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Adaptive Identification of Second-Order Mechanical Systems with Nullspace Parameter Structure: Stability and Parameter Convergence

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***Abstract*— The relationships between persistence of excitation (PE) conditions and asymptotically stable convergence of parameter estimates are well-known for adaptive systems where stability and convergence are derived with respect to the origin of a combined state error and parameter estimation error system. We address a class of second-order mechanical systems in which the true parameters are, rather than one single point in parameter space, members of a nullspace defined by feasible evolutions of a regressor matrix. Differences within the set of true parameters are unobservable, requiring a new characterization of PE and parameter convergence. We report an adaptive identification (AID) approach for this class of systems and show local stability and parameter estimate convergence to the true parameter set under a subspace PE condition. This approach is applicable to many second-order mechanical systems, including robot arms and undersea, land, aerial, and space vehicles, and enables a more complete parameterization of uncertainty in the dynamics, e.g. enabling simultaneous AID of plant and actuator parameters for mechanical systems.**

I. Introduction

Adaptive systems enable compensation for model uncertainty via online parameter learning alongside tasks such as system identification and control. It is well-known that, although it is relatively simple to design adaptive systems wherein a state error, e.g. of an identification plant or trajectory-tracking, converges to zero, additional conditions are required for parameter convergence. Many authors in the 1960s and 1970s connected notions of persistence of excitation (PE) to parameter convergence, including [13], [3], [29], [30], [2], and [20]. In their seminal paper [20], Morgan and Narendra derive necessary and sufficient PE conditions for the uniform asymptotic stability (UAS) of a class of linear time-varying (LTV) adaptive systems of the form

$$\begin{bmatrix}\Delta\dot{v}(t)\\ \Delta\dot{\theta}(t)\end{bmatrix} = \begin{bmatrix}A(t) & -B(t)\\ B(t)^\top & 0\end{bmatrix}\begin{bmatrix}\Delta v(t)\\ \Delta\theta(t)\end{bmatrix} \quad (1)$$

where $\Delta v \in \mathbb{R}^n$ is a state error, $\Delta\theta \in \mathbb{R}^p$ is a parameter estimation error, $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times p}$ are bounded piecewise continuous, and $A(t)$ is a stable matrix. A PE condition on the regressor $B(t)$ is stated as follows: $\exists\ \epsilon_0, \delta_0 > 0$ such that for any $t \geq t_0$ and unit vector $w \in \mathbb{R}^p$,

$$\left\| \int_t^{t+\delta_0} B(\tau) w d\tau \right\| \geq \epsilon_0, \quad (2)$$

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i.e. if $B(t)$ is PE then it has no persistent nullspace.

PE conditions similar to (2) on the regressor matrix $B(t)$ or on exogenous signals have become well-established properties to determine stability and convergence properties of adaptive systems [4], [22], [26]. Studies such as [23], [21], [25], [14] extended the results of [20] to certain nonlinear systems. [12] summarizes many results on the relationship between PE conditions imposed on exogenous input signals (which are more useful from the perspective of control design) to PE of the regressor matrix and the convergence guarantees that can be obtained. Although stronger guarantees of uniform exponential stability are desirable for robustness and equivalent with UAS for LTV systems, they can be difficult to show for nonlinear systems [12]. Recent studies have also addressed the case of time-varying parameters [9].

Herein we consider a new class of nonlinear adaptive systems with skew-symmetric structure similar to (1), in which the regressor $B(t)$ has a time-varying persistent nullspace defined by the parameter estimate, and the true parameters are formulated as members of a persistent nullspace capturing all feasible evolutions of the true system. This dynamics formulation is applicable to a broad class of second-order mechanical systems including robot arms [5], [28] and space [1], aerial [6], terrestrial [27], and underwater [8] vehicles and has the advantage of providing a more complete parameterization of their dynamics. To the best of our knowledge, such nullspace parameterizations were first studied for model identification on underwater vehicles [10], [24], [11] and ground vehicles [7], as well as for adaptive trajectory-tracking control on underwater vehicles in [15], where this approach enabled simultaneous estimation of plant parameters and actuator parameters, of which the latter are typically assumed exactly known in previously reported model identification methods, e.g. [17], [19] and model-based control methods, e.g. [18].

However, as discussed in Section II, this nullspace parameterization results in analytical challenges which previously reported proofs of asymptotic stability and corresponding PE conditions, e.g. in [20]-[23], do not address. In particular, parameter convergence for this class of nullspace systems must be shown with respect to a true parameter set which is contained in a subspace of the parameter space instead of a single equilibrium point, the origin of (1). Although [21], [22] show cases of static and time-varying transformations of non-PE signals to achieve PE, e.g. PE in a lower-dimensional subspace, the reported results are not applicable to the class of systems addressed herein. To the best of our knowledge, [11] reported the first proof of asymptotic parameter conver-

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gence for nullspace adaptive identification (NS-AID), which was applied to an underwater vehicle model. This present paper generalizes the approach of [11] to a broader class of second-order systems. It is organized as follows: Section II defines the general class of dynamical systems with nullspace parameterization and discusses their properties, Section III reports the derivation of a NS-AID algorithm for such systems and discusses the difficulty in applying existing PE approaches to show parameter convergence, and Section IV reports proofs of local stability and parameter convergence for the NS-AID algorithm using a subspace PE condition.

