HYPERGEOMETRIC SHEAVES AND FINITE GENERAL LINEAR GROUPS

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Abstract. We find all irreducible hypergeometric sheaves whose geometric monodromy group is finite, almost quasisimple and has the projective special linear group $\mathrm{PSL}_n(q)$ with $n \geq 3$ as a composition factor. We use the classification of semisimple elements with specific spectra in irreducible Weil representations to prove that if an irreducible hypergeometric sheaf has such geometric monodromy group, then it must be of certain form. Then we extend results of Katz and Tiep on a prototypical family of such sheaves to full generality to show that these hypergeometric sheaves do have such geometric monodromy groups, and that they have some connection to a construction of Abhyankar.

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Contents

Introduction. Let p be a prime and let q be a power of p. A conjecture of Abhyankar [1], proved by Harbater [3], says that the finite quotient groups of the étale fundamental group of the multiplicative group $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$ over $\overline{\mathbb{F}_p}$ are precisely the finite groups G such that $G/\mathbf{O}^{p'}(G)$ is cyclic, where $\mathbf{O}^{p'}(G)$ is the subgroup generated by all Sylow p-subgroups of G. Since this proof is nonconstructive, one might want to realize each of these groups in a faithful complex representation as the monodromy group of an explicitly written $\overline{\mathbb{Q}_\ell}$-local system over $\mathbb{G}_m$.

Hypergeometric sheaves are the simplest rigid local systems over $\mathbb{G}_m$, in the sense that they have the sum of Swan conductors equal to 1, which is the lowest possible nonzero value. Katz, Rojas-Leon and Tiep [13], [15], [12] used hypergeometric sheaves to realize many quotient groups of $\pi_1^{et}(\mathbb{G}_m/\mathbb{F}_q)$. In [14], Katz and Tiep studied the converse direction: they showed that if the geometric monodromy group of a hypergeometric sheaves satisfying a mild condition $(\mathbf{S}+)$ is finite, then it is either almost quasisimple or an "extraspecial normalizer". For the almost quasisimple case, they also gave a list of all possible pairs (S, V) of finite simple groups S and their complex representations V which can occur as the unique nonabelian simple factor of the geometric monodromy group of irreducible hypergeometric sheaves, cf. [14, Section 10].

In this paper, we return to the original viewpoint of constructing local systems that realize given finite groups as their monodromy groups. We focus on one of the five "generic" families of finite almost quasisimple groups listed in [14, Section 10], namely those with the unique nonabelian composition factor $\mathrm{PSL}_n(\mathbb{F}_q)$ with $n \geq 3$. Katz and Tiep [13] found some examples of hypergeometric sheaves realizing some of these groups. However, whether these are the only hypergeometric sheaves realizing these groups, and the analogous statements for other families of finite almost quasisimple groups, were not known. Here we give the first result in this direction: we give a complete list of irreducible hypergeometric sheaves for groups "coming from" $\mathrm{PSL}_n(\mathbb{F}_q)$.

The main results of this paper, namely Theorem 4.1, Theorem 4.2 and Corollary 5.10, says that for $n \geq 3$, the irreducible hypergeometric sheaves whose geometric monodromy group is finite almost quasisimple having $\mathrm{PSL}_n(\mathbb{F}_q)$ as the unique nonabelian composition factor are precisely those of the form

$$\mathcal{H}yp_\psi(\mathsf{Char}(\frac{q^n-1}{q-1}, \chi^{(b+c)j}); \mathsf{Char}(\frac{q^m-1}{q-1}, \chi^{bj}) \cup \mathsf{Char}(\frac{q^{n-m}-1}{q-1}, \chi^{cj})) \otimes \mathcal{L}_\varphi$$

or

$$\mathcal{H}yp_\psi(\mathsf{Char}(\frac{q^n-1}{q-1}) \setminus \{\mathbb{1}\}; \mathsf{Char}(\frac{q^m-1}{q-1}) \cup (\mathsf{Char}(\frac{q^{n-m}-1}{q-1}) \setminus \{\mathbb{1}\})) \otimes \mathcal{L}_\varphi$$

where φ is a multiplicative character of $\overline{\mathbb{F}_p}$ of finite order, χ is that of order precisely $q-1$, ψ is a nontrivial additive character of $\mathbb{F}_p$, and m, b, c are integers satisfying certain conditions.

In section 1, we fix notations and review some known facts about irreducible hypergeometric sheaves and their monodromy groups, which will be needed in the subsequent sections. In section 2, we set up some notations regarding the Weil representations of $\mathrm{GL}_n(\mathbb{F}_q)$, and make some observations which will be used in subsequent sections. Section 3 studies the action of certain p-subgroups of $\mathrm{GL}_n(\mathbb{F}_q)$ on the irreducible Weil modules. Together with some facts we review in section 1, the results of section 2 and 3 completely determine the possible local monodromies at 0 and ∞ of the hypergeometric sheaves we want to study. However, the local pictures at these two points are studied separately, so we need to determine which pairs of them can actually arise as local monodromy of a hypergeometric sheaf with the desired geometric monodromy group. In section 4, we use two new ideas to achieve this: the use of determinant sheaves of irreducible hypergeometric sheaves to find a connection between the local monodromies at 0 and ∞, and some new techniques to find counterexamples for certain inequality called the "V-test". Thanks to these, we obtain short lists Theorem 4.1 and Theorem 4.2 of candidate hypergeometric sheaves. In section 5, we prove that these sheaves do have the desired geometric monodromy groups. This is done by extending the method used in [13] to study a smaller family of hypergeometric sheaves. This family in [13] also had some connection to a work of Abhyankar [2]. We briefly discuss a generalization of this connection to the sheaves in Theorem 4.1 and Theorem 4.2 in the final section.

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1. Preliminary Results and the Basic Set-up

Let $\overline{\mathbb{F}_p}$ be the algebraic closure of the finite field of characteristic p. Fix another prime $\ell \neq p$, and let $\overline{\mathbb{Q}}_\ell$ be the algebraic closure of the field of ℓ-adic numbers. Throughout this paper, we will not distinguish between the $\mathbb{C}$-representations and $\overline{\mathbb{Q}_\ell}$-representations of finite groups. Let K be a finite subfield of $\overline{\mathbb{F}_p}$. We will understand lisse $\overline{\mathbb{Q}_\ell}$-local systems over $\mathbb{G}_m/K$ as continuous representations of the étale fundamental group $\pi_1^{et}(\mathbb{G}_m/K)$. The Zariski closures of the image of $\pi_1^{et}(\mathbb{G}_m/K)$ and the subgroup $\pi_1^{\mathrm{geom}}(\mathbb{G}_m) := \pi_1^{et}(\mathbb{G}_m/\overline{\mathbb{F}_p})$ under this representation are called the *arithmetic* and *geometric monodromy group*, often denoted by G_{arith} and G_{geom}, respectively, of this sheaf.

To study the monodromy of lisse $\overline{\mathbb{Q}_\ell}$-sheaves over $\mathbb{G}_m$, we need to look at their local monodromy at 0 and ∞, that is, the restrictions to the inertia subgroups $I(0)$ and $I(\infty)$ of $\pi_1^{et}(\mathbb{G}_m)$. The inertia subgroup $I(0)$ has a normal pro-p-subgroup, namely the wild inertia subgroup $P(0)$. The quotient $I(0)/P(0)$ is a pro-cyclic group of pro-order prime to p. Fix an element γ_0 of $I(0)$ of pro-order prime to p, such that $\gamma_0 P(0)$ is a topological generator of $I(0)/P(0)$. Similarly, fix $\gamma_\infty \in I(\infty)$ of pro-order prime to p such that $\gamma_\infty P(\infty)$ is a topological generator of $I(\infty)/P(\infty)$.

Let $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}_\ell}^\times$ be a nontrivial additive character. Given D multiplicative characters $\chi_1, \dots, \chi_D : K^\times \to \overline{\mathbb{Q}_\ell}^\times$ and m multiplicative characters $\rho_1, \dots, \rho_m : K^\times \to \overline{\mathbb{Q}_\ell}^\times$, where $D > m \geq 0$, one can define the *hypergeometric sheaf of type* (D, m):

$$\mathcal{H}yp_\psi(\chi_1, \dots, \chi_D; \rho_1, \dots, \rho_m).$$

We will assume that $\{\chi_1, \dots, \chi_D\} \cap \{\rho_1, \dots, \rho_m\} = \emptyset$. Under this assumption, this is lisse on $\mathbb{G}_m/K$, geometrically irreducible, has rank D, and pure of weight $D + m - 1$. For the details of these facts and other basic theory of hypergeometric sheaves, see [6, Chapter 8].

The characters $\chi_1, \dots, \chi_D$ are called the "upstairs characters" of this hypergeometric sheaf. The local monodromy at 0 of this sheaf is tame, and given by the direct sum of Jordan blocks of the Kummer sheaves $\mathcal{L}_{\chi_i}$ defined by the upstairs characters. In particular, the characters $\chi_1, \dots, \chi_D$ must be pairwise distinct whenever the geometric monodromy group is finite, and in this case the local monodromy is just the direct sum $\bigoplus_{i=1}^D \mathcal{L}_{\chi_i}$. Since it is tame, we can view it as a continuous representation of $I(0)/P(0) = \langle \gamma_0 P(0) \rangle$. In particular, the image of γ_0 will completely determine this representation (up to isomorphism).

The "downstairs characters" $\rho_1, \dots, \rho_m$ have a similar but slightly different property. The hypergeometric sheaf $\mathcal{H}yp_\psi(\chi_1, \dots, \chi_D; \rho_1, \dots, \rho_m)$ is not tame at ∞, so the local monodromy at ∞ can be written as a direct sum $\mathsf{Tame} \oplus \mathsf{Wild}$. Here, Tame is a tame representation of rank m determined by the downstairs characters in the same way as how the upstairs characters determine the local monodromy at 0. In addition to that, we have a totally wild part Wild of dimension $D - m$ and Swan conductor 1. Hence, to determine the downstairs character, looking at the image of γ_∞ alone is insufficient; we should also look at the image of $P(\infty)$ and use the following result.

Proposition 1.1. [14, Proposition 4.10] [12, Proposition 5.9] *Let $\mathcal{H}$ be an irreducible hypergeometric sheaf of type (D, m) with $D > m \geq 0$. If $D - m = p^a W_0$ for some integer $a \geq 0$ and $p \nmid W_0$, then we have the following:*

(i) $\mathsf{Wild}|_{P(\infty)}$ *is a direct sum of W_0 multiplicative translates of $P|_{P(\infty)}$ by μ_{W_0}, where P is an irreducible $I(\infty)$-representation of dimension p^a and Swan conductor* 1.

(ii) γ_∞ *cyclically permutes these W_0 irreducible constituents of* $\mathsf{Wild}|_{P(\infty)}$.

(iii) *If $a = 0$, then the image of $P(\infty)$ is isomorphic to the additive group of the finite field $\mathbb{F}_p(\mu_{D-m})$.*

(iv) *If $a > 0$, then there exists a root of unity ζ whose order is prime to p, such that the spectrum of $\gamma_\infty^{W_0}$ on each irreducible constituents of* $\mathsf{Wild}|_{P(\infty)}$ *is $\zeta \cdot (\mu_{p^a+1} \setminus \{1\})$.*

If an irreducible hypergeometric sheaf is not primitive, then it is either Kummer induced or Belyi induced, and both cases can be easily recognized from the upstairs and downstairs characters, cf. [10, Proposition 1.2]. If we restrict ourselves to the primitive cases, then [16, Theorem 5.2.9] and [14, Lemma 1.1] tells us that if our hypergeometric sheaf is of type (D, m) with $D > m$ and $D \neq 4, 8, 9$, and if the geometric monodromy group G_{geom} of this sheaf is finite, then G_{geom} is either almost quasisimple or an "extraspecial normalizer". Moreover, if D is not a prime power and not 1, then G_{geom} is almost quasisimple, and the unique nonabelian composition factor and its representation are as listed in [14, Section 10].

In this paper, we study the sheaves which realize [14, Section 10, case (b)]: the irreducible (but not necessarily primitive) hypergeometric sheaves whose geometric monodromy groups are finite, almost quasisimple with unique nonabelian composition factor $\mathrm{PSL}_n(\mathbb{F}_q)$, $n \geq 3$. Such hypergeometric sheaves are known to exist, and one construction can be found in [13]. Most of such hypergeometric sheaves are known to satisfy several nice properties. We will need some of these properties, which we list below for convenience.

Proposition 1.2. [6, Proposition 8.15.2] [14, Theorem 6.6(ii), Theorem 8.1, Corollary 8.4] [16, Theorem 3.1.10] *Let $\mathcal{H}yp_\psi(\chi_1, \ldots, \chi_D; \rho_1, \ldots, \rho_m)$ be an irreducible hypergeometric sheaf over $\mathbb{G}_m/K$ for some finite field K. Suppose that the geometric monodromy group G is finite, almost quasisimple with unique nonabelian composition factor $\mathrm{PSL}_n(\mathbb{F}_q)$ for an integer $n \geq 3$ and a power q of a prime p. Then:*

- (a) *The upstairs and downstairs characters are pairwise distinct.*
- (b) *The characteristic of K is p, unless (n, q) is one of $(3, 2), (4, 2), (3, 4)$ and $D \leq 10$.*
- (c) *Suppose that $(n, q) \neq (3, 4)$. Then the quasisimple layer $E(G)$ of G is a quotient of $\mathrm{SL}_n(\mathbb{F}_q)$. Moreover, the monodromy representation as a representation of $E(G)$ is an irreducible Weil representation of this quotient of $\mathrm{SL}_n(\mathbb{F}_q)$.*
- (d) *If $(n, q) \neq (3, 2), (3, 3), (3, 4)$, then the image $\overline{g_0}$ of γ_0 under the map $\pi_1^{et}(\mathbb{G}_m) \to G \to G/\mathbf{Z}(G) \subseteq \mathrm{Aut}(\mathrm{PSL}_n(\mathbb{F}_q))$ lies in $\mathrm{PGL}_n(\mathbb{F}_q)$. If in addition $D - m \geq 2$, then $G/\mathbf{Z}(G) \cong \mathrm{PGL}_n(\mathbb{F}_q)$.*

When $D - m = 1$, [14, Corollary 8.4] (which is the second part of Proposition 1.2(d)) does not apply, since it relies on [14, Theorem 4.1] which requires $D - m \geq 2$. However, we can at least prove the following weaker version.

Proposition 1.3. *Let $\mathcal{H}$ be a hypergeometric sheaf as in Proposition 1.2, and let $G = G_{\text{geom}}$ be the geometric monodromy group of this. Assume that the conclusion of Proposition 1.2(c) holds. Suppose that $D - m = 1$ and $(n, q) \neq (3, 2), (3, 3), (3, 4)$. Let $g_\infty \in G$ be the image of $\gamma_\infty \in I(\infty)$ in G, and let $\overline{g_\infty} \in G/\mathbf{Z}(G)$ be its image. Then $\overline{g_\infty} \in \mathrm{PGL}_n(\mathbb{F}_q)$.*

Proof. The spectrum of g_∞ on $\mathcal{H}$ is the union of the spectrum on Tame and that on Wild. The spectrum on Tame corresponds to the upstairs characters, so in particular g_∞ has at least $m = D - 1$ distinct eigenvalues. Since the restriction of the monodromy representation is an irreducible Weil representation of $\mathrm{SL}_n(\mathbb{F}_q)$, D is either $\frac{q^n - q}{q-1}$ or $\frac{q^n-1}{q-1}$. Therefore, the order of g_∞ is at least $\frac{q^n-1}{q-1} - 2$. Now we can apply the first part of the proof of [14, Theorem 8.1]. ☐

Instead of excluding all the exceptional pairs of (n, q) in Proposition 1.2 and Proposition 1.3, we want to include those hypergeometric sheaves which satisfies the conclusions of the above propositions. Therefore, we will study the hypergeometric sheaves $\mathcal{H}$ with the geometric monodromy group G which satisfies the following conditions:

$(\star)$
$$\left(\begin{array}{l} \mathcal{H} \text{ is irreducible and lisse on } \mathbb{G}_m/K \text{ for a finite extension } K/\mathbb{F}_p. \\ G \text{ is finite, almost quasisimple with unique composition factor } \mathrm{PSL}_n(\mathbb{F}_q) \text{ for some} \\ \text{integer } n \geq 3 \text{ and a power } q \text{ of } p. \\ \text{The images of } \gamma_0, \gamma_\infty \text{ under the map } \pi_1^{et}(\mathbb{G}_m) \to G \to G/\mathbf{Z}(G) \text{ are in } \mathrm{PGL}_n(\mathbb{F}_q). \\ \text{The quasisimple layer } E(G) \text{ is a quotient of } \mathrm{SL}_n(\mathbb{F}_q) \text{, and the restriction of the} \\ \text{monodromy representation of } \mathcal{H} \text{ to } E(G) \text{ is an irreducible Weil representation.} \end{array}\right)$$

By Proposition 1.2 and Proposition 1.3, except when $(n,q) = (3,2),(3,3),(3,4),(4,2)$, the last two conditions in $(\star)$ and that K has characteristic p are redundant, and $(\star)$ is equivalent to:

$$\left(\begin{array}{l}\mathcal{H} \text{ is irreducible and lisse on } \mathbb{G}_m/K \text{ for a finite field } K.\\ G \text{ is finite, almost quasisimple with unique composition factor } \mathrm{PSL}_n(\mathbb{F}_q) \text{ for some}\\ \text{integer } n \geq 3 \text{ and a power } q \text{ of } p.\end{array}\right)$$

Remark 1.4. In the situation of $(\star)$, let g_0, g_∞ be the images of γ_0, γ_∞ in G under the monodromy representation, and let $\overline{g_0}, \overline{g_\infty}$ be their images in $G/\mathbf{Z}(G) \leq \mathrm{Aut}(\mathrm{PSL}_n(\mathbb{F}_q))$, so that $\overline{g_0}, \overline{g_\infty} \in \mathrm{PGL}_n(\mathbb{F}_q)$. The monodromy representation gives a projective representation of $\mathrm{PSL}_n(\mathbb{F}_q) = E(G)/\mathbf{Z}(E(G))$, which comes from an irreducible Weil representation of $\mathrm{SL}_n(\mathbb{F}_q)$. Hence, if we take the restriction of the monodromy representation to the subgroup $\langle E(G), g_0, g_\infty\rangle$ of G, then this gives a projective representation of the subgroup $\langle E(G)/\mathbf{Z}(E(G)), \overline{g_0}, \overline{g_\infty}\rangle \leq \mathrm{PGL}_n(\mathbb{F}_q)$. This can be lifted to an irreducible Weil representation of the corresponding subgroup H of $\mathrm{GL}_n(\mathbb{F}_q)$. Therefore, the spectrum of γ_0 on $\mathcal{H}$ is just a root of unity times the spectrum of an inverse image $h_0 \in H \leq \mathrm{GL}_n(\mathbb{F}_q)$ of $\overline{g_0}$ on this irreducible Weil representation. The same statement holds for γ_∞ with another root of unity.

2. Weil Representations of Finite General Linear Groups

Suppose that we have a hypergeometric sheaf $\mathcal{H}$ with the geometric monodromy group G which satisfies $(\star)$. The discussions in section 1, including Proposition 1.1 and Proposition 1.2, tells us that the spectrum of γ_0 on $\mathcal{H}$ cannot have an eigenvalue with multiplicity larger than 1, and the spectrum of γ_∞ on each of Tame and Wild also have the same property. As we saw in Remark 1.4, these spectra are just a root of unity times the spectra of some elements of $\mathrm{GL}_n(\mathbb{F}_q)$ on some irreducible Weil representation. In this section, we classify the elements of $\mathrm{GL}_n(\mathbb{F}_q)$ whose eigenvalues on some irreducible Weil module have multiplicity at most 2.

Fix a generator α of the cyclic group $\mathbb{F}_q^\times$, and a primitive $(q-1)$th root of unity $\lambda \in \overline{\mathbb{Q}_\ell}^\times$. Let η be a multiplicative character of $\mathbb{F}_q$ that maps α to λ. Consider the natural (left) permutation action of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$. The corresponding $\mathbb{C}\,\mathrm{GL}_n(\mathbb{F}_q)$-module is called the total Weil module. We will denote this by Weil. It has a standard basis $\{e_v \mid v \in \mathbb{F}_q^n\}$, and each $g \in \mathrm{GL}_n(\mathbb{F}_q)$ acts by $e_v \mapsto e_{gv}$. For each $0 \neq v \in \mathbb{F}_q^n$ and each $j = 0, \ldots, q-2$, define $v^{(j)} = \sum_{i=0}^{q-2} \lambda^{-ij} e_{\alpha^i v}$. Then

$$(\alpha I).v^{(j)} = \sum_{i=0}^{q-2} \lambda^{-ij}(\alpha I).e_{\alpha^i v} = \sum_{i=0}^{q-2} \lambda^{-ij} e_{\alpha^{i+1}v} = \sum_{i=0}^{q-2} \lambda^{-(i-1)j} e_{\alpha^i v} = \lambda^j v^{(j)}.$$

In particular, the element $\alpha I \in \mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))$ has an eigenvalue λ^j on Weil, and $v^{(j)}$ is an eigenvector associated to this eigenvalue. Note that $(\alpha v)^{(j)} = \lambda^j v^{(j)}$, so if we choose one nonzero v from each one-dimensional subspace of $\mathbb{F}_q^n$, then the $v^{(j)}$'s for those v together with e_0 form a basis of Weil. Let $\mathsf{Weil}_j = \mathbb{C}\langle v^{(j)} \mid 0 \neq v \in \mathbb{F}_q^n\rangle$. Then we get the direct sum decomposition

$$\mathsf{Weil} = \mathbb{C}e_0 \oplus \bigoplus_{j=0}^{q-1} \mathsf{Weil}_j\,.$$

The submodules Weil_j have dimension $\frac{q^n-1}{q-1}$, and the restriction of Weil_j to $\mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))$ is precisely $\frac{q^n-1}{q-1}\eta^j$; recall that η is the linear character which maps α to λ. Moreover, Weil_j is irreducible unless $j = 0$, in which case $\mathsf{Weil}_0 = \mathbb{1} \oplus \mathsf{Weil}_0'$. The $\mathbb{C}\,\mathrm{GL}_n(\mathbb{F}_q)$-modules Weil_j and Weil_0', together with their tensor products with linear characters of $\mathrm{GL}_n(\mathbb{F}_q)$, are called the irreducible Weil modules.

