $\mathbf{m}(A') = \mathbf{m}'$.*

Proof. It suffices to show that we can decrease any number larger than one in the filtered multiplicity list $\mathbf{m}(A)$. Suppose $A = BJB^{-1}$, where $J = J_1 \oplus \cdots \oplus J_s$ is the Jordan canonical form of A. If a large multiplicity is associated with one of the real eigenvalues λ of A, then consider a Jordan block associated with λ,

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}.$$

Consider the matrix formed by adding a sufficiently small $\delta > 0$ to one of the diagonal entries in this block,

$$M = B(J + \delta E_{kk})B^{-1} = A + \delta B E_{kk} B^{-1},$$

where E_{kk} is the matrix with 1 in the (k,k)-entry and zero elsewhere for an appropriate choice of k, aligning with J_λ. Note that δ can be chosen so that M satisfies $||M - A|| < \epsilon$ for any $\epsilon > 0$ and so applying Lemma 4.6 we know that there is a matrix $A' \in \mathcal{Q}(S)$ with filtered multiplicity list $\mathbf{m}(M)$. Thus, we can find a matrix with any refinement of the real eigenvalue multiplicities by shifting the appropriate number of diagonal entries by a sufficiently small $\delta > 0$.

Now consider a multiplicity in $\mathbf{m}(A)$ associated with a pair of complex conjugate eigenvalues $\lambda_1 = \overline{\lambda_2}$. We can write the following block structure associated with $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$:

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$$C_{\lambda_1,\lambda_2} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Furthermore, without any loss of generality, we can assume the Jordan form of A will include the block structure:

$$\begin{bmatrix} C_{\lambda_1,\lambda_2} & I & \mathrm{O} & \cdots & \mathrm{O} \\ \mathrm{O} & C_{\lambda_1,\lambda_2} & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathrm{O} \\ \vdots & & \ddots & C_{\lambda_1,\lambda_2} & I \\ \mathrm{O} & \cdots & \cdots & \mathrm{O} & C_{\lambda_1,\lambda_2} \end{bmatrix}.$$

Thus, we can obtain a bifurcation of any repeated complex pair of eigenvalues by letting

$$M = B(J + \delta(E_{kk} + E_{k+1,k+1}))B^{-1},$$

for the appropriate choice of k. As for the real eigenvalue multiplicities, we can find a matrix with any refinement of the complex eigenvalue multiplicities by shifting the appropriate number of diagonal blocks, using a sufficiently small $\delta > 0$. ∎

THEOREM 4.11. *The matrix $A = A_1 \oplus A_2$ has the nSMP if and only if both A_1 and A_2 have the nSMP and* $\operatorname{spec}(A_1) \cap \operatorname{spec}(A_2) = \varnothing$.

Proof. Suppose both A_1 and A_2 have the nSMP and $\operatorname{spec}(A_1) \cap \operatorname{spec}(A_2) = \varnothing$. Let $A = A_1 \oplus A_2$. In order to show that A also has the nSMP take X to be an appropriately sized block matrix defined by

$$X = \begin{bmatrix} X_1 & V \\ W & X_2 \end{bmatrix}.$$

Suppose X^T is in the centralizer of A. Then

$$\begin{aligned} [A, X^T] &= \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} \begin{bmatrix} X_1^T & W^T \\ V^T & X_2^T \end{bmatrix} - \begin{bmatrix} X_1^T & W^T \\ V^T & X_2^T \end{bmatrix} \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 X_1^T - X_1^T A_1 & A_1 W^T - W^T A_2 \\ A_2 V^T - V^T A_1 & A_2 X_2^T - X_2^T A_2 \end{bmatrix} = O. \end{aligned}$$

Thus, $A_1 W^T - W^T A_2 = O$ and $A_2 V^T - V^T A_1 = O$. Since the spectra of A_1 and A_2 are disjoint, by [17, Section 2.4, Exercise 9], it follows that $W = V = O$. Therefore, $X = X_1 \oplus X_2$ and $[A_1, X_1^T] = [A_2, X_2^T] = O$. Also, $A \circ X = O$ implies that $A_1 \circ X_1 = 0$ and $A_2 \circ X_2 = 0$.

