THE FIELDS OF VALUES OF THE HEIGHT ZERO CHARACTERS

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Abstract. We determine what are the fields of values of the irreducible p-height zero characters of all finite groups for $p = 2$; we conjecture what they should be for odd primes, and reduce this statement to a problem on blocks of quasi-simple groups.

1. Introduction

Every Abelian number field can be realized as the field of values of a complex irreducible character of a finite group (see, for instance, Theorem 2.2 of [NT]). Motivated by the McKay conjecture and the McKay-Navarro conjecture [N2, Conjecture A], it is of great interest to characterize the fields of values of the irreducible characters of degree not divisible by a fixed prime p. This task was accomplished in [NT] for $p = 2$ following prior work on the fields of values of odd-degree characters in [ILNT]. For instance, the quadratic fields of values of the odd-degree irreducible characters of all finite groups are exactly the fields $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{d})$, where $1 \neq d \equiv 1 \pmod 4$ is a square-free integer.

The characters of degree not divisible by p constitute only a part of Brauer's p-height zero characters, namely those that lie in p-blocks of maximal defect. The McKay conjecture admits a version for Brauer p-blocks (the celebrated Alperin–McKay conjecture) where characters of degree not divisible by p are replaced by p-height zero characters, and this conjecture also admits a version taking into account

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the action of Galois automorphisms [N2, Conjecture B] (which is sometimes referred to as the Alperin-McKay-Navarro conjecture).

The main question now is: What are then the fields of values of the 2-height zero characters? As we have mentioned, $\mathbb{Q}(\sqrt{3})$, say, cannot be the field of values of an irreducible character of odd degree, but it is easy to find many 2-height zero characters having this field of values. (For instance, in a double cover of S_5.) However, $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{-2})$, say, do not appear to be the field of values of any 2-height zero character.

When studying fields of values of characters, character conductors are a fundamental invariant. If χ is a complex character of a group G and $\mathbb{Q}(\chi)$ is the smallest field extension of $\mathbb{Q}$ containing the values of χ, then we define $c(\chi)$, the conductor of χ, to be the smallest positive integer n such that $\mathbb{Q}(\chi)$ is contained in the n-th cyclotomic field $\mathbb{Q}_n = \mathbb{Q}(e^{2\pi i/n})$. If F is any subfield of $\mathbb{C}$, then we write $F(\chi) = \langle F, \mathbb{Q}(\chi)\rangle$.

The following is the main result of this paper. Its proof uses the Classification of Finite Simple Groups, together with the work of [BDR], and its refinement by [KL].

THEOREM A. *Let G be a finite group, and let χ be an irreducible complex character of 2-height zero of G. Write $c(\chi) = 2^a m$, where m is odd and $a \geq 0$. Then $\mathbb{Q}_{2^a} \subseteq \mathbb{Q}_m(\chi)$.*

In fact, the fields of values of the 2-height zero irreducible characters can be characterized in the following way. Let $\mathcal{F}_2$ be the set of abelian number fields F such that $\mathbb{Q}_n = \langle \mathbb{Q}_m, F\rangle$, where n is the conductor of the field F and m is the $2'$-part of n.

THEOREM B. *The set consisting of the fields of values of the 2-height zero irreducible complex characters of finite groups is exactly $\mathcal{F}_2$.*

As a consequence of Theorem B we obtain the following.

COROLLARY C. *Let F be a quadratic number field. Then $F = \mathbb{Q}(\chi)$ for some 2-height zero irreducible complex character χ of a finite group if and only if $F = \mathbb{Q}(\sqrt{d})$ for some odd square-free integer $d \neq 1$.*

In this context, it is natural to wonder what happens for odd primes. The fields of values of the characters of degree not divisible by p are conjectured to be precisely the abelian number fields F such that $[\mathbb{Q}_{p^a} : \mathbb{Q}_{p^a} \cap F]$ is not divisible by p in [NT, Conjecture C], a conjecture that does not seem to follow from the McKay-Navarro conjecture [N2, Conjecture A]. As happens in the case of characters of degree not divisible by p, we can only conjecture what the fields of values of the p-height zero characters should be for odd primes. This is Conjecture D below. The novelty is that we can show that the statement of Conjecture D does follow from the statement of the Alperin-McKay-Navarro conjecture. Both in the case of maximal defect p-blocks considered in [NT], and the general case of any p-blocks considered in the present paper, the fields of values of characters of various quasisimple groups of Lie type, especially for even-degree characters, are still not understood well enough to allow us to make further advances towards a complete proof of Conjecture 6.2, and hence of Conjecture D.

For any prime p, let $\mathcal{F}_p$ now be the set of abelian number fields F with conductor $n = p^a m$, where p does not divide m, such that the degree $|\mathbb{Q}_n : \langle \mathbb{Q}_m, F\rangle|$ is not divisible by p. Notice that the fields F whose p-part of the conductor p^a is such that $[\mathbb{Q}_{p^a} : \mathbb{Q}_{p^a} \cap F]$ is not divisible by p are a subclass contained in $\mathcal{F}_p$.

CONJECTURE D. *The set of fields of values of the p-height zero irreducible complex characters of finite groups is exactly $\mathcal{F}_p$.*

We show that any field in $\mathcal{F}_p$ is the field of values of a p-height zero character in Theorem 5.1, hence settling one of the containments in Conjecture D (and also reducing the proof of Theorem B to proving Theorem A). We reduce the verification of the other containment to quasi-simple groups in Theorem 6.3. We show that the statement of Conjecture D follows from the statement of the Alperin-McKay-Navarro conjecture [N2, Conjecture B] in Theorem 5.4.

A fundamental part of our work is devoted to showing that Theorem A is true for quasi-simple groups. We believe that this will be useful in the final verification of the Alperin-McKay-Navarro conjecture.

The layout of the paper is as follows. In Section 2 we prove some elementary properties of conductors. In Section 3, we study the relationship between conductors, height zero characters and normal subgroups, which will be crucial in the proof of the main reduction Theorem 6.3 in Section 6. In Section 4, we discuss how to use projective representations and character triples, taking into account field of values and height zero characters. In the short Section 5 we prove the easy containments in Theorem B and Conjecture D, and we also prove that Conjecture D is a consequence of the Alperin-McKay-Navarro conjecture. In Section 6 we reduce Conjecture D to quasisimple groups. In Section 7, we handle the quasisimple groups in the case of $p = 2$, thereby completing the proofs of Theorems A, B and Corollary C.

2. Conductors

Let us start by recording some elementary results on characters and conductors that we will frequently use. If G is a finite group, then $\mathrm{Irr}(G)$ is the set of the irreducible complex characters of G. For a positive integer k, we write $\zeta_k = e^{2\pi i/k}$, and $\mathbb{Q}_k = \mathbb{Q}(\zeta_k)$.

Recall that if ψ is a character of a finite group, then $\psi(g)$ is a sum of $o(g)$-th roots of unity for $g \in G$, and therefore the field of values $\mathbb{Q}(\psi)$, which is the smallest field containing $\psi(g)$ for all $g \in G$, is contained in $\mathbb{Q}_{|G|}$. The *conductor* $c(\psi)$ is the smallest positive integer n such that $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_n$. Therefore $c(\psi)$ divides $|G|$. Moreover, $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_m$ if and only if $c(\psi)$ divides m. If F is an Abelian number field, that is $F \subseteq \mathbb{C}$ and $F/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(F/\mathbb{Q})$ abelian, then the Kronecker-Weber theorem implies that $F \subseteq \mathbb{Q}_n$ for some n and $c(F)$, the conductor of F, is the smallest such n. By elementary Galois theory, recall that $c(\langle F_1, F_2\rangle)$ is the least common multiple of $c(F_1)$ and $c(F_2)$.

In this paper, if p is a prime and $n \geq 1$ is an integer, then n_p is the largest power of p dividing n, and $n_{p'} = n/n_p$. We call n_p the p-part of n and $n_{p'}$ the p'-part of n. For a fixed prime p, we are interested in the p-parts of conductors. If ψ is a character and $c(\psi)_p = 1$, then ψ is called *p-rational*. If $p = 2$, then ψ is either 2-rational or $c(\psi)_2 \geq 4$. Notice that if λ is a linear character then $c(\lambda) = o(\lambda)$ unless $o(\lambda)_2 = 2$ in which case $c(\lambda) = o(\lambda)/2$.

Lemma 2.1. *Let p be a prime. Suppose that $\chi \in \mathrm{Irr}(G)$, and write $c(\chi) = p^a m$, where m is not divisible by p and $a \geq 0$. If n is a positive integer not divisible by p and $f \geq 0$ is an integer with $\mathbb{Q}_{p^f} \subseteq \mathbb{Q}_{pn}(\chi)$, then $\mathbb{Q}_{p^f} \subseteq \mathbb{Q}_{pm}(\chi)$. Moreover $f \leq a$ unless possibly when $f = 1$ and $a = 0$.*

Proof. By replacing n by mn, we we may assume that m divides n. We may assume that $f \geq 1$. If $a = 0$ and $f = 1$ then $\mathbb{Q}_p \subseteq \mathbb{Q}_{pm}(\chi)$.

Hence we may assume that $a \geq 1$. Then $a \geq 2$ if $p = 2$. In either case $\mathbb{Q}_{p^f} \subseteq \mathbb{Q}_{pn}(\chi) \subseteq \mathbb{Q}_{p^a n}(\chi) = \mathbb{Q}_{p^a n}$ because m divides n, so $f \leq a$. If $p = 2$, we may also assume that $f \geq 2$, because otherwise the result is trivial.

Write $F = \mathbb{Q}_n$, $K = \mathbb{Q}_m$, $L = \mathbb{Q}_{p^a m}$ and $E = \langle F, L\rangle = \mathbb{Q}_{p^a n}$. We have that $F \cap L = K$. Let $J = \mathbb{Q}_{pm}(\chi)$, so that $K \subseteq J \subseteq L$. Let $M = \mathbb{Q}_{pn}(\chi) = \langle F, J\rangle$. Since $\mathbb{Q}_{p^f} \subseteq M \subseteq \mathbb{Q}_{p^a n}$, we have that $f \leq a$. Now, $\mathbb{Q}_{p^f} \subseteq M \cap L = J$, by Lemma 2.6(i) of [NT], for instance. □

Lemma 2.2. *Let p be a prime. Suppose that χ and ψ are characters of groups G and H. Suppose that $\mathbb{Q}_{pn}(\chi) = \mathbb{Q}_{pn}(\psi)$ for some n not divisible by p.*

(i) *If n divides m, then $\mathbb{Q}_{pm}(\chi) = \mathbb{Q}_{pm}(\psi)$.*
(ii) *If $p = 2$, or p is odd and $c(\chi)_p, c(\psi)_p \geq p$, then $c(\chi)_p = c(\psi)_p$.*

Proof. To prove part (i) just notice that

$$\mathbb{Q}_{pm}(\chi) = \mathbb{Q}(\zeta_p, \zeta_m, \chi) = \mathbb{Q}(\zeta_p, \zeta_n, \zeta_m, \chi) = \mathbb{Q}_{np}(\chi)(\zeta_m) = \mathbb{Q}_{np}(\psi)(\zeta_m) = \mathbb{Q}_{pm}(\psi).$$

To prove part (ii) notice that $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_{pn}(\chi) \subseteq \mathbb{Q}_{c(\chi)_p m}$ with $m = nc(\chi)_{p'}$. In particular $c(\psi)_p$ divides $c(\chi)_p$. By reversing the roles played by χ and ψ we obtain the result. □

3. Fields and Height Zero Characters

Our notation for blocks follows [N3]. We fix p a prime number. An irreducible character $\chi \in \mathrm{Irr}(G)$ lies in a unique Brauer p-block B, and we then write $\chi \in \mathrm{Irr}(B)$. If D is any defect group of B, then $\chi(1)_p = |G : D|_p p^h$, for a unique integer $h \geq 0$, called the *height* of χ. Recall that χ has *height zero* (or *p-height zero*) if $h = 0$. Since the prime p is fixed, we usually simply write that a character has height zero if it has p-height zero.