II. NULLSPACE PARAMETERIZATION OF GENERAL SECOND-ORDER MECHANICAL SYSTEMS

A. A General Class of Second-Order Mechanical Systems

We consider a broad class of second-order mechanical systems whose kinematics are given by

$$\dot{q} = J(q)v, \tag{3}$$

with the position $q(t) \in \mathbb{R}^m$, velocity $v(t) \in \mathbb{R}^n$, combined state $x(t) = [q(t)^\top v(t)^\top]^\top \in \mathbb{R}^{m+n}$, and the Jacobian matrix $J(q) \in \mathbb{R}^m \rightarrow \mathbb{R}^{m\times n}$, and whose dynamics are linear in unknown constant parameters $\theta \in \mathbb{R}^p$, with the general form

$$M(x,\theta)\dot{v} + C(x,\theta)v = F(t,x)\theta, \tag{4}$$

where $M, C \in \mathbb{R}^{m+n} \times \mathbb{R}^p \rightarrow \mathbb{R}^{n\times n}$ are matrix-valued functions (MVFs) parameterized by the parameter vector θ, M is positive definite symmetric (PDS), C has the property that $\dot{M} - 2C$ is skew-symmetric, and $F(t,x) : \mathbb{R} \times \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{n\times p}$ is uniformly continuous in t and x.

B. Example Robot Systems with Nullspace Parameterization

Examples of systems which take this form (3,4) include robot arms [5], [28], and marine [8], aerial [6], ground [27] vehicles. The term $F(t,x)\theta$ specializes the general dynamical model to different plants and actuators.

The dynamics of rigid-body robot arms take the form

$$M(q,\theta)\dot{v} + C(x,\theta) = -g(q,\theta) + \tau(u,\theta), \tag{5}$$

with joint position $q(t) \in \mathbb{R}^n$, and joint velocity $v(t) \in \mathbb{R}^n$, with kinematics (3), often with $J(q) = I$, where g is the gravitational force/moment vector, τ is the joint force/moment vector arising from control input u, and θ contains mass/inertia, link, and actuator parameters, e.g. motor constants.

The dynamics of marine and aerial vehicles are similar and take the form

$$M(\theta)\dot{v} + C(x,\theta)v = -D(v,\theta)v - g(q,\theta) + \tau(u,\theta), \tag{6}$$

with world-frame position and orientation $q(t) \in \mathbb{R}^m$, and body-frame velocty $v(t) \in \mathbb{R}^n$, with kinematics (3), where D captures aero- or hydrodynamic lift and drag, g is a gravity term for aerial vehicles and a gravity/buoyancy term for marine vehicles, and τ arises from actuators such as thrusters and control surfaces, with control inputs u such as thruster motor speed or articulated control surface deflection angle. The parameters θ include plant parameters such as aero-/hydrodynamic mass, hydrodynamic added mass, aero-/hydrodynamic drag coefficients, center of mass, and center of buoyancy, and actuator parameters such as thrust coefficients and control-surface lift and drag coefficients.

The dynamics of ground vehicles take the form

$$M(q,\theta)\dot{v} + C(x,\theta)v = -D(v,\theta)v - h(x,\theta) + \tau(u,\theta), \tag{7}$$

with world-frame position and orientation $q(t) \in \mathbb{R}^m$, and body-frame velocty $v(t) \in \mathbb{R}^n$, with kinematics (3), where, in addition to the plant mass/inertia terms, θ parameterizes an aerodynamic lift and drag term D, a force/moment vector h due to wheel-ground interaction such as rolling resistance and wheel slip, and the control force/moment vector τ arising from steering and wheel actuators with control input u.

More complex articulated-body mobile robots or systems with attached robot arms can also be modeled using the general kinematics and dynamics equations (3,4), where the state is augmented to include additional dimensions of shape-variables representing the articulated joint positions and joint velocities, and a state-dependence is included in the inertia matrix $M(x,\theta)$.

C. Nullspace Parameter Structure

Since the parameters θ enter linearly on the LHS of (4) through the matrices M and C, each term on the LHS may be factored into the product of a regressor matrix and θ

$$M(x,\theta)\dot{v} = W_M(x,\dot{v})\theta \tag{8}$$

$$C(x,\theta)v = W_C(x,v)\theta. \tag{9}$$

Defining the combined regressor $W : \mathbb{R}\times\mathbb{R}^{m+n}\times\mathbb{R}^n\times\mathbb{R}^n \rightarrow \mathbb{R}^{n\times p}$,

$$W(t,x,v,\dot{v}) \triangleq W_M(x,\dot{v}) + W_C(x,v) - F(t,x), \tag{10}$$

we can rearrange (4) to obtain a nullspace relationship

$$W_M(x,\dot{v})\theta + W_C(x,v)\theta = F(t,x)\theta \tag{11}$$

$$[W_M(x,\dot{v}) + W_C(x,v) - F(t,x)]\theta = 0 \tag{12}$$

$$W(t,x,v,\dot{v})\theta = 0. \tag{13}$$

This nullspace structure reveals that the true parameter vector satisfying the equations of motion (4) for this class of systems is not unique. For example, $\forall\ \alpha \in \mathbb{R}$, from (13)

$$W(t,x,v,\dot{v})\theta = 0 \implies W(t,x,v,\dot{v})(\alpha\theta) = 0. \tag{14}$$

Indeed, given θ and all possible resulting bounded evolutions of $t, x(t), v(t), \dot{v}(t)$ from (4), there is a true parameter set consisting of all nonzero vectors θ^* that equivalently satisfy (13) and thus (4) for the same evolutions of $t, x(t), v(t), \dot{v}(t)$. We term this set $P(\theta)$ the "persistent nullspace" of the regressor matrix $W(t,x,v,\dot{v})$, which is defined as

$$P(\theta) \triangleq \{\theta^* \in \mathbb{R}^p : \theta^* \neq 0 \text{ and } W(t,x,v,\dot{v})\theta = 0 \iff W(t,x,v,\dot{v})\theta^* = 0\}. \tag{15}$$

The set $P(\theta)$ is embedded in an at least one-dimensional subspace of $\mathbb{R}^p$, but the subspace can be of higher-dimension when the dynamics are fully or partially decoupled.

