

# A FINITELY CONVERGENT CIRCUMCENTER METHOD FOR THE CONVEX FEASIBILITY PROBLEM\*

ROGER BEHLING<sup>§,†</sup>, YUNIER BELLO-CRUZ<sup>‡</sup>, ALFREDO N. IUSEM<sup>§</sup>, DI LIU<sup>¶</sup>, AND LUIZ-RAFAEL SANTOS<sup>†</sup>

## Abstract.

In this paper, we present a variant of the circumcenter method for the Convex Feasibility Problem (CFP), ensuring finite convergence under a Slater assumption. The method replaces exact projections onto the convex sets with projections onto separating halfspaces, perturbed by positive exogenous parameters that decrease to zero along the iterations. If the perturbation parameters decrease slowly enough, such as the terms of a diverging series, finite convergence is achieved. To the best of our knowledge, this is the first circumcenter method for CFP that guarantees finite convergence.

**Key words.** Convex Feasibility Problem, Finite convergence, Circumcentered-reflection method, Projection methods.

**MSC codes.** 49M27, 65K05, 65B99, 90C25

**1. Introduction.** The convex feasibility problem (CFP) aims at finding a point in the intersection of  $m$  closed and convex sets  $C_i \subset \mathbb{R}^n$ ,  $i = 1, \dots, m$ , *i.e.*,

$$(1.1) \quad \text{find } x^* \in C := \bigcap_{i=1}^m C_i.$$

Convex feasibility represents a modeling paradigm for solving numerous engineering and physics problems, such as image recovery [31], wireless sensor networks localization [42], gene regulatory network inference [60], and many others.

Projection-reflection based methods are widely recognized and effective schemes for solving a diverse range of feasibility problems, including (1.1). This sort of methods continues to gain popularity due to their ability to strike a balance between high performance and simplicity, as evidenced by their extensive utilization (see, *e.g.*, [11]). Among these methods, two particularly renowned and widely adopted methods are the classical Douglas-Rachford method (DRM) and its modifications (see, *e.g.*, [2, 8, 39]), and the famous method of alternating projections (MAP) (see, *e.g.*, [9, 10]). The elementary Euclidean concept of *circumcenter* has recently been employed to improve the convergence of those classical projection-reflection methods for solving the CFP (1.1).

The circumcentered-reflection method (CRM) was first presented in [22] as an acceleration technique for DRM for the two set affine CFP. Since then, CRM has been shown as a valid and powerful new tool for solving (non)convex structured feasibility problems because of its ability

---

\*

**Funding:** **RB** was partially supported by Brazilian agency Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (Grant E-26/201.345/2021); **YBC** was partially supported by the USA agency National Science Foundation (Grant DMS-2307328). **LRS** was partially supported by Brazilian agencies Conselho Nacional de Desenvolvimento Científico e Tecnológico (Grant 310571/2023-5) and Fundação de Amparo à Pesquisa do Estado de São Paulo (Grant 2023/03969-4). **DL** thanks the Brazilian agency Coordenação de Aperfeiçoamento de Pessoal de Nível Superior for the doctoral scholarship.

<sup>†</sup>Department of Mathematics, Federal University of Santa Catarina, Blumenau-SC – 89065-300, Brazil. (roger-behling@gmail.com, l.r.santos@ufsc.br)

<sup>‡</sup>Department of Mathematical Sciences, Northern Illinois University, DeKalb-IL – 60115-2828, USA (yunier-bello@niu.edu)

<sup>§</sup>School of Applied Mathematics, Fundação Getúlio Vargas, Rio de Janeiro-RJ – 22250-900, Brazil (alfredo.iusem@fgv.br)

<sup>¶</sup>Instituto de Matemática Pura e Aplicada, Rio de Janeiro-RJ – 22460-320, Brazil (di.liu@impa.br)

to minimize the inherent *zigzag* behavior of projection-reflection based methods, in particular. In [24], for instance, CRM was connected to the classical MAP. Moreover, CRM obviates the *spiralizing* behavior for the classical DRM [36, 37, 50]. There are already a plethora of works where circumcentered-based schemes were studied; see for instance [3–7, 13–17, 19–21, 23, 24, 26, 52–56].

The circumcenter of three points  $x, y, z \in \mathbb{R}^n$ , denoted as  $\text{circ}(x, y, z)$ , is the point in  $\mathbb{R}^n$  that lies in the affine manifold spanned by  $x, y$  and  $z$  and is equidistant to these three points. Given two closed convex sets  $A, B \subset \mathbb{R}^n$ , the CRM iterates by means of the operator  $\mathcal{C}_{A,B} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined as

$$(1.2) \quad \mathcal{C}_{A,B}(x) := \text{circ}(x, R_A(x), R_B(R_A(x))),$$

where  $P_A : \mathbb{R}^n \rightarrow A, P_B : \mathbb{R}^n \rightarrow B$  are the orthogonal projections onto  $A, B$  respectively,  $R_A = 2P_A - \text{Id}$ ,  $R_B = 2P_B - \text{Id}$  (i.e.,  $R_A, R_B$  are the *reflections* onto  $A, B$ , respectively), and  $\text{Id}$  is the identity operator in  $\mathbb{R}^n$ . Hence, the sequence generated by CRM is defined as

$$(1.3) \quad x^{k+1} = \mathcal{C}_{A,B}(x^k).$$

If  $x^k \in A \cap B$ , then the sequence stops at iteration  $k$ , in which case, we say that the algorithm has finite convergence.

One limitation of CRM is that its convergence theory requires one of the sets to be an affine manifold. In [22], it was pointed out that the iteration defined in (1.3) may fail to be well-defined or to approach  $A \cap B$ , if such an assumption is not met. Subsequently, a specific counterexample was presented in [1] where CRM does not converge for two general convex sets. However, CRM can be used to solve the general CFP with  $m$  general arbitrary closed convex sets by employing Pierra's product space reformulation, as presented in [57]. The product space reformulation relies on the Cartesian product  $\mathbf{C} := C_1 \times \cdots \times C_m \subset \mathbb{R}^{nm}$  and on  $\mathbf{D} := \{(x, x, \dots, x) \in \mathbb{R}^{nm} \mid x \in \mathbb{R}^n\}$ .  $\mathbf{D}$  is said to be the *diagonal subspace* in  $\mathbb{R}^{nm}$ . One can easily see that

$$(1.4) \quad x^* \in C \Leftrightarrow \mathbf{z}^* := (x^*, x^*, \dots, x^*) \in \mathbf{C} \cap \mathbf{D},$$

where  $C = \bigcap_{i=1}^m C_i$ . Due to (1.4), solving (1.1) corresponds to solving

$$\text{find } \mathbf{z}^* \in \mathbf{C} \cap \mathbf{D}.$$

Since  $\mathbf{D}$  is an affine manifold, CRM can be applied to finding a point in  $\mathbf{C} \cap \mathbf{D}$  in the product space  $\mathbb{R}^{nm}$ .