Let d be the largest multiplicity of any root of $m_{A_1}(x) m_{A_2}(x)$. By Hermite interpolation, there exists a polynomial $p(x)$ such that

$$p(\lambda) = \lambda,\ p'(\lambda) = 1,\ p^{(2)}(\lambda) = 0,\ \ldots,\ p^{(d)}(\lambda) = 0,$$

for all $\lambda \in \operatorname{spec}(A_1)$ and

$$p(\lambda) = 0,\ p'(\lambda) = 0,\ p^{(2)}(\lambda) = 0,\ \ldots,\ p^{(d)}(\lambda) = 0,$$

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for all $\lambda \in \operatorname{spec}(A_2)$. As seen in [16, Theorem 11.1.1], if $1 \le t \le d$, for any $t \times t$ Jordan block,

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}, \qquad \text{then} \qquad p(J) = \begin{bmatrix} p(\lambda) & \frac{p'(\lambda)}{1!} & \cdots & \cdots & \frac{p^{(t)}(\lambda)}{t!} \\ 0 & p(\lambda) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{p'(\lambda)}{1!} \\ 0 & \cdots & \cdots & 0 & p(\lambda) \end{bmatrix}.$$

Thus, $p(A_1) = A_1$ and $p(A_2) = 0$. Thus, $p(A) = A_1 \oplus O$.

Similarly we can show that $O \oplus A_2$ is also a polynomial in A. Since $\operatorname{tr}(X^T A^k) = 0$ for all $k = 0, \ldots, n$ and each of $A_1 \oplus O$ and $O \oplus A_2$ are polynomials in A, it follows that $\operatorname{tr}((X_1^T \oplus X_2^T)(A_1 \oplus O)^k) = 0$ and $\operatorname{tr}((X_1^T \oplus X_2^T)(O \oplus A_2)^k) = 0$ for all $k = 0, \ldots, n$. This implies that $\operatorname{tr}(X_1^T A_1^k) = \operatorname{tr}(X_2^T A_2^k) = 0$ for all $k = 1, \ldots, n$. Since I is a polynomial in any nonzero matrix, we can also deduce that $\operatorname{tr}(X_1^T A_1^0) = 0$ and $\operatorname{tr}(X_2^T A_2^0) = 0$. Now since A_1 and A_2 both have the nSMP it follows that $X_1 = X_2 = O$. Thus, the only matrix that satisfies $A \circ X$, $[A, X^T] = O$ and $\operatorname{tr}(X^T A^k) = 0$ for $k = 0, \ldots, n$ is the zero matrix $X = O$, and A has the nSMP.

Conversely, suppose $A = A_1 \oplus A_2$ has the nSMP and $A_j \circ X_j = O$, $[X_j^T, A_j] = O$, and $\operatorname{tr}(X_j^T A_j) = 0$ for all $k = 1, \ldots, n$ and $j = 1, 2$. Then the matrix

$$X := \begin{bmatrix} X_1 & O \\ O & X_2 \end{bmatrix},$$

satisfies $A \circ X = O$, $[A, X^T] = O$ and $\operatorname{tr}(X^T A^k) = 0$ for all $k = 1, \ldots, n$. Since A has the nSMP, $X = O$. This implies that $X_1 = X_2 = O$ and therefore both A_1 and A_2 have the nSMP.

Suppose there exists some $\lambda \in \operatorname{spec}(A_1) \cap \operatorname{spec}(A_2)$. Choose right eigenvectors z_1, z_2 such that $A_1 z_1 = \lambda z_1$, $A_2 z_2 = \lambda z_2$ and left eigenvectors w_1^T, w_2^T such that $w_1^T A_1 = \lambda w_1^T$ and $w_2^T A_2 = \lambda w_2^T$. The matrix

$$Z^T := \begin{bmatrix} O & z_1 w_2^T \\ z_2 w_1^T & O \end{bmatrix},$$

satisfies $A \circ Z = O$, $[A, Z^T] = O$ and $\operatorname{tr}(Z^T A^k) = 0$ for $k = 0, \ldots, n$ and $Z \neq O$. Therefore, if A has the nSMP, then $\operatorname{spec}(A_1) \cap \operatorname{spec}(A_2) = \varnothing$. ∎

Corollary 4.12. *Suppose A_1 and A_2 have the nSMP and A_1 has full rank. If S_1 and S_2 are the sign patterns of A_1 and A_2, then $S_1 \oplus S_2$ allows the nSMP.*

Proof. If A_1 has full rank, then there exists a scalar $c > 0$ such that $\operatorname{spec}(cA_1) \cap \operatorname{spec}(A_2) = \varnothing$. The result then follows from Theorem 4.11. ∎

Recall that a matrix or pattern is *reducible* if it is permutationally similar to a block triangular matrix.