The primary benefit of this nullspace structure is the simultaneous parameterization — thus enabling simultaneous parameter estimation — of all terms in the equation of motion (4). This differs from most conventional parameterization approaches, which represent the true parameters as a single, unique point in parameter space. In the case of the marine vehicle model (6), parameter identification methods, e.g. in [17], [19], commonly assume that $\tau(u(t))$ is a known signal and estimate parameters in the plant terms M, C, D, and g only, thus requiring exact knowledge of the control-actuator parameters. In contrast, the nullspace approaches parameterizes the LHS and entire RHS of (6), and more generally (4), thus more fully capturing uncertainty in the dynamics. The nullspace AID approach reported in [24], [11] for parameter identification on a 6-degree-of-freedom (DOF) underwater vehicle model estimates the aforementioned plant parameters as well as unknown actuator parameters for a thrust and control-surface lift and drag model.

D. Nullspace Parameterization is Not Overparameterization

A natural question is whether the choice to represent the true parameters as members of the set $P(\theta)$ (15) is an overparameterization – i.e. if true parameter vectors are not unique, is there a redundancy in the parameter space?

For systems with very simple plant dynamics and actuator dynamics, it is possible in some cases to reduce the true parameter set to a unique point in a reduced-dimension parameter space via normalization. For example, in the case of 1-DOF underwater vehicle dynamics, the general 6-DOF dynamical equation (6) reduces to

$$m\dot{v}(t) = -dv(t)|v(t)| - g + au(t), \tag{16}$$

which is linear in the plant and actuator parameters $\theta = [m, d, g, a]^\top \in \mathbb{R}^4$. In this special case (16), a reduced-dimension parameter vector can be obtained via normalization by (for example) m, so that 16 can be rewritten as

$$\dot{v}(t) = -\frac{d}{m}v(t)|v(t)| - \frac{g}{m} + \frac{a}{m}u(t), \tag{17}$$

which is not linear in θ, but is linear in the new reduced-dimension parameter vector $\theta_n = [\frac{d}{m}, \frac{g}{m}, \frac{a}{m}]^\top \in \mathbb{R}^3$, which is comprised of elements corresponding to members of the Cartesian (set) product of m^{-1} and the original parameter vector θ. For general 6-DOF dynamics, however, normalizing the dynamics (6) by $M(\theta)$ results in

$$\dot{v} + M(\theta)^{-1}C(x,\theta)v \tag{18}$$

$$= M(\theta)^{-1}[-D(v,\theta)v - g(q,\theta) + \tau(u(t),\theta)], \tag{19}$$

which is not linear in θ, but is linear in a parameter vector (of potentially very large dimension) comprised of elements corresponding to members of the Cartesian (set) product of the entries of M^{-1} and the original parameter vector θ.

In this new parameterization of the model terms $M^{-1}C$, $M^{-1}D$, $M^{-1}g$, and $M^{-1}\tau$ arising in (19), the individual plant parameter terms of mass, inertia, lift, drag, and buoyancy, and the individual actuator parameter terms are no longer distinct, and may not be individually observable. Moreover, the new parameterization may be of larger dimension than the nullspace parameterization θ in (4) and (6).

Another normalization approach is to restrict parameter vectors to a given magnitude, e.g. unit magnitude. When the persistent nullspace $P(\theta)$ is one-dimensional, this has the effect of normalizing the true parameter set to a single unique point in $P(\theta)$, but this approach may not generalize when $P(\theta)$ is multi-dimensional.

We note that the nullspace approach to dynamical model parameterization is relatively new, having first been studied in [10], [24], [11] for underwater vehicle model identification and in [15] for underwater vehicle model-based control, and that its implications have yet to be fully explored. It is apparent, however, that the nullspace approach enables a useful parameterization of uncertainty in the whole dynamical model, including but distinguishing both plant and actuator components, which is not simply an overparameterization. Simultaneously estimating physically meaningful parameters for both plant and actuator models may also support other model-based tasks, e.g. fault detection and isolation based on parameter changes in subsets of parameter space [16].

E. Properties of Nullspace Parameterization

This Section introduces several properties of systems of the form (4,13) with nullspace parameterization.

1) Invariance of Dynamics Under Equivalent Parameters: As a consequence of the definition of $P(\theta)$, all parameter vectors in $P(\theta)$ produce identical dynamics $\dot{v}$, so that differences between parameter vectors in the set $P(\theta)$ are not observable. Solving (4) for $\dot{v}$ as a function of time t, the state x, the velocity v, and the parameters θ, we define

$$\dot{v} = \dot{v}(t,x,v,\theta) \triangleq M(x,\theta)^{-1}[-W_C(x,v) + F(t,x)]\theta. \tag{20}$$

Given another parameter vector $\theta^* \in P(\theta)$, since $\forall\ t, x, v, \dot{v}$ that satisfy (13) it is also true by definition of $P(\theta)$ (15) that

$$W(t,x,v,\dot{v})\theta^* = 0, \tag{21}$$

$$[W_M(x,\dot{v}) + W_C(x,v) - F(t,x)]\theta^* = 0 \tag{22}$$

then from (8,22) there is an equivalent expression for $\dot{v}$ (20)

$$W_M(x,\dot{v})\theta^* = [-W_C(x,v) + F(t,x)]\theta^* \tag{23}$$

$$M(x,\theta^*)\dot{v} = [-W_C(x,v) + F(t,x)]\theta^* \tag{24}$$

$$\dot{v} = M(x,\theta^*)^{-1}[-W_C(x,v) + F(t,x)]\theta^*. \tag{25}$$

Similarly writing (25) as a function of the arguments t, v, x, θ^*, we have from (20,25) that $\forall\theta^* \in P(\theta)$,

$$\dot{v}(t,v,x,\theta^*) \triangleq M(x,\theta^*)^{-1}[-W_C(x,v) + F(t,x)]\theta^* \tag{26}$$

$$= \dot{v} \tag{27}$$

$$= \dot{v}(t,v,x,\theta) \tag{28}$$

2) Nullspace of the Regresssor: The nullspace of the regressor $W(t,x,v,\dot{v})$ can be characterized by the arguments of $\dot{v}$. It can easily be verified by (8,10) that, given $\dot{v}(t,x,\tilde{v},\tilde{\theta})$

$$\dot{v}(t,x,\tilde{v},\tilde{\theta}) \triangleq M(x,\tilde{\theta})^{-1}[-W_C(x,\tilde{v}) + F(t,x)]\tilde{\theta}, \tag{29}$$

then $\tilde{\theta}$ is an element of the persistent nullspace of $W(t,x,\tilde{v},\dot{v}(t,x,\tilde{v},\tilde{\theta}))$, i.e.