In [6], it was proved that the Circumcenter Reflected Method (CRM) with Pierra's product space reformulation achieves a superior convergence rate compared to MAP. The iteration operator of MAP is simply the composition of two orthogonal projections  $P_A$  and  $P_B$ , that is,  $P_A P_B$ . Furthermore, in the same paper, it was shown that for certain special cases, circumcenter schemes such as CRM attain a superlinear rate of convergence [6, 19], and even linear convergence in the absence of an error bound. Notably, no other known method utilizing individual projections achieves such convergence rate even for these particular cases.

More recently, an extension of CRM, called cCRM (acronym for centralized circumcentered-reflection method), was introduced in [20] to overcome the drawback of CRM, namely the requirement that one of the sets be an affine manifold. The cCRM can solve the CFP for any pair of closed and convex sets and converges linearly under an error-bound condition (similar to a transversality hypothesis), and superlinearly if the boundaries of the convex sets are smooth. Yet, for solving the CFP with  $m$  sets, it is necessary to go through the Pierra reformulation in the product space.

For solving the CFP with  $m$  sets more efficiently, a successive extension of cCRM, called ScCRM, was developed in [19]. ScCRM avoids the product space reformulation and inherits

from cCRM the linear and superlinear convergence rates under the error-bound and smoothness hypothesis, respectively.

All the aforementioned methods require several exact orthogonal projections onto some convex sets in each iteration. However, determining the exact orthogonal projection is computationally expensive in practice, except in some very special cases. This limitation was overcome in [4], where an approximate version of CRM, called CARM, is proposed. CARM replaces the exact projections onto the original sets with projections onto sets containing them (*e.g.*, halfspaces or Cartesian products of halfspaces), which are easily computable.

In this paper, we improve upon the aforementioned methods by introducing an algorithm in the product space that uses projections onto perturbed halfspaces, referred to as *Perturbed Approximate Circumcenter Algorithm* (PACA). PACA's iteration reads as follows

$$(1.5) \quad \mathbf{x}^{k+1} = \mathcal{C}_{\mathbf{S}^k, \mathbf{D}}(\mathbf{x}^k) = \text{circ}(\mathbf{x}^k, R_{\mathbf{S}^k}(\mathbf{x}^k), R_{\mathbf{D}}(R_{\mathbf{S}^k}(\mathbf{x}^k))),$$

where  $\mathbf{S}^k$  is a suitable perturbed separating set for  $\mathbf{x}^k$ . For any  $\mathbf{x}^0 \in \mathbf{D}$ , we will be able to obtain a circumcenter sequence in  $\mathbf{D}$  generated by means of (1.5) converging in finitely many steps to a point in  $\mathbf{C} \cap \mathbf{D}$ , if the Slater condition holds, meaning that the intersection of the convex sets has nonempty interior. This is possible by strategically building the perturbed separating set  $\mathbf{S}^k$  (see (3.1) and (4.2)). These perturbed separating sets resemble the surrogate halfspaces used by [45] in the context of CFP, and by [33, 46] in a more general framework.

Finite convergence results for projection-based methods with perturbations applied to CFP, under the Slater condition, have been obtained before [18, 29, 34, 35, 43, 47, 48, 58]; we point out the readers to [48, Tab. 1], a summary where many of these results are compared. We mention that in [12], it is proved that even the Douglas-Rachford method applied to the two-set setting can converge in finitely many steps under several assumptions regarding the underlying sets. Moreover, a generalized version of MAP is proposed and analyzed in [40, 41], and the algorithm can achieve finite convergence under the assumption of some regularity of the underlying convex sets at a solution [41, Thm. 6.1].

The paper is organized as follows: In [section 2](#), we present some definitions and preliminary material. In [section 3](#), we state our algorithm, PACA, [section 4](#), shows the development of the convergence analysis. Finally, [section 5](#) shows the linear convergence rate of the sequence generated by our algorithm. Finite convergence is proved in [section 6](#) under additional assumptions on the perturbation parameters. Finally, [section 7](#) presents numerical experiments for solving ellipsoids intersection problems, comparing PACA with CARM (in the product space), the Simultaneous (Cimmino [30]) subgradient projection method (SSPM) by [43], and the Modified Cyclic (Kaczmarz [44]) subgradient projection method (CSPM) from [35].

**2. Preliminary results.** In this paper, we consider the CFP as in (1.1), assuming that the sets  $C_i$  are given as:

$$(2.1) \quad C_i = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0\},$$

where  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function for  $i = 1, \dots, m$ .

This formulation does not impose any limitations in principle, as we can always choose  $f_i(x) = \|P_{C_i}(x) - x\|$ . However, this choice is not consistent with the later imposition of the Slater hypothesis. Nevertheless, in most instances of the CFP, it is possible to formulate them as in (2.1) using suitable functions  $f_i$ .

A basic assumption for our finite convergence result is the Slater condition, defined as follows:

DEFINITION 2.1 (Slater condition). *There exists  $\hat{x} \in \mathbb{R}^n$  such that*

$$f_i(\hat{x}) < 0,$$

for all  $i = 1, \dots, m$ .

We continue by recalling the explicit formula of the orthogonal projection onto a halfspace.

**PROPOSITION 2.2.** *If  $H = \{y \in \mathbb{R}^n \mid a^\top y \leq \alpha\}$ , with  $a \in \mathbb{R}^n$ ,  $\alpha \in \mathbb{R}$ , then the orthogonal projection onto  $H$  is given by*

$$P_H(x) = x - \frac{\max\{0, \alpha - a^\top x\}}{\|a\|^2} a.$$

*Proof.* The result follows from the definition of orthogonal projection after an elementary calculation.  $\square$

The following proposition is a key result concerning the circumcenter operator  $\mathcal{C}_{A,B}$ , as defined in (1.2). It represents the only result pertaining to circumcenter steps that will be used in our convergence analysis.

**PROPOSITION 2.3.** *Let  $A$  and  $B$  be closed convex subsets of  $\mathbb{R}^n$  with nonempty intersection. Consider the circumcenter operator  $\mathcal{C}_{A,B}$  defined in (1.2). Suppose that  $B$  is an affine manifold. Then, for all  $x \in B$  there exists a closed and convex set  $H(x) \subset \mathbb{R}^n$  such that,*

- (i)  $A \subset H(x)$ ;
- (ii)  $\mathcal{C}_{A,B}(x) = P_{H(x) \cap B}(x)$ ;
- (iii)  $\|\mathcal{C}_{A,B}(x) - s\|^2 \leq \|x - s\|^2 - \|\mathcal{C}_{A,B}(x) - x\|^2$  for all  $s \in A$ .

*Proof.* The result follows from Lemma 3 in [22]; cf. also Lemma 3.3 and Proposition 3.4 in [4]. In fact  $H(x)$  is a halfspace containing  $A$ , with an explicit formula, which ensures that item (i) holds, but this is irrelevant for our purposes. We also mention that item (ii) is the essential result, while item (iii) follows immediately from it.  $\square$

A useful concept for analyzing the convergence of projection-based algorithms is the notion of *Fejér monotonicity*. A sequence  $\{y^k\} \subset \mathbb{R}^n$  is said to be *Fejér monotone* with respect to a set  $M \subset \mathbb{R}^n$  if  $\|y^{k+1} - y\| \leq \|y^k - y\|$  for all  $y \in M$  and all  $k \in \mathbb{N}$ . It is known that if  $\{y^k\}$  is Fejér monotone with respect to  $M$ , then  $\{y^k\}$  is bounded, and if it has a cluster point  $y^* \in M$ , then the entire sequence  $\{y^k\}$  converges to  $y^*$  [11, Thm. 2.16]. In our analysis, we require a slightly weaker notion and also need to handle cases where  $M$  is open, and the cluster points of  $\{y^k\}$  belong to the boundary of  $M$ .