To reduce Theorem A and Conjecture D to quasisimple groups, working by induction, we need to study conductors of height zero characters and their behavior under restriction to normal subgroups. In this section, we first analize this for Clifford induction, which will allow our characters to be primitive. The remainder of the section deals with the case where we have a normal subgroup of index p. (This will take care of Steps 1 and 2 in the key Theorem 6.3.) This includes proving one of the containments in Conjecture D for characters in blocks with a normal defect group.

We will frequently use the following facts on height zero characters.

Theorem 3.1. *Let B be a p-block of a finite group G, and let $\chi \in \mathrm{Irr}(B)$ with height zero.*

(i) *If $\psi^G = \chi$, where $\psi \in \mathrm{Irr}(H)$ of some subgroup H of G, then ψ has height zero, and any defect group of the p-block of ψ is a defect group of B.*
(ii) *If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$ is under χ, then θ has height zero.*

Proof. Part (i) is Proposition 2.5(e) of [NS]. Part (ii) is due to M. Murai, and is Proposition 2.5(a) of [NS]. □

Theorem 3.2. *Let p be a prime, and suppose that $\chi \in \mathrm{Irr}(G)$ has p-height zero. Let $N \trianglelefteq G$, let $\theta \in \mathrm{Irr}(N)$ be under χ, and let $\psi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ over θ. Then ψ and θ have p-height zero. Also, $\mathbb{Q}_{pn}(\chi) = \mathbb{Q}_{pn}(\psi)$, where $n = |G|_{p'}$. Therefore, if $p = 2$ or p is odd and $c(\chi)_p \geq p$, then $c(\chi)_p = c(\psi)_p$.*

Proof. We have that ψ and θ have height zero by Theorem 3.1. We argue by induction on $|G : N|$. Since $\psi^G = \chi$, we have that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\psi)$. Let T^* be the semi-inertia group of θ in G consisting of the elements $g \in G$ for which there is some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ such that $\theta^g = \theta^\sigma$, as in Problem 3.9 of [N4]. Let $\eta = \psi^{T^*}$. Since $\eta^G = \chi$, then we have that η has height zero. Also, $\mathbb{Q}(\chi) = \mathbb{Q}(\eta)$, $T \trianglelefteq T^*$ and T^*/T is abelian.

If $T^* < G$, by induction, we have that $\mathbb{Q}_{p|T^*|_{p'}}(\eta) = \mathbb{Q}_{p|T^*|_{p'}}(\psi)$. By Lemma 2.2(i), we have that $\mathbb{Q}_{pn}(\psi) = \mathbb{Q}_{pn}(\eta) = \mathbb{Q}_{pn}(\chi)$, and we are done. Thus we may assume that $T^* = G$. Then $T \trianglelefteq G$, and G/T is abelian. By induction, we may assume that $T = N$, and $\psi = \theta$. If M is a maximal normal subgroup of G with $N < M < G$, then again by induction (and using Lemma 2.2(i)), $\mathbb{Q}_{pn}(\theta^M) = \mathbb{Q}_{pn}(\theta)$. Hence it is enough to prove the statement in the case where G/N has prime order.

Now G/N has prime order. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{pn}(\theta)/\mathbb{Q}_{pn}(\chi))$. We want to show σ is trivial. Assume that $\sigma \neq 1$. Notice that σ is a p-element, since $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{pn})$ is a p-group. By Clifford's theorem, we have that $\theta^\sigma = \theta^g$ for some $g \in G$. Also $\theta^\sigma \neq \theta$ because σ is not trivial. In particular $\langle gN \rangle = G/N$ is a group of order p. Let b be the block of θ. Since σ fixes p'-roots of unity, it follows that $b^\sigma = b$. (Use, for instance, Theorem 3.19 of [N3].) Then $b^g = b$ and b is G-invariant. Then we apply Corollary 9.6 and Corollary 9.18 of [N3], and conclude that θ is G-invariant, a contradiction.

The second part of the statement follows from Lemma 2.2. Notice if p is odd then $c(\chi)_p \geq p$ implies that $c(\psi)_p \geq 1$ because if $c(\psi)_p = 1$, then ψ is p-rational, and $\chi = \psi^G$ is also p-rational. □

Notice that the hypothesis on the odd case of the second statement of the above theorem is necessary: if $p = 3$, $\chi \in \mathrm{Irr}(\mathsf{S}_3)$ has degree 2 and $\psi \in \mathrm{Irr}(N)$ is under χ with $|N| = 3$, then $c(\chi)_3 = 1$ but $c(\psi)_3 = 3$.

Corollary 3.3. *Let $N \trianglelefteq G$, let $\chi \in \mathrm{Irr}(G)$ of height zero in its p-block, and let θ be an irreducible constituent of χ_N. If p is odd, assume that $c(\chi)_p \geq p$. Then $c(\theta)_p \leq c(\chi)_p$. In particular, if $p = 2$ and χ is 2-rational, then θ is 2-rational.*

Proof. By Theorem 3.2, we may assume that θ is G-invariant. Then $\chi_N = e\theta$, $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\chi)$, and the statement is clear. □

Of course, Corollary 3.3 is about height zero characters. (Consider, for instance, $G = \mathsf{D}_8$, $\chi \in \mathrm{Irr}(G)$ of degree 2, and N a cyclic subgroup of G of order 4.)

Next we prove the normal defect case of Theorem A. We first need a lemma. Suppose that $\chi \in \mathrm{Irr}(G)$ lies in a block B with defect group $D \trianglelefteq G$. Let $C = \mathbf{C}_G(D)$. Let b a block of CD covered by B. By Corollary 9.21 of [N3], we have that $B = b^G$ is the only block of G covering b. By Theorem 9.26 of [N3], we have that b has defect group D. By Theorem 9.12 of [N3], there is a unique irreducible character $\theta \in \mathrm{Irr}(b)$ such that $D \subseteq \ker(\theta)$. This character has defect zero, viewed as a character of CD/D, and it is called the canonical character of b, which is uniquely defined up to G-conjugacy. The irreducible characters of b are described in Theorem 9.12 of [N3].

Lemma 3.4. *Suppose that $\chi \in \mathrm{Irr}(G)$ has height zero and belongs to a p-block B with a normal defect group D. Let $C = \mathbf{C}_G(D)$ and $Z = \mathbf{Z}(D)$. If χ_{CD} is homogeneous, then χ_D is homogeneous, the canonical character $\theta \in \mathrm{Irr}(CD/D)$ of B is G-invariant, and G/CD is a p'-group. If λ is the irreducible constituent of χ_D, then λ is linear and $c(\chi)_p = c(\lambda)$.*

Proof. Let $\eta \in \mathrm{Irr}(CD)$ be the unique irreducible constituent of χ_{CD}. By Theorem 3.1, we have that η has height zero. By Theorem 9.12 of [N3], and using its notation, we know that we can write $\eta = \theta_\lambda$, where $\lambda \in \mathrm{Irr}(D)$ is linear and $\theta \in \mathrm{Irr}(CD/D)$ has defect zero. Moreover, $\eta(x) = 0$ if $x_p \notin D$, and $\eta(x) = \theta(x_{p'})\lambda(x_p)$ if $x_p \in D$. Thus $\eta_D = \theta(1)\lambda$. Since χ_{CD} is homogeneous, then it follows that χ_D is homogeneous. Thus λ and η are G-invariant. We claim that $\theta \in \mathrm{Irr}(CD/D)$ is G-invariant. View θ as a character of C/Z. Let $x \in C$ and $g \in G$. Since $\theta \in \mathrm{Irr}(C/Z)$ has p-defect zero, if xZ is p-singular, then $\theta(x) = 0 = \theta(x^g)$ because x^gZ is also p-singular. If xZ is p-regular, then x^gZ is also p-regular, $xZ = x_{p'}Z$ and $x^gZ = (x_{p'})^gZ$. Since η is G-invariant, we have that $\theta(x_{p'})\lambda(x_p) = \eta(x) = \eta(x^g) = \theta((x_{p'})^g)\lambda((x_{p'})^g)$. Since λ is G-invariant and linear, we deduce that $\theta(x_{p'}) = \theta((x_{p'})^g)$. Now $\theta(x) = \theta(x_{p'}) = \theta((x_{p'})^g) = \theta(x^g)$ because Z is contained in the kernel of θ, and we deduce that θ is G-invariant. Therefore, we have that G/CD has order coprime to p by Theorem 9.22 of [N3].

Since $\chi_{CD} = e\eta$ and $\eta_D = \theta(1)\lambda$, we have that $\mathbb{Q}_{c(\lambda)} \subseteq \mathbb{Q}(\eta)$. Now, θ is p-rational, because it is a defect zero character, and therefore, $\mathbb{Q}(\theta) \subseteq \mathbb{Q}_m$, where m is a p'-number. Then, using the formula for the values of η, we have that $\mathbb{Q}_{c(\lambda)} \subseteq \mathbb{Q}(\eta) \subseteq \mathbb{Q}_{c(\lambda)m}$. Therefore, $c(\lambda)$ divides $c(\eta)$ which divides $c(\lambda)m$, implying that $c(\eta)_p = c(\lambda)$. We apply Lemma 4.2(ii) of [NT], and we get that $c(\chi)_2 = c(\eta)_2$ if $p = 2$ and $c(\chi)_p = c(\eta)_p$ if p is odd and $c(\lambda) = c(\eta)_p > 1$. If p is odd and $c(\lambda) = c(\eta)_p = 1$, then $\lambda = 1_D$. Therefore $\eta = \theta$, and χ has p-defect zero, so χ is p-rational and $c(\chi)_p = 1$. In any case, we conclude that $c(\chi)_p = c(\lambda)$. □

Next we prove Theorem A, and one of the containments of Conjecture D, in the case of blocks with a normal defect group.

Lemma 3.5. *Let $\chi \in \mathrm{Irr}(B)$ of height zero, where B is a p-block with a normal defect group D. Write $c(\chi) = p^a m$, where m is not divisible by p. Then $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm}(\chi)$.*

Proof. First notice that we may assume that $a \geq 2$ as otherwise the result trivially holds. We argue by induction on $|G|$.

Write $C = \mathbf{C}_G(D)$. Let $\eta \in \mathrm{Irr}(CD)$ be an irreducible constituent of χ_{CD}. Let $T = G_\eta$ be the stabilizer of η in G and $\psi \in \mathrm{Irr}(T|\eta)$ be the Clifford correspondent of χ over η. By Theorem 3.1, we know that ψ has height zero and that D is a defect group of its block. By Theorem 3.2, we have that $\mathbb{Q}_{pn}(\chi) = \mathbb{Q}_{pn}(\psi)$ where $n = |G|_{p'}$ and $c(\chi)_p = c(\psi)_p$. Assume that $T < G$. By induction $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pc(\psi)_{p'}}(\psi) \subseteq \mathbb{Q}_{pn}(\psi) = \mathbb{Q}_{pn}(\chi)$. By Lemma 2.1 we conclude that $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm}(\chi)$.