$$\text{span}\{\tilde{\theta}\} \subset \text{null}\big(W(t,x,\tilde{v},\dot{v}(t,x,\tilde{v},\tilde{\theta}))\big). \tag{30}$$

For convenience, we employ the notation

$$W(t,x,\tilde{v},\tilde{\theta}) \triangleq W(t,x(t),\tilde{v}(t),\dot{v}(t,x(t),\tilde{v}(t),\tilde{\theta})). \tag{31}$$

3) Coordinate Transformations: We note that the set $P(\theta)$ does not contain the origin, and that the set $P_s(\theta)$,

$$P_s(\theta) \triangleq P(\theta) \cup \{0\} \tag{32}$$

is a linear vector subspace of $\mathbb{R}^p$ with orthogonal complement $P_\perp(\theta)$. We define $r = \dim(P_s(\theta))$ and an orthonormal basis $\{p_1, ..., p_r\}$ for $P_s(\theta)$, as well as the matrix $\bar{P} \in \mathbb{R}^{p\times r}$,

$$\bar{P} \triangleq \begin{bmatrix} p_1 & ... & p_r \end{bmatrix}, \quad (33)$$

which, since the columns of $\bar{P}$ are by definition basis vectors of the persistent nullspace of $W(t,x,v,\theta)$, has the property

$$W(t,x,v,\theta)\bar{P} = 0_{n\times r}. \quad (34)$$

Similarly, we define an orthonormal basis $\{q_1, ..., q_{p-r}\}$ for $P_\perp(\theta)$ and the matrix $\bar{P}_\perp \in \mathbb{R}^{p\times(p-r)}$,

$$\bar{P}_\perp \triangleq \begin{bmatrix} q_1 & ... & q_{p-r} \end{bmatrix}. \quad (35)$$

Then the orthogonal matrix $Q \in \mathbb{R}^{p\times p}$

$$Q \triangleq \begin{bmatrix} \bar{P} & \bar{P}_\perp \end{bmatrix} \quad (36)$$

defines a change of basis, so that any parameter vector may be decomposed into the component belonging to $P(\theta)$ and to its orthogonal complement $P_\perp(\theta)$.

III. Nullspace Adaptive Identification

The nullspace adaptive identifier (NS-AID) consists of an identification plant with velocity $\hat{v}(t) \in \mathbb{R}^n$ and a parameter estimate $\hat{\theta}(t) \in \mathbb{R}^p$. The task is to estimate the parameters $\theta \in \mathbb{R}^p$ for systems of the form (4,13) using the signals $x(t), F(t,x)$. and the known structure of the regressor (10).

A. Error Coordinates

We define the identification plant error Δv as

$$\Delta v(t) \triangleq \hat{v}(t) - v(t), \quad (37)$$

and the absolute parameter estimate error $\Delta\theta$ as the difference between the parameter estimate $\hat{\theta}$ and any constant true parameter vector $\theta \in P(\theta)$ (15),

$$\Delta\theta(t) \triangleq \hat{\theta}(t) - \theta. \quad (38)$$

To characterize parameter error with respect to $P(\theta)$, we use the change of basis matrix Q (36) to decompose the estimate $\hat{\theta}$ as the sum of two orthogonal components,

$$\hat{\theta} = Q\bar{\theta} \quad (39)$$

$$= \bar{P}_\perp \bar{\theta}_{P_\perp} + \bar{P}\bar{\theta}_P, \quad (40)$$

where $\bar{\theta}_{P_\perp} \in \mathbb{R}^{p-r}$ and $\bar{\theta}_P \in \mathbb{R}^r$ are the coordinates of $\hat{\theta}$ in $P_\perp(\theta)$ and $P_s(\theta)$ (32), respectively.

B. Problem Statement

The NS-AID task is to design an identification plant update law $\dot{\hat{v}}(t)$ and a parameter estimate update law $\dot{\hat{\theta}}(t)$ to achieve the following goals with all signals remaining bounded:

$$\lim_{t\to\infty} \Delta v(t) = 0, \quad (41)$$

$$\lim_{t\to\infty} \hat{\theta}(t) \in P(\theta). \quad (42)$$

Due to the non-uniqueness of the true parameter vector, the goal is *not* to show convergence of the absolute error $\Delta\theta$ to 0, but rather convergence of $\hat{\theta}$ to the larger true parameter set $P(\theta)$ (15). An equivalent statement of the goal (42) is

$$\lim_{t\to\infty} \bar{\theta}_{P_\perp}(t) = 0. \quad (43)$$

C. Update Laws

We choose the identification plant dynamics to be

$$\dot{\hat{v}} = \dot{v}(t,x,\hat{v},\hat{\theta}) - K\Delta v \quad (44)$$

where $\dot{v}(t,x,\hat{v},\hat{\theta})$ uses the function notation (29) to denote the time-derivative of v that would arise from the identification plant velocity $\hat{v}$ and parameter estimate $\hat{\theta}$ in place of the true v and θ, and where $K \in \mathbb{R}^{n\times n}$ is a PDS gain matrix.

The parameter estimate update law, using $\dot{v}(t,x,\hat{v},\hat{\theta})$ and the notation (31), is

$$\dot{\hat{\theta}} = \Gamma W(t,x,\hat{v},\hat{\theta})^\top \Delta v, \quad (45)$$

where $\Gamma \in \mathbb{R}^{p\times p}$ is a PDS adaptation gain matrix.