We now introduce the appropriate tools to address this situation.

**DEFINITION 2.4** (Fejér\* monotonicity). *Let  $M \subset \mathbb{R}^n$  and consider a sequence  $\{y^k\} \subset \mathbb{R}^n$ . We say that  $\{y^k\}$  is Fejér\* monotone with respect to  $M$  if for any point  $y \in M$ , there exists  $N(y) \in \mathbb{N}$  such that*

$$(2.2) \quad \|y^{k+1} - y\| \leq \|y^k - y\|,$$

for all  $k \geq N(y)$ .

The main difference with the usual Fejér monotonicity notion lies in the fact that now the decreasing distance property holds for the tail of the sequence, starting at some index which depends on the considered point  $y \in M$ . We present next some useful properties of Fejér\* monotonicity.

**PROPOSITION 2.5** (Characterization of Fejér\* monotonicity). *Let  $\{y^k\} \subset \mathbb{R}^n$  be a Fejér\* monotone w.r.t. a nonempty set  $M$  in  $\mathbb{R}^n$ . Then,*

- (i)  $\{y^k\}$  is bounded;
- (ii) for every  $y \in M$ , the scalar sequence  $\{\|y^k - y\|\}$  converges;
- (iii)  $\{y^k\}$  is Fejér\* monotone w.r.t.  $\text{conv}(M)$ ;
- (iv) for every  $\bar{y} \in \text{cl}(\text{conv}(M))$ , the closure of  $\text{conv}(M)$ , the scalar sequence  $\{\|y^k - \bar{y}\|\}$  converges.

*Proof.* For proving (i), take any point  $y \in M$ . From the definition of Fejér\* monotonicity, we conclude that  $y^k$  belongs to the ball with center at  $y$  and radius  $\|y^{N(y)} - y\|$  for all  $k \geq N(y)$ . Consequently,  $\{y^k\}$  is bounded.

Item (ii) is a direct consequence of (2.2), since the  $N(y)$ -tail of sequence  $\{\|y^k - y\|\}$  is monotone and bounded, so the sequence converges.

For item (iii), take any  $y \in \text{conv}(M)$ . Thus,  $y$  can be written as  $y = \sum_{i=1}^p \lambda_i y_i$ , where  $y_i \in M$ ,  $\lambda_i \in [0, 1]$  for all  $i = 1, 2, \dots, p$ ,  $\sum_{i=1}^p \lambda_i = 1$  and  $p \in \mathbb{N}$ . Taking into account that  $\{y^k\}$  is Fejér\* monotone w.r.t.  $M$ , for each  $i = 1, 2, \dots, p$ , there exists  $N(y_i)$  such that  $\|y^{k+1} - y_i\| \leq \|y^k - y_i\|$  for all  $k \geq N(y_i)$ . Squaring both sides of the last inequality and rearranging the terms, we get

$$(2.3) \quad 0 \leq \|y^k\|^2 - \|y^{k+1}\|^2 - 2(y^k - y^{k+1})^\top y_i.$$

The above inequality also holds for all  $k \geq K := \max\{N(y_1), N(y_2), \dots, N(y_p)\}$ . Multiplying (2.3), for each  $i = 1, 2, \dots, p$ , by the respective  $\lambda_i$  and adding up, we obtain

$$\begin{aligned} 0 &\leq \|y^k\|^2 - \|y^{k+1}\|^2 - 2(y^k - y^{k+1})^\top \left( \sum_{i=1}^p \lambda_i y_i \right) \\ &= \|y^k\|^2 - \|y^{k+1}\|^2 - 2(y^k - y^{k+1})^\top y, \end{aligned}$$

and therefore,  $\|y^{k+1} - y\| \leq \|y^k - y\|$ , for all  $k \geq K$ . So  $\{y^k\}$  is Fejér\* monotone w.r.t.  $\text{conv}(M)$ .

Regarding item (iv), for any  $w \in \text{conv}(M)$ , since item (iii) implies that  $\{y^k\}$  is Fejér monotone w.r.t.  $\text{conv}(M)$ , item (ii) yields that the scalar sequence  $\{\|y^k - w\|\}$  converges. Take now  $\bar{y} \in \text{cl}(\text{conv}(M))$ , and a sequence  $\{w^\ell\} \subset \text{conv}(M)$  such that  $w^\ell \rightarrow \bar{y}$ .

Using the triangle inequality, we have, for all  $\ell \in \mathbb{N}$ ,

$$-\|w^\ell - \bar{y}\| \leq \|y^k - \bar{y}\| - \|y^k - w^\ell\| \leq \|w^\ell - \bar{y}\|.$$

Taking  $\liminf$  and  $\limsup$  with respect to  $k$  in the above inequalities, we obtain

$$\begin{aligned} -\|w^\ell - \bar{y}\| &\leq \liminf_k \|y^k - \bar{y}\| - \lim_k \|y^k - w^\ell\| \\ &\leq \limsup_k \|y^k - \bar{y}\| - \lim_k \|y^k - w^\ell\| \leq \|w^\ell - \bar{y}\|, \end{aligned}$$

because, as aforementioned, item (ii) says that, for every  $\ell \in \mathbb{N}$ ,  $\lim_k \|w^\ell - y^k\|$  exists. Now, as  $\ell$  goes to infinity, we have  $\|w^\ell - \bar{y}\| \rightarrow 0$ , and we conclude that

$$\liminf_k \|y^k - \bar{y}\| = \limsup_k \|y^k - \bar{y}\| = \lim_k \|y^k - \bar{y}\|,$$

so that we have the desired result.  $\square$

Now, we show the extension of the Fejér monotonicity results needed for our analysis. We recall first the notions of R-linear and Q-linear convergence. A sequence  $\{y^k\} \subset \mathbb{R}^n$  converges Q-linearly to  $y^* \in \mathbb{R}^n$  if

$$\limsup_{k \rightarrow \infty} \frac{\|y^{k+1} - y^*\|}{\|y^k - y^*\|} = \eta < 1,$$

and R-linearly if

$$\limsup_{k \rightarrow \infty} \|y^k - y^*\|^{1/k} = \eta < 1.$$

The value  $\eta$  is said to be the *asymptotic constant*. It is well known that Q-linear convergence implies R-linear convergence with the same asymptotic constant [51].