Hence $T = G$ and we are under the hypotheses of Lemma 3.4. If $\lambda \in \mathrm{Irr}(D)$ lies under χ, then $p^a = c(\chi)_p = c(\lambda)$. Notice that $\chi_D = f\lambda$ and hence $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}(\chi) \subseteq \mathbb{Q}_{pm}(\chi)$. □

The next results are key to understanding the statement of Conjecture D when the group possesses a normal subgroup of index p (a fundamental step in our reduction theorem).

Lemma 3.6. *Suppose that $N \trianglelefteq G$, G/N is a p-group, and b is a G-invariant p-block of N covered by a block B of G with defect group D. Suppose that $D_0 = D \cap N \trianglelefteq G$. Then $G = DN$ and b has a G-invariant height zero p-rational irreducible character.*

Proof. By Corollary 9.6 of [N3], B is the unique p-block covering b. Then Theorem 9.17 of [N3] implies that $G = ND$ and D_0 is the unique defect group of b. Let $C = \mathbf{C}_N(D_0)$. Notice that $CD_0 \trianglelefteq G$. By the Fong-Reynolds correspondence, Theorem 9.14 of [N3], we can find e a block of CD_0 covered by b such that the block b_T of $T = G_e$, the stabilizer of e in G, inducing B and covering e has defect group D. Notice that e has defect group D_0. Since b is G-invariant, notice that $TN = G$, b_T covers a unique block f of $N_e = N \cap T$ that induces b and covers e. By induction and the Fong-Reynolds correspondence, we may assume that e is G-invariant. Then we have that N/CD_0 is a p'-group, by Theorem 9.22 of [N3]. Since e is G-invariant, we have that the canonical character $\theta \in \mathrm{Irr}(CD_0/D_0)$ of e is G-invariant. By Theorem 13.31 of [Is], some irreducible constituent ξ of θ^N is D-invariant. Thus ξ is G-invariant. Since b is the only block of N that covers e, we have that $\xi \in \mathrm{Irr}(b)$. Also, ξ is p-rational, because it has defect zero considered as a character of N/D_0. It also has height zero because θ has height zero and N/CD_0 is a p'-group. □

Lemma 3.7. *Suppose that $N \trianglelefteq G$, G/N is a cyclic p-group, and $\theta \in \mathrm{Irr}(N)$ is G-invariant of p-height zero. Then every $\chi \in \mathrm{Irr}(G|\theta)$ has p-height zero. Also, if D is a defect group of the block of χ, then $DN = G$ and $D \cap N$ is a defect group of the block of θ.*

Proof. Let b be the block of θ. Let B be the unique p-block of G covering b by Corollary 9.6 of [N3]. Let $\chi \in \mathrm{Irr}(G|\theta)$, so that $\chi \in \mathrm{Irr}(B)$. Since b is G-invariant, we have that $G = DN$, where D is a defect group of B, and $D_0 = D \cap N$ is a defect group of b by Theorem 9.17 of [N3]. We have that $\chi_N = \theta$ because G/N is cyclic and θ is G-invariant (using Theorem 5.1 of [N4] and the Gallagher correspondence Corollary 1.23 of [N4]). Then $\chi(1)_p = \theta(1)_p = |N : D_0| = |G : D|$. Thus χ has height zero. □

Lemma 3.8. *Suppose that $N \trianglelefteq G$ and G/N is a p-group. Let $\theta \in \mathrm{Irr}(N)$ be of p-height zero and G-invariant. Let $n = |G|_{p'}$. Let D_0 be a defect group of the block of θ, let $H = \mathbf{N}_G(D_0)$. Then there exists an H-invariant $\varphi \in \mathrm{Irr}(N \cap H)$ of p-height zero such that $[\theta_{H\cap N}, \varphi] \not\equiv 0 \bmod p$, and $\mathbb{Q}_{pn}(\varphi) \subseteq \mathbb{Q}_{pn}(\theta)$.*

Proof. Let b be the p-block of θ and let B be the only p-block of G covering b. Since b is G-invariant, we have that $G = DN$, where D is a defect group of B, and $D_0 = D \cap N$ is a defect group of B_0, by Theorem 9.17 of [N3].

Since $D \subseteq H$, note that $G = HN$. Let $M = H \cap N = \mathbf{N}_N(D_0)$. Then $H = MD$. Let e be the Brauer correspondent block of M (with defect group D_0) inducing b. By the Harris–Knörr Theorem [N3, Theorem 9.28], there is a unique block E of H covering e that induces B. This block E has defect group D.

Let $\mathcal{U} = \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{pn}(\theta))$. Notice that $\mathcal{U} \leq \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{pn})$ which is a p-group. We must then work to show that e has a $D \times \mathcal{U}$-invariant height zero character φ with $[\theta_M, \varphi] \not\equiv 0 \bmod p$. We will use the ˜ construction as in [N3, page 27] and the fact that e possesses an irreducible character ψ which is H-invariant and p-rational by Lemma 3.6.

Let $\delta = \theta_M$. Then $\tilde{\delta}$ defined as $\tilde{\delta}(x) = |M|_p\delta(x)$, if $x \in M$ is p-regular, and 0, otherwise, is a generalized character of M by Lemma 2.15 of [N3]. We can write

$$\tilde{\delta} = \sum_{\xi \in \mathrm{Irr}(M)} [\delta, \xi]\tilde{\xi}\,.$$

By Lemma 6.5.(b) of [N3], we have that

$$\frac{[\tilde{\delta}, \psi]}{\psi(1)} \not\equiv 0 \bmod \mathcal{P}\,,$$

where $\mathcal{P}$ is the maximal ideal of $\mathbf{R}_I$ the localization of the ring of algebraic integers $\mathbf{R}$ at a maximal ideal I containing p (see [N3, page 16]). Therefore

$$\Lambda = \sum_{\xi \in \mathrm{Irr}(M)} [\delta, \xi]\frac{[\tilde{\xi}, \psi]}{\psi(1)} \not\equiv 0 \bmod \mathcal{P}\,.$$

Note that $[\tilde{\xi}, \psi] = [\xi, \tilde{\psi}]$ whenever $\xi \in \mathrm{Irr}(M)$. By Lemma 3.20 of [N3], recall that $[\xi, \tilde{\psi}] = 0$ if ξ is not in e, so that

$$\Lambda = \sum_{\xi \in \mathrm{Irr}(e)} [\delta, \xi]\frac{[\tilde{\xi}, \psi]}{\psi(1)} \not\equiv 0 \bmod \mathcal{P}\,.$$

By Lemma 3.22(a) of [N3] $\frac{[\tilde{\xi}, \psi]}{\psi(1)} \in \mathbf{R}_I \cap \mathbb{Q}$ for every $\xi \in \mathrm{Irr}(e)$, so $\nu(\frac{[\tilde{\xi}, \psi]}{\psi(1)}) \geq 0$, where ν is the valuation function defined in [N3, page 64]. By Theorem 3.24 of [N3], we have that $\nu(\frac{[\tilde{\xi}, \psi]}{\psi(1)}) = 0$ if, and only if, ξ has height zero in the p-block e ($\xi \in \mathrm{Irr}_0(e)$). By Lemma 3.21 of [N3] we have that $\frac{[\tilde{\xi}, \psi]}{\psi(1)} \in \mathcal{P}$ whenever ξ does not have height zero in e. Hence

$$\Lambda \equiv \sum_{\xi \in \mathrm{Irr}_0(e)} [\delta, \xi]\frac{[\tilde{\xi}, \psi]}{\psi(1)} \not\equiv 0 \bmod \mathcal{P}\,.$$

Consider $\Omega = \{\xi \in \mathrm{Irr}_0(e) \mid [\delta, \xi] \not\equiv 0 \bmod\ p\}$. We have that

$$\Lambda \equiv \sum_{\xi \in \Omega} [\delta, \xi]\frac{[\tilde{\xi}, \psi]}{\psi(1)} \not\equiv 0 \bmod \mathcal{P}\,.$$

The p-group $D \times \mathcal{U}$ acts on Ω. Let $\Omega = \Omega_{\xi_1} \cup \ldots \cup \Omega_{\xi_r}$ be the orbit decomposition of Ω. Then, given that δ and ψ are $D \times \mathcal{U}$-invariant, we have that

$$\Lambda \equiv \sum_{i=1}^{r} |\Omega_{\xi_i}|[\delta, \xi_i]\frac{[\tilde{\xi}_i, \psi]}{\psi(1)} \not\equiv 0 \bmod \mathcal{P}\,.$$

In particular there is some $D \times \mathcal{U}$-invariant $\varphi \in \Omega$. The D-invariance of φ implies that φ is H-invariant and the $\mathcal{U}$-invariance of φ implies that $\mathbb{Q}_{pn}(\varphi) \subseteq \mathbb{Q}_{pn}(\theta)$. □

The following is a consequence of an argument of J. Thompson. (We refer the reader to Theorem 6.9 of [N4].)

Lemma 3.9. *Suppose that $N \trianglelefteq G$ and G/N is a p-group, and let $H \leq G$ such that $G = NH$. Write $M = N \cap H$. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant, and let $\varphi \in \mathrm{Irr}(M)$ be H-invariant such that $[\theta_M, \varphi] \not\equiv 0$ mod p.*

(i) *Suppose that $\xi \in \mathrm{Irr}(H)$ extends φ. Then there is an extension $\chi \in \mathrm{Irr}(G)$ of θ such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\xi, \theta)$.*

(ii) *Suppose that $\chi \in \mathrm{Irr}(G)$ extends θ. Then there is an extension $\xi \in \mathrm{Irr}(H)$ of φ such that $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\chi, \varphi)$.*

(iii) *Suppose that G/N has order p. Then $\mathbb{Q}_p(\chi, \varphi) = \mathbb{Q}_p(\theta, \xi)$ for every $\chi \in \mathrm{Irr}(G|\theta)$ and $\xi \in \mathrm{Irr}(H|\varphi)$.*

Proof. Write $m = |G|$. In order to prove part (i), let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q}(\xi, \theta))$. Since $\xi^\sigma = \xi$, in particular $\varphi^\sigma = \varphi$. By Theorem 6.9 of [N4], there is a unique extension $\chi \in \mathrm{Irr}(G)$ of θ such that $\chi_H = \Psi\xi + \Delta$, where Δ is a character of H or zero, with $[\Delta_M, \varphi] = 0$ and Ψ is a character of H/M with trivial determinant. Now, χ^σ is an extension of $\theta = \theta^\sigma$ and $(\chi^\sigma)_H = \Psi^\sigma\xi + \Delta^\sigma$, where $[(\Delta^\sigma)_M, \varphi] = 0$ and Ψ^σ is a character of H/M with trivial determinant. By the uniqueness of χ, we get that $\chi = \chi^\sigma$. Part (ii) is proved in the same way.

We now prove part (iii). We have that θ extends to G by Theorem 5.1 of [N4]. In fact, every character in $\mathrm{Irr}(G|\theta)$ is an extension of θ by the Gallagher correspondence. Let $\chi \in \mathrm{Irr}(G|\theta)$. By part (ii), there is an extension $\xi \in \mathrm{Irr}(H)$ of φ such that $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\chi, \varphi)$. By part (i), there is an extension $\chi' \in \mathrm{Irr}(G)$ of θ such that $\mathbb{Q}(\chi') \subseteq \mathbb{Q}(\xi, \theta)$. Since $\chi' = \lambda\chi$ for some $\lambda \in \mathrm{Irr}(G/N)$, we have that $\mathbb{Q}_p(\chi) = \mathbb{Q}_p(\chi')$. Since $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\chi)$ and $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\xi)$, part (c) follows. $\square$

In order to treat later the case where N is a normal subgroup of G of index p, we need to extend Lemma 3.5, and prove the statement of Conjecture D in a slightly more general case than the normal defect group case.