D. Assumptions

We make the following assumptions.

- $W(t,x,v,\theta)$ (10) is uniformly continuous in t, x, v, θ.
- To assure boundedness and invertibility of $M(x,\hat{\theta})^{-1}$ in (44,45), we assume that $\Delta v(t_0) = 0$ and $\exists\ \epsilon > 0$ s.t.

$$\sqrt{\lambda_{\max}(\Gamma)\Delta\theta(t_0)^\top\Gamma^{-1}\Delta\theta(t_0)} + \epsilon \le \lambda_{\min}(M(x,\theta)) \quad (46)$$

  In practice, reasonable initial estimates are often available from prior experiments or empirical measurement.

E. Error Dynamics

From the identification plant update law (44) and the system dynamics (20), the time-derivative of the identification plant error $\Delta\dot{v}$ is

$$\Delta\dot{v} = \dot{\hat{v}} - \dot{v} \quad (47)$$

$$= -K\Delta v + \dot{v}(t,x,\hat{v},\hat{\theta}) - \dot{v}(t,x,v,\theta) \quad (48)$$

$$= -K\Delta v + \dot{v}(t,x,\hat{v},\hat{\theta}) - M(x,\theta)^{-1}[-W_C(x,v) + F(t,x)]\theta. \quad (49)$$

Factoring $M(x,\theta)^{-1}$ from the last two terms of (48) yields

$$\Delta\dot{v} = -K\Delta v + M(x,\theta)^{-1}\Big[M(x,\theta)\dot{v}(t,x,\hat{v},\hat{\theta}) + [W_C(x,v) - F(t,x)]\theta\Big]. \quad (50)$$

Then using $W_M(x,\dot{v}(t,x,\hat{v},\hat{\theta}))$ (8) to rearrange the second term and the relationship $v = \hat{v} - \Delta v$ (37) to write the third term according to the definition of W_C (9), we can substitute the full regressor $W(t,x,\hat{v},\hat{\theta})$ (10,31),

$$\Delta\dot{v} = -K\Delta v + M(x,\theta)^{-1}\Big[W_M(x,\dot{v}(t,x,\hat{v},\hat{\theta}))\theta + [W_C(x,\hat{v}) - F(t,x)]\theta - W_C(x,\Delta v)\theta\Big] \quad (51)$$

$$= -K\Delta v + M(x,\theta)^{-1}[W(t,x,\hat{v},\hat{\theta})\theta - C(x,\theta)\Delta v]. \quad (52)$$

Since $\hat{\theta} \in \text{null}(W(t,x,\hat{v},\hat{\theta}))$ (30), we may substitute $\hat{\theta}$ (38) into (52) to obtain error dynamics of the form

$$\Delta\dot{v} = -K\Delta v - M(x,\theta)^{-1}[W(t,x,\hat{v},\hat{\theta})(\hat{\theta} - \theta) + C(x,\theta)\Delta v] \quad (53)$$

$$= -(K + M(x,\theta)^{-1}C(x,\theta))\Delta v - M(x,\theta)^{-1}W(t,x,\hat{v},\hat{\theta})\Delta\theta. \quad (54)$$

From (38), we have that $\Delta\dot{\theta} = \dot{\hat{\theta}}$. Thus from (45, 54), we can construct the nonlinear error system

$$\begin{bmatrix}\Delta\dot{v}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} -K - M^{-1}C & -M^{-1}W(t,x,\hat{v},\hat{\theta}) \\ \Gamma W(t,x,\hat{v},\hat{\theta})^\top & 0 \end{bmatrix}\begin{bmatrix}\Delta v\\ \Delta\theta\end{bmatrix} \quad (55)$$

where $M = M(x,\theta)$, $C = C(x,\theta)$ for brevity, which is similar in form to the classical LTV system (1) studied in

[20], as well as the nonlinear extension studied in [23], which were shown to be asymptotically stable about the origin given a PE condition of the form (2) on the upper-right block.

For the class of systems of the form (4,13) addressed herein, however, for which any non-unique true parameter in $P(\theta)$ vector gives rise to identical dynamics (as discussed in Section II-E), a difference between parameter vectors in $P(\theta)$ is unobservable. Consequently, no identification approach (e.g. least squares, recursive least squares, adaptive, etc) can guarantee convergence of $\Delta\theta$ to 0. Moreover, the following lemma shows that a PE condition of the form (2) contradicts parameter convergence to any vector in the set $P(\theta)$, including convergence to $\Delta\theta = 0$ (38).

Proposition 1. Given the system (55), if $\lim_{t\to\infty} \hat{\theta} \in P(\theta)$ then $M(x,\theta)^{-1} W(t,x,\hat{v},\hat{\theta})$ is not PE in the sense of (2).

Proof. Examining the structure of $W(t,x,\hat{v},\hat{\theta})$, we observe from (30) that $\forall\, t$ there is at least the non-trivial time-varying nullspace $\text{span}\{\hat{\theta}\} \subset \text{null}\{W(t,x,\hat{v},\hat{\theta})\}$, and that if $\lim_{t\to\infty} \hat{\theta} = \theta^* \in P(\theta)$ in addition to $\lim_{t\to\infty} \Delta v = 0$, then from the equivalence of $\dot{v}$ arising from any $\theta^* \in P(\theta)$ (28)

$$\lim_{t\to\infty} W(t,x,\hat{v},\hat{\theta})\theta = \lim_{t\to\infty} W(t,x,v,\theta^*)\theta \qquad (56)$$

$$= \lim_{t\to\infty} W(t,x,v,\theta)\theta \qquad (57)$$

$$= 0. \qquad (58)$$

Thus parameter convergence implies that $W(t,x,\hat{v},\hat{\theta})$ has the persistent nullspace $P(\theta)$ in the limit, resulting in failure of the PE condition (2) for any $w \in P(\theta)$. ■

In summary, since the error dynamics (55) and canonical PE condition (2) are not suitable to show parameter convergence for this class of systems, we formulate alternative error dynamics and a modified PE condition. Instead of attempting to show that $\lim_{t\to\infty} \Delta\theta = 0$, we employ the error coordinates $\bar{\theta}_{P_\perp}$ (40) as a more meaningful characterization of the convergence of $\hat{\theta}$ to the true parameter set $P(\theta)$.

