SIMPLE MODULES AS SUBMODULES AND QUOTIENTS OF SYMMETRIC POWERS

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Given a faithful (linear) action of a finite group G on a finite-dimensional vector space V over a field $\mathbb{F}$, it is a classical theme to study the *invariants* $\mathbb{F}[V]^G$, or *semi-invariants*, i.e. the one-dimensional G-submodules in $\mathbb{F}[V]$. One is particularly interested in the first nontrivial occurrence (in $\mathrm{Sym}^m(V)$ with $m \geq 1$) of invariants, or semi-invariants, see e.g. [9].

More generally, one can ask whether, and when, a given irreducible $\mathbb{F}G$-module W occurs, as a submodule or as a quotient, of $\mathrm{Sym}^m(V)$ for suitable m. An affirmative answer to the first question was given in [2, Theorem A], see also [3, Theorem 4.3] and [8, Theorem 1.2], namely, the regular representation of G occurs as a summand of the symmetric algebra $\mathrm{Sym}(V)$. Motivated in particular by the study of automorphism groups of rings of invariants, see e.g. [7, Examples 10, 14], we give an answer to the second question.

First we prove the following statement.

Proposition 1. *Let $\mathbb{F}$ be an algebraically closed field, G any finite group, and $V = \mathbb{F}^n$ any finite-dimensional, faithful $\mathbb{F}G$-module. Then, for any $k \in \mathbb{Z}_{\geq 0}$, the regular $\mathbb{F}G$-module $\mathbf{reg}_G$ is a direct summand of*

$$\bigoplus_{j=1}^{|Z|} \mathrm{Sym}^{|G|-j+k|G|}(V),$$

where $Z := \mathbf{Z}(\mathrm{GL}(V)) \cap G$.

Proof. (i) Fixing a basis $(x_1, \ldots, x_n)$ of V, for any $k \in \mathbb{Z}_{\geq 0}$ we can identify $\mathrm{Sym}^k(V)$ with the space of degree k homogeneous polynomials in n variables $x_1, \ldots, x_n$.

Since Z is isomorphic to a finite subgroup of $\mathbb{F}^\times$, Z is cyclic. The statement is obvious in the case $G = Z$, so we will assume

$$G > Z$$

throughout. Note that any element $g \in G \smallsetminus Z$ does not act as a scalar on V, and hence each of its eigenspaces in V is a proper subspace of V, giving a proper closed subset of V viewed as $\mathbb{A}^n_{\mathbb{F}}$. Since $\mathbb{A}^n_{\mathbb{F}}$ is an irreducible affine variety and G is finite, the union of all these eigenspaces, when g runs over $G \smallsetminus Z$, is a proper closed subset of V. It follows that we can find $0 \neq v \in V$ such that no element $g \in G \smallsetminus Z$ can fix the line $\langle v \rangle_{\mathbb{F}}$.

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For any coset $\bar{h} = hZ$, note that the line $\langle h(v)\rangle_{\mathbb{F}}$ does not depend on the choice of the representative $h \in G$; we will use $\bar{h}(v)$ to denote $h(v)$ for some choice of h. Now we set

$$F(\bar{g}) := \prod_{\bar{h}\in G/Z\smallsetminus\{\bar{g}\}} \bar{h}(v) \in \mathrm{Sym}^{N-1}(V), \tag{1}$$

where

$$N := |G/Z| > 1.$$

By this definition, for any $g, h \in G$ and any $j \in \mathbb{Z}_{\geq 1}$, we see that g sends the line $\langle F(\bar{h})^j\rangle_{\mathbb{F}}$ to $\langle F(\bar{g}\bar{h})^j\rangle_{\mathbb{F}}$. In particular, G/Z permutes the N lines $\langle F(\bar{g})^j\rangle_{\mathbb{F}}$, $\bar{g} \in G/Z$, transitively for any $j \geq 1$.

Next we show that, for any $j \in \mathbb{Z}_{\geq 1}$, the N polynomials $F(\bar{g})^j$, $\bar{g} \in G/Z$, are linearly independent. Assume the contrary:

$$\sum_{\bar{g}\in G/Z} a_{\bar{g}} F(\bar{g})^j = 0 \tag{2}$$

for some constants $a_{\bar{g}} \in \mathbb{F}$, and $a_{\bar{g}_0} \neq 0$. Applying g_0^{-1} to this relation, we may assume that $a_{\bar{1}} \neq 0$. For any $\bar{g} \neq \bar{1}$, the definition (1) implies that the polynomial $F(\bar{g})$ is divisible by v in $k[x_1, \ldots, x_n]$. As $a_{\bar{1}} \neq 0$, it follows from (2) that $F(\bar{1})^j$ is divisible by v in $k[x_1, \ldots, x_n]$ which is a unique factorization domain. Using (1), we see that there is some $h \in G \smallsetminus Z$ such that $h(v)$ is a multiple of v, contrary to the choice of v.