Corollary 4.13. *If a sign pattern S is reducible, but there is a matrix $A \in \mathcal{Q}(S)$ whose irreducible blocks have the nSMP, and do not share eigenvalues, then S allows the nSMP.*

Proof. The block diagonal matrix obtained from the irreducible blocks of A will satisfy the hypotheses of Theorem 4.11. Using permutation equivalence and Theorem 4.3, the pattern S allows the nSMP. ∎

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The definition of nSMP suggests that on average, the more nonzero entries a matrix has, the more likely the matrix will have the nSMP. However, as noted in Example 3.3, an $n \times n$ matrix can have as few as n nonzero entries and still have the nSMP. The next example shows that there are $n \times n$ matrices that have the nSMP with only $n-1$ nonzero entries.

EXAMPLE 4.14. A diagonal matrix with distinct diagonal entries and one diagonal entry zero will have the nSMP by Theorem 4.11.

THEOREM 4.15. *Let P be an $n \times n$ sign pattern, and $A \in \mathcal{Q}(P)$. If A has the nSMP, then for any superpattern S of P there exists a matrix $A' \in \mathcal{Q}(S)$ such that the algebraic and geometric multiplicities of eigenvalues of A' are the same as the respective multiplicities of the eigenvalues of A, and A' has the nSMP.*

Proof. Given the matrix $A \in \mathcal{Q}(P)$ with the nSMP consider the function $H : \mathcal{Q}^{\circledast}(S) \times \mathcal{M}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ defined by

$$H(B, L, \mathbf{c}) = (I + L)^{-1} p_{\mathbf{c}}(A)(I + L) + B,$$

since A has the nSMP we know that $\dot{H}$ is surjective by Proposition 4.5. By applying Theorem 2.2, it follows that there exists an $\epsilon > 0$ such that for any choice of M with $\|M - A\| < \epsilon$ there exist matrices $B' \in \mathcal{Q}^{\circledast}(P)$, $L' \in \mathcal{M}^{n\times n}$ and $\mathbf{c}' \in \mathbb{R}^n$ such that

$$M = (I + L')^{-1} p_{\mathbf{c}'}(A)(I + L') + B'.$$

Therefore, the matrix $A' = M - B'$ is similar to $p_{\mathbf{c}'}(A)$ and by Lemma 4.4, A' has the same list of algebraic and geometric multiplicities as A. By Theorem 2.2 for every $\delta > 0$ we can choose ϵ so that $\|M - A\| < \epsilon$ implies that $\|T(M)\| < \delta$. Now choose M to be a matrix with sign pattern S, some superpattern of sign pattern P. We can choose M such that $\|M - A\| < \epsilon$ and $\|T(M)\| < \delta$. Since $\|B'\| < \|T(M)\| < \delta$ and $B' \in \mathcal{Q}^{\circledast}(P)$, then $A' \in \mathcal{Q}(S)$ for any positive δ less than the magnitude of every nonzero entry of A. Furthermore, $\|A - A'\| \le \|A - M\| + \|M - A'\| < \epsilon + \delta$ and so we can choose $\epsilon > 0$ and $\delta > 0$ small enough so that A' is sufficiently close to A, implying that A' will have the nSMP by Theorem 4.3. ☐

The number of distinct eigenvalues of a matrix A is denoted by $q(A)$. The minimum number of distinct eigenvalues allowed by a sign pattern S, denoted $q(S)$, was studied in [3]. The following is an immediate consequence of Theorem 4.15.