**THEOREM 2.6** (Fejér\* monotonicity and closed convex sets). *Suppose that  $C \subset \mathbb{R}^n$  is a closed convex set with nonempty interior, that is,  $\text{int}(C) \neq \emptyset$ . Assume that  $\{y^k\} \subset \mathbb{R}^n$  is Fejér\* monotone with respect to  $\text{int}(C)$ . Then,*

- (i) *if there exists a cluster point  $\bar{y}$  of  $\{y^k\}$  which belongs to  $C$ , then the whole sequence  $\{y^k\}$  converges to  $\bar{y}$ ;*
- (ii) *if  $\{y^k\}$  converges to  $y^* \in C$  and  $\{\text{dist}(y^k, C)\}$  converges Q- or R-linearly to 0 with asymptotic constant  $\eta$ , then  $\{y^k\}$  converges R-linearly to  $y^*$  with the same asymptotic constant.*

*Proof.* Item (i) is a direct consequence of [Proposition 2.5\(iv\)](#), with  $\text{int}(C)$  playing the role of  $M$ . Indeed, let  $\bar{y} \in C$  be a cluster point of the bounded (due to [Proposition 2.5\(i\)](#)) sequence  $\{y^k\}$  and say  $\{y^{\ell_k}\}$  is a subsequence of  $\{y^k\}$  converging to  $\bar{y}$ , that is,  $\|y^{\ell_k} - \bar{y}\| \rightarrow 0$ . Note that  $C = \text{cl}(\text{conv}(\text{int}(C)))$ , and in view of [Proposition 2.5\(iv\)](#), the sequence  $\{\|y^k - \bar{y}\|\}$  converges. Therefore,

$$\lim_{k \rightarrow \infty} \|y^k - \bar{y}\| = \lim_{\ell_k \rightarrow \infty} \|y^{\ell_k} - \bar{y}\| = 0,$$

so, the whole sequence  $\{y^k\}$  converges to  $\bar{y}$ .

For proving (ii) note that Fejér\* monotonicity of  $\{y^k\}$  implies that for each  $\hat{y} \in \text{int}(C)$  there exists  $N(\hat{y})$  such that  $\|y^{k+m} - \hat{y}\| \leq \|y^k - \hat{y}\|$  for all  $k \geq N(\hat{y})$  and all  $m \in \mathbb{N}$ . Hence,

$$(2.4) \quad \|y^{k+m} - y^k\| \leq \|y^{k+m} - \hat{y}\| + \|y^k - \hat{y}\| \leq 2\|y^k - \hat{y}\|,$$

for all  $\hat{y} \in \text{int}(C)$ , all  $k \geq N(\hat{y})$  and all  $m \in \mathbb{N}$ . Suppose  $y_k \rightarrow y^*$ . Taking limits in (2.4) with  $m \rightarrow \infty$ , we get

$$(2.5) \quad \|y^k - y^*\| \leq 2\|y^k - \hat{y}\|,$$

for all  $\hat{y} \in \text{int}(C)$  and  $k \geq N(\hat{y})$ . Let  $z^k = P_C(y^k)$ , so that  $z^k \in C$ . Convexity of  $C$  ensures the existence of  $\hat{y} \in \text{int}(C)$  such that  $\|\hat{y} - z^k\| \leq \|y^k - z^k\|$ . Hence, in view of (2.5),

$$\begin{aligned} \|y^k - y^*\| &\leq 2\|y^k - \hat{y}\| \leq 2\|y^k - z^k\| \\ &\leq 2\|y^k - z^k\| + 2\|\hat{y} - z^k\| \\ &\leq 2\|y^k - z^k\| + 2\|y^k - z^k\| \\ &= 4\|y^k - z^k\| = 4\text{dist}(y^k, C). \end{aligned}$$

Taking  $k$ -th roots in both sides of the last inequality, we get

$$\|y^k - y^*\|^{1/k} \leq 4^{1/k} \text{dist}(y^k, C)^{1/k}.$$

In view of the relation between Q-linear and R-linear convergence,

$$\limsup_{k \rightarrow \infty} \|y^k - y^*\|^{1/k} \leq \limsup_{k \rightarrow \infty} \text{dist}(y^k, C)^{1/k} = \eta < 1,$$

and we are done.  $\square$

**3. Statement of the algorithm.** We are now ready to introduce the Perturbed Approximate Circumcenter Algorithm (PACA) for the CFP defined in (2.1). Let us assume that the sets  $C_i$ ,  $i = 1, \dots, m$  in the CFP (1.1) take the form described in (2.1). Given an arbitrary initial point  $x^0 \in \mathbb{R}^n$  and a decreasing sequence of perturbation parameters  $\{\epsilon_k\}$  which converges to 0, the PACA scheme for solving CFP is defined in [Algorithm 3.1](#).

Next, we provide a somewhat informal explanation of several properties of the sequence  $\{x^k\}$  generated by PACA. By examining (3.3) with  $\epsilon_k = 0$ , and considering [Proposition 2.2](#), it becomes

evident that  $x^k - v_i^k$  represents the orthogonal projection of  $x^k$  onto a halfspace containing  $C_i$ , while  $x^k - w^k$  is a convex combination of such projections. With the presence of perturbation parameters  $\epsilon_k$ ,  $x^k - v_i^k$  corresponds to the projection of  $x^k$  onto the perturbed set  $S_i^k$ , defined as:

$$(3.1) \quad S_i^k = \begin{cases} C_i^k, & \text{if } x^k \in C_i^k, \\ \{z \in \mathbb{R}^n \mid (u_i)^\top(z - x^k) + f_i(x^k) + \epsilon_k \leq 0\}, & \text{otherwise,} \end{cases}$$

which contains the perturbed convex set

$$(3.2) \quad C_i^k := \{x \in \mathbb{R}^n \mid f_i(x) + \epsilon_k \leq 0\},$$

as we will prove in [Proposition 4.1](#).

---

**Algorithm 3.1** Perturbed Approximate Circumcenter Algorithm (PACA)

---

**1) Initialization.**

Take  $x^0 \in \mathbb{R}^n$ .

**2) Iterative step.**

Given  $x^k$ , take  $u_i^k \in \partial f_i(x^k)$ , the subdifferential of  $f_i$  at  $x^k$  (so,  $u_i^k$  is a subgradient of  $f_i$  at  $x^k$ ), and define:

$$(3.3) \quad v_i^k := \left[ \frac{\max\{0, f_i(x^k) + \epsilon_k\}}{\|u_i^k\|^2} \right] u_i^k,$$

$$(3.4) \quad w^k := \frac{1}{m} \sum_{i=1}^m v_i^k.$$

If  $w^k = 0$ , then take  $x^{k+1} = x^k$  and proceed to the  $(k+1)$ -th iteration. Otherwise, define

$$(3.5) \quad \alpha_k := \frac{\frac{1}{m} \sum_{i=1}^m \|v_i^k\|^2}{\|w^k\|^2},$$

$$(3.6) \quad x^{k+1} := x^k - \alpha_k w^k.$$

**3) Stopping criterion.**

If  $x^k \in C := \bigcap_{i=1}^m C_i$ , then stop (in this case, we say that PACA has finite convergence).