We have shown that the $\mathbb{F}G$-submodule

$$A_j := \langle F(\bar{g})^j \mid \bar{g} \in G/Z\rangle_{\mathbb{F}}$$

of $\mathrm{Sym}^{(N-1)j}(V)$ has dimension N. Since G acts transitively on the basis $\{F(\bar{g})^j \mid \bar{g} \in G/Z\}$ of A_j, with Z being the stabilizer of $F(\bar{1})^j$, A_j is induced from the one-dimensional $\mathbb{F}Z$-module $\langle F(\bar{1})^j\rangle_{\mathbb{F}}$. In fact,

$$A_j = \mathrm{Ind}_Z^G\big(\lambda^{(N-1)j}\big),$$

if λ denotes the linear character via which Z acts on V.

(ii) We will now consider A_j in the range

$$1 \leq j \leq |Z|.$$

For each such j, we choose

$$i := |Z| - j$$

and consider

$$B_i := \langle \prod_{\bar{h}\in G/Z} (\bar{h}(v)^i\rangle_{\mathbb{F}},$$

which is a one-dimensional $\mathbb{F}G$-submodule of $\mathrm{Sym}^{Ni}(V)$. Then Z acts on B_i via the character λ^{Ni}. It follows that Z acts on the $\mathbb{F}G$-module A_jB_i with character $\lambda^m = \lambda^i$, where

$$m := Ni + (N-1)j = N|Z| - j = |G| - j.$$

Moreover,

$$A_jB_i = \mathrm{Ind}_Z^G\big(\lambda^i\big), \tag{3}$$

and it is a submodule of $\mathrm{Sym}^{|G|-j}(V)$.

(iii) For any $k \in \mathbb{Z}_{\geq 0}$, we note that

$$C = \langle \prod_{h\in G} h(v)^k\rangle_{\mathbb{F}},$$

is a trivial $\mathbb{F}G$-submodule in $\mathrm{Sym}^{k|G|}(V)$. Hence,

$$A_jB_iC \cong \mathrm{Ind}_Z^G(\lambda^i),$$

and it is a submodule of $\mathrm{Sym}^{|G|-j+k|G|}(V)$.

We also note that the regular $\mathbb{F}Z$-module $\mathbf{reg}_Z$ is semisimple. (Indeed, this is obvious if $\mathrm{char}(\mathbb{F}) = 0$. If $\mathrm{char}(\mathbb{F}) = p > 0$, then, since the finite subgroup Z embeds in $\mathbb{F}^\times$, $p \nmid |Z|$.) Furthermore, $\mathbf{reg}_Z$ is isomorphic to the direct sum

$$\bigoplus_{t=0}^{|Z|-1} \lambda^t$$

(where we have identified one-dimensional modules with their Brauer characters). In turn, this implies by (3) that

$$\mathbf{reg}_G = \mathrm{Ind}_1^G(\mathbb{F}) \cong \mathrm{Ind}_Z^G(\mathrm{Ind}_1^Z(\mathbb{F})) = \mathrm{Ind}_Z^G(\mathbf{reg}_Z)$$

is isomorphic to the direct sum

$$\bigoplus_{t=0}^{|Z|-1} \mathrm{Ind}_Z^G(\lambda^t) \cong \bigoplus_{j=1}^{|Z|} A_jB_iC.$$

It follows from (ii) that $\mathbf{reg}_G$ is a submodule of

$$M_k := \bigoplus_{j=1}^{|Z|} \mathrm{Sym}^{|G|-j+k|G|}(V).$$

Clearly, $\mathbf{reg}_G$ is projective and injective, see [4, Lemma 45.3]. Hence $\mathbf{reg}_G$ is a direct summand of M_k, as stated. □

Corollary 2. *Let $\mathbb{F}$ be any field, G any finite group, and $V = \mathbb{F}^n$ any finite-dimensional, faithful $\mathbb{F}G$-module. Then, for any $k \in \mathbb{Z}_{\geq 0}$, the regular $\mathbb{F}G$-module $\mathbf{reg}_G$ is a direct summand of*

$$\bigoplus_{j=1}^{|Z|} \mathrm{Sym}^{|G|-j+k|G|}(V),$$

where $Z := \mathbf{Z}(G)$.