Fix an element g of $\mathrm{GL}_n(\mathbb{F}_q)$. To study the spectrum of the action of g on Weil_j, it is convenient to make the following definitions. For a nonzero vector $v \in \mathbb{F}_q^n \setminus \{0\}$, let s_v be the smallest positive integer which satisfies $g^{s_v} v = \alpha^t v$ for some integer t, and let t_v be this t. For two nonzero vectors $v, w \in \mathbb{F}_q^n \setminus \{0\}$, we will say $v \sim_g w$ if $w = \beta g^r v$ for some $\beta \in \mathbb{F}_q^\times$ and some nonnegative integer r, that is, if v and w lie in the same $\langle g, \mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))\rangle$-orbit. This defines an equivalence relation on $\mathbb{F}_q^n \setminus \{0\}$. Also, let $V(g; v)$ and $\mathsf{Weil}_j(g; v)$ denote the g-cyclic subspaces of $\mathbb{F}_q^n$ and Weil_j generated by v and $v^{(j)}$, respectively. Then we can easily see the following properties.

Lemma 2.1. *Let $v, w \in \mathbb{F}_q^n \setminus \{0\}$.*

(1) *$v \sim_g w$ if and only if $\mathsf{Weil}_j(g; v) = \mathsf{Weil}_j(g; w)$. Moreover, if $v \not\sim_g w$, then $\mathsf{Weil}_j(g; v) \cap \mathsf{Weil}_j(g; w) = 0$.*
(2) *Weil_j is the direct sum of subspaces of the form $\mathsf{Weil}_j(g; v)$, one for each equivalence class of $\sim_g$.*
(3) *s_v and t_v only depends on the $\sim_g$-equivalence class of v.*
(4) *The eigenvalues of the action of g on $\mathsf{Weil}_j(g; v)$ are the s_vth roots of $\lambda^{t_v j}$.*

Proof. For (4), let $\xi \in \mathbb{C}$ be a number such that $\xi^{s_v} = \lambda^t$. The vector $\sum_{i=0}^{s_v-1} \xi^{-ij} g^i.v^{(j)} \in \mathsf{Weil}_j(g; v)$ then satisfies

$$\begin{aligned} g.\left(\sum_{i=0}^{s_v-1} \xi^{-ij} g^i.v^{(j)}\right) &= \sum_{i=0}^{s_v-1} \xi^{-ij} g^{i+1}.v^{(j)} = \sum_{i=1}^{s_v-1} \xi^{-(i-1)j} g^i.v^{(j)} + \xi^{-(s_v-1)j} g^{s_v}.v^{(j)} \\ &= \sum_{i=1}^{s_v-1} \xi^{-(i-1)j} g^i.v^{(j)} + \xi^j \left(\sum_{k=0}^{q-2} \lambda^{-(k+t)j} e_{\alpha^{k+t}v}\right) \\ &= \sum_{i=1}^{s_v-1} \xi^{-(i-1)j} g^i.v^{(j)} + \xi^j v^{(j)} = \xi^j \sum_{i=0}^{s_v-1} \xi^{-ij} g^i.v^{(j)}. \end{aligned}$$

Therefore, this vector is an eigenvector of the action of g on $\mathsf{Weil}_j(g; v)$ with eigenvalue ξ^j. Since we can choose s_v distinct ξ's and $\dim \mathsf{Weil}_j(g; v) = s_v$, these vectors form an eigenbasis of the action of g on this cyclic subspace.

□

Lemma 2.1 implies that on each $\mathsf{Weil}_j(g; v)$, the action of g has no repeated eigenvalues. Repeated eigenvalues can only occur by appearing in more than one g-cyclic subspaces; this is equivalent to saying that there exists $v \not\sim_g w \in \mathbb{F}_q^n \setminus \{0\}$ such that a s_vth root of $\lambda^{t_v j}$ is also a s_wth root of $\lambda^{t_w j}$. One important situation where this happens is the following.

Lemma 2.2. *If $v, w \in \mathbb{F}_q^n \setminus \{0\}$ satisfy $v \in V(g; w)$ (or equivalently $V(g; v) \subseteq V(g; w)$), then s_v divides s_w, and $t_v(s_w/s_v) \equiv t_w \bmod q-1$. Consequently, the spectrum of the action of g on $\mathsf{Weil}_j(g; v)$ is included in the spectrum of the action of g on $\mathsf{Weil}_j(g; w)$.*

Proof. Since g^{s_w} acts on $V(g; w)$ as a scalar multiplication by α^{t_v}, $g^{s_w} v = \alpha^{t_w} v$. By the definition of s_v, it must divide s_w, and $\alpha^{t_w} v = g^{s_w} v = g^{s_v(s_w/s_v)} v = \alpha^{t_v(s_w/s_v)} v$. Since α has order $q-1$, it follows that $t_v(s_w/s_v) \equiv t_w \bmod q-1$. Also, the set of s_wth roots of $\lambda^{t_w j}$ includes the set of s_vth roots of $\lambda^{t_v j}$th roots. By Lemma 2.1, they are precisely the spectra of the action of g on $\mathsf{Weil}_j(g; w)$ and $\mathsf{Weil}_j(g; v)$, respectively. □

Now we focus on the situation where the action of g on Weil_j has no eigenvalue of multiplicity larger than 2. The next result is an immediate consequence of Lemma 2.2.

Corollary 2.3. *Suppose that the action of g on Weil_j has no eigenvalue of multiplicity larger than 2. Then*

(a) *There are no $u, v, w \in \mathbb{F}_q^n \setminus \{0\}$ in distinct $\sim_g$-equivalence classes such that $V(g;u) \cap V(g;v) \supseteq V(g;w)$. In particular, all g-cyclic subspaces are either minimal or maximal.*
(b) *If $V(g;v)$ is minimal among the g-cyclic subspaces of $\mathbb{F}_q^n$, then $V(g;v) \setminus \{0\}$ is the union of at most two $\sim_g$-equivalence classes.*
(c) *If a g-cyclic subspace W is a union of $\{0\}$ and exactly two $\sim_g$-equivalence classes, then it intersects $V(g;v)$ trivially for all $v \in \mathbb{F}_q^n \setminus W$. In particular, it is both minimal and maximal among the g-cyclic subspaces of $\mathbb{F}_q^n$. Also, the two $\sim_g$-equivalence classes in W have the same s and t.*

The condition of having no eigenvalues of multiplicity larger than 2 not only limits the number of $\sim_g$-equivalence classes with "compatible" s and t, but also says something about those with ss and ts which do not directly give common eigenvalues.

Lemma 2.4. *Suppose that the action of g on Weil_j has no eigenvalue of multiplicity larger than 2. Suppose that $v, w \in \mathbb{F}_q^n \setminus \{0\}$ are such that $\frac{s_v t_w}{\gcd(s_v,s_w)} \not\equiv \frac{s_w t_v}{\gcd(s_v,s_w)} \bmod q-1$. Then*

(a) *$V(g;v)$ and $V(g;w)$ are minimal but not maximal. In particular, $V(g;v) \cap V(g;w) = 0$, $s_v = \frac{q^{\dim V(g;v)}-1}{q-1}$, and $s_w = \frac{q^{\dim V(g;w)}-1}{q-1}$.*
(b) *$(V(g;v) + V(g;w)) \setminus (V(g;v) \cup V(g;w))$ is a single $\sim_g$-equivalence class, and it generates $V(g;v) \oplus V(g;w)$.*
(c) *If $u \in V(g;v) \oplus V(g;w) \setminus (V(g;v) \cup V(g;w))$, then $s_u = \frac{(q^{\dim V(g;v)}-1)(q^{\dim V(g;w)}-1)}{q-1}$ and $t_u = 0$.*
(d) *$\dim V(g;v)$ is relatively prime to $\dim V(g;w)$, and $t_v \dim V(g;w) - t_w \dim V(g;v)$ is relatively prime to $q-1$.*

Proof. Note that if $V(g;v) \cap V(g;w) \neq 0$, then by Corollary 2.3(a), one of $V(g;v)$ and $V(g;w)$ contains the other. If $V(g;v) \subseteq V(g;w)$, then by Lemma 2.2, $\frac{t_v s_w}{\gcd(s_v,s_w)} = t_v(s_w/s_v) \equiv t_w = \frac{t_w s_v}{\gcd(s_v,s_w)}$ mod $q-1$, which contradicts our assumption. Similarly we cannot have $V(g;w) \subseteq V(g;v)$, so $V(g;v) \cap V(g;w) = 0$.

Let $u \in V(g;v) + V(g;w) \setminus (V(g;v) \cup V(g;w))$. Then we can write $u = u_v + u_w$ for some (unique) $u_v \in V(g;v) \setminus \{0\}$ and $u_w \in V(g;w) \setminus \{0\}$. Hence,

$$\begin{aligned}
g^{\mathrm{lcm}(s_v,s_w)}u - \alpha^{\frac{t_v s_w}{\gcd(s_v,s_w)}}u &= g^{s_v \frac{s_w}{\gcd(s_v,s_w)}}u_v + g^{s_w \frac{s_v}{\gcd(s_v,s_w)}}u_w - \alpha^{\frac{t_v s_w}{\gcd(s_v,s_w)}}(u_v + u_w) \\
&= \alpha^{t_v \frac{s_w}{\gcd(s_v,s_w)}}u_v + \alpha^{t_w \frac{s_v}{\gcd(s_v,s_w)}}u_w - \alpha^{\frac{t_v s_w}{\gcd(s_v,s_w)}}(u_v + u_w) \\
&= (\alpha^{t_w \frac{s_v}{\gcd(s_v,s_w)}} - \alpha^{\frac{t_v s_w}{\gcd(s_v,s_w)}})u_w \in V(g;u) \cap V(g;w).
\end{aligned}$$

By assumption, this is nonzero, so $V(g;u) \cap V(g;w) \neq 0$. Since $u \notin V(g;w)$, by Corollary 2.3(a), $V(g;w) \subseteq V(g;u)$, and similarly $V(g;v) \subseteq V(g;u)$. Therefore $V(g;v) \oplus V(g;w) \subseteq V(g;u) = V(g;u_v + u_w) \subseteq V(g;v) + V(g;w) = V(g;v) \oplus V(g;w)$, so the equality holds. Moreover, by Corollary 2.3(a), $V(g;u)$ is generated as a g-cyclic subspace by only one $\sim_g$-equivalence class, so $V(g;u) \setminus (V(g;v) \cup V(g;w))$ is a single $\sim_g$-equivalence class.

Corollary 2.3(a) also tells us that both $V(g;v)$ and $V(g;w)$ are minimal g-cyclic subspaces, and since they are also not maximal, by Corollary 2.3(c) both $V(g;v) \setminus \{0\}$ and $V(g;w) \setminus \{0\}$ are single $\sim_g$-equivalence classes, so in particular $s_v = \frac{q^{\dim V(g;v)}-1}{q-1}$ and $s_w = \frac{q^{\dim V(g;w)}-1}{q-1}$. Also,

$$s_u = \frac{q^{\dim V(g;u)} - 1}{q-1} - s_v - s_w = \frac{(q^{\dim V(g;v)} - 1)(q^{\dim V(g;w)} - 1)}{q-1} = (q-1)s_v s_w.$$

On the other hand, $g^{s_v} = g^{\frac{q^{\dim V(g;v)}-1}{q-1}}$ acts as α^{t_v} on $V(g;v)$, and $g^{s_w} = g^{\frac{q^{\dim V(g;w)}-1}{q-1}}$ acts as α^{t_w} on $V(g;w)$. Hence, $g^{s_v s_w}$ acts as $\alpha^{t_v \frac{q^{\dim V(g;w)}-1}{q-1}} = \alpha^{t_v \dim V(g;w)}$ on $V(g;v)$ and as $\alpha^{t_w \frac{q^{\dim V(g;v)}-1}{q-1}} = \alpha^{t_w \dim V(g;v)}$ on $V(g;w)$. Since $s_u = (q-1)s_v s_w$ is the smallest integer s such that g^s acts on $V(g;u)$ as a scalar (which is α^{t_u}), it follows that the order of $\alpha^{t_v \dim V(g;w) - t_w \dim V(g;v)}$ is exactly $q-1$. Therefore $t_v \dim V(g;w) - t_w \dim V(g;v)$ is relatively prime to $q-1$, and $t_u = 0$.

Since $g^{s_v(q-1)}$ acts trivially on $V(g;v)$ and $g^{s_w(q-1)}$ acts trivially on $V(g;w)$, so $g^{\mathrm{lcm}(s_v(q-1), s_w(q-1))} = g^{(q-1)\,\mathrm{lcm}(s_v,s_w)}$ acts trivially on $V(g;u)$. Since $t_u = 0$, $g^{s_u} = g^{(q-1)s_v s_w}$ is the smallest power of g acting trivially on $V(g;u)$, we must have $\mathrm{lcm}(s_v, s_w) = s_v s_w$. This just means that $s_v = \frac{q^{\dim V(g;v)}-1}{q-1}$ is relatively prime to $s_w = \frac{q^{\dim V(g;w)}-1}{q-1}$, or equivalently, $\dim V(g;v)$ is relatively prime to $\dim V(g;w)$. □

Proposition 2.5. *If the action of g on Weil_j has no eigenvalue of multiplicity larger than 2, then $\mathbb{F}_q^n$ is g-cyclic.*

Proof. Suppose that we have two nonzero vectors $v, w \in \mathbb{F}_q^n$ such that $V(g;w)$ is a maximal g-cyclic subspace of $\mathbb{F}_q^n$ and $V(g;v) \not\subseteq V(g;w)$ (so in particular $V(g;w) \neq \mathbb{F}_q^n$). By the previous results and the maximality of $V(g;w)$, we must have $V(g;v) \cap V(g;w) = 0$ and $\frac{s_v t_w}{\gcd(s_v,s_w)} \equiv \frac{s_w t_v}{\gcd(s_v,s_w)} \bmod q-1$. Let a, b be integers such that $as_v - bs_w = \gcd(s_v, s_w)$, and let $\xi \in \mathbb{C}$ be a $\gcd(s_v, s_w)$th root of $\lambda^{(at_v - bt_w)j}$. Then

$$\xi^{s_v} = \xi^{\gcd(s_v,s_w)\frac{s_v}{\gcd(s_v,s_w)}} = \lambda^{\frac{at_v s_v j - bt_w s_v j}{\gcd(s_v,s_w)}} = \lambda^{j\frac{at_v s_v - bt_v s_w}{\gcd(s_v,s_w)}} = \lambda^{t_v j}$$

and

$$\xi^{s_w} = \xi^{\gcd(s_v,s_w)\frac{s_w}{\gcd(s_v,s_w)}} = \lambda^{\frac{at_v s_w j - bt_w s_w j}{\gcd(s_v,s_w)}} = \lambda^{j\frac{at_w s_v - bt_w s_w}{\gcd(s_v,s_w)}} = \lambda^{t_w j}.$$

Therefore, the spectra of the actions of g on both $\mathsf{Weil}_j(g;v)$ and $\mathsf{Weil}_j(g;w)$ contain ξ. In particular, if $V(g;v)$ is not maximal among the g-cyclic subspaces of $\mathbb{F}_q^n$, then ξ must have multiplicity at least 3 as an eigenvalue of the action of g on Weil_j. This is impossible by assumption, so $V(g;v)$ is maximal. Also, both $V(g;v)$ and $V(g;w)$ are generated as g-cyclic subspaces by unique $\sim_g$-equivalence class. This is true for all such pairs v, w, so it follows that every g-cyclic subspace is maximal (so they are also all minimal) and generated by unique $\sim_g$-equivalence class. Consequently, $V(g;v) \setminus \{0\}$ is the $\sim_g$-equivalence class containing v, so in particular $s_v = \frac{q^{\dim V(g;v)}-1}{q-1}$. Similarly, $s_w = \frac{q^{\dim V(g;w)}-1}{q-1}$.

If $\dim V(g;v) = \dim V(g;w)$, then $s_v = s_w$ and

$$t_w = \frac{s_v t_w}{\gcd(s_v, s_w)} \equiv \frac{s_w t_v}{\gcd(s_v, s_w)} = t_v \bmod q-1.$$

Therefore, two $\sim_g$-equivalence classes generating g-cyclic subspaces of $\mathbb{F}_q^n$ of same dimension gives subspaces of Weil_j on which the actions of g have the same spectra. In particular, there are at most two g-cyclic subspaces of $\mathbb{F}_q^n$ of the same dimension. Since the g-cyclic subspaces intersect trivially, we can also see that there is at most one g-cyclic subspace of dimension larger than $n/2$.

If there is some $u \in \mathbb{F}_q^n \setminus \{0\}$ with $\dim V(g;u) = 1$, then $s_u = 1$ and

$$s_v t_u = \frac{s_v t_u}{\gcd(s_v, s_u)} \equiv \frac{s_u t_v}{\gcd(s_v, s_u)} = t_v \bmod q-1$$

for any $v \in \mathbb{F}_q^n \setminus \{0\}$. In particular, the spectrum of the action of g on $\mathsf{Weil}_j(g;u)$ is contained in the spectrum of the action of g on $\mathsf{Weil}_j(g;v)$. Therefore, there are at most two $\sim_g$-equivalence classes, so there are two g-cyclic subspaces of $\mathbb{F}_q^n$, and both of them are proper (otherwise one of them is not maximal). This is impossible, since no nontrivial vector space is a union of two proper subspaces, while the union of all g-cyclic subspaces is $\mathbb{F}_q^n$.

Therefore, there is no g-cyclic subspace of dimension 1, and the number of elements in the union of all g-cyclic subspaces of $\mathbb{F}_q^n$ cannot exceed

$$q^{n-1} + 2\sum_{i=2}^{\lfloor n/2\rfloor}(q^i - 1) = q^{n-1} + 2\frac{q^{\lfloor n/2\rfloor+1} - q^2}{q-1} - 2\lfloor n/2\rfloor < q^n \ (n \geq 3).$$

Since the union of all g-cyclic subspaces of $\mathbb{F}_q^n$ is $\mathbb{F}_q^n$, this is impossible. Therefore, no such pair v, w exist, so there is only one maximal g-cyclic subspace of $\mathbb{F}_q^n$, and it must be $\mathbb{F}_q^n$ itself.

□

In the case of $j = 0$, we can also have some situations where the multiplicity of 1 as an eigenvalue of the action of g on Weil_0 is 3, and all other eigenvalues have multiplicity at most 2. Fortunately, this situation is not very complicated, since when $j = 0$, Lemma 2.1 shows that the action of g on $\mathsf{Weil}_0(g; v)$ always has 1 as an eigenvalue regardless of the choice of v.

Proposition 2.6. *Suppose that the action of g on Weil_0 has no eigenvalue of multiplicity larger than 2 except for the eigenvalue 1 which has multiplicity exactly 3. Then there are exactly 3 distinct $\sim_g$-equivalence classes in $\mathbb{F}_q^n \setminus \{0\}$, and at least one of them generates $\mathbb{F}_q^n$. Moreover, if the three nontrivial g-cyclic subspaces of $\mathbb{F}_q^n$ form a chain, then the smallest g-cyclic subspace has dimension 1.*

Proof. By the discussion above, the multiplicity of the eigenvalue 1 is precisely the number of distinct $\sim_g$-equivalence classes of $\mathbb{F}_q^n \setminus \{0\}$. Let u, v, w be representatives of these equivalence classes.

If $V(g; u) \subseteq V(g; v) \subseteq V(g; w)$, then by Lemma 2.2, the eigenvalues of the action of g on $\mathsf{Weil}_0(g; u)$ has multiplicity 3 as eigenvalues of the action on Weil_0. Since 1 is the only such eigenvalue, it follows that

$$\frac{q^{\dim V(g;u)} - 1}{q-1} = s_u = \dim \mathsf{Weil}_0(g; u) = 1$$

so that $\dim V(g; u) = 1$. Also, $V(g; w)$ is the union of all g-cyclic subspaces of $\mathbb{F}_q^n$, so $V(g; w) = \mathbb{F}_q^n$.