---

We mention that, depending on the value of  $\epsilon_k$ , the set  $C_i^k$  may be empty. On the other hand,  $S_i^k$  is always nonempty, so that the sequence  $\{x^k\}$  is always well-defined, independently of the value of  $\epsilon_k$ . However, the set  $S^k := \bigcap_{i=1}^m S_i^k$  may still be empty, in which case the sequence  $\{x^k\}$  may exhibit an erratic behavior. It could even happen that  $w^k = 0$ , in which case, as stated in the Iterative Step, we get  $x^{k+1} = x^k$ . This only occurs if, by chance,  $x^k$  minimizes the function  $\sum_{i=1}^m \|x - P_{S_i^k}(x)\|^2$ . In this case the sequence does not stop: it proceeds to the next iteration, with perturbation parameter  $\epsilon_{k+1} < \epsilon_k$ . For large enough  $k$ , however, the set  $S^k$  will become nonempty; this is ensured to happen when the Slater point  $\hat{x}$  gets inside  $S^k$  (more precisely, when  $k$  is large enough so that  $\epsilon_k < \min_{1 \leq i \leq m} \{f_i(\hat{x})\}$ ). At that point, the sequence starts moving toward  $S^k$ , and *a fortiori* toward  $C^k := \bigcap_{i=1}^m C_i^k \subset S^k$ , so that eventually it converges to a point in  $C$ . We also observe that  $x^k - w^k$  is a convex combination, with equal weights, of the projections onto the perturbed sets  $S_i^k$ .

Note that, when  $\epsilon_k = 0$ , the set  $S^k$  is related to the surrogate halfspaces studied and used in the algorithms proposed in [33, 45, 46]. In particular, the original algorithm proposed in [45] used a relaxed projection onto the surrogate (and unperturbed) block  $S_k$ , with only a part of the  $S_i^k$  used at each iteration. Moreover, in the case of PACA, the surrogate sets  $S_i^k$  are perturbed, and the next iteration is computed using the circumcenter scheme. Indeed, the factor  $\alpha_k$  in the definition of  $x^{k+1}$  (*i.e.*, in (3.6)) encapsulates the contribution of the circumcentered-reflection approach, as we will discuss in the next section. It is easy to verify that if  $\epsilon_k = 0$  for all  $k$ , then the PACA sequence coincides with the CARM sequence in [4], using the separating operator given in [4, Ex. 2.7]. If we remove the factor  $\alpha_k$  in (3.6), PACA reduces to Algorithm 4 in [43] with relaxation parameters equal to 1. In this case,  $x^{k+1}$  is simply a convex combination of perturbed approximate projections of  $x^k$  onto the sets  $C_i$ . We mention that the algorithm in [43] also enjoys finite convergence, under the same assumptions as in [Theorem 6.2](#) in this paper. It is worth mention that, analogous algorithms to ours and the one in [43] were considered in [47, Ex. 3.2] and in [48, Ex. 4.9].

**PROPOSITION 3.1.** *Consider the PACA sequence defined by [Algorithm 3.1](#). Then,  $\alpha_k \geq 1$  for all  $k$ .*

*Proof.* The result follows directly from (3.4) and (3.5), and the convexity of the function  $\|\cdot\|^2$ .  $\square$

[Proposition 3.1](#) indicates that the PACA sequence can be seen as an overrelaxed version of the sequence given by  $x^{k+1} = x^k - w^k$ , studied in [43]. We comment that the particular value of  $\alpha_k$  determined in (3.5), related to the circumcenter approach, has special consequences in terms of the performance of the method. Moreover, the relaxation  $\alpha_k$  can be equivalently interpreted as the extrapolation parameter for the simultaneous projection  $\frac{1}{m} \sum_{i=1}^m P_{S_i^k}$ , where the  $S_i^k$ 's are perturbed versions of the original sublevel sets; see (3.1). Indeed, we have  $v_i^k = x^k - P_{S_i^k}(x^k)$ . Consequently,

$$w^k = \frac{1}{m} \sum_{i=1}^m v_i^k = \frac{1}{m} \sum_{i=1}^m (x^k - P_{S_i^k}(x^k)),$$

and

$$\alpha_k = \frac{\frac{1}{m} \sum_{i=1}^m \|v_i^k\|^2}{\|w^k\|^2} = \frac{\frac{1}{m} \sum_{i=1}^m \|x^k - P_{S_i^k}(x^k)\|^2}{\left\| \frac{1}{m} \sum_{i=1}^m (x^k - P_{S_i^k}(x^k)) \right\|^2}.$$

An analogous expression appears in the extrapolated method of parallel projections; see [32, eq. (1.9)]. A comprehensive study of extrapolation techniques for iterative methods can be found in [28, sects. 4.9 and 5.10].

**4. Convergence Analysis.** For the convergence analysis, it is necessary to establish a connection between PACA and the CRM method applied to a CFP in the product space  $\mathbb{R}^{nm}$ . This allows us to utilize Pierra's formulation, which transforms a CFP with  $m$  sets in  $\mathbb{R}^n$  into a CFP with only two sets in  $\mathbb{R}^{nm}$ , with one of them being a linear subspace.

Consider  $C_i^k$  as defined in (3.2) (once again, we recall that  $C_i^k$  may be empty for some values of  $\epsilon_k$ ) and  $S_i^k$ , as defined in (3.1). Using [Proposition 2.2](#), the projection of  $x^k$  onto  $S_i^k$  is given by:

$$(4.1) \quad P_{S_i^k}(x^k) = x^k - \frac{\max\{0, f_i(x^k) + \epsilon_k\}}{\|u_i^k\|^2} u_i^k,$$

with  $u_i^k \in \partial f_i(x^k)$ . *i.e.*,  $u_i^k$  is a subgradient of  $f_i$  at  $x^k$ . We remark that (4.1) also works when  $x^k$  belongs to  $C_i^k$ , in which case  $S_i^k$  is not a halfspace, but just the set  $C_i^k$ , and  $P_{S_i^k}(x^k) = x^k$ .

Now we introduce the appropriate sets in  $\mathbb{R}^{nm}$  for using Pierra's formulation. Define:

$$(4.2) \quad \mathbf{S}^k = S_1^k \times \cdots \times S_m^k,$$

and

$$(4.3) \quad \mathbf{C}^k = C_1^k \times \cdots \times C_m^k.$$

We need an elementary result on the relation between  $\mathbf{C}^k$  and  $\mathbf{S}^k$ .

PROPOSITION 4.1. *Take  $\mathbf{S}^k, \mathbf{C}^k \subset \mathbb{R}^{nm}$  as defined in (4.2) and (4.3), respectively. Then,*

$$\mathbf{C}^k \subset \mathbf{S}^k.$$

*Proof.* The result holds trivially if  $\mathbf{C}^k$  is empty (*i.e.*, if  $C_i^k = \emptyset$  for some  $i$ ). Otherwise, it suffices that  $C_i^k \subset S_i^k$  for all  $i$ . Take any point  $z \in C_i^k$ , so that  $f_i(z) + \epsilon_k \leq 0$ . If  $x^k \in C_i^k$ , then the result is satisfied by the definition of  $S_i^k$ . If  $x^k \notin S_i^k$ , since the  $f_i$ 's are convex, then we have

$$(4.4) \quad (u_i^k)^\top (z - x^k) + f_i(x^k) + \epsilon_k \leq f_i(z) + \epsilon_k \leq 0,$$

for all  $z \in \mathbb{R}^n$ , using (3.2) and the definition of subgradient. The result follows from (4.4) and the definition of  $S_i^k$  (*i.e.*, (3.1)), since  $z \in S_i^k$  for all  $i$ .  $\square$

Recall that  $\mathbf{D} \subset \mathbb{R}^{nm}$  is the *diagonal subspace* in  $\mathbb{R}^{nm}$ . In the next lemma, we will prove that the sequence  $\{\mathbf{x}^k\} \subset \mathbb{R}^{nm}$  generated by CRM with the sets  $\mathbf{S}^k$  and  $\mathbf{D}$  is related to the sequence  $\{x^k\} \subset \mathbb{R}^n$  generated by PACA with the sets  $C_i^k$ . In fact, we will get that  $\mathbf{x}^k = (x^k, x^k, \dots, x^k)$  for all  $k$ .