Lemma 3.10. *Suppose that $N \trianglelefteq G$ and G/N has order p. Let $n = |G|_{p'}$. Let $\chi \in \mathrm{Irr}(G)$ have p-height zero. Suppose that $\chi_N = \theta \in \mathrm{Irr}(N)$ and the defect group D_0 of the block of θ is normal in G. Then $\mathbb{Q}_{c(\chi)_p} \subseteq \mathbb{Q}_{pn}(\chi)$.*

Proof. Write $p^a = c(\chi)_p$. We may assume that $a \geq 2$ as the statement is trivially satisfied otherwise. We argue by induction on $|G|$.

Write $K = \mathbf{C}_N(D_0)D_0 \trianglelefteq G$. Let η be an irreducible constituent of χ_K and $\psi \in \mathrm{Irr}(G_\eta|\eta)$ be the Clifford correspondent of χ. Using Theorem 3.2 and induction, we may assume that η is G-invariant. In particular, $\theta_K = \chi_K$ is homogeneous. By Lemma 3.4, $\chi_{D_0} = \theta_{D_0}$ is homogeneous. Write $\theta_{D_0} = \theta(1)\lambda$, where $\lambda \in \mathrm{Irr}(D_0)$ is linear and G-invariant.

Let D be a defect group of the block of χ. We have that $G = ND$ and $D_0 = N \cap D$, using for instance Lemma 3.7. Let $H = \mathbf{N}_G(D)$ and $C = \mathbf{N}_N(D) = N \cap H$. If $H = G$

then χ lies in a block of normal defect, and we are done by Lemma 3.5. Hence we may assume that $H < G$. By Theorem A of [NS], there are a block b' of C with defect group D_0 and a D-invariant character θ' in b' satisfying that $[\theta_C, \theta'] \equiv \pm 1 \bmod p$. Since λ is G-invariant θ' is the unique irreducible constituent of θ_C such that $\theta'(1)_p = |C : D_0|_p$, and $[\theta_C, \theta'] \not\equiv 0 \bmod p$. The block B' of $H = CD$ covering θ' has defect group D, the block of θ' has defect group D_0, and θ' has height zero.

Given $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_n)$, we have that $b^\sigma = b$ and $(b')^\sigma = b'$ because σ fixes p'-roots of unity. Moreover $(\theta^\sigma)' = (\theta')^\sigma$ under the canonical correspondence given by Theorem A of [NS] (because of part (c) in that statement, noticing that λ^σ is also G-invariant). This implies that $\mathbb{Q}_n(\theta') = \mathbb{Q}_n(\theta)$ by elementary Galois theory.

Let $\xi \in \mathrm{Irr}(H|\theta')$. Then ξ extends θ' and, by Lemma 3.7, notice that ξ has height zero. By Lemma 3.9(iii), we have that $\mathbb{Q}_p(\chi, \theta') = \mathbb{Q}_p(\theta, \xi)$. Since $\mathbb{Q}_n(\theta) = \mathbb{Q}_n(\theta')$ we have that $\mathbb{Q}_{pn}(\chi, \theta) = \mathbb{Q}_{pn}(\chi, \theta') = \mathbb{Q}_{pn}(\xi, \theta) = \mathbb{Q}_{pn}(\xi, \theta')$. We easily deduce that $\mathbb{Q}_{pn}(\chi) = \mathbb{Q}_{pn}(\xi)$ using that $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\chi)$ and $\mathbb{Q}(\theta') \subseteq \mathbb{Q}(\xi)$. We want to apply Lemma 2.2(ii). If $c(\xi)_p = 1$ then $\mathbb{Q}(\xi) \subseteq \mathbb{Q}_n$ and consequently $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{pn}(\chi) \subseteq \mathbb{Q}_{pn}$, but this is impossible as $c(\chi)_p \geq p^2$. Hence $c(\chi)_p \geq p$ and by Lemma 2.2(ii) $c(\chi)_p = c(\xi)_p$. Recall that $H < G$. Write $k = |H|_{p'}$. By induction, $\mathbb{Q}_{c(\chi)_p} = \mathbb{Q}_{c(\xi)_p} \subseteq \mathbb{Q}_{pk}(\xi) \subseteq \mathbb{Q}_{pn}(\xi) = \mathbb{Q}_{pn}(\chi)$, then we are done. $\square$

4. Character Triples and Fields

The goal of this section is to prove Theorem 4.3. This will allow us, in the proof of the key Theorem 6.3, to assume that a certain maximal normal subgroup with non-abelian simple group quotient is central, therefore bringing quasisimple groups into the problem.

If $\chi \in \mathrm{Irr}(G)$ lies in a p-block B, we denote $h(\chi)$ the p-height of χ (we will sometimes just refer to $h(\chi)$ as the height of χ). We remind the reader that if $N \subseteq \ker(\chi)$, then the height of χ as a character of G and as a character of G/N can be different. For instance, the character of degree 2 of S_3 has 2-height zero, but as a character of S_4 has 2-height 1. The next result clarifies this situation.

Lemma 4.1. *Suppose that $\chi \in \mathrm{Irr}(B)$, where B is a p-block of G with defect group P. Suppose that $K \subseteq \ker(\chi)$ and let $\bar{\chi} \in \mathrm{Irr}(G/K)$ the character χ viewed as a character of G/K. Let $\bar{B}$ be the p-block of G/K containing $\bar{\chi}$.*

(i) *There is a defect group $\bar{D}$ of $\bar{B}$ such that $\bar{D} \leq PK/K$.*
(ii) *We have that*

$$p^{h(\chi)} = \frac{|PK/K|}{|\bar{D}|} p^{h(\bar{\chi})}.$$

In particular, $h(\chi) \geq h(\bar{\chi})$ and if $h(\chi) = 0$ then $PK/K = \bar{D}$ and $h(\bar{\chi}) = 0$.

(iii) *If $K \subseteq \mathbf{Z}(G)$, then $PK/K = \bar{D}$ and $h(\chi) = h(\bar{\chi})$.*

Proof. The first part is Theorem 9.9 of [N3]. Since χ lies over 1_K, it follows that B covers the principal block of K, by Theorem 9.2 of [N3]. By Theorem 9.26 of [N3], we have that $P \cap K \in \mathrm{Syl}_p(K)$, so $|K : K \cap P|_p = 1$. Now,

$$\chi(1)_p = \bar{\chi}(1)_p = \frac{|G/K|_p}{|PK/K|}\frac{|PK/K|}{|\bar{D}|}p^{h(\bar{\chi})} = \frac{|G|_p}{|P|}\frac{|PK/K|}{|\bar{D}|}p^{h(\bar{\chi})}\,,$$

and we use the definition of $h(\chi)$. The third part follows from Lemma 2.2 of [R]. $\square$

Lemma 4.2. *Suppose that G^* is a finite group, $N, Z \trianglelefteq G^*$ such that $N \cap Z = 1$, where $Z \subseteq \mathbf{Z}(G^*)$. Let $N^* = N \times Z$. Let $\theta \in \mathrm{Irr}(N)$ be G^*-invariant, and $\lambda \in \mathrm{Irr}(Z)$. Let $\theta^* = \theta \times 1_Z$, $\lambda^* = 1_N \times \lambda$, and assume that $(\lambda^*)^{-1}\theta^*$ extends to some $\tau \in \mathrm{Irr}(G^*)$. Then the map $\chi^* \mapsto \chi^*\tau$ defines a character triple isomorphism $(G^*, N^*, \lambda^*) \to (G^*, N^*, \theta^*)$. Let $\chi^* \in \mathrm{Irr}(G^*|\lambda^*)$. If $\chi = \chi^*\tau$ has height zero in G^*, then χ^* has height zero in G^*/N.*

Proof. We have that $\tau_N = \theta$. The fact that $\chi^* \mapsto \chi^*\tau$ defines a character triple isomorphism follows from Lemma 11.27 of [Is].

Let $\chi^* \in \mathrm{Irr}(G^*|\lambda^*)$. Then $N \subseteq \ker(\chi^*)$ and we can see χ^* has a character of G^*/N lying over λ (identified with a character of N^*/N). Assume that $\chi = \chi^*\tau \in \mathrm{Irr}(G^*|\theta^*)$ has height zero in G^*, we want to prove that χ^* has height zero in G^*/N. Let B be the p-block of G^*/N that contains χ^*, and let D^*/N be a defect group of B. By Proposition 2.5(b) of [NS], D^*/N is contained in DN/N, where D is a defect group of χ. Since θ is an irreducible constituent of χ_N and χ has height zero, by Theorem 3.1, we know that θ has height zero. By Theorem 9.26 of [N3], we know that $D \cap N$ is a defect group of the block of θ. Therefore:

$$\chi^*(1)_p = (\chi(1)/\theta(1))_p = |G^* : DN|_p\,.$$

By definition,

$$\chi^*(1)_p = |G^*/N : D^*/N|_p p^{h(\chi^*)} \geq |G^*/N : DN/N|_p p^{h(\chi^*)} = |G^* : DN|_p p^{h(\chi^*)}\,.$$

We conclude that $p^{h(\chi^*)} = 1$, as wanted. $\square$

Next we use the theory of character triples, as developed in [Is, Chapter 11]. Recall that if G/N is a group, by [Is, Theorem 11.17] there exists a finite central extension (Γ, π) of G/N such that $A = \ker(\pi) \cong M(G/N)$ and the standard map $\eta\colon \mathrm{Irr}(A) \to M(G/N)$ is an isomorphism. In particular, by [Is, Theorem 11.19], if G/N is perfect then Γ is perfect. We will also make use of the results contained in [GrP, Section 3].

Theorem 4.3. *Suppose that (G, N, θ) is a character triple, where $\theta \in \mathrm{Irr}(N)$ and G/N is perfect. Then there exists a character triple isomorphism*

$$(G, N, \theta) \to (\Gamma, A, \lambda),$$

where Γ is perfect, $A = \mathbf{Z}(\Gamma)$, $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}(\theta)$ and $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*, \theta)$ for every $\chi \in \mathrm{Irr}(G|\theta)$, where χ^* corresponds to χ under the character triple isomorphism. Furthermore, if χ has height zero in G, then χ^* has height zero in Γ.