F. Nullspace Error Dynamics

Revisiting (48), there is an equivalent derivation for the error dynamics $\Delta\dot{v}$ obtained by substituting $\dot{v}(t,x,\hat{v},\hat{\theta})$ (29) and factoring out $M(x,\hat{\theta})^{-1}$ instead of $M(x,\theta)^{-1}$, yielding

$$\Delta\dot{v} = -K\Delta v + M(x,\hat{\theta})^{-1}[-W_C(x,\hat{v}) + F(t,x)]\hat{\theta} - \dot{v}(t,x,v,\theta) \qquad (59)$$

$$= -K\Delta v - M(x,\hat{\theta})^{-1}\Big[[W_C(x,\hat{v}) - F(t,x)]\hat{\theta} + M(x,\hat{\theta})\dot{v}(t,x,v,\theta)\Big]. \qquad (60)$$

Again using the function W_M (8) to rearrange the last term, decomposing the second term with the function W_C (9), and substituting $W(t,x,v,\theta)$ (10), we have

$$\Delta\dot{v} = -K\Delta v - M(x,\hat{\theta})^{-1}\Big[W_M(\dot{v}(t,x,v,\theta))\hat{\theta} + [W_C(x,v) - F(t,x)]\hat{\theta} - W_C(x,\Delta v)\hat{\theta}\Big] \qquad (61)$$

$$= -K\Delta v - M(x,\hat{\theta})^{-1}[W(t,x,v,\theta)\hat{\theta} - C(x,\hat{\theta})\Delta v]. \qquad (62)$$

Decomposing $\hat{\theta}$ using the projection matrix Q (36,40) into the component belonging to $P(\theta)$ (15) and the component belonging to its orthogonal complement $P_\perp(\theta)$ and using the property (34) results in

$$\Delta\dot{v} = -(K - M(x,\hat{\theta})^{-1}C(x,\hat{\theta}))\Delta v - M(x,\hat{\theta})^{-1}W(t,x,v,\theta)(\bar{P}_\perp\bar{\theta}_{P_\perp} + \bar{P}\bar{\theta}_P) \qquad (63)$$

$$= -(K - M(x,\hat{\theta})^{-1}C(x,\hat{\theta}))\Delta v - M(x,\hat{\theta})^{-1}W(t,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}. \qquad (64)$$

The error system, where $z \triangleq [\Delta v^\top\ \bar{\theta}_{P_\perp}^\top\ \bar{\theta}_P^\top]^\top \in \mathbb{R}^{n+p}$ is

$$\dot{z} = \begin{bmatrix} -K+\hat{M}^{-1}\hat{C} & -\hat{M}^{-1}W(t,x,v,\theta)\bar{P}_\perp & 0 \\ \bar{P}_\perp^\top \Gamma W(t,x,\hat{v},\hat{\theta})^\top & 0 & 0 \\ \bar{P}^\top \Gamma W(t,x,\hat{v},\hat{\theta})^\top & 0 & 0 \end{bmatrix} z, \qquad (65)$$

where $\hat{M} = M(x,\hat{\theta})$, $\hat{C} = C(x,\hat{\theta})$ for brevity. Comparing (64,65) to the equivalent expressions (54,55), we see that the upper-right block now contains of the product of $W(t,x,v,\theta)$ and $\bar{P}_\perp$. The following Section contains a theorem of parameter convergence based on PE of the term $W(t,x,v,\theta)\bar{P}_\perp$, showing that, under a modified PE condition, the system converges to $\{\Delta v = 0,\ \bar{\theta}_{P_\perp} = 0\}$.

IV. Theorems of Stability and Parameter Convergence

Theorem 1 in this Section gives a standard result guaranteeing uniform stability about the origin of (55), boundedness of all signals, and convergence to zero of the identification plant error $\Delta v(t)$. We then present Lemma 1, followed by Theorem 2, a proof of parameter convergence for this class of NS-AID methods with respect to the set $P(\theta)$, using the alternative error dynamics (65), the new error term $\bar{\theta}_{P_\perp}(t)$ (40), and a subspace PE condition.

Theorem 1. Under the assumptions in Section III-D, the system (55) is locally uniformly stable about the origin $(\Delta v, \Delta\theta) = 0$, all signals remain bounded, and $\lim_{t\to\infty} \Delta v(t) = 0$

Proof. Consider the Lyapunov function candidate

$$V(\Delta v, \Delta\theta) = \frac{1}{2}\Delta v^\top M(x,\theta)\Delta v + \frac{1}{2}\Delta\theta^\top \Gamma^{-1}\Delta\theta, \qquad (66)$$

which is $\mathcal{C}^1$, positive-definite, radially unbounded in $\Delta v, \Delta\theta$, and equal to zero if and only if $(\Delta v, \Delta\theta) = 0$. From the error dynamics (55), the time-derivative of $V(\Delta v, \Delta\theta)$ is

$$\dot{V} = \frac{1}{2}\Delta v^\top \dot{M}\Delta v + \Delta v^\top M\Delta\dot{v} + \Delta\theta^\top \Gamma^{-1}\Delta\dot{\theta} \qquad (67)$$

$$= \frac{1}{2}\Delta v^\top(\dot{M} - 2C)\Delta v - \Delta v^\top MK\Delta v \qquad (68)$$

$$= -\Delta v^\top MK\Delta v. \qquad (69)$$

Thus $\dot{V}$ is negative-definite in Δv and negative semi-definite in $\{\Delta v,\ \Delta\theta\}$, satisfying the requirements on a Lyapunov function to show that the system (55) is uniformly stable about the origin and that Δv and $\Delta\theta$ are bounded. From (66,69) and the assumption (46), which limits this proof to local stability, it is easy to show that all other signals $\hat{v}, \hat{\theta}, M(x,\hat{\theta})^{-1}, \dot{x}, \dot{\hat{v}}, \Delta\dot{v}, \dot{\hat{\theta}}$ are bounded, and additionally that $\Delta v(t) \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ (Definition 2.11, [22]). By Barbalat's Lemma (Corollary 2.9, [22]), this together with bounded $\Delta\dot{v}$ this implies that

$$\lim_{t\to\infty} \Delta v(t) = 0. \qquad (70)$$

However, concerning the parameter estimates, from (45,70) and bounded $W(t,x,\hat{v},\hat{\theta})$ we can only conclude that $\lim_{t\to\infty}\dot{\hat{\theta}}(t)=0$. ■