LEMMA 4.2. *Assume that, for  $i = 1, \dots, m$ ,  $C_i \subset \mathbb{R}^n$  is in the form of (2.1) with each  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  convex. Consider the set  $\mathbf{S}^k$ , as defined in (4.2), the diagonal set  $\mathbf{D}$ , and the sequence  $\{\mathbf{x}^k\} \subset \mathbb{R}^{nm}$  generated by CRM given in (1.2) and starting from a point  $\mathbf{x}^0 \in \mathbf{D}$ , i.e.,*

$$(4.5) \quad \mathbf{x}^{k+1} := \mathcal{C}_{\mathbf{S}^k, \mathbf{D}}(\mathbf{x}^k) = \text{circ}(\mathbf{x}^k, R_{\mathbf{S}^k}(\mathbf{x}^k), R_{\mathbf{D}}(R_{\mathbf{S}^k}(\mathbf{x}^k))),$$

where  $R_{\mathbf{S}^k} := 2P_{\mathbf{S}^k} - \text{Id}$ ,  $R_{\mathbf{D}} := 2P_{\mathbf{D}} - \text{Id}$ . Let  $\{x^k\} \subset \mathbb{R}^n$  be generated by PACA as in (3.6) starting from  $x^0 \in \mathbb{R}^n$ . If  $\mathbf{x}^0 = (x^0, \dots, x^0)$ , then  $\mathbf{x}^k = (x^k, \dots, x^k)$  for all  $k \geq 1$ . Also, PACA stops at iteration  $k$  if and only if CRM does so.

*Proof.* Since  $\mathbf{x}^0 \in \mathbf{D}$ , which is an affine subspace, and  $\mathbf{S}^k$  is a closed convex set, we get (inductively) that, for all  $k \geq 0$ , not only  $\mathbf{x}^k$  is computable (therefore, sequence  $\{\mathbf{x}^k\}$  is well-defined) but also  $\mathbf{x}^k \in \mathbf{D}$ ; see Proposition 2.3(ii).

Suppose now that  $\mathbf{x}^k := (y^k, \dots, y^k)$  with  $y^k \in \mathbb{R}^n$ . Since  $\mathbf{x}^0 := (x^0, \dots, x^0)$ , we assume inductively that  $y^k = x^k$ , and we must prove that  $\mathbf{x}^{k+1} = (x^{k+1}, \dots, x^{k+1})$ . We proceed to compute the arguments of the right-hand side of (4.5) in order to get  $\mathcal{C}_{\mathbf{S}^k, \mathbf{D}}(\mathbf{x}^k)$ . By (4.2), we get

$$(4.6) \quad P_{\mathbf{S}^k}(\mathbf{x}^k) = \left( P_{S_1^k}(x^k), \dots, P_{S_m^k}(x^k) \right).$$

Moreover, (4.1) and (3.3) yields

$$(4.7) \quad P_{S_i^k}(x^k) = x^k - \frac{\max\{0, f_i(x^k) + \epsilon_k\}}{\|u_i^k\|^2} u_i^k = x^k - v_i^k,$$

with  $u_i^k \in \partial f_i(x^k)$ . It follows from (4.6) and (4.7) that

$$(4.8) \quad P_{\mathbf{S}^k}(\mathbf{x}^k) = (x^k - v_1^k, \dots, x^k - v_m^k).$$

In view of (4.8) and of the definition of the reflection operator, we have

$$(4.9) \quad R_{\mathbf{S}^k}(\mathbf{x}^k) = (x^k - 2v_1^k, \dots, x^k - 2v_m^k).$$

It follows from the definition of the diagonal subspace  $\mathbf{D}$  that  $P_{\mathbf{D}}(x^1, \dots, x^m) = (z, \dots, z)$  with  $z = (1/m) \sum_{i=1}^m x^i$ . Hence, in view of (3.4),

$$(4.10) \quad P_{\mathbf{D}}(\mathbf{x}^k) = (x^k - 2w^k, \dots, x^k - 2w^k).$$

Finally, in view of (4.10),

$$(4.11) \quad R_{\mathbf{D}}(R_{\mathbf{S}^k}(\mathbf{x}^k)) = (x^k - 4w^k, \dots, x^k - 4w^k).$$

Now, by the definition of circumcenters,  $\mathcal{C}_{\mathbf{S}^k, \mathbf{D}}(\mathbf{x}^k)$  must be equidistant to  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$ , with  $\mathbf{y} := R_{\mathbf{S}^k}(\mathbf{x})$  as in (4.9) and  $\mathbf{z} := R_{\mathbf{D}}(R_{\mathbf{S}^k}(\mathbf{x}^k))$  as in (4.11). An elementary but somewhat lengthy calculation using (4.9) and (4.11) shows that any vector of the form  $\mathbf{e}(\alpha) = (x^k - \alpha w^k, \dots, x^k - \alpha w^k)$  with  $\alpha \in \mathbb{R}$  is equidistant from  $\mathbf{y}$  and  $\mathbf{z}$ .

Note that the equation  $\|\mathbf{e}(\alpha) - \mathbf{x}\|^2 = \|\mathbf{e}(\alpha) - \mathbf{z}\|^2$ , leads, after some elementary calculations, to a linear equation in  $\alpha$ , whose solution is  $\alpha = \alpha_k$ , with  $\alpha_k$  as in (3.5). Hence, taking into account (3.6), we get

$$\mathbf{x}^{k+1} = \mathcal{C}_{\mathbf{S}^k, \mathbf{D}}(\mathbf{x}^k) = (x^k - \alpha_k w^k, \dots, x^k - \alpha_k w^k) = (x^{k+1}, \dots, x^{k+1}),$$

completing the inductive step. The equivalence between the stopping criteria of both algorithms is immediate.  $\square$

Another useful result is the quasi-nonexpansiveness of the CRM operator  $\mathcal{C}_{\mathbf{S}^k, \mathbf{D}}$ , with respect to  $\mathbf{C}^k$ .

LEMMA 4.3 (Quasi-nonexpansiveness of  $\mathcal{C}_{\mathbf{S}^k, \mathbf{D}}$ ). *Let  $\mathbf{S}^k, \mathbf{C}^k \subset \mathbb{R}^{nm}$  be defined as in (4.2) and (4.3), respectively. Consider the operator  $\mathcal{C}_{\mathbf{S}^k, \mathbf{D}}$  as defined in (1.3) and assume that  $\mathbf{C}^k \neq \emptyset$ . Then,*

$$\|\mathcal{C}_{\mathbf{S}^k, \mathbf{D}}(\mathbf{x}) - \mathbf{s}\|^2 \leq \|\mathbf{x} - \mathbf{s}\|^2 - \|\mathcal{C}_{\mathbf{S}^k, \mathbf{D}}(\mathbf{x}) - \mathbf{x}\|^2,$$

for all  $\mathbf{x} \in \mathbb{R}^{nm}$  and all  $\mathbf{s} \in \mathbf{C}^k$ .