Proof. We consider a *canonically constructed* character triple (Γ, A, λ) isomorphic to (G, N, θ) in the sense of [GrP]. Notice that the values of any $\psi \in \mathrm{Irr}(\Gamma|\lambda)$ are in $\mathbb{Q}_{|G|}$. (See the paragraph before gary 3.3 of [GrP].) By Theorem 3.6 of [GrP], we have that whenever $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}(\theta))$, then $(\chi^*)^\sigma = (\chi^\sigma)^*$. Hence $\mathbb{Q}(\chi) = \mathbb{Q}(\chi, \theta) = \mathbb{Q}(\chi^*, \theta)$, as wanted. We do notice that Γ is perfect using Theorem 11.19 of [Is]. For the second part, we notice that the construction of the character triple isomorphism in [GrP] follows the construction in Theorem 11.28 of [Is]. We have that (G, N, θ) is isomorphic to (G^*, N^*, θ^*) where $G^* \subseteq G \times \Gamma$, $N^* = N \times A$, $A \subseteq \mathbf{Z}(G^*)$ and $\theta^* = \theta \times 1_A$; in fact $G \cong G^*/A$. Also (G^*, N^*, θ^*) is isomorphic to (G^*, N^*, λ^*) where $\lambda^* = 1_N \times \lambda$ (here $\theta^*(\lambda^*)^{-1}$ extends to $\tau \in \mathrm{Irr}(G^*)$). Finally (G^*, N^*, λ^*) is isomorphic to (Γ, A, λ) using that $\Gamma \cong G^*/N$. Given $\chi \in \mathrm{Irr}(G|\theta)$ of height zero in $G \cong G^*/A$, the first character triple isomorphism just sends χ to χ viewed as a character of G^*. By Lemma 4.1(iii), we have that χ has height zero as a character of G^*. By Lemma 4.2 we have that $\chi = \chi^*\tau$, and $\chi^* \in \mathrm{Irr}(G^*|\lambda^*)$ has height zero in G^*/N. Since the last character triple isomorphism just sends χ^* to χ^* seen as a character of $\Gamma \cong G^*/N$, we have that χ^* has height zero as a character of Γ, and the second part of the statement follows. □

We believe that Theorem 4.4 below might be useful in the future. (It can be used, together with the ideas of the proof of Theorem 5.1 of [NTT], as an alternative to Theorem 4.3 in the proof of our main result.)

Theorem 4.4. *Suppose that $\chi \in \mathrm{Irr}(G)$ has 2-height zero. Let $n = |G|_{2'}$ and $F = \mathbb{Q}_n(\chi)$. Then χ can be afforded by an absolutely irreducible F-representation.*

Proof. We argue by induction on $|G|$. We want to show that the Schur index $m_F(\chi)$ of χ over F is 1. We know by Corollary 10.13 of [Is] that $m_F(\chi) \leq 2$. Suppose that D is a defect group of the block of χ, and let $H = \mathbf{N}_G(D)$. Let $C = \mathbf{C}_G(D)$ and $Z = \mathbf{Z}(D)$. By Lemma 3.8, we have that χ_H contains some $\psi \in \mathrm{Irr}(H)$ with 2-height zero, $\mathbb{Q}(\psi) \subseteq F$ and $[\chi_H, \psi]$ is odd. If we can show that ψ is afforded by an F-representation, then so is ψ^G and by Corollary 10.2(c) of [Is] we have that $m_F(\chi)$ divides $[\psi^G, \chi] = [\psi, \chi_H]$ which is odd.

If $H < G$, by induction, we have that ψ can be afforded by an absolutely irreducible $\mathbb{Q}_{|H|_{2'}}(\psi)$-representation. Since $\mathbb{Q}_{|H|_{2'}}(\psi) \subseteq F$, then we are done in this case. So we may assume that $H = G$.

By Theorem 3.2, and arguing by induction on $|G|$, we may assume that χ is quasiprimitive. In particular, as $DC \trianglelefteq G$, we have that χ_{DC} is homogeneous. Write $\chi_{DC} = e\eta$, where $\eta \in \mathrm{Irr}(CD)$ is an F-valued character. By Lemma 3.4 we have that G/CD is an odd-order group, and the canonical character $\theta \in \mathrm{Irr}(C/Z)$ of the block of ψ is G-invariant. Write $\eta_D = \eta(1)\lambda$, where $\lambda \in \mathrm{Irr}(D)$ is linear (using that χ has height zero). Notice that by Corollary 11.29 of [Is] e is odd. Using again Lemma 3.4 and its notation, we have that $\eta = \theta_\lambda$. Recall that $\eta(x) = 0$ if $x_p \notin D$, and $\eta(x) = \theta(x_{p'})\lambda(x_p)$ if $x_p \in D$. Let $\mu = \lambda_Z \in \mathrm{Irr}(Z)$. By Isaacs' restriction lemma (Lemma 6.8(d) of [N4]), we have that $\nu = \eta_C \in \mathrm{Irr}(C)$. Notice that ν has defect group Z, by Theorem 9.26 of [N3], for instance. By Theorem 3.6 of [W], we have that ν is afforded by an irreducible F-representation. Since $\chi_C = e\nu$ and e is odd, we have that $m_F(\chi)$ is odd, by Corollary 10.2(c) of [Is], and this completes the proof. □

According to M. Geline, Theorem 4.4 can also be proved by using the main theorem of [GeGl] and some number theoretical arguments. We notice that the case where χ has odd degree of Theorem 4.4 follows from a theorem of Fong [Is, Corollary 10.13] using [Is, Corollary 10.2(h)].

5. Fields of Characters in $\mathcal{F}_p$

We briefly pause in our journey to the proof of Theorem A and the reduction theorem for Conjecture D to show the easy containments in Theorem B and Conjecture D. In other words, we show that every number field in $\mathcal{F}_p$ is the field of values of some irreducible character of p-height zero. We will also show that the statement of Conjecture D is implied by the statement of [N2, Conjecture B] and that Corollary C easily follows from Theorem B.

Theorem 5.1. *Suppose that $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}_n$, where n is the conductor of F. Write $n = p^a m$, where m is not divisible by p. If $|\mathbb{Q}_n : \langle F, \mathbb{Q}_m\rangle|$ is not divisible by p, then there is a solvable group G and $\chi \in \mathrm{Irr}(G)$ of p-height zero such that $\mathbb{Q}(\chi) = F$.*

Proof. Let ζ_n be a primitive n-th root of unity, and let $C_n = \langle \zeta_n \rangle$ be the cyclic group of order n, which is acted on faithfully by $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. Let G be the semidirect product of C_n with $H = \mathrm{Gal}(\mathbb{Q}_n/F) \le \mathcal{G}$. If $\lambda \in \mathrm{Irr}(C_n)$ has order n, then $\lambda^G = \chi \in \mathrm{Irr}(G)$ has field of values F. Let $\nu = \lambda_{C_m}$, where $C_m \le C_n$ has order m. Then ν has order m. Notice that $H_\nu = \mathrm{Gal}(\mathbb{Q}_n/\langle F, \mathbb{Q}_m\rangle)$ has order not divisible by p by hypothesis. Thus $G_\nu = C_n H_\nu$. By the Fong-Reynolds Theorem 9.14 of [N3], it follows that χ has height zero if and only if λ^{G_ν} has height zero, which it has, because it has degree not divisible by p. □

Next we prove one of the containments in Corollary C. Recall that if d is a square free integer and n is the conductor of $\mathbb{Q}(\sqrt{d})$, then $n = |d|$ if $d \equiv 1 \bmod 4$, and $n = 4|d|$, otherwise.

Corollary 5.2. *Suppose that $d \neq 1$ is an odd square-free integer. Then there is a group G and a 2-height zero character $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{d})$.*

Proof. By considering the cyclic group of order 4, we may assume that $d \neq -1$. Let $F = \mathbb{Q}(\sqrt{d})$. If $d \equiv 1 \bmod 4$, then $F \subseteq \mathbb{Q}_{|d|}$, $|d|$ is the conductor of F, and we are done by Theorem 5.1. Suppose that $d \equiv 3 \bmod 4$. Let $n = 4|d|$. By Theorem 5.1, we only need to show that $\langle \mathbb{Q}_{|d|}, \mathbb{Q}(\sqrt{d})\rangle = \mathbb{Q}_n$. Since $|\mathbb{Q}_n : \mathbb{Q}_{|d|}| = 2$, this can only fail if $\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{Q}_{|d|}$, which is not possible because the conductor of $\mathbb{Q}(\sqrt{d})$ is n. □

Next, we prove the rest of Corollary C assuming Theorem B.

Theorem 5.3. *Let F be a quadratic number field. Then $F = \mathbb{Q}(\chi)$ for some 2-height zero irreducible complex character χ of a finite group if and only if $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{d})$ where $d \neq 1$ is an odd square-free integer.*

Proof. (Assuming Theorem B.) Suppose that $F = \mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{d})$, where $\chi \in \mathrm{Irr}(G)$ has 2-height zero and $1 \neq d$ is some square-free integer. We prove that d is odd. Write $n = c(\chi) = 2^a m$, where $a \geq 0$ and m is odd. It is well-known that $n = |d|$ if $d \equiv 1 \bmod 4$, and $n = 4|d|$, otherwise. By Theorem B we have that $\mathbb{Q}_m(\sqrt{d}) = \mathbb{Q}_m(\chi) = \mathbb{Q}_n$. By Natural Irrationalities, we then have that $|\mathbb{Q}_n : \mathbb{Q}_m| \leq 2$. However, $|\mathbb{Q}_n : \mathbb{Q}_m| = |\mathbb{Q}_{2^a} : \mathbb{Q}|$. Therefore, $|\mathbb{Q}_{2^a} : \mathbb{Q}| \leq 2$ and thus $a \leq 2$. If d is even, then $n = 4|d|$ and therefore $a \geq 3$. This is a contradiction. □

We finish this section by showing that Conjecture D follows from the Alperin-McKay-Navarro conjecture [N2, Conjecture B]. We recall that the Alperin-McKay-Navarro conjecture implies, for a p-block B of a finite group G and its Brauer first main correspondent b, that there is a bijection between the sets of height zero characters of B and b such that, if χ corresponds to χ^* then $\mathbb{Q}_n(\chi) = \mathbb{Q}_n(\chi^*)$ where $n = |G|_{p'}$. We care to mention that the currently accepted and most studied form of this conjecture is more general, predicting the existence of an $\mathcal{H}$-equivariant bijection between height zero characters of B and b, where the Galois group $\mathcal{H}$, which contains $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_n)$, is defined in [N2, Section 2]. When we write *the Alperin-McKay-Navarro conjecture* we refer to the statement predicting $\mathcal{H}$-equivariant character bijections.

Theorem 5.4. *Conjecture D follows from the Alperin-McKay-Navarro conjecture.*

Proof. Let $\chi \in \mathrm{Irr}(B)$ be of p-height zero, where B has defect group D. By the Alperin-McKay-Navarro conjecture, there is $\tau \in \mathrm{Irr}(\mathbf{N}_G(D))$ of height zero, in the Brauer correspondent block b of $\mathbf{N}_G(D)$ such that $\mathbb{Q}_n(\chi) = \mathbb{Q}_n(\tau)$, where $n = |G|_{p'}$. In particular $c(\chi)_p = c(\tau)_p$ reasoning as in Lemma 2.2. Hence, it is enough to prove that $\mathbb{Q}_{c(\tau)_p} \subseteq \mathbb{Q}_{pm}(\tau)$ with $m = |\mathbf{N}_G(D)|_{p'}$. We may then assume that $D \trianglelefteq G$. In this case, we can apply Lemma 3.5. □

6. The Reduction

There is one more issue that we have to solve before proving a reduction of Conjecture D to quasi-simple groups. If G/N is a p'-group, $\theta \in \mathrm{Irr}(N)$ is G-invariant and p-rational, it is not necessarily true that the characters of G over θ are p-rational (even if they extend θ and p is odd). This can be seen, for example, when $p = 3$ and $G = \texttt{SmallGroup}(24, 4)$. Indeed, we have that $G = \mathsf{C}_3 \rtimes \mathsf{Q}_8$. If N is any of the two normal subgroups $\mathsf{C}_3 \rtimes \mathsf{C}_4$, then N has a rational-valued irreducible character θ of degree 2. Then θ extends to G, and the two extensions have conductor 12.

We need the following statement.