If $\mathbb{F}_q^n$ is not g-cyclic, then $V(g; u), V(g; v), V(g; w)$ are all proper, and $\mathbb{F}_q^n = V(g; u) \cup V(g; v) \cup V(g; w)$. Since $q^n > q^{n-1} + (q^{n-2} - 1) + (q^{n-2} - 1)$, at least two of the g-cyclic subspaces must have dimension $n - 1$. If these two intersect trivially, then $n - 1 \leq n/2$, which is impossible since $n \geq 3$. Hence, these two must intersect nontrivially, and the intersection must contain another g-cyclic subspace, so we have only two maximal g-cyclic subspaces. Since $\mathbb{F}_q^n$ is not a union of two proper subspaces, this is impossible. Therefore, $\mathbb{F}_q^n$ is g-cyclic. □

By combining the previous three results, we get the main result of this section.

Theorem 2.7. *Suppose that either $j > 0$ and the action of g on Weil_j has no eigenvalue of multiplicity larger than 2, or the action of g on Weil_0 has no eigenvalue of multiplicity larger than 2 other than 1 which has multiplicity at most 3. Then g is one of the following.*

(a) *$g = \alpha_n^a$, where α_n is a generator of $\mathbb{F}_{q^n}^\times$ such that $\alpha_n^{\frac{q^n-1}{q-1}} = \alpha$, viewed as an element of $\mathrm{GL}_n(\mathbb{F}_q)$ via some embedding $\mathrm{GL}_1(\mathbb{F}_{q^n}) \hookrightarrow \mathrm{GL}_n(\mathbb{F}_q)$, and a is an integer relatively prime to $|\alpha_n|/(q-1) = (q^n-1)/(q-1)$. The spectrum of the action of g on Weil_j is the $\frac{q^n-1}{q-1}$th roots of λ^{aj}.*
(b) *The squares of the elements described in (a), when $(q^n - 1)/(q - 1)$ is even.*
(c) *$g = \alpha_m^b \oplus \alpha_{n-m}^c$. Here, m is a positive integer relatively prime to n, and α_m and α_{n-m} are as in (c). b is an integer relatively prime to $\frac{q^m-1}{q-1}$, c is an integer relatively prime to $\frac{q^{n-m}-1}{q-1}$, and $b(n-m) - cm$ must be relatively prime to $q - 1$. The spectrum of the action*

of g on Weil_j is the $\frac{q^m-1}{q-1}$th roots of λ^{bj}, the $\frac{q^{n-m}-1}{q-1}$th roots of λ^{cj}, and the $\frac{(q^m-1)(q^{n-m}-1)}{q-1}$th roots of unity.

(d) $g = \begin{pmatrix} \alpha_m^b & X \\ 0 & \alpha_{n-m}^c \end{pmatrix}$ *for some positive integer m dividing n. Here, α_m and α_{n-m} are defined in the same way α_n is defined in (a), and we view them as elements of $\mathrm{GL}_n(\mathbb{F}_q)$ via some embedding $\mathrm{GL}_1(\mathbb{F}_{q^m}) \oplus \mathrm{GL}_1(\mathbb{F}_{q^{n-m}}) \hookrightarrow \mathrm{GL}_m(\mathbb{F}_q) \oplus \mathrm{GL}_{n-m}(\mathbb{F}_q) \hookrightarrow \mathrm{GL}_n(\mathbb{F}_q)$. b is an integer relatively prime to $\frac{q^m-1}{q-1}$, c is an integer relatively prime to $\frac{q^{n-m}-1}{q-1}$, and $c \equiv b\frac{n-m}{m} \mod q-1$. X is a nonzero $m \times (n-m)$ matrix. The spectrum of the action of g on Weil_j is the $\frac{q^m-1}{q-1}$th roots of λ^{bj} and the $\frac{q^n-q^m}{q-1}$th roots of λ^{cj}.*

(e) $j = 0$, *and* $g = \begin{pmatrix} \alpha^a & X_{12} & X_{13} \\ 0 & \alpha_{m-1}^b & X_{23} \\ 0 & 0 & \alpha_{n-m}^c \end{pmatrix}$. *Here, m is an integer larger than 1 such that $m-1$ divides $n-1$. α_{m-1} and α_{n-m} are as in (c). a, b, c are integers such that b is relatively prime to $\frac{q^{m-1}-1}{q-1}$, c is relatively prime to $\frac{q^{n-m}-1}{q-1}$, $c \equiv b\frac{n-m}{m} \mod q-1$, and $b \equiv a(m-1) \mod q-1$. X_{12} is a nonzero $1 \times (m-1)$ matrix, and X_{13}, X_{23} are $1 \times (n-m)$ and $(m-1) \times (n-m)$ matrices, respectively, such that at least one of them is nonzero. The spectrum of g on Weil_0 is the $\frac{q^m-q}{q-1}$th roots of unity, the $\frac{q^n-q^m}{q-1}$th roots of unity, and an additional 1.*

(f) $j = 0$, *q is even, and* $g = \begin{pmatrix} \alpha^a & X \\ 0 & \alpha_{n-1}^b \end{pmatrix}$. *Here, α_{n-1} is as in (c), a is an integer, and b is an integer relatively prime to $\frac{q^{n-1}-1}{q-1}$. The spectrum of the action of h on Weil_0 is the $\frac{q^n-q}{2(q-1)}$th roots of 1, each of them having multiplicity 2 except for the eigenvalue 1 which has multiplicity 3.*

(g) $j = 0$, *q is odd, and* $g = \begin{pmatrix} \alpha^a & X \\ 0 & \alpha_{n-1}^{2b} \end{pmatrix}$. *Here, α_{n-1} is as in (c), a is an integer, and b is an integer relatively prime to $\frac{q^{n-1}-1}{q-1}$. The spectrum of the action of h on Weil_0 is the $\frac{q^n-q}{2(q-1)}$th roots of 1, each of them having multiplicity 2 except for the eigenvalue 1 which has multiplicity 3.*

Remark 2.8. In cases (d), (e), (f) and (g), the converse for these cases may or may not hold, i.e. there might be some elements of $\mathrm{GL}_n(\mathbb{F}_q)$ which satisfy the conditions of one of these cases and still have some eigenvalues of multiplicity larger than 2. To make the converse true, we might need more conditions for the submatrices on the upper right corners. However, we will only be interested about the cases where g has order prime to p, which are precisely the cases (a), (b) and (c). For these cases, the converse holds.

proof of Theorem 2.7. By Proposition 2.5 and Proposition 2.6, we always have one $\sim_g$-equivalence class which generates $\mathbb{F}_q^n$. If there is no proper g-cyclic subspace of $\mathbb{F}_q^n$, then by Corollary 2.3, there are at most 2 distinct $\sim_g$-equivalence classes in $\mathbb{F}_q^n \setminus \{0\}$. If $j = 0$ and there are some proper g-cyclic subspaces, then there are at most 3 distinct $\sim_g$-equivalence classes in $\mathbb{F}_q^n \setminus \{0\}$ by Proposition 2.6.

The same is true when $j > 0$; to see this, note that if $u, v, w \in \mathbb{F}_q^n \setminus \{0\}$ are such that $v \not\sim_g w$, $V(g; v), V(g; w)$ are both proper, $V(g; u) = \mathbb{F}_q^n$, and $\frac{s_v t_w}{\gcd(s_v, s_w)} \equiv \frac{s_w t_v}{\gcd(s_v, s_w)} \mod q-1$, then $\mathsf{Weil}_j(g; v), \mathsf{Weil}_j(g; w)$ and $\mathsf{Weil}_j(g; u)$ have common eigenvalue by the proof of Proposition 2.5 together with Lemma 2.2. This is impossible, so if there is a pair of distinct $\sim_g$-equivalence classes generating proper g-cyclic subspaces, then they satisfy the condition of Lemma 2.4. In particular, the direct sum of these two proper g-cyclic subspaces is also a g-cyclic subspace of $\mathbb{F}_q^n$. Since we have no eigenvalue of multiplicity 2, while the g-cyclic space $\mathbb{F}_q^n$ contains this direct sum, Lemma 2.2

shows that the direct sum must be $\mathbb{F}_q^n$, and there are exactly 3 equivalence classes. If there is at most one $\sim_g$-equivalence class generating proper g-cyclic subspace, then by Corollary 2.3 there are at most two equivalence classes generating $\mathbb{F}_q^n$, so we still have at most 3 distinct equivalence classes. Moreover, when $j > 0$ and there are 3 distinct equivalence classes, then by Corollary 2.3, only one of them generates $\mathbb{F}_q^n$. Therefore, we have the following possibilities.

(a) There is exactly one $\sim_g$-equivalence class, which is just $\mathbb{F}_q^n \setminus \{0\}$.
(b) There are exactly two $\sim_g$-equivalence classes, and both of them generates $\mathbb{F}_q^n$.
(c) There are exactly three $\sim_g$-equivalence classes, only one of them generates $\mathbb{F}_q^n$, and the two proper g-cyclic subspaces generated by other classes intersect trivially.
(d) There are exactly two $\sim_g$-equivalence classes, and only one of them generates $\mathbb{F}_q^n$.
(e) $j = 0$, there are exactly three $\sim_g$-equivalence classes, exactly one of them generates $\mathbb{F}_q^n$, and the g-cyclic subspaces of $\mathbb{F}_q^n$ form a chain.
(f) $j = 0$, there are exactly three $\sim_g$-equivalence classes, and exactly two of them generates $\mathbb{F}_q^n$.

We start with case (a). In this case, let v be any nonzero vector in $\mathbb{F}_q^n$. Then since $V(g; v) = \mathbb{F}_q^n$, the vectors $v, gv, g^2v, \dots, g^{n-1}v$ form an ordered basis of $\mathbb{F}_q^n$, and the matrix of g with respect to this basis takes the form

$$(*) \qquad g = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\beta_0 \\ 1 & 0 & 0 & \cdots & 0 & -\beta_1 \\ 0 & 1 & 0 & \cdots & 0 & -\beta_2 \\ 0 & 0 & 1 & \cdots & 0 & -\beta_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -\beta_{n-1} \end{pmatrix}$$

where $\beta_1, \dots, \beta_n \in \mathbb{F}_q$ are the numbers such that $g^n v = -\beta_0 v - \cdots - \beta_{n-1} g^{n-1} v$. Let $f(t) \in \mathbb{F}_q[t]$ be the characteristic polynomial of the above matrix:

$$f(t) = t^n + \beta_{n-1} t^{n-1} + \cdots + \beta_0.$$

Suppose that $f(t) = f_1(t) f_2(t)$ for some nonconstant polynomials $f_1(t), f_2(t) \in \mathbb{F}_q[t]$. Then the set $\{f_1(g)v, g f_1(g)v, \dots, g^{\deg f_2} f_1(g)v\}$ is linearly dependent, because $f_2(g) f_1(g) v = f(g)v = 0$ is a linear combination of them. In particular, $V(g; f_1(g)v)$ is a proper g-cyclic subspace, which cannot exist in case (a). Therefore $f(t)$ is irreducible, so it is also the minimal polynomial of g.

Since $g^{s_v} = \alpha^{t_v} I$ on $V(g; v) = \mathbb{F}_q^n$, the order of g divides $(q-1)s_v = q^n - 1$. It follows that $f(t)$ divides the polynomial $t^{q^n-1} - 1 \in \mathbb{F}_q[t]$. Therefore, the roots of $f(t)$ lies in $\mathbb{F}_q^n$, and we can write one of them as α_n^a for some positive integer a. The action of α_n^a by left multiplication on $\mathbb{F}_q^n$, viewed as an $\mathbb{F}_q$-vector space, has minimal and characteristic polynomial $f(t)$. Therefore, with respect to some basis, it can be represented by the matrix $(*)$, so we can say $g = \alpha_n^a$ under a good choice of embedding $\mathrm{GL}_1(\mathbb{F}_{q^n}) \hookrightarrow \mathrm{GL}_n(\mathbb{F}_q)$. If a is not relatively prime to $\frac{q^n-1}{q-1}$, then the image of α_n^a in $\mathrm{PGL}_n(\mathbb{F}_q)$ has order strictly less than $\frac{q^n-1}{q-1} = s_v$. However, we know that this is the order of g in $\mathrm{PGL}_n(\mathbb{F}_q)$. Therefore a is relatively prime to $\frac{q^n-1}{q-1}$.

For case (b), by Corollary 2.3(c), if v and w are representatives of the $\sim_g$-equivalence classes, then $\frac{q^n-1}{q-1}$ is even and $s_v = s_w = \frac{q^n-1}{2(q-1)}$. The argument in (a) shows that the characteristic polynomial of g divides the polynomial $t^{(q^n-1)/2} - 1$, so that $g = \alpha_n^2 a$ for some a relatively prime to $\frac{q^n-1}{q-1}$, again under a good choice of embedding $\mathrm{GL}_1(\mathbb{F}_{q^n}) \hookrightarrow \mathrm{GL}_n(\mathbb{F}_q)$.

For (c), let $v, w \in \mathbb{F}_q^n \setminus \{0\}$ be such that $V(g; v)$ and $V(g; w)$ are proper. Since they are minimal g-cyclic subspaces, we have $s_v = \frac{q^{\dim V(g;v)}-1}{q-1}$ and $s_w = \frac{q^{\dim V(g;w)}-1}{q-1}$. As we discussed earlier in this proof, v and w satisfy the condition of Lemma 2.4, that is, $\frac{s_v t_w}{\gcd(s_v,s_w)} \not\equiv \frac{s_w t_v}{\gcd(s_v,s_w)} \bmod q-1$. In particular, $\mathbb{F}_q^n = V(g;v) \oplus V(g;w)$, and $\dim V(g;v)$ is relatively prime to $\dim V(g;w)$. By arguing as in (a), we can see that $g = \alpha_m^b \oplus \alpha_{n-m}^c$ under some choice of embedding $\mathrm{GL}_1(\mathbb{F}_{q^m}) \oplus \mathrm{GL}_1(\mathbb{F}_{q^{n-m}}) \hookrightarrow \mathrm{GL}_n(\mathbb{F}_q)$, where $m = \dim V(g;v)$, $b = t_v$ is relatively prime to $\frac{q^m-1}{q-1}$, and $c = t_w$ is relatively prime to $\frac{q^{n-m}-1}{q-1}$. The conditions about m, b, c and the statement about the spectrum follows from the conclusions of Lemma 2.4 together with Lemma 2.1.

In case (d), let $v, w \in \mathbb{F}_q^n \setminus \{0\}$ be vectors such that $V(g;v) = \mathbb{F}_q^n \neq V(g;w)$. Then $V(g;w)$ is a minimal g-cyclic subspace, so we can argue as in (a) to see that the action of g on $V(g;w)$ is given by α_m^b, where $m = \dim V(g;w)$, and b is an integer relatively prime to $\frac{q^m-1}{q-1}$. Also, the action of g on $\mathbb{F}_q^n/V(g;w)$ has only one $\sim_g$-equivalence class, so it can be viewed as α_{n-m}^c for some integer c relatively prime to $\frac{q^{n-m}-1}{q-1}$. Therefore, $g = \begin{pmatrix} \alpha_m^b & X \\ 0 & \alpha_{n-m}^c \end{pmatrix}$. X is nonzero, since if $X = 0$, then the lower right diagonal block gives rise to a proper g-cyclic subspace not equal to $V(g;w)$, which cannot exist by assumption. Also, since $s_w = \frac{q^m-1}{q-1}$, $s_v = \frac{q^n-1}{q-1} - s_w = \frac{q^n-q^m}{q-1}$, and $V(g;w) \subseteq V(g;v)$, Lemma 2.2 shows that s_w divides s_v and $t_w(s_v/s_w) \equiv t_v \bmod q-1$, or equivalently, m divides n and $c \equiv b\frac{n-m}{m} \bmod q-1$.

In case (e), let $u, v, w \in \mathbb{F}_q^n \setminus \{0\}$ be the representatives of the $\sim_g$-equivalence classes, so that $V(g;w) \subseteq V(g;v) \subseteq V(g;u)$. By Proposition 2.6, $\dim V(g;w) = 1$, so the action of g on $V(g;w)$ is just a scalar α^a. By arguing as in case (c), we can see that the action of g on $\mathbb{F}_q^n/V(g;v)$ and $V(g;v)/V(g;w)$ are given by α_{n-m}^c and α_{m-1}^b for $m = \dim V(g;v)$ and some integers b, c satisfying the conditions similar to those in (c). Therefore, with respect to some basis, $g = \begin{pmatrix} \alpha^a & X_{12} & X_{13} \\ 0 & \alpha_{m-1}^b & X_{23} \\ 0 & 0 & \alpha_{n-m}^c \end{pmatrix}$. The submatrices X_{12} and $\begin{pmatrix} X_{13} \\ X_{23} \end{pmatrix}$ are nonzero since otherwise there will be nontrivial proper g-cyclic subspaces other than $V(g;v)$ and $V(g;w)$.

Finally, for case (f), let $u, v, w \in \mathbb{F}_q^n \setminus \{0\}$ be representatives of the equivalence classes, so that $V(g;u) = V(g;v) = \mathbb{F}_q^n \supsetneq V(g;w)$ and $u \not\sim_g v$. We again have $\dim V(g;w) = 1$ by Lemma 2.2 and the fact that 1 is the only eigenvalue that can have multiplicity 3. Also, the action of g on $\mathbb{F}_q^n/V(g;w)$ is as in case (a) or (b). If it is as in case (a), then with respect to some basis, $g = \begin{pmatrix} \alpha^a & X \\ 0 & \alpha_{n-1}^b \end{pmatrix}$, where $a = t_w$, b is an integer relatively prime to $\frac{q^{n-1}-1}{q-1}$, and X is some nonzero matrix. Also, if $\overline{u}$ is the image of u in $\mathbb{F}_q^n/V(g;w)$, then $s_{\overline{u}} = \frac{q^{n-1}-1}{q-1}$ (for the action of g on $\mathbb{F}_q^n/V(g;w)$) divides s_u, since $g^{s_u}\overline{u} = \overline{g^{s_u}u}$ is a scalar multiple of u. Note that Lemma 2.2, $s_u = s_v$, so $s_u = (\frac{q^n-1}{q-1} - s_w)/2 = \frac{(q^{n-1}-1)q}{2(q-1)}$. Therefore, q is even in this case.

If the action of g on $\mathbb{F}_q^n/V(g;w)$ is as in case (b), then by (b) q must be odd and $g = \begin{pmatrix} \alpha^a & X \\ 0 & \alpha_{n-1}^{2b} \end{pmatrix}$, where $a = t_w$, b is an integer relatively prime to $\frac{q^{n-1}-1}{q-1}$, and X is some nonzero matrix. □

Remark 2.9. Before other parts of this paper were completed, Katz and Tiep [16, Proposition 10.3.6] found another proof of Theorem 2.7 for p'-elements (cases (a), (b) and (c)). Compared to the proof given here, their proof has an advantage of being much shorter, but it cannot be extended to cover elements of order divisible by p.

3. The Action of Wild Inertia Subgroup

Let $\mathcal{H}$ be an irreducible hypergeometric sheaf with the geometric monodromy group $G = G_{\mathrm{geom}}$ which satisfies ($\star$). The goal of this section is to completely determine the possible sets of upstairs characters and downstairs characters $\mathcal{H}$ can have. Equivalently, we want to find all possible spectra of the action of γ_0 on $\mathcal{H}$ and the action of γ_∞ on Tame and Wild. In Proposition 3.1, we will see that Theorem 2.7 already gives the answer for γ_0, but for γ_∞, it only gives the possible spectra of the action on $\mathcal{H} = \mathsf{Tame} \oplus \mathsf{Wild}$. To see how these spectra splits into the spectra on Tame and Wild, we will need to see how the action of γ_∞ interacts with the action of $P(\infty)$.

Let g_0 and g_∞ be the images of γ_0 and γ_∞ in G. Let $\overline{g_0}$ and $\overline{g_\infty}$ be the images of g_0 and g_∞ in $G/\mathbf{Z}(G)$. By ($\star$), $\overline{g_0}, \overline{g_\infty} \in \mathrm{PGL}_n(\mathbb{F}_q)$. We can choose representatives h_0 and h_∞ of $\overline{g_0}$ and $\overline{g_\infty}$, respectively, in $\mathrm{GL}_n(\mathbb{F}_q)$. Since γ_0 and γ_∞ have pro-order prime to p, both h_0 and h_∞ also have order prime to p. As explained in Remark 1.4, the spectrum of γ_0 is a root of unity times the spectrum of h_0 on an irreducible Weil representation, and the same holds for γ_∞ and h_∞.

Proposition 3.1. (1) *h_0 is an element described in Theorem 2.7(a).*
(2) *If $\dim \mathsf{Wild} > 1$, then h_∞ is an element described in Theorem 2.7(c).*
(3) *If $\dim \mathsf{Wild} = 1$, then h_∞ is an element described in Theorem 2.7(a).*

Proof. Since the upstairs characters are pairwise distinct, the action of γ_0 on $\mathcal{H}$ has simple spectrum. Hence, the action of h_0 on $\mathcal{H}$ also has simple spectrum. Among the elements described in Theorem 2.7, (a) is the only case with simple spectrum. For h_∞, we know that the downstairs characters are pairwise distinct, so the action on Tame has simple spectrum. By Proposition 1.1, the action on Wild also has simple spectrum. Therefore, h_∞ must be as in Theorem 2.7. If in addition $\dim \mathsf{Wild} > 1$, then we will see in Lemma 3.2(c) below that h_∞ stabilizes a nontrivial proper subspace of $\mathbb{F}_q^n$. Only (c) of Theorem 2.7 has this property.