*Proof.* Since  $S_i^k$  is closed and convex by definition, then  $\mathbf{S}^k$  is also closed and convex. The result follows immediately from Proposition 2.3(iii) and Proposition 4.1, with  $\mathbf{S}^k$  playing the role of  $A$  and  $\mathbf{D}$ , the role of  $B$ .  $\square$

Next, we prove that the sequence generated by PACA is Fejér\* monotone with respect to  $\text{int}(C)$ .

LEMMA 4.4. *Consider the CFP with the  $C_i$ 's of the form (2.1). Suppose that  $\{x^k\} \subset \mathbb{R}^n$  is the sequence generated by PACA starting at  $x^0 \in \mathbb{R}^n$ . If the sequence  $\{x^k\}$  is infinite, then,*

- (i)  $\{x^k\}$  is Fejér\* monotone with respect to  $\text{int}(C)$ ;
- (ii)  $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$ ;
- (iii) The sequence  $\{x^k\}$  is bounded.

*Proof.* For item (i), let  $\{\mathbf{x}^k\} \subset \mathbb{R}^{nm}$  be the sequence generated by CRM as in (4.5) starting at  $\mathbf{x}^0 = (x^0, \dots, x^0)$ . By definition of  $\mathbf{x}^0$ , it belongs to  $\mathbf{D} \subset \mathbb{R}^{nm}$ . By Proposition 2.3(ii) and (4.5),  $\mathbf{x}^k \in \mathbf{D}$  for all  $k$ . Define

$$(4.12) \quad C^k := \bigcap_{i=1}^m C_i^k.$$

From [Lemma 4.3](#) and [\(4.5\)](#), we get

$$(4.13) \quad \|\mathbf{x}^{k+1} - \mathbf{s}\|^2 \leq \|\mathbf{x}^k - \mathbf{s}\|^2 - \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2,$$

for all  $\mathbf{s} \in \mathbf{C}^k \cap \mathbf{D}$ . Since  $\{\mathbf{x}^k\} \subset \mathbf{D}$  and  $\mathbf{s} \in \mathbf{D}$ , we have, in view of [Lemma 4.2](#),  $\mathbf{x}^k = (x^k, \dots, x^k)$ ,  $\mathbf{x}^{k+1} = (x^{k+1}, \dots, x^{k+1})$  and  $\mathbf{s} = (s, \dots, s)$  with  $s \in C^k \subset \mathbb{R}^n$  as defined in [\(4.12\)](#), so that  $\|\mathbf{x}^{k+1} - \mathbf{s}\|^2 = m\|x^{k+1} - s\|^2$ ,  $\|\mathbf{x}^k - \mathbf{s}\|^2 = m\|x^k - s\|^2$  and  $\|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 = m\|x^{k+1} - x^k\|^2$ . Hence, it follows from [\(4.13\)](#) that

$$(4.14) \quad \|x^{k+1} - s\|^2 \leq \|x^k - s\|^2 - \|x^{k+1} - x^k\|^2,$$

for all  $s \in C^k$  as defined in [\(4.12\)](#). The Slater condition implies that  $\text{int}(C) \neq \emptyset$ . Take any  $s \in \text{int}(C)$ , so that  $f_i(s) < 0$  for all  $i$ . Since  $\{\epsilon_k\}$  converges to 0, there exists  $N(s) \in \mathbb{N}$ , such that  $\epsilon_k < \min_{1 \leq i \leq m} |f_i(s)|$  for all  $k \geq N(s)$ . Therefore, we have  $s \in C^k$  for all  $k \geq N(s)$ . Hence, in view of [\(4.14\)](#)

$$\|x^{k+1} - s\|^2 \leq \|x^k - s\|^2,$$

for all  $s \in \text{int}(C)$  and all  $k \geq N(s)$ . In view of the definition of Fejér\* monotonicity,  $\{x^k\}$  is Fejér\* monotone with respect to  $\text{int}(C)$ .

For (ii), note that [\(4.14\)](#) implies that  $\{\|x^k - s\|\}$  is decreasing and non-negative. Hence,  $\{\|x^k - s\|\}$  is convergent, and also

$$\|x^{k+1} - x^k\|^2 \leq \|x^k - s\|^2 - \|x^{k+1} - s\|^2,$$

which immediately implies the result.

For item (iii), notice that the Slater condition implies that  $\text{int}(C) \neq \emptyset$ . Then the result follows from item (i) and [Proposition 2.5\(i\)](#).  $\square$

Next we prove that the cluster points of the sequence generated by PACA solve the CFP [\(1.1\)](#).

**PROPOSITION 4.5.** *Let  $\{x^k\}$  be the sequence generated by PACA. If  $\{x^k\}$  is infinite and  $\bar{x}$  is a cluster point of  $\{x^k\}$ , then  $\bar{x} \in C$ .*

*Proof.* First, note that the existence of cluster points of  $\{x^k\}$  follows from [Lemma 4.4](#). Let  $\hat{x}$  be a Slater point for the CFP. Since all the  $f_i$ 's are convex and  $f_i(\hat{x}) < 0$  for all  $i$ , we get from the definition of subgradient that

$$(u_i^k)^\top (\hat{x} - x^k) \leq f_i(\hat{x}) - f_i(x^k) \leq -f_i(x^k),$$

Let  $\{x^{\ell_k}\}$  be a subsequence of  $\{x^k\}$  which converges to the cluster point  $\bar{x}$ . By [\(3.6\)](#),

$$x^k - x^{k+1} = \alpha_k w^k.$$

By [Proposition 3.1](#), we know that  $\alpha_k \geq 1$ , so

$$(4.15) \quad \|x^k - x^{k+1}\|^2 = \alpha_k^2 \|w^k\|^2 = \alpha_k \left( \frac{1}{m} \sum_{i=1}^m \|v_i^k\|^2 \right) \geq \frac{1}{m} \sum_{i=1}^m \left( \frac{\max \{0, f_i(x^k) + \epsilon_k\}}{\|u_i^k\|^2} \right)^2.$$

Since  $\{x^k\}$  is bounded by [Theorem 2.6\(i\)](#) and the subdifferential is locally bounded (see, e.g., [\[59, Thm. 24.7\]](#)), we refine the subsequence  $\{x^{\ell_k}\}$ , if needed, in order to ensure that for each

$i$ ,  $\{u_i^{\ell_k}\}$  converges, say to  $\bar{u}_i$ . Let  $I := \{i \in \{1, \dots, m\} \mid f_i(\bar{x}) \geq 0\}$ . We take limits with  $k \rightarrow \infty$  on both sides of (4.15) along the refined subsequence  $\{x^{\ell_k}\}$ , getting

$$(4.16) \quad 0 \geq \frac{1}{m} \sum_{i \in I} \left( \frac{f_i(\bar{x})}{\|\bar{u}_i\|^2} \right)^2,$$

because  $\lim_{k \rightarrow \infty} \epsilon_k = 0$ , and  $\lim_{k \rightarrow \infty} (x^{k+1} - x^k) = 0$  by Lemma 4.4(ii). It follows from (4.16) that  $f_i(\bar{x}) = 0$  for all  $i \in I$ . Since, by the definition of  $I$ ,  $f_i(\bar{x}) < 0$  for all  $i \notin I$ , we get that  $f_i(\bar{x}) \leq 0$  for  $i = 1, \dots, m$ . Therefore, we conclude that  $\bar{x} \in C$ .  $\square$

We close this section by establishing convergence of the full sequence generated by PACA to a solution of the CFP (1.1).