Theorem 6.1. *Suppose that $N \trianglelefteq G$ and that G/N is a simple group of order coprime to p. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant and p-rational. If $\chi \in \mathrm{Irr}(G|\theta)$, then $c(\chi)_p \leq p$.*

Proof. Suppose first that G/N has order a prime q. Then this follows from Theorem B of [V1]. Suppose now that G/N is perfect. By Theorem 4.3, there is a character triple isomorphism $(G, N, \theta) \to (G^*, N^*, \theta^*)$ such that $N^* = \mathbf{Z}(G^*)$, $\mathbb{Q}(\theta^*) \subseteq \mathbb{Q}(\theta)$ and $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*, \theta^*)$ whenever $\chi \in \mathrm{Irr}(G|\theta)$ and χ^* corresponds to χ under the character triple isomorphism. In particular, we have that G^*/N^* is a simple non abelian p'-group, and θ^* is p-rational. Let $x \in G^*$, and let $\delta \in \mathrm{Irr}(N^*\langle x\rangle)$ be over θ^*. Since $N^*\langle x\rangle/N^*$ is cyclic and θ^* is G^*-invariant, $\delta_{N^*} = \theta^*$. Since $x_p \in N^*$ and δ is linear, we have that $\delta(x) = \delta(x_p)\delta(x_{p'}) = \theta^*(x_p)\delta(x_{p'}) \in \mathbb{Q}_{|G^*|_{p'}}$. Then every $\delta \in \mathrm{Irr}(N^*\langle x\rangle|\theta^*)$ is p-rational. Since $\chi_{N^*\langle x\rangle}$ is a sum of irreducible characters lying over θ^* and the choice of $x \in G^*$ was arbitrary, we are done. □

Conjecture 6.2. *Let $\chi \in \mathrm{Irr}(G)$ of p-height zero, where G is a quasi-simple group. Assume in addition that the p-block B containing χ is not (virtual) Morita equivalent over an absolutely unramified complete discrete valuation ring to a p-block of any group H with $|H : \mathbf{Z}(H)| < |G : \mathbf{Z}(G)|$. Write $c(\chi) = p^a m$. Then $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm}(\chi)$.*

Notice that if the p-block B of χ is virtual Morita equivalent over an absolutely unramified complete discrete valuation ring to a p-block $\tilde{B}$, by Theorem 1.6 of [KL], there is a height zero character $\tilde{\chi}$ in $\tilde{B}$ and an integer l not divisible by p such that $\mathbb{Q}_l(\chi) = \mathbb{Q}_l(\tilde{\chi})$. By Lemma 2.2, we have that $c(\chi)_p = c(\tilde{\chi})_p = p^a$, and therefore $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pl}(\chi)$ if and only if $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pl}(\tilde{\chi})$. Hence, if $c(\chi)_{p'} = m$ and $c(\tilde{\chi})_{p'} = m_1$, then $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm}(\chi)$ if, and only if, $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm_1}(\tilde{\chi})$, by Lemma 2.1. We will use this argument in the following proof of our main reduction theorem.

Theorem 6.3. *Let G be a finite group, and let p be a prime. Let $\chi \in \mathrm{Irr}(G)$ be of p-height zero, and write $c(\chi) = p^a m$, where $a \geq 0$ and p does not divide m. If Conjecture 6.2 is true, then $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm}(\chi)$.*

Proof. We argue by induction on $|G : \mathbf{Z}(G)|$. We may assume that $a \geq 2$, because otherwise the statement is trivially satisfied. Let $n = |G|_{p'}$. By Lemma 2.1, it is enough to show that $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pn}(\chi)$.

Step 1. If $N \trianglelefteq G$ is a proper normal subgroup, then we may assume that $\chi_N = e\theta$ for some $\theta \in \mathrm{Irr}(N)$, with $c(\theta)_p < c(\chi)_p$.

Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N, let T be the stabilizer of θ in G, and let $\psi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ over θ. By Theorem 3.2, we have that $\mathbb{Q}_{pn}(\psi) = \mathbb{Q}_{pn}(\chi)$ and $c(\chi)_p = p^a = c(\psi)_p$. Assume that $T < G$. Then $|T : \mathbf{Z}(T)| < |G : \mathbf{Z}(G)|$, and by induction, if m_1 is the p'-part of the conductor of ψ, we have that $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm_1}(\psi) \subseteq \mathbb{Q}_{pn}(\psi) = \mathbb{Q}_{pn}(\chi)$. By Lemma 2.1, we deduce that $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm}(\chi)$. Hence, we may assume that $\chi_N = e\theta$. In particular, $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\chi)$, and $c(\theta)$ divides $c(\chi)$. If $c(\theta)_p = c(\chi)_p$, and m_2 is the p'-part of the conductor of θ, then by induction $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pm_2}(\theta) \subseteq \mathbb{Q}_{pm_2}(\chi)$. Again by Lemma 2.1, we deduce what we want.

Step 2. We may assume that G does not have a normal subgroup of index p.

Otherwise, let N be a normal subgroup of G with G/N of order p. By Step 1, χ_N is homogeneous. Since G/N is cyclic we can write $\chi_N = \theta \in \mathrm{Irr}(N)$.

Suppose that there exists a subgroup H such that $G = NH$ such that, if $M = N \cap H$, then there exists some H-invariant $\varphi \in \mathrm{Irr}(M)$ of p-height zero such that $[\theta_M, \varphi]$ is not divisible by p and $\mathbb{Q}_{pn}(\varphi) \subseteq \mathbb{Q}_{pn}(\theta)$. Let $\xi \in \mathrm{Irr}(H)$ be an extension of φ. By Lemma 3.7, we have that ξ has p-height zero. By Lemma 3.9(iii), we have that $\mathbb{Q}_p(\xi, \theta) = \mathbb{Q}_p(\chi, \varphi)$. Notice that $\mathbb{Q}_{pn}(\chi) \subseteq \mathbb{Q}_{pn}(\chi, \varphi) \subseteq \mathbb{Q}_{pn}(\chi, \theta) = \mathbb{Q}_{pn}(\chi)$. Therefore $\mathbb{Q}_{pn}(\chi) = \mathbb{Q}_{pn}(\chi, \varphi) = \mathbb{Q}_{pn}(\theta, \xi)$. Also, $\mathbb{Q}_{pn}(\xi) \subseteq \mathbb{Q}_{pn}(\chi)$. Write $c(\theta)_p = p^b$ and $c(\xi)_p = p^c$. We have that $\mathbb{Q}(\xi) \subseteq \mathbb{Q}_{pn}(\xi) \subseteq \mathbb{Q}_{pn}(\chi) \subseteq \mathbb{Q}_{p^a n}$. Therefore $c \leq a$. Notice that if θ and ξ are p-rational, then $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{pn}(\chi) = \mathbb{Q}_{pn}(\chi, \varphi) = \mathbb{Q}_{pn}(\theta, \xi) = \mathbb{Q}_{pn}$, but we are assuming that $a \geq 2$. Hence $b, c \geq 1$. Now $\mathbb{Q}_{pn}(\theta) \subseteq \langle \mathbb{Q}_{p^b n}, \mathbb{Q}_p \rangle$, and $\mathbb{Q}_{pn}(\xi) \subseteq \langle \mathbb{Q}_{p^c n}, \mathbb{Q}_p \rangle$. Thus

$$\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{pn}(\chi) = \langle \mathbb{Q}_{pn}(\theta), \mathbb{Q}_{pn}(\xi) \rangle \subseteq \langle \mathbb{Q}_{p^b n}, \mathbb{Q}_{p^c n}, \mathbb{Q}_p \rangle \subseteq \mathbb{Q}_{p^d n}\,,$$

where $d = \max(b, c)$. Hence $p^a \leq p^d$. Since $b < a$ by Step 1 and $c \leq a$, we conclude that $d = c = a$. If $|H : \mathbf{Z}(H)| < |G : \mathbf{Z}(G)|$ then, by induction, we have that $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pn}(\xi) \subseteq \mathbb{Q}_{pn}(\chi)$, and we are done (using Lemma 2.1).

Else, by Lemma 3.8, we may assume that the defect group D_0 of the block of θ is normal in G. By Lemma 3.10, we are done in this case.

Step 3. We may assume that G does not have a proper normal subgroup of index not divisible by p.

Otherwise, let $N \trianglelefteq G$ such that G/N is simple of order not divisible by p. By Step 1, we have that $\chi_N = e\theta$ for some $\theta \in \mathrm{Irr}(N)$. Recall that $c(\chi)_p \geq p^2$. Hence, by Theorem 6.1, we have that $c(\theta)_p \geq p$, in this case. By Lemma 4.2.(ii) of [NT], we conclude that $c(\chi)_p = c(\theta)_p$, contradicting Step 1.

Final Step. Let N be a maximal normal subgroup of G. By Steps 2 and 3, G/N is simple non-abelian of order divisible by p. By Step 1, write $\chi_N = e\theta$, where $c(\theta)_p = p^b$ and $b < a$. By Theorem 4.3, there is a quasi-simple group G^* and a p-height zero character χ^* of G^* such that $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*, \theta)$. Write $c(\chi^*) = p^c k$, where k is not divisible by p. Then the conductor of the field $\mathbb{Q}(\chi^*, \theta) = \langle \mathbb{Q}(\chi^*), \mathbb{Q}(\theta) \rangle$ is the least common multiple of the conductors of χ^* and θ. In particular, its p-part is $p^{\max(c,b)}$. As $c(\chi) = c(\mathbb{Q}(\chi^*, \theta))$, we have that $a = \max(c, b)$. Since $b < a$, we have that $a = c$. Thus $c(\chi)_p = c(\chi^*)_p$. If the p-block B^* of χ^* is virtual Morita equivalent over an absolutely unramified complete discrete valuation ring to a p-block $\tilde{B}$ of a group H with $|H : \mathbf{Z}(H)| < |G : \mathbf{Z}(G)|$, then $c(\chi^*)_p = c(\tilde{\chi})_p$ for some $\tilde{\chi} \in \tilde{B}$. As explained before the statement of this theorem, we are done in this case, using induction. Therefore, by Conjecture 6.2, we may assume that $\mathbb{Q}_{p^a} \subseteq \mathbb{Q}_{pk}(\chi^*) \subseteq \mathbb{Q}_{pk}(\chi)$. By using Lemma 2.1, the proof of the theorem follows. □

7. Theorem A for quasisimple groups

The aim of this section is to prove Conjecture 6.2 in the case that $p = 2$, see Theorem 7.2 below. In view of Theorem 6.3 (and Theorem 5.1), this will complete the proof of Theorem A (and Theorem B).

The following result is useful in working with extensions of quasi-simple groups.