Lemma 1. Given the system (55) and its equivalent expression (65), then $\lim_{t\to\infty} W(t,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}(t)=0$

Proof. From Theorem 1, $\lim_{t\to\infty}\Delta v(t)=0$. Again by Barbalat's Lemma (Lemma 2.12, [22]), if $\lim_{t\to\infty}\Delta v(t)$ is finite and $\Delta\dot{v}(t)$ is uniformly continuous in time, then

$$\lim_{t\to\infty}\Delta\dot{v}(t)=0. \tag{71}$$

The boundedness results of Theorem 1 guarantee the uniform continuity (UC) of $\Delta\dot{v}(t)$ in t through the UC of its component signals (62), as well as the the existence and boundedness of $M(x,\hat{\theta})^{-1}$. Thus from (65,70,71), we have

$$\lim_{t\to\infty} -(K-\hat{M}^{-1}\hat{C})\Delta v-\hat{M}^{-1}W(t,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}=0 \tag{72}$$

$$\lim_{t\to\infty}\hat{M}^{-1}W(t,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}=0 \tag{73}$$

$$\lim_{t\to\infty}W(t,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}=0, \tag{74}$$

thus concluding the proof. ■

Examining the product $W(t,x,v,\theta)\bar{P}_\perp$ in (74), we observe that the nullspace of $W(t,x,v,\theta)$ by definition always contains $P(\theta)$ (15), but that $\bar{P}_\perp$ (35) is a projection matrix of basis vectors orthogonal to $P(\theta)$ and thus eliminates $P(\theta)$ from the persistent nullspace of $W(t,x,v,\theta)\bar{P}_\perp$. Intuitively, if $W(t,x,v,\theta)\bar{P}_\perp$ has no persistent nullspace, then to satisfy (74) it must be true that $\bar{\theta}_{P_\perp}(t)$ converges to 0.

Theorem 2. Given the system (55,65), if there exist $\epsilon_0,\delta_0>0$ such that for any $t\geq t_0$ and any unit $w\in\mathbb{R}^p$,

$$\left\|\int_t^{t+\delta_0}W(\tau,x(\tau),v(\tau),\theta)\bar{P}_\perp w d\tau\right\|\geq\epsilon_0, \tag{75}$$

i.e. if the regressor $W(t,x,v,\theta)$ is PE in the subspace orthogonal to the set $P(\theta)$ (15), then $\lim_{t\to\infty}\hat{\theta}\in P(\theta)$.

Proof.

1) Let all assumptions of Theorem 1 be satisfied such that $\lim_{t\to\infty}\Delta v(t)=0$, let Lemma 1 hold such that $\lim_{t\to\infty}W(t,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}(t)=0$, let the PE condition (75) hold, and let the bounds $b_\theta,b_{\hat{\theta}}$ satisfy

$$\forall\, t\geq t_0, b_\theta\geq\|W(t,x,v,\theta)\bar{P}_\perp\| \tag{76}$$

$$\forall\, t\geq t_0, b_{\hat{\theta}}\geq\|W(t,x,\hat{v},\hat{\theta})\|. \tag{77}$$

2) Let us suppose that $\lim_{t\to\infty}\bar{\theta}_{P_\perp}(t)\neq 0$. Then there exists an $\epsilon>0$ and an unbounded sequence of times $\{t_i\}_{i=1,\ldots,\infty}$ such that $\|\bar{\theta}_{P_\perp}(t_i)\|\geq\epsilon$.

3) We make use of the property that, given $T',\phi>0$, $\exists$ $\epsilon'>0$ such that if $\|\Delta v(t)\|\leq\epsilon'$ $\forall t\in[t_i,t_i+T']$, then

$$\|\bar{\theta}_{P_\perp}(t)-\bar{\theta}_{P_\perp}(t_i)\|\leq\ \phi\ \forall t\in[t_i,t_i+T'], \tag{78}$$

i.e. "when Δv is small, $\bar{\theta}_{P_\perp}$ is flat" [20].
This property can be verified for given T' and ϕ by choosing $\epsilon'=\phi/T'\lambda_{\max}(\Gamma)b_{\hat{\theta}}$ and observing that, $\forall\, t\in[t_i,t_i+T']$, using the expression for $\dot{\bar{\theta}}_{P_\perp}(t)$ in (65), the bound $b_{\hat{\theta}}$ (77), the orthogonality of $\bar{P}_\perp$ (i.e. that $\|\bar{P}_\perp\|=1$), and the assumption on $\|\Delta v(t)\|$ over the time interval, we have

$$\|\bar{\theta}_{P_\perp}(t)-\bar{\theta}_{P_\perp}(t_i)\|=\left\|\int_{t_i}^t\dot{\bar{\theta}}_{P_\perp}(\tau)d\tau\right\| \tag{79}$$

$$\leq\int_{t_i}^t\|\bar{P}_\perp^\top\|\|\Gamma\|\|W(\tau,x,\hat{v},\hat{\theta})\|\|\Delta v(\tau)\|d\tau \tag{80}$$