Proof. Let $\tilde{V} := V \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. By Proposition 1, the regular $\overline{\mathbb{F}}G$-module is a direct summand of

$$\bigoplus_{j=1}^{t} \mathrm{Sym}^{|G|-j+k|G|}(\tilde{V}),$$

where $t := |\mathbf{Z}(\mathrm{GL}(\tilde{V}) \cap G| \leq |Z|$. By (a variation of) the Noether–Deuring theorem, cf. [5, Problem 3.8.4], the condition of whether or not a finite-dimensional module contains the regular module as a direct summand does not change under field extensions. Hence the statement follows. □

Now we can prove the main result of the paper.

Theorem 3. *Let $\mathbb{F}$ be any field, G any finite group, and $V = \mathbb{F}^n$ any finite-dimensional, faithful $\mathbb{F}G$-module. Let W be any irreducible $\mathbb{F}G$-module. Then there is some integer $1 \leq m \leq |G|$ (depending on W) such that the following statements hold.*

(i) *W is isomorphic to a submodule of $\mathrm{Sym}^m(V)$.*
(ii) *W is isomorphic to a quotient of $\mathrm{Sym}^m(V)$.*

(iii) *In fact, for any $k \in \mathbb{Z}_{\geq 0}$, W is isomorphic to a submodule of* $\mathrm{Sym}^{m+k|G|}(V)$*, and to a quotient of* $\mathrm{Sym}^{m+k|G|}(V)$.

Proof. By Frobenius reciprocity, W is both a submodule and a quotient of the regular $\mathbb{F}G$-module $\mathbf{reg}_G$. Hence the theorem follows from Corollary 2. $\square$

We conclude with some remarks concerning the range of the integers m in Theorem 3 needed to cover all irreducible $\mathbb{F}G$-modules.

Remark 4. (i) Let $V = \mathbb{F} = \overline{\mathbb{F}}$ and $G \leq \mathrm{GL}(V) \cong \mathrm{GL}_1(\mathbb{F})$ be a finite cyclic subgroup. Then, to cover all irreducible $\mathbb{F}G$-modules W, we need to use all $\mathrm{Sym}^m(V)$ with m up to $|G| - 1$.

(ii) Let $\mathrm{char}(\mathbb{F}) = p > 0$, and consider the natural 2-dimensional module $V = \mathbb{F}^2$ of $G \cong \mathrm{SL}_2(p)$. A full set of non-isomorphic irreducible $\mathbb{F}G$-modules is given by $\mathrm{Sym}^m(V)$, $0 \leq m \leq p-1$, see e.g. [1, §10.2]. Thus in Theorem 3, we need to use m up to $\approx |G|^{1/3}$.

(iii) Let $\mathbb{F} = \overline{\mathbb{F}}$ be of characteristic $p > 0$, $f \geq 2$ any integer, and consider the natural 2-dimensional module $V = \mathbb{F}^2$ of $G \cong \mathrm{SL}_2(p^f)$. For any integer $0 \leq m \leq p^{f-1} - 1$, the module $\mathrm{Sym}^m(V)$ has highest weight $m\varpi_1$, whereas the $(p^{f-1})^{\mathrm{th}}$ Frobenius twist $V^{(p^{f-1})}$ of V has highest weight $p^{f-1}\varpi_1$, see e.g. [1, §10.2] or [6, §5.4], and is irreducible of dimension 2. So again in Theorem 3 we have to use m up to $\approx |G|^{1/3}$.

(iv) In general, $\dim(W)$ can grow unboundedly compared to $\dim(V)$. Hence, the m in Theorem 3 cannot be bounded in terms of $\dim(V)$, cf. the examples in (i)–(iii). In fact the example in (i) shows that m cannot be bounded in terms of both $\dim(V)$ and $\dim(W)$, whereas (ii) shows m cannot be bounded in terms of $\dim(V)$ and $|\mathbf{Z}(G)|$. Furthermore, the example in (iii) shows that m cannot be bounded in terms of $\dim(V)$, $\dim(W)$, $|\mathbf{Z}(G)|$, and $p = \mathrm{char}(\mathbb{F})$. One may compare this to the first nontrivial occurrence in $\mathrm{Sym}^m(V)$ of semi-invariants for $\mathbb{C}G$-modules V considered in [9], where it is shown that m can be taken to be at most $1184036(\dim V)$.

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