Theorem 7.1. *Suppose that G/N is abelian. Write $n = |G|_{2'}$. Suppose that $\chi \in \mathrm{Irr}(G)$ has 2-height zero and suppose that $\mathbb{Q}_{2^a} \subseteq \mathbb{Q}_n(\chi)$, where $c(\chi)_2 = 2^a$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N, and write $c(\theta)_2 = 2^b$. Then $\mathbb{Q}_{2^b} \subseteq \mathbb{Q}_n(\theta)$.*

Proof. We argue by induction on $|G : N|$. We may assume that G/N has prime order. By Theorem 3.2, we may assume that $\chi_N = \theta$ is irreducible. By Lemma 4.2.(ii) of [NT], we may assume that G/N has order 2. Let D be a defect group of the block of χ, such that $DN = G$ and $D_0 = D \cap N$ is a defect group of the block of θ. Let $H = \mathbf{N}_G(D_0)$ and $M = H \cap N$. By Lemma 3.8, there is $\varphi \in \mathrm{Irr}(M)$ of height zero, $\mathbb{Q}_n(\varphi) \subseteq \mathbb{Q}_n(\theta)$ with an extension $\xi \in \mathrm{Irr}(H)$ such that $\mathbb{Q}(\chi, \varphi) = \mathbb{Q}(\theta, \xi)$. Thus $\mathbb{Q}_n(\chi, \varphi) = \mathbb{Q}_n(\theta, \xi)$. Hence $\mathbb{Q}_{2^a} \subseteq \mathbb{Q}_n(\theta, \xi)$. We may assume that $b \geq 2$. Suppose that $c(\xi)_2 = 2^c$. We know that $\mathbb{Q}_{2^c} \subseteq \mathbb{Q}_n(\xi)$ by Lemma 3.10. If ξ is 2-rational, then we are done. So we may assume that $c \geq 2$, and thus $i \in \mathbb{Q}_n(\xi)$. Then $\mathbb{Q}_n(i) \subseteq \mathbb{Q}_n(\xi) \cap \mathbb{Q}_n(\theta)$. Since $\mathbb{Q}_n(\xi), \mathbb{Q}_n(\theta) \subseteq \mathbb{Q}_{|G|}$, and $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_n(i))$ is a cyclic 2-group, we then have that $\mathbb{Q}_n(\xi) \subseteq \mathbb{Q}_n(\theta)$ or $\mathbb{Q}_n(\theta) \subseteq \mathbb{Q}_n(\xi)$. Since $\mathbb{Q}_{2^a} \subseteq \mathbb{Q}_n(\theta, \xi)$, we may assume the second. Then $\mathbb{Q}_{2^a} \subseteq \mathbb{Q}_n(\xi)$. Thus $a \leq c$. If $H < G$, then by induction $\mathbb{Q}_{2^c} \subseteq \mathbb{Q}_n(\varphi) \subseteq \mathbb{Q}_n(\theta)$. Thus we may assume that $D_0 \trianglelefteq G$. But in this case, we are done by Lemma 3.5. □

The remainder of the section is devoted to the proof of the following theorem.

Theorem 7.2. *Let $\chi \in \mathrm{Irr}(G)$ of 2-height zero, where G is a quasi-simple group. Assume in addition that the 2-block B containing χ is not (virtual) Morita equivalent over an absolutely unramified complete discrete valuation ring to a 2-block of any group H with $|H : \mathbf{Z}(H)| < |G : \mathbf{Z}(G)|$. If $c(\chi) = 2^a m$, where m is odd, then $\mathbb{Q}_{2^a} \subseteq \mathbb{Q}_m(\chi)$.*

Theorem 7.3. *Theorem 7.2 is true in the case $G/\mathbf{Z}(G)$ is a simple group of Lie type in characteristic* 2.

Proof. In the case $S := G/\mathbf{Z}(G)$ is isomorphic to A_5, A_6, A_8, $\mathrm{SL}_3(2)$, $\mathrm{SU}_4(2)$, $\mathrm{Sp}_6(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_6(2)$, $\Omega_8^+(2)$, ${}^2B_2(8)$, $G_2(4)$, $F_4(2)$, ${}^2F_4(2)'$, or ${}^2E_6(2)$, the statement is checked using [GAP]. Hence we may assume that S is not isomorphic to any of these simple groups. This implies that G is a quotient (by a central subgroup) of $\mathcal{G}^F$, where $\mathcal{G}$ is a simple, simply connected, algebraic group in characteristic 2 and $F : \mathcal{G} \to \mathcal{G}$ a Steinberg endomorphism. It follows from the main result of [Hum] that any 2-block B of G is either of defect 0, or of maximal defect. Moreover, in the former case $\chi \in \mathrm{Irr}(B)$ is just the Steinberg character; in particular it is rational and so we are done. In the latter case, $\chi(1)$ is odd, and the statement follows from [NT, Theorem A1]. □

Theorem 7.4. *Theorem 7.2 is true in the case $G/\mathbf{Z}(G)$ is an alternating or sporadic simple group.*

Proof. In the case $S := G/\mathbf{Z}(G)$ is isomorphic to A_n with $5 \le n \le 8$ or one of the 26 sporadic simple groups, the statement is checked using [GAP]. Hence we may assume that $S = \mathsf{A}_n$ with $n \ge 9$.

(a) First we consider the case $G = S$. If χ extends to S_n then χ is rational. Otherwise, [JK, Theorem 2.5.13] shows that the S_n-character lying above χ is labeled by a self-associated partition of n, with hook lengths along the main diagonal of the Young diagram being the $k \ge 1$ odd integers $2h_1+1 > 2h_2+1 > \ldots > 2h_k+1$, in which case the only possible irrational values of χ are

$$\frac{(-1)^{(n-k)/2} \pm \sqrt{(-1)^{(n-k)/2}\prod_{i=1}^k (2h_i+1)}}{2}.$$

Since $n = \sum_{i=1}^k (2h_i+1)$, we see that $(-1)^{(n-k)/2} = 1$ if and only if $\prod_{i=1}^k (2h_i+1) \equiv 1 \pmod 4$. It follows that χ is 2-rational, and we are done in this case.

(b) It remains to handle the case $G = 2\mathsf{A}_n$. We change the notation, and let $\tilde{B}$ the 2-block of G containing χ. We also embed G in a double cover $\tilde{G} = 2\mathsf{S}_n$ of S_n. By [D, Lemma 2.2], $\tilde{B}$ contains a unique block B of $G/\mathbf{Z}(G) \cong \mathsf{A}_n$. Similarly, by [D, Lemma 2.1], $\tilde{B}$ is covered by a unique block $\tilde{B}_s$ of $\tilde{G}$, and B is covered by a unique block B_S of $(\tilde{G})/\mathbf{Z}(G) \cong \mathsf{S}_n$. All these blocks $\tilde{B}$, $\tilde{B}_S$, B, and B_S have the same weight $w \ge 0$.

If χ is trivial on $\mathbf{Z}(G)$, then we are done by (a). Hence we may assume that χ is a spin character of G. Since $\tilde{B}$ contains a spin character of height zero, by [D, Proposition 3.1] we must have that $w \in \{0,1\}$, and χ is the unique spin character of height zero in $\tilde{B}$. Now, fix a character $\theta \in \mathrm{Irr}(\mathsf{A}_n)$ lying in B which is contained in $\tilde{B}$, and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{2'}})$. By part (a), σ fixes θ, whence it fixes B. In turn, this implies that σ fixes $\tilde{B}$ (the unique block containing B), and hence it fixes χ by its uniqueness. It follows that χ is 2-rational, and the statement follows in this case as well. $\square$

Theorem 7.5. *Theorem 7.2 is true in the case $G/\mathbf{Z}(G)$ is a simple classical group in odd characteristic.*

Proof. In the case $S := G/\mathbf{Z}(G)$ is isomorphic to $\mathrm{PSU}_4(3)$ or $\Omega_7(3)$, the statement is checked using [GAP]. (Note that in Theorem 7.2 we are dealing with a 2-block B of G, and so we may assume $\mathbf{O}_{2'}(\mathbf{Z}(G))$ is cyclic. Hence in the case of covers of $\mathrm{PSU}_4(3)$, it suffices to handle the two covers $12_1 \cdot \mathrm{PSU}_4(3)$ and $12_2 \cdot \mathrm{PSU}_4(3)$, which are given in [GAP].)

Hence we may assume that S is not isomorphic to any of these simple groups, as well as any Lie type group in characteristic 2. This implies that G is a quotient (by a central subgroup) of $\mathcal{G}^F$, where $\mathcal{G}$ is a simple, simply connected, algebraic group in odd characteristic $r \neq 2$ and $F : \mathcal{G} \to \mathcal{G}$ a Steinberg endomorphism. Without any loss, we may replace G by $\mathcal{G}^F$. Let $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$; in particular, $\mathcal{G}^*$ is of adjoint type, and let $G^* := (\mathcal{G}^*)^{F^*}$.

By the main result of [BrM], the set $\mathrm{Irr}(B)$ of complex characters in the 2-block B containing χ is contained in $\mathcal{E}_2(G, s)$ for some semisimple element $s \in G^*$ of odd order. Suppose that s is not quasi-isolated (in the sense of [B]). Then, by the main result of [BR], B is Morita equivalent to a 2-block of a group H with $|H : \mathbf{Z}(H)| < |G : \mathbf{Z}(G)|$. Moreover, by [FK, Proposition 4.2] this Morita equivalence descends to an absolutely unramified discrete valuation ring contrary to our assumption. Hence we may assume that s is quasi-isolated.

Assume in addition that $\mathcal{G}$ is not of type A. By the classification result of Bonnafé [B, Table 2], the odd-order assumption on s implies that $s = 1$. In this case, by [CE, Theorem 21.14], $\mathcal{E}_2(G, s)$ is just the set of irreducible characters in the principal 2-block B_0 of G. In such a case, $\chi(1)$ is odd, and the statement follows from [NT, Theorem A1].

It remains to consider the case $\mathcal{G}$ is of type A. The same arguments as in the preceding paragraph allow us to assume that $s \neq 1$, and so s is **not** isolated, see [B, Table 2]. The main result of [BDR] together with [FK, Proposition 4.2] now shows that B is again Morita equivalent over an absolutely unramified complete discrete valuation ring to a 2-block of a group H with $|H : \mathbf{Z}(H)| < |G : \mathbf{Z}(G)|$, contrary to our assumption. $\square$

Theorem 7.6. *Theorem 7.2 is true in the case $G/\mathbf{Z}(G)$ is a simple exceptional group in odd characteristic.*

Proof. In the case $S := G/\mathbf{Z}(G) \cong G_2(3)$, the statement is checked using [GAP]. We may now assume that G is a (quotient by a central subgroup) of $\mathcal{G}^F$, where $\mathcal{G}$ is a simple, simply connected, algebraic group in odd characteristic $r \neq 2$ and $F : \mathcal{G} \to \mathcal{G}$ is a Steinberg endomorphism. Arguing as in Theorem 7.5, we may also assume that s

is isolated. By Lemma 2.1, we observe that it suffices to prove that $\mathbb{Q}_{c(\psi)_2} \subset \mathbb{Q}_{|G|_{2'}}(\psi)$ for every height zero character $\psi \in \mathrm{Irr}_0(B)$. We may also assume $c(\psi)_2 \geq 4$ since otherwise $\mathbb{Q}_{c(\psi)_2} = \mathbb{Q}$ and the statement is trivally true.

(a) Let us first assume that the defect group of B has order $|G^* : \mathbf{C}_{G^*}(s)|_2$ and $\mathbf{C}^\circ_{\mathcal{G}^*}(s)$ has only components of classical type. Let ψ be a height zero character of B and $t \in \mathbf{C}_{G^*}(s)_2$ such that $\psi \in \mathcal{E}(G, st)$, the (rational) Lusztig series labeled by the G^*-conjugacy class of st [GM, Definition 2.6.1]. Then the degree formula for Jordan decomposition (see for example [M, (2.1)]) shows that $|G^* : \mathbf{C}_{G^*}(st)|_2$ divides $\psi(1)_2$. Hence, the element $t \in G^*$ is 2-central in the group $H := \mathbf{C}^\circ_{\mathcal{G}^*}(s)^{F^*}$.