COROLLARY 4.16. *If A has nSMP, then $q(S) \le q(A)$ for every superpattern S of the sign pattern of A.*

EXAMPLE 4.17. Let

$$A = \begin{bmatrix} \frac{5}{3} & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4}{3} & 0 & 0 & 0 \end{bmatrix}.$$

Then $\operatorname{spec}(A) = \left\{1, 1, \frac{4}{3} \pm \frac{2\sqrt{7}}{3}\right\}$ with characteristic polynomial $x^4 - \frac{14}{3}x^3 + 5x^2 - \frac{4}{3}$. Further, if $X \circ A = \mathrm{O}$, $AX^T - X^T A = \mathrm{O}$ and $\mathrm{tr}(\mathrm{X}^T\mathrm{A}^2) = 0$, then $X = \mathrm{O}$. Hence, A has nSMP. Since A has three distinct eigenvalues, Corollary 4.16 implies that $q(S) \le 3$ for each superpattern of the sign pattern of A.

5. Further examples and application. We first give a couple of examples demonstrating that having the nSMP is different then having the nSSP. Recall that a matrix A *has the nSSP* if $X = \mathrm{O}$ is the only matrix such that $A \circ X = \mathrm{O}$ and $[A, X^T] = \mathrm{O}$ (see e.g. [12]).

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EXAMPLE 5.1. While matrix A in Example 4.17 has the nSMP, it does not have the nSSP since if

$$Y = \begin{bmatrix} 0 & 0 & 4 & -12 \\ 0 & 0 & 0 & 4 \\ 3 & -5 & 15 & 0 \\ 0 & 3 & -14 & 15 \end{bmatrix},$$

then $AY^T = Y^T A$ and $A \circ Y = O$, but $Y \neq O$. Alternatively, [12, Theorem 5.4] can also be used to show A does not have the nSSP. This theorem states that if A has the nSSP, then every superpattern of the sign pattern of A allows a matrix similar to A. Consider

$$\hat{A} = \begin{bmatrix} a & 1 & 0 & d \\ 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{S} = \begin{bmatrix} + & + & 0 & + \\ 0 & + & + & 0 \\ 0 & 0 & 0 & + \\ + & 0 & 0 & 0 \end{bmatrix},$$

with $a, b, c,$ and d positive and $\hat{A} \in Q(S)$. Note that every matrix in $Q(A)$ is similar to a matrix in the form of $\hat{A}$ via a nonsingular diagonal matrix. Further, the pattern S is a superpattern of the sign pattern of A. However, the coefficient of x in the characteristic polynomial of $\hat{A}$ is $bcd \neq 0$ and so $\mathrm{spec}(\hat{A}) \neq \mathrm{spec}(\mathrm{A})$. Thus, no matrix in $Q(S)$ will be similar to A. Therefore, A does not have the nSSP by [12, Theorem 5.4].

EXAMPLE 5.2. The pattern

$$\mathcal{H}_6 = \begin{bmatrix} 0 & + & 0 \\ 0 & 0 & + \\ - & + & 0 \end{bmatrix},$$

was noted in [4] to not allow the nSSP. In fact, no matrix with all zeros on the diagonal allows the nSSP since $X = I$ satisfies $X \circ A = O$ and $[A, X^T] = O$. In [4], a derivative Jacobian method was used to show that each superpattern of $\mathcal{H}_6$ allows a repeated eigenvalue. The matrix

$$H_6 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 0 \end{bmatrix},$$

has the nSMP and $\mathrm{spec}(H_6) = \{-2, 1, 1\}$, thus Theorem 4.3 gives an alternate technique for showing that all superpatterns of $\mathcal{H}_6$ allow a repeated eigenvalue.

The *allow sequence* of an $n \times n$ sign pattern S is a binary vector q_{seq} of length n with ith entry equal to 1 if and only if S allows a matrix with i distinct eigenvalues. Initial results on allow sequences were developed in [4]. Theorem 4.10, the bifurcation theorem for nSMP, along with Theorem 4.15, implies that if a matrix A has the nSMP, then the ith entry of $q_{seq}(S)$ is 1 for all $i \geq q(A)$ for every superpattern S of the sign pattern of A.