If $\dim \mathsf{Wild} = 1$, then h_∞ has at most one eigenvalue with multiplicity 2 on an irreducible Weil representation, and all other eigenvalues have multiplicity 1. The elements in Theorem 2.7(b) and (c) have more than one eigenvalues with multiplicity 2 on every irreducible Weil representation, so h_∞ must be as in Theorem 2.7(a). □

For the rest of this section, assume that $\dim \mathsf{Wild} > 1$, so that $G/\mathbf{Z}(G) \cong \mathrm{PGL}_n(\mathbb{F}_q)$ by Proposition 1.2(d). For the exceptional cases $(n,q) = (3,2), (3,3), (3,4)$ where we cannot directly apply Proposition 1.2(d), we still have this property since Proposition 3.1(1) and the assumption $\dim \mathsf{Wild} > 1$ are sufficient to make the proof of [14, Corollary 8.4] valid. Let J and Q be the images of $I(\infty)$ and $P(\infty)$, respectively, in G. Let $\overline{J}$ and $\overline{Q}$ be their image in $\mathrm{PGL}_n(\mathbb{F}_q) = G/\mathbf{Z}(G)$. Let $\tilde{J}$ be the preimage of $\overline{J}$ in $\mathrm{GL}_n(\mathbb{F}_q)$, and let R be a Sylow p-subgroup of the preimage of $\overline{Q}$ in $\tilde{J}$. Since $P(\infty)$ is a pro-p-group, Q and $\overline{Q}$ are p-groups.

Lemma 3.2. (a) *R is a nontrivial normal Sylow p-subgroup of $\tilde{J}$, and $R \cong \overline{Q}$.*
(b) *The subspace $(\mathbb{F}_q^n)^R$ of points fixed by R is nontrivial and proper.*
(c) *There is a nontrivial proper h_∞-stable subspace of $\mathbb{F}_q^n$.*

Proof. (a) Since $P(\infty)$ is a normal subgroup of $I(\infty)$ such that $I(\infty)/P(\infty)$ has pro-order prime to p, Q is a normal Sylow subgroup of J. Also, since the wild part of $\mathcal{H}$ is nontrivial, Q cannot be trivial. In fact, $Q \not\leq \mathbf{Z}(G_{\mathrm{geom}})$ by [12], so $\overline{Q}$ is a nontrivial normal Sylow subgroup of $\overline{J}$. Since $|\mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))| = q-1$ is relatively prime to p, it follows that $|\tilde{J}|_p = |\overline{J}|_p = |\overline{Q}|$.

Let $\tilde{Q}$ be the preimage of $\overline{Q}$ in $\tilde{J}$. Then $|\tilde{Q}| = (q-1)|\overline{Q}|$, so $|R| = |\tilde{Q}|_p = |\overline{Q}|$. Therefore, R is a nontrivial Sylow p-subgroup of $\tilde{J}$, and the quotient map $\mathrm{GL}_n(\mathbb{F}_q) \to \mathrm{PGL}_n(\mathbb{F}_q)$ restricts to an isomorphism $R \to Q$. To see that R is normal in $\tilde{J}$, note that $\tilde{Q} = R\mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))$. Since $\mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))$

normalizes R, R is the normal Sylow p-subgroup of $\tilde{Q}$. In particular, R is characteristic in $\tilde{Q}$, which is a normal subgroup of $\tilde{J}$ since $\overline{Q}$ is normal in $\overline{J}$. Therefore, R is normal in $\tilde{J}$.

(b) The subgroup of upper triangular matrices with 1's on the diagonal is a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$, and it is obvious that every nontrivial subgroup of this Sylow p-subgroup has this property. Since R is a nontrivial p-subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$, it is conjugate to a nontrivial subgroup of this Sylow p-subgroup, so it also has this property.

(c) If $u \in (\mathbb{F}_q^n)^R$, then for any $r \in R$, we have $rh_\infty u = h_\infty(h_\infty^{-1} r h_\infty)u = h_\infty u$ since $h_\infty^{-1} r h_\infty \in R^{h_\infty} = R$. Therefore $h_\infty(\mathbb{F}_q^n)^R = (\mathbb{F}_q^n)^R$. □

This completes the proof of Proposition 3.1(b), so h_∞ is as in Theorem 2.7(c). To see which part of the spectrum of h_∞ comes from the tame part, we will use Proposition 1.1 and compare it to certain $\mathbb{C}R$-submodules of irreducible Weil modules Weil_j which we introduce in the next proposition. To see how these submodules interact with h_∞, it is enough to work with a larger elementary abelian p-group containing R. For our h_∞, choose $v, w \in \mathbb{F}_q^n \setminus \{0\}$ and $b, c, m \in \mathbb{Z}$ as in Theorem 2.7(c), so that in the notation of section 2,

$$h_\infty = \alpha_m^b \oplus \alpha_{n-m}^c \in \mathrm{GL}(V(h_\infty; v)) \oplus \mathrm{GL}(V(h_\infty; w)) \subset \mathrm{GL}_n(\mathbb{F}_q).$$

By exchanging v and w if necessary, we can assume that $(\mathbb{F}_q^n)^R = V(h_\infty; v)$. Define

$$\begin{aligned} E : &= \{r \in \mathrm{GL}_n(\mathbb{F}_q) \mid r \text{ acts trivially on both } V(h_\infty; v) \text{ and } \mathbb{F}_q^n / V(h_\infty; v)\} \\ &= \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \mid X \in M_{m \times (n-m)}(\mathbb{F}_q) \right\}. \end{aligned}$$

Note that E is an elementary abelian p-group containing R, and that h_∞ normalizes E.

Proposition 3.3. *Each of the restrictions to $\langle h_\infty, E\rangle$ of Weil_0' and Weil_j, $j = 1, \ldots, q-2$, has a $\frac{(q^m-1)(q^{n-m}-1)}{q-1}$-dimensional submodule induced by a nontrivial irreducible $\mathbb{C}E$-module.*

Proof. Let v, w, b, c, m be as above. We may view $V(h_\infty; v)$ as an additive elementary abelian p-group of order q^m. For each irreducible $\mathbb{C}$-character $\varphi \in \mathrm{Irr}(V(h_\infty; v))$ and each $y \in V(h_\infty; w) \setminus \{0\}$, define

$$y_{\varphi,j} := \sum_{x \in V(h_\infty; v)} \varphi(x)(x+y)^{(j)} \in \mathsf{Weil}_j\,.$$

Note that the map $E \to V(h_\infty; v)$ given by $r \mapsto ry - y$ is a surjective group homomorphism: if $r, r' \in E$, then they fix $V(h_\infty; v) = (\mathbb{F}_q^n)^R$ pointwise, and $r'y - y \in V(h_\infty, v)$, so

$$rr'y - y = rr'y - ry + ry - y = r(r'y - y) + ry - y = (r'y - y) + (ry - y).$$

Therefore, $r \mapsto \varphi(ry - y)^{-1}$ is a one-dimensional $\mathbb{C}$-representation of E, and non-isomorphic pairs of φ give non-isomorphic pairs of representations of E. Also, for each $r \in E$, we have

$$\begin{aligned} ry_{\varphi,j} &= r \sum_{x \in V(h_\infty; v)} \varphi(x)(x+y)^{(j)} = \sum_{x \in V(h_\infty; v)} \varphi(x)(r(x+y))^{(j)} = \sum_{x \in V(h_\infty; v)} \varphi(x)(x + (ry - y) + y)^{(j)} \\ &= \sum_{x \in V(h_\infty; v)} \varphi(x - (ry - y))(x+y)^{(j)} = \varphi(ry - y)^{-1} \sum_{x \in V(h_\infty; v)} \varphi(x)(x+y)^{(j)} = \varphi(ry - y)^{-1} y_{\varphi,j}. \end{aligned}$$

Hence, $\mathbb{C}y_{\varphi,j}$ is a $\mathbb{C}E$-module on which E acts by the representation described above.

For each positive integer d, we have

$$h_\infty^d y_{\varphi,j} = h_\infty^d \sum_{x\in V(h_\infty;v)} \varphi(x)(x+y)^{(j)} = \sum_{x\in V(h_\infty;v)} \varphi(x)(h_\infty^d(x+y))^{(j)}$$
$$= \sum_{x\in V(h_\infty;v)} \varphi(x)(\alpha_m^{bd}x+\alpha_{n-m}^{cd}y)^{(j)} = \sum_{x\in V(h_\infty;v)} \varphi(\alpha_m^{-bd}x)(x+\alpha_{n-m}^{cd}y)^{(j)} = (\alpha_{n-m}^{cd}y)_{\varphi\circ\alpha_m^{-bd},j}$$

which makes sense because $\varphi\circ\alpha_m^{-bd} : x\mapsto \varphi(\alpha_m^{-bd}x)$ is an irreducible character of $V(h_\infty;v)$. Of course, $\varphi\circ\alpha_m^{-bd}$ is nontrivial if and only if φ is nontrivial. Therefore, h_∞ permutes the set of $\frac{(q^m-1)(q^{n-m}-1)}{q-1}$ pairwise non-isomorphic one-dimensional $\mathbb{C}E$-modules

$$\{\mathbb{C}y_{\varphi,j} \mid y\in V(h_\infty;w)\setminus\{0\},\ \varphi\in \mathrm{Irr}(V(h_\infty;v))\setminus\{\mathbb{1}\}\}.$$

Note that they are all nontrivial, so when $j=0$, we can view them as submodules of Weil_0' (recall that Weil_0 is a direct sum of a trivial module and the irreducible module Weil_0'.) Also, since $\mathrm{Irr}(V(h_\infty;v))$ is linearly independent, so are the vectors $v_{\varphi,j}$.

I claim that this action of h_∞ has only one orbit, so that it cyclically permutes these modules. Fix $y\in V(h_\infty;w)\setminus\{0\}$ and $\varphi\in\mathrm{Irr}(V(h_\infty;v))\setminus\{\mathbb{1}\}$. Let d be the size of the orbit containing $\mathbb{C}y_{\varphi,j}$. It is the smallest positive integer such that $h_\infty^d y_{\varphi,j} = (\alpha_{n-m}^{cd}y)_{\varphi\circ\alpha_m^{-bd},j}\in\mathbb{C}y_{\varphi,j}$. If we write this as a linear combination of the usual basis vectors of the form $u^{(j)}$, $u\in\mathbb{F}_q^n\setminus\{0\}$, then the coefficient of $y^{(j)}$ is nonzero if and only if $\alpha_{n-m}^{cd}y\in\mathbb{F}_q y$. Since $y_{\varphi,j}$ has nonzero coefficient for $y^{(j)}$ and $0\neq h_\infty^d y_{\varphi,j}\in\mathbb{C}y_{\varphi,j}$, the coefficient is indeed nonzero, and $\alpha_{n-m}^{cd}y\in\mathbb{F}_q y$. Recall that $\alpha=\alpha_{n-m}^{\frac{q^{n-m}-1}{q-1}}$ is the lowest power of α_{n-m} contained in $\mathbb{F}_q$. Since c is relatively prime to $\frac{q^{n-m}-1}{q-1}$, we must have $d=d'\frac{q^{n-m}-1}{q-1}$ for some positive integer d'. Hence

$$h_\infty^d y_{\varphi,j} = (\alpha_{n-m}^{cd}y)_{\varphi\circ\alpha_m^{-bd},j} = (\alpha^{cd'}y)_{\varphi\circ\alpha_m^{-bd'\frac{q^{n-m}-1}{q-1}},j} = \sum_{x\in V(h_\infty;v)} \varphi(\alpha_m^{-bd'\frac{q^{n-m}-1}{q-1}}x)(x+\alpha^{cd'}y)^{(j)}$$
$$=\lambda^{cd'j}\sum_{x\in V(h_\infty;v)} \varphi(\alpha_m^{-bd'\frac{q^{n-m}-1}{q-1}}x)(\alpha_m^{-cd'\frac{q^m-1}{q-1}}x+y)^{(j)} = \lambda^{cd'j}y_{\varphi\circ\alpha_m^{-bd'\frac{q^{n-m}-1}{q-1}+cd'\frac{q^m-1}{q-1}},j}\in\mathbb{C}y_{\varphi,j}.$$

Therefore, d' must satisfy $\varphi\circ\alpha_m^{-bd'\frac{q^{n-m}-1}{q-1}+cd'\frac{q^m-1}{q-1}}=\varphi$, or equivalently,

$$\alpha_m^{-bd'\frac{q^{n-m}-1}{q-1}+cd'\frac{q^m-1}{q-1}}x-x\in\ker\varphi \text{ for all } x\in V(h_\infty;v).$$

Since $\ker\varphi$ is a proper subgroup of $V(h_\infty;v)$, the linear transformation $\alpha_m^{-bd'\frac{q^{n-m}-1}{q-1}+cd'\frac{q^m-1}{q-1}}-I$: $V(h_\infty;v)\to\ker\varphi\subsetneq V(h_\infty;v)$ is not invertible. Therefore, 1 is an eigenvalue of $\alpha_m^{-bd'\frac{q^{n-m}-1}{q-1}+cd'\frac{q^m-1}{q-1}}$ as an element of $\mathrm{GL}_m(\mathbb{F}_q)$. Since α_m is a generator of $\mathbb{F}_{q^m}^\times$, the eigenvalues of $\alpha_m\in\mathrm{GL}_m(\mathbb{F}_q)$ are some primitive q^m-1th roots of unity. Therefore, q^m-1 must divide $(\frac{-b(q^{n-m}-1)+c(q^m-1)}{q-1})d'$. Recall that $\frac{b(q^{n-m}-1)-c(q^m-1)}{q-1}$ is relatively prime to $q-1$. It follows that $d'=d''(q-1)$ for some positive integer d'', and $\frac{q^m-1}{q-1}$ divides $(\frac{-b(q^{n-m}-1)+c(q^m-1)}{q-1})\frac{d'}{q-1} = -\frac{bd''(q^{n-m}-1)}{q-1}+cd''\frac{q^m-1}{q-1}$. We also know that $\frac{q^m-1}{q-1}$ is relatively prime to both b and $\frac{q^{n-m}-1}{q-1}$, so $\frac{q^m-1}{q-1}$ divides d''. Therefore, $d=\frac{q^{n-m}-1}{q-1}d'=(q^{n-m}-1)d''$ is divisible by $\frac{(q^m-1)(q^{n-m}-1)}{q-1}$. On the other hand, $h_\infty^{\frac{(q^m-1)(q^{n-m}-1)}{q-1}}=1$, so d is exactly this number. Therefore, h_∞ cyclically permutes the d one-dimensional $\mathbb{C}E$-modules

$\mathbb{C}y_{\varphi,j}$ with $y \in V(h_\infty; w) \setminus \{0\}$ and $\varphi \in \mathrm{Irr}(V(h_\infty; v)) \setminus \{\mathbb{1}\}$. The direct sum of them is therefore a $\mathbb{C}\langle h_\infty, E\rangle$-module induced by one of these submodules. $\square$

Theorem 3.4. $\dim \mathsf{Wild} = \frac{(q^m-1)(q^{n-m}-1)}{q-1}$*, and the spectrum of the action of* γ_∞ *on* Wild *is the* $(\dim \mathsf{Wild})$*th roots of some number. The eigenvalues of the action of* γ_∞ *on* Tame *have multiplicity* 1*, and they are precisely the eigenvalues of the action of* γ_∞ *on* $\mathcal{H}$ *with multiplicity* 2*.*

Proof. Recall that $\mathcal{H}$ and an irreducible Weil module W of $\mathrm{GL}_n(\mathbb{F}_q)$ give the same projective representation of $\mathrm{PGL}_n(\mathbb{F}_q)$. Since R is abelian, the irreducible representations of R are one-dimensional. By Proposition 1.1, $\dim \mathsf{Wild}$ cannot be divisible by p, so Wild is a direct sum of $\dim \mathsf{Wild}$ one-dimensional $P(\infty)$-representations cyclically permuted by γ_∞. In particular, the spectrum of the action of γ_∞ on Wild is the set of $(\dim \mathsf{Wild})$th roots of some number.

By Proposition 3.3, we know that there is a $\frac{(q^m-1)(q^{n-m}-1)}{q-1}$-dimensional $\mathbb{C}\tilde{J}$-submodule of W induced by a one-dimensional $\mathbb{C}R$-submodule. Since this one-dimensional submodule must be contained in either Tame or Wild, so is the induced submodule. In particular, either Tame or Wild has dimension at least $\frac{(q^m-1)(q^{n-m}-1)}{q-1}$. On the other hand, the spectrum of the action of γ_∞ on $\mathcal{H}$ (which is just a root of unity times the spectrum of the action of h_∞ on W) has $\frac{q^m-1}{q-1} + \frac{q^{n-m}-1}{q-1} - \delta_{j,0}$ eigenvalues of multiplicity 2, and each of them must appear exactly once on both Tame and Wild. Therefore, one of Tame and Wild has dimension $\frac{(q^m-1)(q^{n-m}-1)}{q-1}$ and the other has dimension $\frac{q^m-1}{q-1} + \frac{q^{n-m}-1}{q-1} - \delta_{j,0}$. Moreover, these eigenvalues are some root of unity times the $\frac{q^m-1}{q-1}$th roots of λ^{bj} and the $\frac{q^{n-m}-1}{q-1}$th roots of λ^{cj}. By the observation in the previous paragraph, they all have the same $(\dim \mathsf{Wild})$th power. In particular, $\dim \mathsf{Wild}$ is divisible by $\mathrm{lcm}(\frac{q^m-1}{q-1}, \frac{q^{n-m}-1}{q-1}) = \frac{(q^m-1)(q^{n-m}-1)}{(q-1)^2}$, so $\dim \mathsf{Wild} > \frac{q^m-1}{q-1} + \frac{q^{n-m}-1}{q-1}$. Therefore $\dim \mathsf{Wild} = \frac{(q^m-1)(q^{n-m}-1)}{q-1}$. $\square$

4. Candidate Hypergeometric Sheaves

In the previous two sections, we found a complete list of possible spectra of γ_0 on $\mathcal{H}$ and that of γ_∞ on Tame and Wild. In this section, we will show that only a small number of pairs of these spectra for γ_0 and γ_∞ can occur together as the spectra of γ_0 and γ_∞ on $\mathcal{H}$.

Let $\mathcal{H} = \mathcal{H}yp_\psi(\chi_1, \ldots, \chi_D; \rho_1, \ldots, \rho_{D-W})$ be a hypergeometric sheaf with the wild part Wild at ∞ of dimension W and the geometric monodromy group G, and suppose that $(\star)$ holds. The geometric determinant of $\mathcal{H}$ is given in [6, Lemma 8.11.6] as

$$\det(\mathcal{H}) = \begin{cases} \mathcal{L}_{\prod_{i=1}^D \chi_i} & \text{if } W > 1, \\ \mathcal{L}_{\prod_{i=1}^D \chi_i} \otimes \mathcal{L}_\psi = \mathcal{K}l_\psi(\prod_{i=1}^D \chi_i) & \text{if } W = 1. \end{cases} \tag{4.1}$$

Here, $\mathcal{L}_{\prod_{i=1}^D \chi_i}$ is the Kummer sheaf defined by the multiplicative character $\prod_{i=1}^D \chi_i$, and $\mathcal{L}_\psi$ is the Artin-Schreier sheaf defined by the additive character ψ. In the next proposition, we use this determinant to see how the upstairs and downstairs characters are related.

We first treat the cases where $D = \frac{q^n-1}{q-1}$. Note that the irreducible Weil representations of these dimensions are imprimitive, so the corresponding sheaves will also be imprimitive. It turns out that they must be Belyi induced.

For a positive integer N and a multiplicative character χ, we define

$$\mathsf{Char}(N, \chi) := \{\text{multiplicative characters } \varphi \text{ of } \overline{\mathbb{F}_p} \text{ with } \varphi^N = \chi\}.$$

If $\chi = \mathbb{1}$, we will also write $\mathsf{Char}(N)$ instead of $\mathsf{Char}(N, \mathbb{1})$.

Theorem 4.1. *Let $\mathcal{H}$ and G be as above, and suppose that $D = \frac{q^n-1}{q-1}$. Then $\mathcal{H}$ must be of the form*

$$\begin{aligned}&\mathcal{H}yp_\psi(\varphi\,\mathsf{Char}(\frac{q^n-1}{q-1},\chi^{(b+c)j});\varphi\,\mathsf{Char}(\frac{q^m-1}{q-1},\chi^{bj})\cup\varphi\,\mathsf{Char}(\frac{q^{n-m}-1}{q-1},\chi^{cj}))\\ \cong&\mathcal{H}yp_\psi(\mathsf{Char}(\frac{q^n-1}{q-1},\chi^{(b+c)j});\mathsf{Char}(\frac{q^m-1}{q-1},\chi^{bj})\cup\mathsf{Char}(\frac{q^{n-m}-1}{q-1},\chi^{cj}))\otimes\mathcal{L}_\varphi\end{aligned}$$

for some nontrivial additive character ψ of $\mathbb{F}_p$, some multiplicative character φ of a finite extension of $\mathbb{F}_q$, an integer $1 \le j \le q-2$, integers m, b, c as in Theorem 2.7(c), and a multiplicative character χ of $\mathbb{F}_q^\times$ of order $q-1$. Moreover, we can assume that $b\frac{q^{n-m}-1}{q-1} - c\frac{q^m-1}{q-1} = 1$ and $\gcd(\frac{q^n-1}{q-1}, \frac{q-1}{\gcd(q-1,c)}) = 1$.