If $\mathbf{Z}(\mathcal{G}) \neq 1$ then we let $\mathcal{G} \lhd \tilde{\mathcal{G}}$ a regular embedding, see [GM, Definition 1.7.1], with dual surjective morphism $\iota^* : \tilde{\mathcal{G}}^* \to \mathcal{G}^*$. Otherwise set $\tilde{\mathcal{G}} := \mathcal{G}$. There exists a semisimple element $\tilde{s} \in \tilde{G}^* := (\tilde{\mathcal{G}}^*)^{F^*}$ of $2'$-order such that $\iota^*(\tilde{s}) = s$ and $\tilde{t} \in \mathbf{C}_{\tilde{G}^*}(\tilde{s})_2$ with $\iota^*(\tilde{t}) = t$. We let $\chi \in \mathcal{E}(\tilde{G}, \tilde{s}\tilde{t})$ be a character covering ψ. By [GM, Theorem 4.7.9] (whose assumption is satisfied by [GM, Theorem 3.3.7] in our case), [GM, Proposition 4.5.5] and using that $\mathbf{C}_{\tilde{\mathcal{G}}^*}(\tilde{s}\tilde{t})$ has only components of classical type we have $\mathbb{Q}(\chi) \subset \mathbb{Q}_{\mathrm{o}(\tilde{s}\tilde{t})}$ and so $c(\chi)_2 \leq \mathrm{o}(\tilde{t})$.

(a1) Assume now first that $|\chi(1) : \psi(1)|_2 > 1$. Then $G = E_7(q)$ and $\chi = (\psi')^{\tilde{G}}$ for some $\psi' \in \mathrm{Irr}(G\mathbf{Z}(\tilde{G}) \mid \psi)$. In this case, $c(\chi)_2 = c(\psi')_2$ by Theorem 3.2 and $\mathbb{Q}(\chi) \subset \mathbb{Q}(\psi')$. Since $\psi \in \mathrm{Irr}(G)$ has height zero, it follows that $\psi_{\mathbf{Z}(G)}$ is trivial by [Ruh, Lemma 8.7]. Hence, we can choose $\chi \in \mathrm{Irr}(\tilde{G} \mid \psi)$ with the additional property that χ is trivial on $\mathbf{Z}(\tilde{G})$. A consequence of this choice is that $c(\psi') = c(\psi)$ and $\mathbb{Q}(\psi') = \mathbb{Q}(\psi)$.

Since χ is the unique character in its $\mathrm{Irr}(\tilde{G}/G)$-orbit which is trivial on $\mathbf{Z}(\tilde{G})$, it follows that any Galois automorphism that stabilizes the $\mathrm{Irr}(\tilde{G}/G)$-orbit of χ also stabilizes χ. Hence, [GM, Theorem 4.7.9] and [GM, Proposition 4.5.5] show therefore that $\mathbb{Q}(\chi) \subset \mathbb{Q}_{\mathrm{o}(\tilde{s})\mathrm{o}(t)}$.

Recall that $\psi \in \mathcal{E}(G, st)$ has height zero and the defect group of B has order $|G^* : \mathbf{C}_{G^*}(s)|_2$. Assume that e is an integer coprime to $\mathrm{o}(t)$ such that t is H-conjugate to t^e. We claim that $t = t^e$. Consulting the relevant tables in [KM1] shows that if $\pi : \mathbf{C}^\circ_{\mathcal{G}^*}(s)_{\mathrm{sc}} \to \mathbf{C}^\circ_{\mathcal{G}^*}(s)$ is the universal covering of $\mathbf{C}^\circ_{\mathcal{G}^*}(s)$, then $|\ker(\pi)|_2 \leq 2$. Thus, the argument in case (3) of [M, Theorem 5.9] (for $|\ker(\pi)| \leq 2$) now works in our slightly more general situtation and we see that t lies in the centralizer of a Sylow d-torus $\mathcal{S}_d$ of $\mathbf{C}^\circ_{\mathcal{G}^*}(s)$ for d the order of q modulo 4.

By the proof of [BroR, Corollary 2.4], any Sylow 2-subgroup of the Weyl group $W := \mathbf{N}_H(\mathcal{S}_d)/\mathbf{C}_H(\mathcal{S}_d)$ is self-normalizing. Moreover, since t is 2-central in H its centralizer $W(t)$ in W contains a Sylow 2-subgroup W_2 of W.

Since $\mathbf{N}_H(\mathcal{S}_d)$ controls H-fusion in $\mathbf{C}_H(\mathcal{S}_d)$ by [M, Proposition 5.11] it follows that ${}^w t = t^e$ for some $w \in \mathbf{N}_W(W(t))$. Hence, ${}^w W_2 \subset W(t^e) = W(t)$ and so ${}^w W_2$ and W_2 are both Sylow 2-subgroups of $W(t)$. In particular, by conjugacy of Sylow subgroups in $W(t)$ we can assume that $w \in \mathbf{N}_W(W_2) = W_2$ and so $t = t^e$ as claimed. This implies that $\mathbb{Q}_{\mathrm{o}(t)} \subset \mathbb{Q}_{o(s)}(\psi)$ by [GM, Proposition 3.3.15] and thus $c(\psi)_2 \geq \mathrm{o}(t)_2$ by Lemma 2.1. On the other hand,

$$c(\chi)_2 = c(\psi)_2 \geq \mathrm{o}(t) \geq c(\chi)_2,$$

and so $\mathrm{o}(t)_2 = c(\psi)_2$. Hence, $\mathbb{Q}_{c(\psi)_2} \subset \mathbb{Q}_{o(s)}(\psi)$.

(a2) Assume now that $|\chi(1) : \psi(1)|_2 = 1$. In this case, χ is a height zero character of $\tilde{G}$. The arguments from above therefore show that $\tilde{t}$ is 2-central in $\mathbf{C}_{\tilde{G}^*}(\tilde{s})$. The arguments from the first case now show that $\mathbb{Q}_{c(\chi)_2} \subset \mathbb{Q}_{\mathrm{o}(\tilde{t})_2} \subset \mathbb{Q}_{o(\tilde{s})}(\chi)$. Hence, the claim follows in this case from Theorem 7.1.

(b) Suppose now that $s = 1$, i.e. that B is a unipotent block. We can assume that B has non-maximal defect since otherwise the statement follows from [NT, Theorem A.1]. By Lemma 3.5 we can also assume that B has non-central defect. In this case, B is one of the blocks considered in [Ruh, Lemma 7.1]. In case (i) of [Ruh, Lemma 7.1], the defect group of B is dihedral, so every character in $\mathrm{Irr}_0(B)$ is 2-rational by [Sam, Theorem 8.1]. Hence, the claim holds. In case (ii), $G = E_8(q)$ and the height zero characters were explicitly described in [Ruh, Lemma 7.4]. It follows from this description that $\mathrm{Irr}_0(B) \subset \cup_t \mathcal{E}(G, t)$, where $t \in G^*$ runs over elements with $t^2 = 1$, and all height zero characters of $\mathrm{Irr}_0(B)$ have $\mathbf{Z}(G)$ in their kernel. Now [GM, Theorem 4.7.9] and [GM, Proposition 4.5.5] show that $\mathbb{Q}(\chi)$ is a cyclotomic field or $\mathbb{Q}(\chi) \subset \mathbb{Q}(\sqrt{r})$ which in both cases implies the statement.

(c) An analysis of the tables in [KM1] shows that in the remaining cases $G = E_8(q)$ and $\mathbf{C}_{\mathcal{G}^*}(s)$ is of type E_6A_2. Let $G(s)$ be the F-fixed points of the connected reductive group in duality with $\mathbf{C}_{\mathcal{G}^*}(s)$. For $t \in \mathbf{C}_{G^*}(s)_2$ let $\psi_{G,st} : \mathcal{E}(G, st) \to \mathcal{E}(\mathbf{C}_{G^*}(st), 1)$ be Digne–Michel's unique Jordan decomposition for groups with connected center as in in [GM, Theorem 4.7.1]. For this, note that the centralizer of st in $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected. Hence, there exists a bijection $\psi_{G(s),st} : \mathcal{E}(G(s), st) \to \mathcal{E}(\mathbf{C}_{G^*}(st), 1)$ which is obtained by restricting Digne–Michel's unique Jordan decomposition in a regular embedding of $G(s)$. We have a bijection $\mathcal{J} : \mathcal{E}_2(G, s) \to \mathcal{E}_2(G(s), s)$ which is the union of the bijections $\psi^{-1}_{G(s),st} \circ \psi_{G,st}$ with $t \in \mathbf{C}_{G^*}(s)_2$, see [Ruh, Lemma 2.3]. By [GM, Theorem 4.7.9] and the construction of $\mathcal{J}$ it follows that $\mathcal{J}$ is $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{\mathrm{o}(s)})$-equivariant. Moreover, by [Ruh, Proposition D] there exists a bijection $c \mapsto b$ between blocks contained in $\mathcal{E}_2(G(s), s)$ and the blocks contained in $\mathcal{E}_2(G, s)$ such that $\mathcal{J}(\mathrm{Irr}_0(b)) = \mathrm{Irr}_0(c)$. From this it follows that $\mathbb{Q}_{o(s)}(\mathcal{J}(\chi)) = \mathbb{Q}_{o(s)}(\chi)$ and $c(\chi)_2 = c(\mathcal{J}(\chi))_2$. Hence, it suffices to consider the height zero characters of the unipotent blocks of $G(s)$. However, by [CE, Theorem 17.7] the unipotent blocks of $G(s)$ are isomorphic in a natural way to the blocks of $G(s)_{\mathrm{sc}}$, the F-fixed points of the simply connected covering of $G(s)$. Using the explicit description in [CE, Proposition 17.4] shows that the associated character bijection is Galois equivariant.

By what we have established about unipotent blocks, $\mathbb{Q}_{c(\mathcal{J}(\chi))_2} \subset \mathbb{Q}_{|G|_{2'}}(\mathcal{J}(\chi)_2)$. Therefore, $\mathbb{Q}_{c(\chi)_2} \subset \mathbb{Q}_{|G|_{2'}}(\psi)$ for all height zero characters ψ. □

Notice that the proof of Corollary C is now elementary. We finish this paper by proving a consequence of Theorem A. For an integer $e \geq 1$, let σ_e be the Galois automorphism in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ fixing $2'$-roots of unity and sending ξ to ξ^{1+2^e}, where ξ is any 2-power root of unity. If $n \geq 1$ is any integer, then

$$\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_{n_{2'}}) = \langle \tau_1, \tau_2 \rangle,$$

where τ_i is the restriction of σ_i to $\mathbb{Q}_n$ (since the cosets of 3 and of 5 generate the unit group of $\mathbb{Z}/2^k\mathbb{Z}$ for any positive integer k). Notice that if G is a finite group, a character $\chi \in \mathrm{Irr}(G)$ is 2-rational if, and only if, χ is $\mathcal{G}$-fixed, where $n = |G|$. The set of 2-height zero characters fixed under the action of $\langle \sigma_1 \rangle$ has been recently studied in connection with the number of generators of 2-defect groups (see [RSV, NRSV, V2]).

We have the following.

Theorem 7.7. *Let $\chi \in \mathrm{Irr}(G)$ of 2-height zero. Then χ is 2-rational if, and only if, χ is σ_1-fixed.*

Proof. Let $m = |G|_{2'}$. Suppose that χ is σ_1-fixed. Then $\mathbb{Q}_m(\chi)$ is also fixed by σ_1. If χ is not 2-rational, then $i \in \mathbb{Q}_m(\chi)$ by Theorem A. However, $\sigma_1(i) = i^3 \neq i$, a contradiction. □

The conclusion of Theorem 7.7 is not true for characters which do not have height zero. The smallest example is an irreducible character χ of degree 2 of a semidihedral group of order 16 with field of values $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-2})$.

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