$$\leq\phi. \tag{81}$$

4) Since $\lim_{t\to\infty}\Delta v(t)=0$, for any $\epsilon'>0$ we can always find a $t'\geq t_0$ such that $\|\Delta v(t)\|\leq\epsilon'\ \forall\ t\geq t'$. Thus using ϵ_0,δ_0 from the PE condition (75), ϵ from the assumption on $\|\bar{\theta}_{P_\perp}(t)\|$ made in Part 2 of this proof, and the bound b_θ (76), we can choose

$$\phi=\frac{\epsilon_0\epsilon}{2\delta_0 b_\theta} \tag{82}$$

$$T'=\delta_0 \tag{83}$$

and find a $t'\geq t_0$ such that the flatness property (78) holds for each interval $[t_i,t_i+\delta_0]$, where $t_i\geq t'$.
We will now show that for each $t_i\geq t'$, there is a $t_j\in[t_i,t_i+\delta_0]$ where $\|W(t_j,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}(t_j)\|>0$. For each $t_i\geq t'$, we use the reverse triangle inequality and the flatness property (78) to show

$$\left\|\int_{t_i}^{t_i+\delta_0}W(\tau,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}(t_i)d\tau\right\| -\left\|\int_{t_i}^{t_i+\delta_0}W(\tau,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}(\tau)d\tau\right\| \tag{84}$$

$$\leq\left\|\int_{t_i}^{t_i+\delta_0}W(\tau,x,v,\theta)\bar{P}_\perp[\bar{\theta}_{P_\perp}(t_i)-\bar{\theta}_{P_\perp}(\tau)]d\tau\right\| \tag{85}$$

$$\leq\int_{t_i}^{t_i+\delta_0}\|W(\tau,x,v,\theta)\bar{P}_\perp\|\|[\bar{\theta}_{P_\perp}(t_i)-\bar{\theta}_{P_\perp}(\tau)]\|d\tau \tag{86}$$

$$\leq\delta_0 b_\theta\phi. \tag{87}$$

Rearranging (87) and choosing the unit vector in the PE condition (75) to be $w\triangleq\bar{\theta}_{P_\perp}(t_i)/\|\bar{\theta}_{P_\perp}(t_i)\|$, we have

$$\left\|\int_{t_i}^{t_i+\delta_0}W(\tau,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}(\tau)d\tau\right\| \geq\left\|\int_{t_i}^{t_i+\delta_0}W(\tau,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}(t_i)d\tau\right\|-\delta_0 b_\theta\phi \tag{88}$$

$$\geq\|\bar{\theta}_{P_\perp}(t_i)\|\left\|\int_{t_i}^{t_i+\delta_0}W(\tau,x,v,\theta)\bar{P}_\perp w d\tau\right\|-\delta_0 b_\theta\phi. \tag{89}$$

We invoke the PE condition (75), substitute our choice of ϕ (82), and recall the assumption made in Part 2 that $\|\bar{\theta}_{P_\perp}(t_i)\|\geq\epsilon$. This yields

$$\left\|\int_{t_i}^{t_i+\delta_0}W(\tau,x,v,\theta)\bar{P}_\perp\bar{\theta}_{P_\perp}(\tau)d\tau\right\| \geq\|\bar{\theta}_{P_\perp}(t_i)\|\epsilon_0-\delta_0 b_\theta\phi \tag{90}$$

$$\geq\frac{\epsilon_0\epsilon}{2}. \tag{91}$$

Furthermore, we have

$$\int_{t_i}^{t_i+\delta_0} \|W(\tau,x,v,\theta)\bar{P}_\perp \bar{\theta}_{P_\perp}(\tau)\| d\tau$$

$$\geq \left\| \int_{t_i}^{t_i+\delta_0} W(\tau,x,v,\theta)\bar{P}_\perp \bar{\theta}_{P_\perp}(\tau) d\tau \right\| \quad (92)$$

$$\geq \frac{\epsilon_0 \epsilon}{2}. \quad (93)$$

By the mean value theorem for integrals, there must be a $t_j \in [t_i, t_i + \delta_0]$ such that

$$\int_{t_i}^{t_i+\delta_0} \|W(\tau,x,v,\theta)\bar{P}_\perp \bar{\theta}_{P_\perp}(\tau)\| d\tau$$

$$= \|W(t_j,x,v,\theta)\bar{P}_\perp \bar{\theta}_{P_\perp}(t_j)\| \delta_0 \quad (94)$$

and therefore from (93,94)

$$\|W(t_j,x,v,\theta)\bar{P}_\perp \bar{\theta}_{P_\perp}(t_j)\| \geq \frac{\epsilon_0 \epsilon}{2\delta_0} \quad (95)$$

$$> 0. \quad (96)$$

Thus we have found an unbounded sequence of times $\{t_j\}_{j=1,..,\infty}$ where $\|W(t_j,x,v,\theta)\bar{P}_\perp \bar{\theta}_{P_\perp}(t_j)\| > 0$, which contradicts Lemma 1 and implies that the assumption $\lim_{t\to\infty} \bar{\theta}_{P_\perp}(t) \neq 0$ must be false. Therefore $\lim_{t\to\infty} \bar{\theta}_{P_\perp}(t) = 0$ and, equivalently, $\lim_{t\to\infty} \hat{\theta}(t) \in P(\theta)$, achieving parameter convergence (42) under the sufficient persistence of excitation condition (75). ■

V. Conclusion

This paper addresses a broad class of second-order mechanical systems with nullspace parameter structure, which enables a more complete parameterization of uncertainty in the dynamics. We report an adaptive identifier for this class of systems with proofs of local stability and a new characterization of asymptotic parameter convergence with respect to the true nullspace parameter set under a subspace PE condition on the regressor matrix. Future work includes simulation and experimental evaluation, stronger stability and convergence guarantees with less restrictive assumptions, the relationship of subspace PE to exogenous signals, and other adaptive tasks such as trajectory-tracking control.

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