Proof. Let K be a finite extension of $\mathbb{F}_q$ over which all upstairs characters χ_i and downstairs characters ρ_j are defined, that is, $\#K - 1$ is divisible by the orders of all χ_i and ρ_j. Let ϑ be a multiplicative character of order $\#K - 1$ of K, so that every upstairs and downstairs character is a power of ϑ. Let $\Theta : \pi_1^{et}(\mathbb{G}_m/K) \to \overline{\mathbb{Q}_\ell}$ be the monodromy representation of the Kummer sheaf $\mathcal{L}_\vartheta$. Then both $\Theta(\gamma_0)$ and $\Theta(\gamma_\infty)$ are primitive $(\#K-1)$th roots of unity, so we can fix an integer r relatively prime to $\#K - 1$ such that $\Theta(\gamma_0) = \Theta(\gamma_\infty)^r$. The same relation holds (with the same r) for the values at γ_0 and γ_∞ of the monodromy representations of the Kummer sheaves for all powers of ϑ.

To apply the results of the previous section, we first prove that $W = \dim \mathsf{Wild} > 1$. So suppose that $W = 1$. By (4.1), the geometric determinant of $\mathcal{H}$ is $\mathcal{L}_{\prod_{i=1}^D \chi_i} \otimes \mathcal{L}_\psi$. Note that ψ has order p while both γ_0 and γ_∞ have pro-order prime to p. Also, since each of χ_i is a power of ϑ, so is $\prod_{i=1}^D \chi_i$. Therefore, for the images $g_0, g_\infty \in G \le \mathrm{GL}_D(\overline{\mathbb{Q}_\ell})$ of γ_0, γ_∞, we have

$$\det g_0 = (\det g_\infty)^r.$$

Since $\mathcal{H}$ satisfies $(\star)$ and $D = \frac{q^n-1}{q-1}$, the restriction of $\mathcal{H}$ to $E(G)$ comes from Weil_j for some $j \in \{1, \dots, q-2\}$. By Proposition 3.1 and Theorem 2.7, the eigenvalues of g_0 and g_∞ are given by

$$\{\mu\zeta \mid \zeta^{\frac{q^n-1}{q-1}} = \lambda^{aj}\} \text{ and } \{\nu\zeta \mid \zeta^{\frac{q^n-1}{q-1}} = \lambda^{bj}\}, \text{ respectively,}$$

where a, b are integers relatively prime to $\frac{q^n-1}{q-1}$, and $\mu, \nu \in \overline{\mathbb{Q}_\ell}^\times$ are some roots of unity. From the above equality of determinants, we get

$$\mu^{\frac{q^n-1}{q-1}}\lambda^{aj} = (\nu^{\frac{q^n-1}{q-1}}\lambda^{bj})^r, \text{ so that } \{(\nu\zeta)^r \mid \zeta^{\frac{q^n-1}{q-1}} = \{\mu\zeta \mid \zeta^{\frac{q^n-1}{q-1}} = \lambda^{aj}\}.$$

Note that the set on the left-hand side contains the rth powers of values at γ_∞, which are equal to the values at γ_0, of the monodromy representations of Kummer sheaves obtained from the downstairs characters. The set on the right-hand side is precisely the set obtained similarly from the upstairs characters. Therefore, the set of downstairs characters is contained in the set of upstairs characters, which is impossible by Proposition 1.2(a). Therefore $W > 1$.

Now we can apply Proposition 1.2, Proposition 3.1 and Theorem 3.4 to find the possible spectra of g_0 and g_∞, as well as the possible sets of upstairs and downstairs characters: the spectra must be

$$\{\mu\zeta \mid \zeta^{\frac{q^n-1}{q-1}} = \lambda^{aj}\}$$

and

$$\{\nu\zeta \mid \zeta^{\frac{q^m-1}{q-1}} = \lambda^{bj}\} \sqcup \{\nu\zeta \mid \zeta^{\frac{q^{n-m}-1}{q-1}} = \lambda^{cj}\} \sqcup \{\nu\zeta \mid \zeta^{\frac{(q^m-1)(q^{n-m}-1)}{q-1}} = 1\},$$

and the sets of upstairs and downstairs characters are

$$\eta\,\mathsf{Char}(\frac{q^n-1}{q-1},\chi^{aj}) \text{ and } \varphi\,\mathsf{Char}(\frac{q^m-1}{q-1},\chi^{bj})\cup\varphi\,\mathsf{Char}(\frac{q^{n-m}-1}{q-1},\chi^{cj}),$$

respectively, for some integers a,b,c,m satisfying the conditions in Theorem 2.7 and some roots of unity $\mu,\nu\in\overline{\mathbb{Q}_\ell}^\times$ and the corresponding multiplicative characters η,φ of some finite extension of $\mathbb{F}_q$. By computing $\det g_0=(\det g_\infty)^r$ as above, we can see that

$$\eta^{\frac{q^n-1}{q-1}}\chi^{aj}=\varphi^{\frac{q^n-1}{q-1}}\chi^{(b+c)j}$$

so that the upstairs characters can be written as

$$\eta\,\mathsf{Char}(\frac{q^n-1}{q-1},\chi^{aj})=\varphi\,\mathsf{Char}(\frac{q^n-1}{q-1},\chi^{(b+c)j}).$$

Therefore $\mathcal{H}$ must be of the form

$$\mathcal{H}yp_\psi(\varphi\,\mathsf{Char}(\frac{q^n-1}{q-1},\chi^{(b+c)j});\varphi\,\mathsf{Char}(\frac{q^m-1}{q-1},\chi^{bj})\cup\varphi\,\mathsf{Char}(\frac{q^{n-m}-1}{q-1},\chi^{cj})).$$

By [6, 8.2.14] this is geometrically isomorphic to

$$\mathcal{H}yp_\psi(\mathsf{Char}(\frac{q^n-1}{q-1},\chi^{(b+c)j});\mathsf{Char}(\frac{q^m-1}{q-1},\chi^{bj})\cup\mathsf{Char}(\frac{q^{n-m}-1}{q-1},\chi^{cj}))\otimes\mathcal{L}_\varphi.$$

Let d be a positive integer such that $q-1$ is not divisible by the dth power of any prime. Since m is relatively prime to n, we can choose integers e,x,y such that

$$e(n-m)\equiv c \bmod n^d, \text{ and } x\frac{q^{n-m}-1}{q-1}-y\frac{q^m-1}{q-1}=1.$$

Since $b\frac{q^{n-m}-1}{q-1}-c\frac{q^m-1}{q-1}$ is relatively prime to $q-1$, we can also choose integers z,w such that

$$z(b\frac{q^{n-m}-1}{q-1}-c\frac{q^m-1}{q-1})-w(q-1)=1.$$

Let $b':=z(b-e\frac{q^m-1}{q-1})-(q-1)xw$, $c':=z(c-e\frac{q^{n-m}-1}{q-1})-(q-1)yw$, $j':=(b(n-m)-cm)j$, and $\varphi':=\varphi\chi^{ej}$. One can easily check that in the above expression of $\mathcal{H}$, replacing the parameters (b,c,j,φ) with (b',c',j',φ') does not change the set of upstairs and downstairs characters, and these new parameters satisfy $b'\frac{q^{n-m}-1}{q-1}-c'\frac{q^m-1}{q-1}=1$ and $\gcd(\frac{q^n-1}{q-1},\frac{q-1}{\gcd(q-1,c')})=\gcd(n,\frac{q-1}{\gcd(q-1,c-e(n-m))})=1$. ☐

The sheaves of rank $\frac{q^n-q}{q-1}$ require more work. In particular, we will need to use the so-called "V-test", which is a criterion determining whether the geometric monodromy group of an irreducible hypergeometric sheaf is finite or not, based on an inequality involving the upstairs and downstairs characters and Kubert's V function. For details and the proof of this test, see [7, Section 13] and [6, Section 8.16]. We shall also use the basic properties [7, Section 13, p. 206] of the function V, without explicit mention, to simplify expressions involving V.

Theorem 4.2. *Let $\mathcal{H}$ and G be as in the beginning of this section, so that $(\star)$ holds. Suppose that $D=\frac{q^n-q}{q-1}$. Then $\mathcal{H}$ must be of the form*

$$\begin{aligned}&\mathcal{H}yp_\psi(\varphi(\mathsf{Char}(\frac{q^n-1}{q-1})\setminus\{\mathbb{1}\});\varphi\,\mathsf{Char}(\frac{q^m-1}{q-1})\cup\varphi(\mathsf{Char}(\frac{q^{n-m}-1}{q-1})\setminus\{\mathbb{1}\}))\\ \cong&\mathcal{H}yp_\psi(\mathsf{Char}(\frac{q^n-1}{q-1})\setminus\{\mathbb{1}\};\mathsf{Char}(\frac{q^m-1}{q-1})\cup(\mathsf{Char}(\frac{q^{n-m}-1}{q-1})\setminus\{\mathbb{1}\}))\otimes\mathcal{L}_\varphi\end{aligned}$$

for some nontrivial additive character ψ of $\mathbb{F}_p$, some multiplicative character φ of a finite extension of $\mathbb{F}_q$, and an integer m satisfying the conditions in Theorem 2.7(c).

Proof. (1) We first prove that $W > 1$.

Choose K, ϑ, Θ and r as in the proof of Theorem 4.1. We can assume that $\#K - 1$ is also divisible by $\frac{q^{n-1}-1}{q-1}$. As before, the elements g_0 and g_∞ must be as in Theorem 2.7(a). However, Theorem 2.7 describes the spectrum of $\mathsf{Weil}_0 = \overline{\mathbb{Q}_\ell} \oplus \mathsf{Weil}_0'$, while $\mathcal{H}$ gives Weil_0'. Therefore, the spectra of g_0 and g_∞ can be written as

$$\{\mu\zeta \mid \zeta^{\frac{q^n-1}{q-1}} = 1, \zeta \neq 1\} \text{ and } \{\nu\zeta \mid \zeta^{\frac{q^n-1}{q-1}} = 1, \zeta \neq 1\}$$

for some roots of unity $\mu, \nu \in \overline{\mathbb{Q}_\ell}^\times$ of order not divisible by p. Now from the equality $\det g_0 = (\det g_\infty)^r$, we get

$$\mu^{\frac{q^n-q}{q-1}} = \nu^{\frac{(q^n-q)r}{q-1}}, \text{ or equivalently } (\mu\nu^{-r})^{\frac{q^{n-1}-1}{q-1}} = 1.$$

Therefore, $\mathcal{H}$ must be geometrically isomorphic to

$$\mathcal{H}yp_\psi(\tau(\mathsf{Char}(\frac{q^n-1}{q-1}) \setminus \{\mathbb{1}\}); \mathsf{Char}(\frac{q^n-1}{q-1}) \setminus \{\mathbb{1}, \rho\}) \otimes \mathcal{L}_\varphi \tag{4.2}$$

for some nontrivial multiplicative character τ of order dividing $\frac{q^{n-1}-1}{q-1}$, some $\mathbb{1} \neq \rho \in \mathsf{Char}(\frac{q^n-1}{q-1})$, and some multiplicative character φ. Here, τ is nontrivial since $\mathcal{H}$ is irreducible so that the upstairs and downstairs characters must be disjoint.

I claim that the hypergeometric sheaf (4.2) does not satisfy ($\star$). More specifically, I will show that the geometric monodromy group is not finite. For the sheaf (4.2), the V-test, after a simplification, says that the geometric monodromy group is finite if and only if for every integer N relatively prime to $\#K - 1$ and for every $x \in \mathbb{Q}/\mathbb{Z}$ whose denominator is not divisible by p, the following inequality holds:

(4.3)

$$V(Nt + Ax) + V(-Ax) + \frac{3}{2} \geq \frac{(A-2)V(Nt) + V(Nt - \frac{Ns}{A})}{A-1} + V(Nt + x) + V(-x) + V(-\frac{Ns}{A} - x)$$

where $A = \frac{q^n-1}{q-1}$, and $t \in \frac{q-1}{q^{n-1}-1}\mathbb{Z} \setminus \mathbb{Z}$ is the number such that $\vartheta^{t(\#K-1)} = \tau$, and $s \in \{1, \dots, A-1\}$ is the number such that $\vartheta^{s\frac{(\#K-1)}{A}} = \rho$.

Suppose that (4.3) holds for all pairs (N, x). Recall from [7, Section 13] that for any $x \in \mathbb{Q}/\mathbb{Z}\setminus\mathbb{Z}$, we have $V(x) + V(-x) = 1$, and for any $x \in \mathbb{Z}$ we have $V(x) = 0$. Note that for every N relatively prime to $\#K - 1$ (which is divisible by both A and $\frac{q^{n-1}-1}{q-1}$ by the choice of K), if we let $x = -Nt + \frac{u}{A}$ for an integer u not divisible by A, then we have $Nt + Ax = -(A-1)Nt + u \in \mathbb{Z}$ but $Ax, Nt, Nt - N\frac{s}{A}, Nt + x, x, N\frac{s}{A} - x$ are not integers, so that the sum of (4.3) for the pairs (N, x) and $(-N, -x)$ becomes the inequality $1 \geq 4 - 3$. Therefore the equality holds in (4.3) for each of these pairs:

$$\frac{V(Nt) - V(Nt - \frac{Ns}{A})}{A-1} = V(\frac{u}{A}) + V(Nt - \frac{u}{A}) + V(Nt - \frac{Ns+u}{A}) - \frac{3}{2}.$$

The explicit formula [7, Theorem 13.4] of the function V tells us that the right-hand side of the equality lies in $\frac{1}{(p-1)n(n-1)\log_p q}\mathbb{Z}$. On the other hand, the left-hand side lies in $(-\frac{1}{A-1}, \frac{1}{A-1})$. Since $n \geq 3$, we have $A - 1 = \frac{q^n-q}{q-1} \geq (p-1)n(n-1)\log_p q$, so that $\frac{1}{(p-1)n(n-1)\log_p q}\mathbb{Z} \cap (-\frac{1}{A-1}, \frac{1}{A-1}) = \{0\}$.

Therefore, the equality forces that

$$V(Nt) = V(Nt - \frac{Ns}{A}), \text{ and } V(\frac{u}{A}) + V(Nt - \frac{u}{A}) + V(Nt - \frac{Ns+u}{A}) = \frac{3}{2}.$$

These are true for all integers N relatively prime to $\#K - 1$, so (4.3) implies

$$V(Nt + Ax) + V(-Ax) + \frac{3}{2} \geq V(Nt) + V(Nt + x) + V(-x) + V(-\frac{Ns}{A} - x) \tag{4.3$'$}$$

for all pairs (N, x).

The equality in (4.3$'$) holds not only for $x = -Nt + \frac{u}{A}$ for $u \in \mathbb{Z} \setminus A\mathbb{Z}$, but also for $x = \frac{u}{A}$ for $u \in \mathbb{Z}$ with $u \not\equiv 0, -Ns \bmod A$ by the same reason. For $x = -Nt + \frac{u}{A}$ and $-\frac{u}{A}$ with $u \not\equiv 0, Ns \bmod A$, these become

$$\frac{3}{2} = V(\frac{u}{A}) + V(Nt - \frac{u}{A}) + V(Nt - \frac{u+Ns}{A}) = V(Nt - \frac{u}{A}) + V(\frac{u}{A}) + V(\frac{u-Ns}{A}) \tag{4.4}$$

so that

$$V(Nt - \frac{u+Ns}{A}) = V(\frac{u-Ns}{A}). \tag{4.5}$$

Take the sum of (4.5) for $u \in \{1, \dots, A-1\}$ except for $Ns \bmod A$, and use the fact $V(Nt) = V(Nt - \frac{Ns}{A})$ we saw above to get:

$$V(Nt - \frac{2Ns}{A}) = V(-\frac{Ns}{A}).$$

On the other hand, choose $u = Ns + p^f$ for either $f = 0$ or 1, so that $Ns + p^f \not\equiv 0 \bmod A$. Then from (4.5) we get

$$\frac{2}{n} = 2V(\frac{p^f}{A}) = V(Nt - \frac{u+Ns}{A}) + V(\frac{u-Ns}{A}) \geq V(Nt - \frac{2Ns}{A}) = V(-\frac{Ns}{A}).$$

Similarly $\frac{2}{n} \geq V(\frac{Ns}{A})$, so we must have $\frac{4}{n} \geq V(-\frac{Ns}{A}) + V(\frac{Ns}{A}) = 1$. Therefore, either $n = 4$ and $V(\frac{Ns}{A}) = 1/2$, or $n = 3$ and $1/3 \leq V(\frac{Ns}{A}) \leq 2/3$.

Suppose that $n = 3$. Let $N = 1$, $t' = t(q+1) \in \{1, \dots, q\}$, and $x = \frac{(q+1-t')(q-1)(q^{n(n-2)}-1)}{(q^n-1)(q^{(n-1)^2}-1)} = \frac{(q-t')q+t'-1}{q^4-1}$. Then

$$\begin{aligned} 2V(t) &= V(t) + V(qt) = V(t) + V(t'-t) = 1, \\ V(x) &= V(\frac{(q-t')q+t'-1}{q^4-1}) = V(\frac{q-t'}{q^4-1}) + V(\frac{t'-1}{q^4-1}) = \frac{1}{4}, \\ V(Ax) &= V(\frac{(q-t')q^3 + (q^2-1)q + (t'-1)}{q^4-1}) = \frac{3}{4}, \\ V(t+Ax) &= V(\frac{(q-t')q^2 + (t'-1)q}{q^4-1}) = \frac{1}{4}, \\ V(t+x) &= V(\frac{(t'-1)q^3 + (q-t')q^2 + q^2 - 1}{q^4-1}) = \frac{3}{4}. \end{aligned}$$

Therefore (4.3$'$) for (N, x) becomes $2 \geq 2 + V(-\frac{s}{A} - x)$, which forces $x + \frac{s}{A} \in \mathbb{Z}$. However, this would imply $V(\frac{s}{A}) = V(-x) = \frac{3}{4}$, but we saw above that $\frac{1}{3} \leq V(\frac{Ns}{A}) \leq \frac{2}{3}$. Therefore $n \neq 3$.

Finally, suppose that $n = 4$. If $u \in \mathbb{Z}$ with $u \not\equiv 0, -Ns, -2Ns, -3Ns \bmod A$, then by the same argument used in (4.4) and (4.5) with $x = -Nt + \frac{u}{A}, -\frac{u+Ns}{A}, -Nt + \frac{u+Ns}{A}, -Nt + \frac{u+2Ns}{A}$, and

$-\frac{u+3Ns}{A}$, we get

$$V(\frac{u}{A}) = V(Nt - \frac{u+2Ns}{A}) = V(\frac{u+3Ns}{A}) \text{ and } V(Nt - \frac{u}{A}) = V(\frac{u+Ns}{A}).$$

Suppose that $d := \gcd(3s, A) > 1$. Then we can set $u = 1, 1+3Ns, 1+6Ns, \ldots$ to get

$$\frac{1}{4} = V(\frac{1}{A}) = V(\frac{1+d}{A}) = \cdots = V(\frac{1+(A-d)}{A})$$

so that

$$\frac{A}{4d} = \sum_{i=0}^{\frac{A}{d}-1} V(\frac{1+id}{A}) = V(\frac{1}{d}) + \frac{A-d}{2d}.$$

Therefore, $2d = 4dV(\frac{1}{d}) + A$, so we must have $2d > A$. Since d divides A, we get $d = A$, so that A divides $3s$. Therefore A is divisible by 3, and $s = A/3$ or $2A/3$. By (4.4), (4.5) and the above observations, we get

$$\frac{3}{2} = V(\frac{1}{A}) + V(Nt - \frac{1}{A}) + V(Nt - \frac{1+Ns}{A}) = V(\frac{1}{A}) + V(\frac{1}{A} + \frac{1}{3}) + V(\frac{1}{A} + \frac{2}{3}) = V(\frac{3}{A}) + 1.$$

Therefore $\frac{1}{2} = V(\frac{3}{A}) = V(\frac{3(q-1)}{q^4-1})$, so we must have $q = 2$. Now one can manually check that for $q = 2$ and $n = 4$, the inequality (4.3′) fails for all possible pairs of t and s; for instance one can choose $(N, x) = (1, \frac{1}{15})$ for $s = 5$ and all t, and $(N, x) = (1, \frac{2}{15})$ for $s = 10$ and all t.

If $d = 1$, then A is not divisible by 3, and s is relatively prime to A. Hence either $q = 3$ or $q \equiv 1 \mod 3$, so $A \equiv 1 \mod 3$. Then as above we get

$$\begin{aligned}
&V(\frac{3s}{A}) = V(\frac{6s}{A}) = \cdots = V(\frac{(A-1)s}{A}) = 1 - V(\frac{(3A-3)s}{A}),\\
&V(\frac{(A+2)s}{A}) = V(\frac{(A+5)s}{A}) = \cdots = V(\frac{(2A-2)s}{A}) = 1 - V(-\frac{2s}{A}) = \frac{1}{2},\\
&V(\frac{(2A+1)s}{A}) = V(\frac{(2A+4)s}{A}) = \cdots = V(\frac{(3A-3)s}{A}).
\end{aligned}$$

Moreover, since $V(\frac{1}{A}) = \frac{1}{4}$ must be one of the values, the top and bottom rows are either $\frac{1}{4}$ or $\frac{3}{4}$. However $V(\frac{(2A+1)s}{A}) = V(\frac{s}{A}) = \frac{1}{2}$ as we discussed below (4.5), so this case cannot happen. This completes the proof that $W > 1$.