EXAMPLE 5.3. Consider the realization A of the nonnegative loopless companion pattern $\mathcal{A}$ defined in [3, Example 2.17] with $q(A) = 2$. Using an analytic technique with a rank $n-1$ Jacobian, it was noted in [4, Example 3.6] that $q_{seq}(\hat{\mathcal{A}}) = \langle 0, 1, 1, \ldots, 1 \rangle$ or $q_{seq}(\hat{\mathcal{A}}) = \langle 1, 1, \ldots, 1 \rangle$ for every superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$. Here we show that $\mathcal{A}$ allows the nSMP as an alternate technique to obtain the same information about superpatterns of $\mathcal{A}$.

Let X be a matrix that satisfies $X \circ A = 0$, $[A, X^T] = 0$ and $\mathrm{tr}(X^T A^k) = 0$ for $k = 0, \ldots, n$. Since A is a companion matrix,

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$$(AX^T)_{i,j} = (X^T)_{i+1,j} \quad \text{for} \quad 1 \le \mathrm{i} \le \mathrm{n}-2 \quad \text{and} \quad \mathrm{i}+1 \le \mathrm{j} \le \mathrm{n}, \quad \text{and}$$
$$(X^TA)_{i,j} = (X^T)_{i,j-1} \quad \text{for} \quad 2 \le \mathrm{j} \le \mathrm{n} \quad \text{and} \quad 1 \le \mathrm{i} \le \mathrm{j}-1.$$

This can be used to show that the only matrix X^T in the commutator of A with $X \circ A = 0$ is a diagonal matrix $X = wI$ for some scalar w. However, with $X = wI$, $\operatorname{tr}(X^TA^2) = 2wA_{n,n-1}$. Thus, $w = 0$ if $\operatorname{tr}(X^TA^2) = 0$. Therefore, $X = 0$ and A has the nSMP and so $\mathcal{A}$ allows the nSMP.

6. Conclusions and open questions. We developed a new strong property that is useful in determining possible eigenvalue multiplicities of not-necessarily-symmetric sign patterns. A key to developing this property was carefully choosing multiplicity preserving functions, equations (4.1) and (4.2). We observe that while having the nSSP is sufficient for having the nSMP, it is not necessary.

The results developed here naturally raise a couple questions yet to be answered.

Question 6.1. The particular pattern in Example 3.3, which was shown to require the nSMP, is a sign pattern that requires distinct eigenvalues. Is requiring distinct eigenvalues a sufficient condition for requiring the nSMP? If so, this might provide some insight into the problem of determining patterns that require distinct eigenvalues (see e.g.[14]).

Question 6.2. Considering another extreme, when does a pattern that allows a single eigenvalue also allow the nSMP? An answer to this question would help characterize the sign patterns that have a full allow sequence (see [4]).

The above questions both entail determining sufficient conditions for a pattern to allow or require the nSMP. It would be interesting to develop other conditions, if not full characterizations, based on combinatorial matrix properties.

It is possible to automate the process of checking whether a given matrix has a specified strong property. In the Appendix, we have included *Sage* code [18] that will check whether a specified matrix has the nSMP. Note that this code can be adapted to check for the nSSP by deleting the loop that checks the trace conditions.

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Appendix: *Sage* code for checking the nSMP.

```
def nSMP(A):
# receives a square matrix A and determines if it has the nSMP property
# Returns a matrix X in the commutator that satisfies the trace conditions
# and a statement indicating whether or not A has the nSMP

n=A.nrows();
L=list(var( 'x%d' % i) for i in range(n^2));
X=matrix(n,n,L)
H=list();
for r in range(n):          # create a variable matrix X whose transpose
    for c in range(n):      # has a zero Hadamard product with A
        if A[r,c]!=0:
            H=H+[X[c,r]]    # H is a list of variables not in X
            X[c,r]=0
C=X*A-A*X;                  # calculate the commutator of A and X
F=C.list()+H                # F is list of equations that must be zero
for i in range(n):
    P=(X*A^i).trace();
    F=F+[P];                # include the trace conditions in F

for W in solve(F,L):        # solve the commutator and trace conditions
    for i in range(n^2):
        L[i]=L[i].subs(W)
X=matrix(n,n,L)             # insert solutions back into matrix
show(X.transpose())         # print a matrix in the commutator of A satisfying
                            # the trace conditions

if X==zero_matrix(n):       # if X=0, A has the nSMP
    return "A has the nSMP";
if X!=zero_matrix(n):       # if X!=0, A does not have the nSMP
    return "A does not have the nSMP";
```