(2) Now we consider the sheaves with $W > 1$.

As in Theorem 4.1, $\mathcal{H}$ must be geometrically isomorphic to

$$\mathcal{H}yp_\psi(\tau(\mathsf{Char}(\frac{q^n-1}{q-1}) \setminus \{\mathbb{1}\}); \mathsf{Char}(\frac{q^m-1}{q-1}) \cup (\mathsf{Char}(\frac{q^{n-m}-1}{q-1}) \setminus \{\mathbb{1}\})) \otimes \mathcal{L}_\varphi$$

for some multiplicative character τ of order dividing $\frac{q^{n-1}-1}{q-1}$ and some multiplicative character φ. We want to show that $\mathcal{H}$ must be of this form with $\tau = \mathbb{1}$. If $m = n-1$ or 1, then we can replace τ with $\mathbb{1}$ and φ with $\varphi\tau$ without changing the sheaf itself, since the set of downstairs characters $\mathsf{Char}(\frac{q^{n-1}-1}{q-1})$ remains the same when multiplied by τ^{-1}.

Suppose that $1 < m < n-1$ and $\tau \neq \mathbb{1}$. We may replace m with $n-m$ without changing the sheaf, so assume that $n/2 < m < n-1$. The V-test for this sheaf (with $\varphi = \mathbb{1}$) is

$$V(Nt + Ax) + V(-Bx) + V(-Cx) \geq V(Nt + x) + V(-x) \tag{4.6}$$

for the pairs (N,x), where $A=\frac{q^n-1}{q-1}$, $B=\frac{q^m-1}{q-1}$, $C=\frac{q^{n-m}-1}{q-1}$, and $t\in\frac{q-1}{q^{n-1}-1}\mathbb{Z}\setminus\mathbb{Z}$ is the number corresponding to τ. Note that $A=B+Cq^m=Bq^{n-m}+C$. Suppose that this holds for all pairs (N,x).

Let $x=\frac{1}{B}$ and $N=q^f$ for $f=0,1,\dots,m-1$. Then (4.6) for (N,x) becomes

$$V(q^ft+\frac{C}{B})\geq V(q^ft+\frac{1}{B})+\frac{n-m-1}{m}. \tag{4.7}$$

On the other hand, by the basic properties of V,

$$V(q^ft+\frac{C}{B})\leq V(q^ft+\frac{1}{B})+V(\frac{C-1}{B})=V(q^ft+\frac{1}{B})+\frac{n-m-1}{m}.$$

Therefore, the equality must hold.

Let $t'\in\{1,\dots,\frac{q^{n-1}-1}{q-1}-1\}$ be the number such that $t'\equiv t\frac{q^{n-1}-1}{q-1} \bmod \frac{q^{n-1}-1}{q-1}$. Let $a_i\in\{1,\dots,p-1\}$ be the base p digits of $t'(q-1)$, that is, the unique numbers such that $\sum_{i=0}^{(n-1)\log_p q-1}a_ip^i=t'(q-1)$. Now from (4.7) (with $f=0$) and the formula for the function V (cf. [8, Sections 2 and 4] and [9, Discussion above Theorem 2.9]), one can see that if $b_i\in\{0,\dots,p-1\}, 0\leq i\leq(n-1)m\log_p q-1$ are the base p digits of

$$\frac{q^{(n-1)m}-1}{q^{n-1}-1}(\sum_{i=0}^{(n-1)\log_p q-1}a_ip^i)+(q-1)\frac{q^{(n-1)m}-1}{q^m-1}=\sum_{i=0}^{(n-1)m\log_p q-1}b_ip^i,$$

then for each $j=0,\dots,n-2$, we must have

$$b_{(jm+1)\log_p q}=\cdots=b_{(jm+n-m)\log_p q-1}=0.$$

It follows that either

- $a_0=\cdots=a_{(n-m)\log_p q-1}=0$, or
- $a_{\log_p q}=\cdots=a_{(n-m)\log_p q-1}=p-1$ and at least one of $a_0,\dots,a_{\log_p q-1}$ is nonzero.

Note that if we choose a larger power of p as N, then it "circularly shifts" the base p digits of $t'(q-1)$. Hence, the above argument for different values of f shows that either $a_0=\cdots=a_{(n-1)\log_p q-1}=0$ or $a_0=\cdots=a_{(n-1)\log_p q-1}=p-1$. But then $t'(q-1)=0$ or $\frac{q^{n-1}-1}{q-1}$, which contradicts our choice of t. Therefore (4.7) cannot hold for $(N,x)=(q^f,\frac{1}{B})$ for some f, and hence (4.6) fails for this pair, contradicting our assumption. This proves that if $1<m<n-1$ and $\mathcal{H}$ has finite geometric monodromy group, then $\tau=\mathbb{1}$. □

Remark 4.3. Note that in (4.6), if $\tau=\mathbb{1}$, then $t=0$ and the inequality becomes

$$V(Ax)+V(-Bx)+V(-Cx)\geq 1$$

for $x\notin\mathbb{Z}$. Since $V(Ax)+V(-Bx)+V(-Cx)\geq V(Ax)+V(-Bx-q^mCx)=V(Ax)+V(-Ax)=1$, this inequality holds. Therefore, the sheaves in Theorem 4.2 do have finite geometric monodromy groups.

By Theorem 4.1 and Theorem 4.2, we are left with a small family of hypergeometric sheaves. Is it possible to further reduce this family? First, taking tensor product by a Kummer sheaf $\mathcal{L}_\varphi$ wouldn't make much difference on the geometric monodromy group, so we can't remove this. Indeed, if G is the geometric monodromy group of the hypergeometric sheaf without $\otimes\mathcal{L}_\varphi$ and G' is that of the tensor product, then $\langle G,Z\rangle=\langle G',Z\rangle$ as subgroups of $\mathrm{GL}_D(\overline{\mathbb{Q}_\ell})$, where Z is the central subgroup of $\mathrm{GL}_D(\overline{\mathbb{Q}_\ell})$ of order equal to the order of φ. In particular, if G is finite almost quasisimple, then G' is also finite almost quasisimple with the same nonabelian composition factor.

It is also impossible to put any additional restriction on n and q (recall that we do assume that $n \geq 3$). For all pairs (q, n) the sheaves we obtain from Theorem 4.1 and Theorem 4.2 by setting $m = 1$ and $\varphi = \mathbb{1}$ were already studied by Katz and Tiep [13]. All of them do have the desired geometric monodromy groups in irreducible Weil representations, cf. [13, Corollary 8.2]. Since we already have some restrictions on m, further reduction is not likely at this point. It turns out that this is indeed the case, as we will see in the next section.

5. Computing the Geometric Monodromy Groups

Although the sheaves in Theorem 4.1 and Theorem 4.2 survived our attempts to remove the non-examples of $(\star)$, we still need to show that these sheaves do have such geometric monodromy groups. The method we will use to compute these monodromy groups is based on the arguments in [13], which discusses the sheaves in Theorem 4.1 and Theorem 4.2 with $m = 1$ and $\varphi = \mathbb{1}$.

The plan is as follows. We first form a directed sum of appropriately chosen sheaves from Theorem 4.1 and Theorem 4.2. This direct sum will have a "nice" trace function, which becomes even nicer if we take a Kummer pullback. Then we can use results in [13] to see that the monodromy representation must be the restriction of the total Weil representation of $\mathrm{GL}_n(\mathbb{F}_q)$ to some subgroup containing $\mathrm{SL}_n(\mathbb{F}_q)$. We use this fact to prove that the geometric monodromy group of the original direct sum before taking pullback is a quotient of $\mathrm{GL}_n(\mathbb{F}_q)$, and that the monodromy representation is a direct sum of certain irreducible Weil representations.

For the rest of this section, let n, q, χ, ψ be defined as in the previous section. Fix $b, c, m \in \mathbb{Z}$ and a multiplicative character φ which satisfy the conditions in Theorem 4.1, namely

(i) $1 \leq m \leq n - 1$,
(ii) $bC - cB = 1$ (so that n and m, or equivalently $\frac{q^{n-m}-1}{q-1}$ and $\frac{q^m-1}{q-1}$, are coprime to each other), and
(iii) $\gcd(A, \frac{q-1}{\gcd(c,q-1)}) = 1$,

where as in the proof of Theorem 4.2, we set

$$A := \frac{q^n - 1}{q - 1},\ B := \frac{q^m - 1}{q - 1},\ C := \frac{q^{n-m} - 1}{q - 1}.$$

Consider the following irreducible hypergeometric sheaves as in Theorem 4.1:

$$\mathcal{H}_j := \mathcal{H}yp_\psi(\mathsf{Char}(A, \chi^{(b+c)j}); \mathsf{Char}(B, \chi^{bj}) \cup \mathsf{Char}(C, \chi^{cj})) \text{ for } j = 1, \dots, q - 2$$

and one from Theorem 4.2:

$$\mathcal{H}_0 := \mathcal{H}yp_\psi(\mathsf{Char}(A) \setminus \{\mathbb{1}\}; \mathsf{Char}(B) \cup (\mathsf{Char}(C) \setminus \{\mathbb{1}\})).$$

We first compute the trace functions of these sheaves.

Proposition 5.1. *$\mathcal{H}_0$ is geometrically isomorphic to the lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{G}_0$ over $\mathbb{G}_m/\mathbb{F}_q$, which is pure of weight 0 and whose trace function at each point $u \in K^\times$ of each finite extension $K/\mathbb{F}_q$ is given by*

$$u \in K^\times \mapsto -1 + |\{v \in K^\times \mid u^{-b}v^B + u^{cq^m}v^{-Cq^m} = 1\}|.$$

Proof. $\mathcal{H}_0$ is, by definition, the multiplicative ! convolution of three Kloosterman sheaves $\mathcal{K}l_\psi(\mathsf{Char}(A) \setminus \{\mathbb{1}\})$, $\mathsf{inv}^*\, \mathcal{K}l_{\overline{\psi}}(\mathsf{Char}(B))$, and $\mathsf{inv}^*\, \mathcal{K}l_{\overline{\psi}}(\mathsf{Char}(C) \setminus \{\mathbb{1}\})$. By [11, Lemma 1.1 and 1.2], $\mathcal{H}_0$ is geometrically isomorphic to the lisse $\overline{\mathbb{Q}_\ell}$-sheaf over $\mathbb{G}_m$, pure of weight 4, whose trace function at $u \in K^\times$

is given by the convolution

$$\sum_{\substack{r,s,t\in K^\times \\ rst=u}} \left(-\sum_{x\in K}\psi_K(Ax-\frac{1}{r}x^A)\right)\left(\sum_{\substack{y\in K \\ y^B=s^{-1}}}\psi_K(-By)\right)\left(-\sum_{z\in K}\psi_K(tz^C-Cz)\right)$$
$$=\sum_{x,z\in K,y,t\in K^\times}\psi_K(-\frac{t}{u}x^Ay^{-B}+tz^C+x-y-z)$$

where the equality follows from the fact that $A\equiv B\equiv C\equiv 1 \bmod q$. Since ψ is nontrivial and irreducible, we have $\sum_{x\in K}\psi_K(x)=0$. Using this, we can rewrite the above number as

$$\sum_{x,z\in K,y\in K^\times}\left(\psi_K(x-y-z)\sum_{t\in K^\times}\psi_K(t(-\frac{1}{u}x^Ay^{-B}+z^C))\right)$$
$$=\sum_{x,z\in K,y\in K^\times}\psi_K(x-y-z)\left(-1+\sum_{t\in K}\psi_K(t(-\frac{1}{u}x^Ay^{-B}+z^C))\right)$$
$$=-\sum_{z\in K,y\in K^\times}\left(\psi_K(-y-z)\sum_{x\in K}\psi_K(x)\right)+(\#K)\sum_{\substack{x,z\in K,y\in K^\times \\ x^A=uy^Bz^C}}\psi_K(x-y-z)$$
$$=(\#K)(-1+\sum_{\substack{x,y,z\in K^\times \\ x^A=uy^Bz^C}}\psi_K(x-y-z)).$$

There is a bijection between the sets $\{(x,y,z)\mid x,y,z\in K^\times, x^A=uy^Bz^C\}$ and $K^\times\times K^\times$ given by

$$(x,y,z)\mapsto(x,x^{b+cq^m}y^{-b}z^{-c})$$
$$(x,xu^cv^{-C},x^{q^m}u^{-b}v^B)\leftarrow\!\shortmid(x,v).$$

By applying this change of variables, we can rewrite the above expression as

$$(\#K)(-1+\sum_{x,v\in K^\times}\psi_K(x-xu^cv^{-C}-x^{q^m}u^{-b}v^B))$$
$$=(\#K)(-1+\sum_{v\in K^\times}\sum_{x\in K^\times}\psi_K(x^{q^m}(1-u^{cq^m}v^{-Cq^m}-u^{-b}v^B)))$$
$$=(\#K)(-1+\sum_{v\in K^\times}(-1)+\sum_{\substack{v\in K^\times \\ 1-u^{cq^m}v^{-Cq^m}-u^{-b}v^B=0}}(\#K))$$
$$=(\#K)^2(-1+|\{v\in K^\times\mid u^{-b}v^B+u^{cq^m}v^{-Cq^m}=1\}|).$$

Now apply a Tate twist $(\mathbf{2})$ to obtain a lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{G}_0$ on $\mathbb{G}_m/\mathbb{F}_p$, pure of weight 0 and geometrically isomorphic to $\mathcal{H}_0$, whose trace function is given by

$$u\in K^\times\mapsto -1+|\{v\in K^\times\mid u^{-b}v^B+u^{cq^m}v^{-Cq^m}=1\}|.$$

□

By a similar argument and [5, 5.6.2], we get the following:

Proposition 5.2. *$\mathcal{H}_j$ is geometrically isomorphic to the lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{G}_j$ on $\mathbb{G}_m/\mathbb{F}_q$, which is pure of weight 0 and whose trace function at each point $u \in K^\times$ of each finite extension $K/\mathbb{F}_q$ is given by*

$$u \in K^\times \mapsto \sum_{\substack{v \in K^\times \\ u^{-b}v^B + u^{cq^m}v^{-Cq^m} = 1}} \chi_K(v^j).$$

Corollary 5.3. *The sheaves $\mathcal{H}_j$, $0 \leq j \leq q-2$, have finite geometric monodromy groups.*

Proof. By Proposition 5.1 and Proposition 5.2, they are geometrically isomorphic to the sheaves $\mathcal{G}_j$, which are pure of weight 0 and whose trace functions have algebraic integer values. By [6, Theorem 8.14.4], their geometric monodromy groups are finite. □

Let $\mathcal{W}$ be the direct sum of the sheaves $\mathcal{G}_0$ in Proposition 5.1 and $\mathcal{G}_j$ in Proposition 5.2:

$$\mathcal{W} := \bigoplus_{j=0}^{q-2} \mathcal{G}_j.$$

Proposition 5.4. *$\mathcal{W}$ is geometrically isomorphic to the lisse $\overline{\mathbb{Q}_\ell}$-sheaf whose trace function is given by*

$$u \in K^\times \mapsto -2 + |\{w \in K \mid u^b w^{q^m} - u^{b+cq^m} w^{q^n} = w\}|.$$

In particular, the values of this trace function is -2 plus either 1 or a power of q.

Proof. The trace function of $\mathcal{W}$ is given by the sum of trace functions of $\mathcal{G}_j$ computed in Proposition 5.1 and Proposition 5.2:

$$u \in K^\times \mapsto -1 + \sum_{j=0}^{q-2} \sum_{\substack{v \in K^\times \\ u^{-b}v^B + u^{cq^m}v^{-Cq^m} = 1}} \chi_K(v^j).$$

The sum $\sum_{j=0}^{q-2} \chi_K(v^j)$ is $q-1$ if $\mathrm{Norm}_{K/\mathbb{F}_q}(v) = 1$, that is, if v is a $q-1$th power in $K^\times$; it is 0 otherwise. Also, if v is a $q-1$th power in $K^\times$, then it has exactly $q-1$ distinct $q-1$th roots in $K^\times$, since $\mathbb{F}_q \subseteq K$ has all $q-1$ distinct $q-1$th roots of unity. Therefore, the trace becomes

$$\begin{aligned} &-1 + \sum_{\substack{v \in K^\times \\ u^{cq^m}v^{-Cq^m} + u^{-b}v^B = 1, \mathrm{Norm}_{K/\mathbb{F}_q}(v) = 1}} (q-1) \\ &= -1 + (q-1)|\{v \in K^\times \mid u^{-b}v^B + u^{cq^m}v^{-Cq^m} = 1,\ \mathrm{Norm}_{K/\mathbb{F}_q}(v) = 1\}| \\ &= -1 + |\{w \in K^\times \mid u^{-b}w^{(q-1)B} + u^{cq^m}w^{-(q-1)Cq^m} = 1\}| \\ &= -1 + |\{w \in K^\times \mid 1 + u^{b+cq^m}w^{-(q-1)A} = u^b w^{-(q-1)B}\}|. \end{aligned}$$

By mapping w to w^{-1} we can write this as

$$\begin{aligned} &-1 + |\{w \in K^\times \mid u^b w^{(q-1)B} - u^{b+cq^m} w^{(q-1)A} = 1\}| \\ =& -2 + |\{w \in K \mid u^b w^{q^m} - u^{b+cq^m} w^{q^n} = w\}|. \end{aligned}$$

Note that the set $\{w \in K \mid u^b w^{q^m} - u^{b+cq^m} w^{q^n} = w\}$ forms an $\mathbb{F}_q$-vector subspace of K. Therefore, its size is either 1 or a power of q. □

To apply the results in [13], we take a Kummer pullback of $\mathcal{W}$.

Proposition 5.5. *The trace of* $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W}$ *is*

$$u \in K^\times \mapsto -1 + |\{w \in K^\times \mid w^{q^m-1} - w^{q^n-1} = u^{\frac{q-1}{\gcd(q-1,c)}}\}|.$$

This is -2 *plus either* 1 *or a power of* q*. Also, the trace of* $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_0$ *is*

$$u \in K^\times \mapsto -1 + |\{w \in K^\times \mid w^B - w^A = u^{\frac{q-1}{\gcd(q-1,c)}}\}|.$$

Proof. From Proposition 5.4, we can compute the trace function of $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W} = [\frac{(q-1)C}{\gcd(q-1,c)}]^* \bigoplus_{j=0}^{q-2} \mathcal{G}_j$:

$$\begin{aligned} u \in K^\times \mapsto & -2 + |\{w \in K \mid u^{\frac{(q-1)Cb}{\gcd(q-1,c)}} w^{q^m} - u^{\frac{(q-1)C(b+cq^m)}{\gcd(q-1,c)}} w^{q^n} = w\}| \\ &= -1 + |\{w \in K^\times \mid u^{\frac{(q-1)Cb}{\gcd(q-1,c)}} w^{q^m-1} - u^{\frac{(q-1)C(b+cq^m)}{\gcd(q-1,c)}} w^{q^n-1} = 1\}|. \end{aligned}$$

The map $w \mapsto wu^{-c/\gcd(q-1,c)}$ is a bijection from K to itself, so we can rewrite the set in the above trace function as

$$\begin{aligned} &\{w \in K^\times \mid u^{\frac{(q-1)Cb}{\gcd(q-1,c)}} (wu^{-\frac{c}{\gcd(q-1,c)}})^{q^m-1} - u^{\frac{(q-1)C(b+cq^m)}{\gcd(q-1,c)}} (wu^{-\frac{c}{\gcd(q-1,c)}})^{q^n-1} = 1\} \\ =&\{w \in K^\times \mid u^{\frac{(q-1)(Cb-Bc)}{\gcd(q-1,b)}} w^{q^m-1} - u^{\frac{(q-1)(Cb+Ccq^m-cA)}{\gcd(q-1,c)}} w^{q^n-1} = 1\} \\ =&\{w \in K^\times \mid u^{-\frac{q-1}{\gcd(q-1,c)}} w^{q^m-1} - u^{-\frac{q-1}{\gcd(q-1,c)}} w^{q^n-1} = 1\} \\ =&\{w \in K^\times \mid w^{q^m-1} - w^{q^n-1} = u^{\frac{q-1}{\gcd(q-1,c)}}\}. \end{aligned}$$

Again, this set together with 0 form a $\mathbb{F}_q$-vector space, so its size is a power of q. The expression for the trace of $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_0$ at $u \in K^\times$ can be obtained using a similar argument and Proposition 5.1. □

Corollary 5.6. $\overline{\mathbb{Q}_\ell} \oplus \mathsf{inv}^*[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{H}_0$ *is geometrically isomorphic to* $[\frac{q-1}{\gcd(q-1,c)}]^* f_*\overline{\mathbb{Q}_\ell}$*, where* $f(t) \in \mathbb{F}_q[t]$ *is the polynomial* $f(t) = t^B - t^A$*. Also,* $\overline{\mathbb{Q}_\ell} \oplus \mathsf{inv}^*[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W}$ *is geometrically isomorphic to* $[\frac{q-1}{\gcd(q-1,c)}]^* F_*\overline{\mathbb{Q}_\ell}$*, where* $F(t) \in \mathbb{F}_q[t]$ *is the polynomial* $F(t) = t^{(q-1)B} - t^{(q-1)A}$.

Now we are ready to prove the main results of this section.

Theorem 5.7. *The geometric monodromy group* $G = G_{\mathrm{geom}}$ *of* $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W}$ *satisfies* $\mathrm{SL}_n(\mathbb{F}_q) = G^{(\infty)} \lhd G \lhd \mathrm{GL}_n(\mathbb{F}_q)$*. The monodromy representation of* $\overline{\mathbb{Q}_\ell} \oplus [\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W}$ *as a representation of* G *is the restriction of the permutation representation of* $\mathrm{GL}_n(\mathbb{F}_q)$ *acting on* $\mathbb{F}_q^n \setminus \{0\}$*, that is,* $\bigoplus_{j=0}^{q-2} \mathsf{Weil}_j$*.*

Proof. We mimic the proof of [13, Theorem 8.1]. To use [13, Theorem 6.8], we check the conditions for this theorem. We start with the condition (a) of [13, Theorem 6.8]. By Corollary 5.6 and [13, Lemma 5.1], the geometric monodromy group $G = G_{\mathrm{geom}}$ of $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W}$ can be embedded in S_{q^n-1} in a way such that the monodromy representation of $\overline{\mathbb{Q}_\ell} \oplus [\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W}$, viewed as a representation of G, is the restriction of the natural permutation representation of S_{q^n-1}. Also, $\mathcal{W}$ is geometrically isomorphic to $\bigoplus_{j=0}^{q-2} [\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_j$. We need to show that each $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_j$ is irreducible.

For $j = 0$, the monodromy representation of the Kummer pullback $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_0$ is the restriction of the monodromy representation of $\mathcal{G}_0$ to a normal subgroup H of G such that G/H is cyclic of order dividing $\frac{(q-1)C}{\gcd(q-1,c)}$. We also know that by [10, Proposition 1.2], $\mathcal{H}_0$ is not geometrically induced.

Clifford correspondence now shows that $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_0$ is isotypic, that is, all of its irreducible constituents are isomorphic to each other. Now Gallagher's theorem [4, Corollary 6.17] together with the extendibility of invariant characters of normal subgroups with cyclic quotient [4, Corollary 11.22] shows that $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_0$ is irreducible.

For $j \neq 0$, we know that the dimension of the monodromy representation of $\mathcal{G}_j$ is A, which is relatively prime to $\frac{(q-1)C}{\gcd(q-1,c)}$. On the other hand, $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_j$ is the restriction of $\mathcal{G}_j$ to a normal subgroup of index dividing $\frac{(q-1)C}{\gcd(q-1,c)}$. By [4, Corollary 11.29], $\frac{(q-1)C}{\gcd(q-1,c)}$ times the dimension of an irreducible constituent of this restriction must be divisible by A. Therefore, $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_j$ is irreducible, and condition (a) of [13, Theorem 6.8] holds.

By Corollary 5.6 and [13, Lemma 5.1], the geometric monodromy group of $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_0$ can be embedded in S_A in a way such that $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{G}_0$, as a representation of this group, is the restriction of the natural deleted permutation representation of S_A. Also, the image of γ_0 in this monodromy group has simple spectrum, and its order is the least common multiple of the orders of upstairs characters of $\mathcal{H}_0$, possibly divided by a divisor of $\frac{(q-1)C}{\gcd(q-1,c)}$. Since $\frac{(q-1)C}{\gcd(q-1,c)}$ and A are relatively prime, the order of the image of γ_0 is A. We also know that the image of $P(\infty)$ in the geometric monodromy group is a p-subgroup of order at least q^{n-1} by [14, Proposition 4.10]. Thus we have condition (b) of [13, Theorem 6.8].

Proposition 5.5 shows that the trace plus 1 is always a power of q, so condition (c) is also satisfied. [13, Theorem 6.8] now tells us that we must have

$$\mathrm{SL}_n(\mathbb{F}_q) \cong G^{(\infty)} \lhd G \lhd \mathrm{GL}_n(\mathbb{F}_q),$$

and $\overline{\mathbb{Q}_\ell} \oplus [\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W}$ as a representation of G is the restriction of $\bigoplus_{j=0}^{q-2} \mathsf{Weil}_j$ to G. □

Theorem 5.8. *In the situation of Theorem 5.7, the geometric monodromy group $G = G_{\mathrm{geom}}$ of $\mathcal{W}$ is isomorphic to $\mathrm{GL}_n(\mathbb{F}_q)/\langle \alpha^d I\rangle$.*

Proof. Let H be the geometric monodromy group of $[\frac{(q-1)C}{\gcd(q-1,c)}]^*\mathcal{W}$, so that $H \trianglelefteq \langle g_0, H\rangle = G$. By Theorem 5.7, H is the image under the representation $\mathsf{Weil}_0' \oplus \bigoplus_{j=1}^{q-2} \mathsf{Weil}_j$ of a subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ containing $\mathrm{SL}_n(\mathbb{F}_q)$. Let $L := E(H) = \mathrm{SL}_n(\mathbb{F}_q) = [H,H]$ be the quasisimple layer of H. Also for each $j = 0, \dots, q-2$, let G_j be the geometric monodromy group of $\mathcal{G}_j$.

(1) We first prove that $\mathbf{Z}(G) = \mathbf{C}_G(L)$.

Since $\mathcal{W}$ is the direct sum of irreducible representations $\mathcal{G}_j$ and each of them restricts to an irreducible representation of L, we can view G and L as subgroups of $\mathrm{GL}_{A-1}(\overline{\mathbb{Q}_\ell}) \oplus \mathrm{GL}_A(\overline{\mathbb{Q}_\ell}) \oplus \cdots \oplus \mathrm{GL}_A(\overline{\mathbb{Q}_\ell})$. By Schur's Lemma, we get

$$\mathbf{C}_{\mathrm{GL}_{q^n-2}(\overline{\mathbb{Q}_\ell})}(L) = \mathbf{C}_{\mathrm{GL}_{q^n-2}(\overline{\mathbb{Q}_\ell})}(G) = \mathbf{Z}(\mathrm{GL}_{A-1}(\overline{\mathbb{Q}_\ell})) \oplus \mathbf{Z}(\mathrm{GL}_A(\overline{\mathbb{Q}_\ell})) \oplus \cdots \oplus \mathbf{Z}(\mathrm{GL}_A(\overline{\mathbb{Q}_\ell})).$$

Therefore

$$\mathbf{C}_G(L) = \mathbf{C}_{\mathrm{GL}_{q^n-2}(\overline{\mathbb{Q}_\ell})}(L) \cap G = \mathbf{C}_{\mathrm{GL}_{q^n-2}(\overline{\mathbb{Q}_\ell})}(G) \cap G = \mathbf{Z}(G).$$

(2) Next, we show that $G/\mathbf{Z}(G) \cong \mathrm{PGL}_n(\mathbb{F}_q)$.

Since L is normal in G, the restriction of the monodromy representation of each $\mathcal{G}_j$ to L is invariant under conjugation by elements of G. We already saw that they are precisely Weil_0' and Weil_j, $j = 1, \dots, q-2$. The only automorphisms of $\mathrm{SL}_n(\mathbb{F}_q)$ fixing each of these representations are the inner and diagonal automorphisms. Therefore we get

$$G/\mathbf{Z}(G) = G/\mathbf{C}_G(L) \le \mathrm{PGL}_n(\mathbb{F}_q).$$

On the other hand, the geometric monodromy group G_j is the image of G under the projection from $\mathrm{GL}_{A-1}(\overline{\mathbb{Q}_\ell}) \oplus \mathrm{GL}_A(\overline{\mathbb{Q}_\ell}) \oplus \cdots \oplus \mathrm{GL}_A(\overline{\mathbb{Q}_\ell})$ onto the jth summand. Therefore, G_j is a quotient of G, and G_j contains the image of $\mathrm{SL}_n(\mathbb{F}_q)$ acting on an irreducible Weil representation. In particular, G_j is finite, almost quasisimple with unique nonabelian composition factor $\mathrm{PSL}_n(\mathbb{F}_q)$, so by Proposition 1.2(d), $G_j/\mathbf{Z}(G_j) \cong \mathrm{PGL}_n(\mathbb{F}_q)$, except for the exceptional pairs $(n, q) = (3, 2), (3, 3), (3, 4)$ of Proposition 1.2(d). For $(n, q) = (3, 2)$ and $(3, 3)$, we have $\mathrm{PGL}_n(\mathbb{F}_q) = \mathrm{PSL}_n(\mathbb{F}_q)$, so we still get $G_j/\mathbf{Z}(G_j) \cong \mathrm{PGL}_n(\mathbb{F}_q)$. Since we already know that $G/\mathbf{Z}(G) \le \mathrm{PGL}_n(\mathbb{F}_q)$, we get $G/\mathbf{Z}(G) \cong \mathrm{PGL}_n(\mathbb{F}_q)$.

For $(n, q) = (3, 4)$, $A = 21$ is divisible by $|\mathbb{F}_q^\times| = q - 1 = 3$, so by Theorem 2.7, no p'-element (in fact, no element at all) of $\mathrm{SL}_n(\mathbb{F}_q)$ has simple spectrum, which shows that $G_j/\mathbf{Z}(G_j) \ne \mathrm{PSL}_3(\mathbb{F}_4)$. Therefore, $G/\mathbf{Z}(G)$ is also not equal to $\mathrm{PSL}_3(\mathbb{F}_q)$. Since $\mathrm{PGL}_3(\mathbb{F}_4)/\mathrm{PSL}_3(\mathbb{F}_4)$ is simple and $G/\mathbf{Z}(G) \le \mathrm{PGL}_3(\mathbb{F}_4)$, the equality must hold.

(3) We compute the order of $\mathbf{Z}(G)$.

Let $z \in \mathbf{Z}(G)$. By (1),

$$z = (\epsilon_0 I, \epsilon_1 I, \dots, \epsilon_{q-2} I) \in \mathrm{GL}_{A-1}(\overline{\mathbb{Q}_\ell}) \oplus \mathrm{GL}_A(\overline{\mathbb{Q}_\ell}) \oplus \cdots \oplus \mathrm{GL}_A(\overline{\mathbb{Q}_\ell})$$

for some roots of unity $\epsilon_j \in \overline{\mathbb{Q}_\ell}^\times$, where I denotes the identity matrix of appropriate size. Thus for each j, the trace of the action of z on $\mathcal{G}_j$ is just $(A - \delta_{0,j})\epsilon_j$.

Let K be a finite extension of $\mathbb{F}_q$ on which $\mathcal{W}$ is defined and such that $[K : \mathbb{F}_q]$ is even, so that $\mathrm{Norm}_{K/\mathbb{F}_q}(-1) = 1$. Since $\mathcal{G}_j$ is pure of weight 0 and its geometric monodromy group G_j is finite, it follows that the arithmetic monodromy group $G_{j,K}$ over K of $\mathcal{G}_j$ is also finite, and hence the arithmetic monodromy group G_K of $\mathcal{W}$ is also finite. By Chebotarev density theorem, every element of G_K comes from a Frobenius in the arithmetic fundamental group. In particular, z is the image of the Frobenius at some $u \in K^\times$.

The trace of the action of z on $\mathcal{G}_j$, which is $(A - \delta_{0,j})\epsilon_j$, is also the value at u of the trace functions we computed in Proposition 5.1 and Proposition 5.2:

$$(A - \delta_{0,j})\epsilon_j = -\delta_{0,j} + \sum_{\substack{v \in K^\times \\ u^{-b}v^A - v^{Cq^m} + u^{cq^m} = 0}} \chi_K(v^j).$$

Note that there are at most A elements $v \in K^\times$ satisfying $u^{-b}v^A - v^{Cq^m} + u^{cq^m} = 0$, and each $\chi_K(v^j)$ is a root of unity. Therefore, the above equality forces that there are precisely A such v in $K^\times$ and that $\chi_K(v)^j = \epsilon_j$ for all such v. In particular, ϵ_1 is a $(q-1)$th root of unity and $\epsilon_j = \epsilon_1^j$ for all j. Also, all such v has the same $\mathrm{Norm}_{K/\mathbb{F}_q}(v)$, which is precisely the inverse image $\nu \in \mathbb{F}_q^\times$ of ϵ_1 under χ.

Since v are the roots of the polynomial $u^{-b}v^A - v^{Cq^m} + u^{cq^m}$, the product of all v is exactly $(-1)^A u^{b+cq^m}$. Hence

$$\begin{aligned}\nu^n = \nu^A &= \prod_{\substack{v\in K^\times \\ u^{-b}v^A - v^{Cq^m}+u^{cq^m}=0}} \mathrm{Norm}_{K/\mathbb{F}_q}(v) = \mathrm{Norm}_{K/\mathbb{F}_q}(\prod_v (v)) \\ &= \mathrm{Norm}_{K/\mathbb{F}_q}((-1)^A u^{b+cq^m}) = \mathrm{Norm}_{K/\mathbb{F}_q}(u)^{b+c}\end{aligned}$$

where the last equality follows from $\mathrm{Norm}_{K/\mathbb{F}_q}(-1) = 1$.

Let $d = \frac{q-1}{\gcd(q-1,b+c)}$, which is the order of $b+c \in \mathbb{Z}/(q-1)\mathbb{Z}$. Since $\nu \in \mathbb{F}_q^\times$ has order dividing $q-1$, we have

$$\nu^d = \nu^{d(bC-cB)} = \nu^{d(b(n-m)-cm)} = \nu^{bdn-(b+c)dm} = \nu^{bdn} = \mathrm{Norm}_{K/\mathbb{F}_q}(u)^{bd(b+c)} = 1.$$

Therefore $\epsilon_1 = \chi(\nu)$ has order dividing d, so it must be a power of λ^{b+c}. Since this holds for all $z \in \mathbf{Z}(G)$, we get

$$\mathbf{Z}(G) \leq \langle (I, \lambda^{b+c}I, \lambda^{2(b+c)}I, \dots, \lambda^{(q-2)(b+c)}I) \rangle \cong \mathbb{Z}/d\mathbb{Z}.$$

Recall that the geometric determinant of $\mathcal{H}_1$ is $\mathcal{L}_{\chi^{b+c}}$ if A is odd, and $\mathcal{L}_{\chi^{b+c}\chi_2}$ if A is even. Also note that if A is even, then d is also even: we know that $bC - cB = 1$, so that $bn - (b+c)m \equiv 1 \bmod q-1$. Since A is even if and only if n is even and q is odd, it follows that $(b+c)m$ is odd, whence d is even. Therefore, if A is even, then we can choose a j_0 such that $\chi^{(b+c)j_0} = \chi_2$. Then the geometric determinant of $\mathcal{H}_{1+j_0}$ is $\mathcal{L}_{\chi^{b+c}}$. In particular, G has $\mathbb{Z}/d\mathbb{Z}$ as a quotient. Since L is perfect, this quotient map factors through G/L. Since $G/\mathbf{Z}(G) \cong \mathrm{PGL}_n(\mathbb{F}_q)$ and $|L| = |\mathrm{SL}_n(\mathbb{F}_q)| = |\mathrm{PGL}_n(\mathbb{F}_q)|$, we have $|\mathbf{Z}(G)| = |G/L|$. Therefore, $|\mathbf{Z}(G)|$ is divisible by d. Together with the above observations, this implies $\mathbf{Z}(G) \cong \mathbb{Z}/d\mathbb{Z}$ and $G/L \cong \mathbb{Z}/d\mathbb{Z}$.

(4) Now we prove that $G \cong \mathrm{GL}_n(\mathbb{F}_q)/\langle \alpha^d I \rangle$.

To prove this, we will find a surjective group homomorphism $F : \mathrm{GL}_n(\mathbb{F}_q) \to G$ with kernel $\langle \alpha^d I \rangle$. Since we already embedded these groups into $\mathrm{GL}_{q^n-2}(\overline{\mathbb{Q}_\ell})$ in a way such that $\mathrm{SL}_n(\mathbb{F}_q) = L$, we only need to extend this to $\mathrm{GL}_n(\mathbb{F}_q) = \langle \mathrm{diag}(\alpha, 1, \dots, 1) \rangle \mathrm{SL}_n(\mathbb{F}_q)$.

As we saw above, $\mathcal{L}_{\chi^{b+c}}$ is the geometric determinant of some $\mathcal{G}_j$. We can view this as a one-dimensional representation of G. Since $L \leq [G, G]$, L lies in the kernel of this representation. Also, the image of g_0 under this representation has order equal to the order of this representation, which is $d = |G/L|$. Therefore $G = \langle g_0, L \rangle$.

Since g_0L generates G/L, $g_0\mathbf{Z}(G)L$ generates $G/\mathbf{Z}(G)L$. Note that

$$G/\mathbf{Z}(G)L \cong (G/\mathbf{Z}(G))/(\mathbf{Z}(G)L/\mathbf{Z}(G)) \cong \mathrm{PGL}_n(\mathbb{F}_q)/\mathrm{PSL}_n(\mathbb{F}_q) \cong \mathrm{GL}_n(\mathbb{F}_q)/(\mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))\,\mathrm{SL}_n(\mathbb{F}_q)).$$

Each generator of this quotient group is of the form

$$\mathrm{diag}(\alpha^t, 1, \dots, 1)\mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))\,\mathrm{SL}_n(\mathbb{F}_q)$$

for some $t \in \mathbb{Z}$ relatively prime to $q-1$. Therefore, we can choose an integer t_0 relatively prime to $q-1$ and elements $z_0 \in \mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q))$, $s_0 \in \mathrm{SL}_n(\mathbb{F}_q)$ such that

$$g_0 = z_0 h_0 s_0, \text{ where } h_0 = \mathrm{diag}(\alpha^{t_0}, 1, \dots, 1).$$

Let $F : \mathrm{GL}_n(\mathbb{F}_q) \to G$ be the map defined as

$$F(h_0^t s) = (g_0 s_0^{-1})^t s = z_0^t h_0^t s \text{ for each } t \in \mathbb{Z} \text{ and } s \in \mathrm{SL}_n(\mathbb{F}_q).$$

I claim that this map has the desired properties. First, we check the well-definedness: if $t_1, t_2 \in \mathbb{Z}$ and $s_1, s_2 \in \mathrm{SL}_n(\mathbb{F}_q)$ are such that $h_0^{t_1} s_1 = h_0^{t_2} s_2$, then $\mathrm{diag}(\alpha^{t_0(t_1-t_2)}, 1, \dots, 1) = h_0^{t_1-t_2} = s_2 s_1^{-1} \in \mathrm{SL}_n(\mathbb{F}_q)$, so $t_1 - t_2$ must be divisible by $q-1$. Then $z_0^{t_1-t_2} = 1$, so

$$F(h_0^{t_1} s_1) = z_0^{t_1} h_0^{t_1} s_1 = z_0^{t_2-t_1} z_0^{t_1} h_0^{t_2} s_2 = z_0^{t_2} h_0^{t_2} s_2 = F(h_0^{t_2} s_2).$$

Therefore, F is well-defined. We next check that it is a group homomorphism: for $t_1, t_2 \in \mathbb{Z}$ and $s_1, s_2 \in \mathrm{SL}_n(\mathbb{F}_q)$, we have

$$\begin{aligned} F(h_0^{t_1} s_1 h_0^{t_2} s_2) &= F(h_0^{t_1+t_2}(h_0^{-t_2} s_1 h_0^{t_2} s_2)) = z_0^{t_1+t_2} h_0^{t_1+t_2}(h_0^{-t_2} s_1 h_0^{t_2} s_2) \\ &= z_0^{t_1} z_0^{t_2} h_0^{t_1} s_1 h_0^{t_2} s_2 = z_0^{t_1} h_0^{t_1} s_1 z_0^{t_2} h_0^{t_2} s_2 = F(h_0^{t_1} s_1) F(h_0^{t_2} s_2). \end{aligned}$$

Since $G = \langle g_0, L \rangle = \langle g_0 s_0^{-1}, L \rangle$, every element of G can be written in the form $(g_0 s_0^{-1})^t s$ for some $t \in \mathbb{Z}$ and $s \in L = \mathrm{SL}_n(\mathbb{F}_q)$. Therefore F is surjective.

Finally, we check that $\ker F = \langle \alpha^d I \rangle$. If $h_0^t s \in \ker F$ for $t \in \mathbb{Z}$ and $s \in \mathrm{SL}_n(\mathbb{F}_q)$, then

$$(g_0 s_0^{-1})^t = (g_0 s_0^{-1})^t s s^{-1} = F(h_0^t s) s^{-1} = s^{-1} \in L.$$

Since $G = \langle g_0 s_0^{-1}, L \rangle$, the element $(g_0 s_0^{-1})L \in G/L$ has order exactly d. Hence d must divide t, so we may write $t = dt'$ for some $t' \in \mathbb{Z}$. Then

$$h_0^t s = z_0^{-t}(g_0 s_0^{-1})^t s = z_0^{-dt'} F(h_0^t s) = z_0^{-dt'} \in \mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q)) = \langle \alpha I \rangle.$$

Since z_0^d has order dividing $\frac{q-1}{d}$, it follows that $z_0^{-dt'} \in \langle \alpha^d I \rangle$, so that $\ker F \le \langle \alpha^d I \rangle$. Also,

$$|\ker F| = \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{|G|} = \frac{(q-1)|\mathrm{SL}_n(\mathbb{F}_q)|}{d|L|} = \frac{q-1}{d} = |\langle \alpha^d I \rangle|.$$

Therefore $\ker F = \langle \alpha^d I \rangle$, so $\mathrm{GL}_n(\mathbb{F}_q)/\langle \alpha^d I \rangle \cong G$. ☐

To determine the geometric monodromy groups of the summands $\mathcal{G}_j$, we need to know what are the irreducible constituents of the monodromy representation of $\mathcal{W}$ as a representation of $\mathrm{GL}_n(\mathbb{F}_q)$. Motivated by [15, Theorem 16.6] for $\mathrm{SU}_n(\mathbb{F}_q)$ and its extension [17, Theorem 1.2] to $\mathrm{GL}_n(\mathbb{F}_q)$, we will prove an analogous result for $\mathrm{GL}_n(\mathbb{F}_q)$ and use it to study this representation. By a slight abuse of notation, we will also denote by Weil and Weil_j the representations of $\mathrm{GL}_n(\mathbb{F}_q)$ corresponding to these modules.

Theorem 5.9. *Suppose that a complex representation Φ of $\mathrm{GL}_n(\mathbb{F}_q)$ has the following properties:*

(a) $\mathsf{Weil}_0 = \mathsf{Weil}_0' \oplus \mathbb{1}$ *is isomorphic to a subrepresentation of* Φ,
(b) $\Phi|_{\mathrm{SL}_n(\mathbb{F}_q)} \oplus \mathbb{1} \cong \mathsf{Weil}\,|_{\mathrm{SL}_n(\mathbb{F}_q)}$, *and*
(c) *The values of the character afforded by $\Phi \oplus \mathbb{1}$ are in* $\{1, q, q^2, \dots, q^n\}$.

Then $\Phi = \bigoplus_{j=0}^{q-2} \mathsf{Weil}_j \otimes X^{ej}$ *for some integer e, where* $X := \mathrm{GL}_n(\mathbb{F}_q) \xrightarrow{\det} \mathbb{F}_q^\times \xrightarrow{\alpha \mapsto \lambda} \mathbb{C}^\times$ *is a one-dimensional representation of order $q-1$.*

Proof. For each integer b', c' such that $b' \not\equiv c' \bmod \frac{q^{n-1}-1}{q-1}$, let

$$x_{b',c'} := \begin{pmatrix} \alpha^{b'} & 0 \\ 0 & \alpha_{n-1}^{c'-b'} \end{pmatrix} \in \mathrm{GL}_1(\mathbb{F}_q) \oplus \mathrm{GL}_{n-1}(\mathbb{F}_q) < \mathrm{GL}_n(\mathbb{F}_q).$$

If $c' - b'$ is divisible by $\frac{q^{n-1}-1}{q-1}$, then instead define

$$x_{b',c'} := \begin{pmatrix} \alpha^{b'} & 0 \\ 0 & \alpha_{n-1}^{c'-b'+q-1} \end{pmatrix} \in \mathrm{GL}_1(\mathbb{F}_q) \oplus \mathrm{GL}_{n-1}(\mathbb{F}_q) < \mathrm{GL}_n(\mathbb{F}_q).$$

Since $n \geq 3$, we have $\frac{q^{n-1}-1}{q-1} \geq \frac{q^2-1}{q-1} = q+1 > q-1$, so if $c'-b'$ is divisible by $\frac{q^{n-1}-1}{q-1}$ then $c'-b'+q-1$ is not. Recall that α_{n-1} permutes all $q^{n-1}-1$ nonzero vectors of $\mathbb{F}_q^{n-1}$ cyclically, and that $\alpha_{n-1}^{\frac{q^{n-1}-1}{q-1}} = \alpha I$ acts on $\mathbb{F}_q^{n-1}$ as a scalar. Therefore, the lower diagonal block of $x_{b',c'}$ cannot have any eigenvector in $\mathbb{F}_q^{n-1}$. It follows that the only eigenvectors of $x_{b',c'}$ in $\mathbb{F}_q^n$ are those of the form $(u,0) \in \mathbb{F}_q \oplus \mathbb{F}_q^{n-1}$ for nonzero $u \in \mathbb{F}_q$, and they have eigenvalue $\alpha^{b'}$.

Lemma 2.1 tells us that the spectrum of the action of $x_{b',c'}$ on Weil_j can be partitioned into subsets, and each of this subsets is, in the notations of section 2, the set of all s_vth roots of $\lambda^{t_v j}$, where v is a representative of the $\sim_{x_{b',c'}}$-equivalence class corresponding to this subset. The sum of the eigenvalues in a such subset is 0 unless $s_v = 1$, in which case the sum is simply $\lambda^{t_v j}$. But $s_v = 1$ means that v is an eigenvectors of $x_{b',c'}$, and as we saw above, the only eigenvectors of $x_{b',c'}$ are those of the form $(u,0) \in \mathbb{F}_q \oplus \mathbb{F}_q^{n-1}$ with $u \neq 0$. Therefore, the trace of the action of $x_{b',c'}$ on Weil_j is $\lambda^{b'j}$.

The determinant of $x_{b',c'}$ is

$$\det x_{b',c'} = \alpha^{b'} N_{\mathbb{F}_{q^{n-1}}/\mathbb{F}_q}(\alpha_{n-1}^{c'-b'}) = \alpha^{b'} (\alpha_{n-1}^{c'-b'})^{\frac{q^{n-1}-1}{q-1}} = \alpha^{b'+(c'-b')} = \alpha^{c'}$$

if $c' \not\equiv b' \bmod \frac{q^{n-1}-1}{q-1}$, and for the other cases we also get

$$\det x_{b',c'} = \alpha^{b'} N_{\mathbb{F}_{q^{n-1}}/\mathbb{F}_q}(\alpha_{n-1}^{c'-b'+q-1}) = \alpha^{b'+(c'-b'+q-1)} = \alpha^{c'}.$$

Thus we get $X(x_{b',c'}) = \lambda^{c'}$.

Since $\Phi|_{\mathrm{SL}_n(\mathbb{F}_q)} \oplus \mathbb{1} \cong \mathsf{Weil}\,|_{\mathrm{SL}_n(\mathbb{F}_q)}$, and Φ has a subrepresentation isomorphic to Weil_0, we must have

$$\Phi \cong \mathsf{Weil}_0 \oplus \bigoplus_{j=1}^{q-2} \mathsf{Weil}_j \otimes X^{i_j} = \bigoplus_{j=0}^{q-2} \mathsf{Weil}_j \otimes X^{i_j}$$

for some integers i_j with $i_0 = 0$. Then by the assumptions and the above calculations, for each pair of integers b', c' we get

$$1 + \operatorname{Trace} \Phi(x_{b',c'}) = 1 + \sum_{j=0}^{q-2} \lambda^{b'j + c'i_j} \in \{1, q, \ldots, q^n\}.$$

This is a sum of q roots of unity, so this is actually in $\{1, q\}$.

For each $b', c' \in \mathbb{Z}$ and $j \in \{0, \ldots, q-2\}$, let $d_{b',c',j}$ be the unique integer in $\{0, \ldots, q-2\}$ such that $d_{b',c',j} \equiv b'j + c'i_j \bmod q-1$. Consider the polynomial

$$P_{b',c'}(T) := \sum_{j=0}^{q-2} T^{d_{b',c',j}} \in \mathbb{Z}[T].$$

Then every coefficient is nonnegative, and the constant term is positive since $d_{b',c',0} = 0$. Clearly $P_{b',c'}(1) = q-1$, and for each integer r, we have $P_{b',c'}(\lambda^r) \in \{0, q-1\}$. By [17, Theorem 3.1 and Lemma 3.2], there exists $e \in \mathbb{Z}$ such that $i_j \equiv ej \bmod q-1$ for all $j = 0, \ldots, q-2$. Since X has order $q-1$, we get

$$\Phi \cong \bigoplus_{j=0}^{q-2} \mathsf{Weil}_j \otimes X^{ej}.$$

□

Corollary 5.10. *The monodromy representation of $\mathcal{G}_0$ as a representation of $\mathrm{GL}_n(\mathbb{F}_q)$ is Weil_0'. The set of monodromy representations of $\mathcal{G}_j$ for $j = 1, \ldots, q-2$ is equal to the set $\{\mathsf{Weil}_j \otimes X^{ej} \mid j = 1, \ldots, q-2\}$, where e is an integer such that $ne+1$ has order exactly d in $\mathbb{Z}/(q-1)\mathbb{Z}$, and X is as in Theorem 5.9.*

Proof. We know that the restriction of the monodromy representation of $\mathcal{W}$ to L is the same as the restriction of $\mathsf{Weil}_0' \oplus \bigoplus_{j=1}^{q-2} \mathsf{Weil}_j$ to $\mathrm{SL}_n(\mathbb{F}_q)$. The only irreducible constituent of rank $A-1$ of them are $\mathcal{G}_0$ and Weil_0'. Therefore, $\mathcal{G}_0$ as a representation of $\mathrm{GL}_n(\mathbb{F}_q)$ is $\mathsf{Weil}_0' \otimes Y$ for some one-dimensional representation Y of $\mathrm{GL}_n(\mathbb{F}_q)$ which is trivial on $\langle \alpha^d I \rangle$. Since g_0 has simple spectrum on $\mathcal{G}_0$, it also has simple spectrum on Weil_0'. By Theorem 2.7, g_0 as an element of $\mathrm{GL}_n(\mathbb{F}_q)/\langle \alpha^d I \rangle$ is the image of $\alpha_n^a \in \mathrm{GL}_n(\mathbb{F}_q)$ for some integer a relatively prime to A. The spectrum of the action of α_n^a on Weil_0' is the set of Ath roots of unity other than 1, which is exactly the spectrum of g_0 on $\mathcal{G}_0$. Therefore, as a representation of $\mathrm{GL}_n(\mathbb{F}_q)/\langle \alpha^d I \rangle$, Y contains both g_0 and L in the kernel. Since $G = \langle g_0, L \rangle$, Y is trivial. Therefore, $\overline{\mathbb{Q}_\ell} \oplus \mathcal{W}$ as a representation of $\mathrm{GL}_n(\mathbb{F}_q)$ satisfies the condition of Theorem 5.9. By Theorem 5.9, $\mathcal{W}$ must be $\mathsf{Weil}_0' \oplus \bigoplus_{j=1}^{q-2} \mathsf{Weil}_j \otimes X^{ej}$ for some integer e, where X is as in Theorem 5.9.

The restriction of $\mathsf{Weil}_j \otimes X^{ej}$ to $\mathbf{Z}(\mathrm{GL}_n(\mathbb{F}_q)) = \langle \alpha I \rangle$ maps αI to $\lambda^{j+nej} = \lambda^{(ne+1)j}$. Since the kernel of the monodromy representation of $\mathcal{W}$ is $\langle \alpha^d I \rangle$, it follows that $\lambda^{(ne+1)j}$ has order dividing d for all j, and order exactly d for at least one j. Therefore $ne+1$ has order exactly d in $\mathbb{Z}/(q-1)\mathbb{Z}$. $\square$

We finish this section with the following result on the geometric monodromy group of a Kummer pullback of $\mathcal{W}$. The proof is entirely analogous to [13, Corollary 8.4].

Corollary 5.11. *Let f be a divisor of d. The geometric monodromy group G_f of the Kummer pullback $[f]^*\mathcal{W}$ is $(\mathrm{SL}_n(\mathbb{F}_q) \rtimes \langle \mathrm{diag}(\alpha^f, 1, \ldots, 1) \rangle)/\langle \alpha^d I \rangle$.*

6. Abhyankar's Theorem on Galois Groups of Trinomials

In [13, Section 9], Katz and Tiep related their hypergeometric sheaves to Abhyankar's result [2, Theorem 1.2] on the Galois groups of certain polynomials. Since those sheaves are precisely the sheaf $\mathcal{W}$ in the previous section with $m = n-1$, $b = 1$ and $c = 0$, it is natural to ask if this connection can be generalized to other values of m.

Consider the polynomial described in [2, Theorem 1.2]:

$$F(T, U) := T^{q^n-1} - xU^r T^{q^m-1} + yU^s \in \overline{\mathbb{F}_q}(U)[T]$$

where n, m are integers relatively prime to each other, x, y are nonzero elements in $\overline{\mathbb{F}_q}$, and r, s are nonnegative integers such that $r(q^n-1) \neq s(q^n - q^m)$. Let $y', z \in \overline{\mathbb{F}_q}$ be numbers such that $(y')^{q^n-1} = y$, $z^{r(q^n-1)-s(q^n-q^m)} = x^{-1}(y')^{q^n-q^m}$ and let K_0 be a finite extension of $\mathbb{F}_q$ such that $x, y', z \in K_0$. The Galois group of this polynomial over $\overline{\mathbb{F}_q}(U)$ is the geometric monodromy group $G_{\mathrm{geom},\mathcal{A}}$ of the lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{A}$ over $\mathbb{G}_m/K_0$ whose trace at $u \in K^\times$ for a finite extension K of K_0 is the number of solutions of the equation

$$F(T, u) = T^{q^n-1} - xu^r T^{q^m-1} + yu^s = 0.$$

If we take the $[q^n-1]^*$ Kummer pullback, then we get a lisse sheaf over $\mathbb{G}_m/K_0$ whose trace at $u \in K^\times$ is

$$\begin{aligned} &|\{w \in K^\times \mid w^{q^n-1} - xu^{(q^n-1)r} w^{q^m-1} + yu^{(q^n-1)s} = 0\}| \\ =&|\{w \in K^\times \mid (b'u^s w)^{q^n-1} - xu^{(q^n-1)r} (b'u^s w)^{q^m-1} + yu^{(q^n-1)s} = 0\}| \\ =&|\{w \in K^\times \mid w^{q^n-1} - x(y')^{-q^n+q^m} u^{r(q^n-1)-s(q^n-q^m)} w^{q^m-1} + 1 = 0\}| \end{aligned}$$

The geometric monodromy group $G_{\mathrm{geom},[q^n-1]^*\mathcal{A}}$ of this sheaf satisfies

$$G_{\mathrm{geom},[q^n-1]^*\mathcal{A}} \trianglelefteq G_{\mathrm{geom},\mathcal{A}}, G_{\mathrm{geom},\mathcal{A}}/G_{\mathrm{geom},[q^n-1]^*\mathcal{A}} \text{ is cyclic of order dividing } q^n-1.$$

Let $\mathcal{B}$ be the lisse sheaf over K_0 obtained by taking multiplicative translate $[u \mapsto zu]^*[q^n-1]^*\mathcal{A}$. The geometric monodromy group $G_{\mathrm{geom},\mathcal{B}}$ is the same as $G_{\mathrm{geom},[q^n-1]^*\mathcal{A}}$, and the trace of $\mathcal{B}$ at $u \in K^\times$ is

$$\begin{aligned}&|\{w \in K^\times \mid w^{q^n-1} - x(y')^{-q^n+q^m}(zu)^{r(q^n-1)-s(q^n-q^m)}w^{q^m-1}+1=0\}|\\ =&|\{w \in K^\times \mid w^{q^n-1} - u^{r(q^n-1)-s(q^n-q^m)}w^{q^m-1}+1=0\}|.\end{aligned}$$

Since $r(q^n-1)-s(q^n-q^m)$ is a nonzero multiple of $q-1$, $\mathcal{B}$ is geometrically isomorphic to the $[(r(q^n-1)-s(q^n-q^m))/(q-1)]^*$ Kummer pullback of the lisse sheaf $\mathcal{C}$ over $\mathbb{G}_m/K_0$ whose trace at $u \in K^\times$ is

$$|\{w \in K^\times \mid w^{q^n-1} - u^{q-1}w^{q^m-1}+1=0\}|.$$

On the other hand, if we choose integer b, c such that n, m, b, c satisfies the conditions of Theorem 4.1, and use the notations from the previous section, then the $[q^m]^*$ Frobenius pullback of $\mathcal{C}$ has trace

$$\begin{aligned}&|\{w \in K^\times \mid w^{q^n-1} - u^{(q-1)q^m}w^{q^m-1}+1=0\}|\\ =&|\{w \in K^\times \mid w^{q^n-1} - u^{b(q^n-1)-(b+cq^m)(q^m-1)}w^{q^m-1}+1=0\}|\\ =&|\{w \in K^\times \mid (u^{b+cq^m}w)^{q^n-1} - u^{b(q^n-1)-(b+cq^m)(q^m-1)}(u^{b+cq^m}w)^{q^m-1}+1=0\}|\\ =&|\{w \in K^\times \mid u^{(b+cq^m)(q^n-1)}w^{q^n-1} - u^{b(q^n-1)}w^{q^m-1}+1=0\}|.\end{aligned}$$

Therefore, by Proposition 5.4, $[q^m]^*\mathcal{C}$ is geometrically isomorphic to $[q^n-1]^*(\overline{\mathbb{Q}_\ell} \oplus \mathcal{W})$.

Using these geometric isomorphisms, we can show that the geometric monodromy group of $\mathcal{C}$ is isomorphic to that of $[q-1]^*\overline{\mathbb{Q}_\ell} \oplus \mathcal{W}$, which is $\mathrm{SL}_n(\mathbb{F}_q)$ by Corollary 5.11. Hence, $G_{\mathrm{geom},\mathcal{B}}$ are also isomorphic to $\mathrm{SL}_n(\mathbb{F}_q)$, since $\mathcal{B}$ is geometrically isomorphic to $[(r(q^n-1)-s(q^n-q^m))/(q-1)]^*\mathcal{C}$ as we saw above. Since $G_{\mathrm{geom},[q^n-1]^*\mathcal{A}}$ is isomorphic to $G_{\mathrm{geom},\mathcal{B}}$, it follows that $G_{\mathrm{geom},\mathcal{A}}$ contains a normal subgroup $\mathrm{SL}_n(\mathbb{F}_q)$, and $G_{\mathrm{geom},\mathcal{A}}/\mathrm{SL}_n(\mathbb{F}_q)$ is cyclic of order dividing q^n-1. On the other hand, the q^n-1 roots of $F(T,U) \in \overline{\mathbb{F}_q}(U)[T]$ together with 0 form an n-dimensional $\mathbb{F}_q$-vector space. Each element of the Galois group acts $\mathbb{F}_q$-linearly on this set, so the Galois group $G_{\mathrm{geom},\mathcal{A}}$ is contained in $\mathrm{GL}_n(\mathbb{F}_q)$. Thus we get the conclusion of case (3) of [2, Theorem 1.2].

References

[1] S. S. Abhyankar, Coverings of algebraic curves, Amer. J. Math. **79** (1957), 825–856. MR0094354

[2] S. S. Abhyankar, Nice equations for nice groups, Israel J. Math. **88** (1994), no. 1-3, 1–23. MR1303488

[3] D. Harbater, Abhyankar's conjecture on Galois groups over curves, Invent. Math. **117** (1994), no. 1, 1–25. MR1269423

[4] I. M. Isaacs, *Character theory of finite groups*, corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)], Dover Publications, Inc., New York, 1994. MR1280461

[5] N. M. Katz, *Gauss sums, Kloosterman sums, and monodromy groups*, Annals of Mathematics Studies, 116, Princeton University Press, Princeton, NJ, 1988. MR0955052

[6] N. M. Katz, *Exponential sums and differential equations*, Annals of Mathematics Studies, 124, Princeton University Press, Princeton, NJ, 1990. MR1081536

[7] N. M. Katz, G_2 and hypergeometric sheaves, Finite Fields Appl. **13** (2007), no. 2, 175–223. MR2307123

[8] N. M. Katz and A. Rojas-León, A rigid local system with monodromy group $2.J_2$, Finite Fields Appl. **57** (2019), 276–286. MR3922515

[9] N. M. Katz, A. Rojas-León and P. H. Tiep, *Rigid local systems with monodromy group the Conway group Co_3*, J. Number Theory **206** (2020), 1–23. MR4013161

[10] N. M. Katz, A. Rojas-León and P. H. Tiep, *A rigid local system with monodromy group the big Conway group* $2.Co_1$ *and two others with monodromy group the Suzuki group* $6.Suz$, Trans. Amer. Math. Soc. **373** (2020), no. 3, 2007–2044. MR4068288

[11] N. M. Katz, A. Rojas-León and P. H. Tiep, *Rigid local systems with monodromy group the Conway group* Co_2, Int. J. Number Theory **16** (2020), no. 2, 341–360. MR4077426

[12] N. M. Katz, A. Rojas-León and P. H. Tiep, *Rigid local systems and Sporadic Simple groups*, Mem. Amer. Math. Soc. (to appear).

[13] N. M. Katz and P. H. Tiep, *Rigid local systems and finite general linear groups*, Math. Z. **298** (2021), no. 3-4, 1293–1321. MR4282130

[14] N. M. Katz and P. H. Tiep, *Monodromy groups of Kloosterman and hypergeometric sheaves*, Geom. Funct. Anal. **31** (2021), no. 3, 562–662. MR4311580

[15] N. M. Katz and P. H. Tiep, Hypergeometric sheaves and finite symplectic and unitary groups, Camb. J. Math. **9** (2021), no. 3, 577–691. MR4400735

[16] N. M. Katz and P. H. Tiep, *Exponential sums, hypergeometric sheaves, and monodromy groups,* preprint.

[17] L. Tae Young, A question of Katz and Tiep on representations of finite general unitary groups, Comm. Algebra **50** (2022), no. 8, 3426–3446. MR4429473

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