**THEOREM 4.6.** *Let  $\{x^k\}$  be the sequence generated by PACA starting from an arbitrary point  $x^0 \in \mathbb{R}^n$ . If  $\{x^k\}$  is infinite, then it converges to some  $x^* \in C$ .*

*Proof.* By Lemma 4.4(iii),  $\{x^k\}$  is bounded, so that it has cluster points. Proposition 4.5 yields that its cluster points belong to  $C$ . Finally, Theorem 2.6(i) implies that  $\{x^k\}$  converges to some  $x^* \in C$ .  $\square$

**5. Linear convergence rate.** In this section we will prove that if the sequence generated by PACA is infinite, then under the Slater condition it enjoys a linear convergence rate. It is worth mentioning that in other papers addressing circumcentered-reflection methods, such as [4, 6, 19, 23], a linear rate of the generated sequence was proved under an error bound (or transversality) assumption [25, 49]. This assumption is somewhat weaker than our Slater condition (see Proposition 3.2, Theorem 3.3 and the last paragraph of Subsection 3.1 in [19]) but under the Slater condition we can achieve a better result, namely finite convergence. In the above-mentioned references the asymptotic constant related to the linear convergence depends on some constants linked to the error bound assumption; as it could be expected, our asymptotic constant is given in terms of the Slater point.

**PROPOSITION 5.1.** *Let  $\{x^k\}$  be the sequence generated by PACA. If  $\{x^k\}$  is infinite, then there exist  $N \in \mathbb{N}$  such that*

$$\text{dist}^2(x^{k+1}, C^k) \leq \text{dist}^2(x^k, C^k) - \|x^{k+1} - x^k\|^2,$$

for all  $k \geq N$ .

*Proof.* We have proved in (4.14) that

$$(5.1) \quad \|x^{k+1} - s\|^2 \leq \|x^k - s\|^2 - \|x^{k+1} - x^k\|^2,$$

for all  $k \geq N$  and all  $s \in C^k$ . The result follows by taking minimum with  $s \in C^k$  in both sides of (5.1).  $\square$

**PROPOSITION 5.2.** *Let  $\{x^k\}$  be the sequence generated by PACA. If  $\{x^k\}$  is infinite, then there exist  $N \in \mathbb{N}$  and  $\beta \in \mathbb{R}$  such that*

$$\text{dist}(x^k, C^k) \leq \beta(f(x^k) + \epsilon_k),$$

for all  $k \geq N$ , where  $f(x) := \max_{1 \leq i \leq m} f_i(x)$ .

*Proof.* It follows from the definition of  $f$  that  $C^k = \{x \in \mathbb{R}^n \mid f(x) + \epsilon_k \leq 0\}$ . Since  $\{\epsilon_k\}$  is decreasing, there exists  $N \in \mathbb{N}$  such that  $C^k \neq \emptyset$ , for all  $k \geq N$ . Moreover, knowing that  $\{x^k\}$  is infinite, we must have  $f(x^k) > 0$ . Consequently, thanks to [47, Lem. 2.1], we get

$$\frac{\text{dist}(x^k, C^k)}{f(x^k) + \epsilon_k} \leq \frac{\text{dist}(x^k, C^N)}{f(x^k) + \epsilon_N}.$$

The inclusion  $C^N \subset C^k$ , when combined with (4.14), leads to  $\|x^k - s\| \leq \|x^N - s\|$  for all  $k \geq N$  and for all  $s \in C^N$ . In particular, we get  $\text{dist}(x^k, C^N) \leq \text{dist}(x^N, C^N)$ .

On the other hand,  $f(x^k) + \epsilon_N > \epsilon_N$ . Consequently, we arrive at  $\square$

$$\text{dist}(x^k, C^k) \leq \underbrace{\frac{\text{dist}(x^N, C^N)}{\epsilon_N}}_{\beta} (f(x^k) + \epsilon_k).$$

**PROPOSITION 5.3.** *Let  $\{x^k\}$  be the sequence generated by PACA, starting from  $x^0 \in \mathbb{R}^n$ . If  $\{x^k\}$  is infinite, then there exists  $N \in \mathbb{N}$  and  $\sigma > 0$  such that*

$$\text{dist}^2(x^k, C^k) \leq m(\beta\sigma)^2 \|x^{k+1} - x^k\|^2,$$

for all  $k \geq N$ .

*Proof.* Since  $\{x^k\}$  is bounded, and the subdifferential of a convex function defined on the whole  $\mathbb{R}^n$  is locally bounded (again, by [59, Thm. 24.7]), we can take  $\sigma \in \mathbb{R}$  such that

$$(5.2) \quad \|u_i^k\| \leq \sigma,$$

for all  $k$  and all  $i$ . If needed, we also take  $\sigma$  large enough so that  $m(\beta\sigma)^2 > 1$ .

Let  $\{\mathbf{x}^k\} \subset \mathbb{R}^{nm}$  be the sequence generated by CRM as in (4.5) starting at  $\mathbf{x}^0 = (x^0, \dots, x^0)$ . Then,

$$(5.3) \quad \begin{aligned} 2 \|\mathbf{x}^k - P_{\mathbf{S}^k}(\mathbf{x}^k)\| &= \|\mathbf{x}^k - R_{\mathbf{S}^k}(\mathbf{x}^k)\| \leq \|\mathbf{x}^k - \mathbf{x}^{k+1}\| + \|\mathbf{x}^{k+1} - R_{\mathbf{S}^k}(\mathbf{x}^k)\| \\ &= 2 \|\mathbf{x}^{k+1} - \mathbf{x}^k\|, \end{aligned}$$

using the definition of reflection in the first equality, and the definition of circumcenter, together with (1.2) and (1.3), in the second one. It follows from (5.3) that

$$(5.4) \quad \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 \geq \|\mathbf{x}^k - P_{\mathbf{S}^k}(\mathbf{x}^k)\|^2.$$

Now, define

$$(5.5) \quad I_k := \{i \in \{1, \dots, m\} \mid f_i(x^k) + \epsilon_k > 0\}.$$

In view of Lemma 4.2, we pass from  $\mathbb{R}^{nm}$  to  $\mathbb{R}^n$  in (5.4), obtaining

$$(5.6) \quad \begin{aligned} m \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 &\geq \sum_{i=1}^m \left\| x^k - P_{S_i^k}(x^k) \right\|^2 = \sum_{i=1}^m \left\| \frac{\max\{0, f_i(x^k) + \epsilon_k\}}{\|u_i^k\|^2} u_i^k \right\|^2 \\ &= \sum_{i=1}^m \left[ \frac{\max\{0, f_i(x^k) + \epsilon_k\}}{\|u_i^k\|} \right]^2 = \sum_{i \in I_k} \left[ \frac{f_i(x^k) + \epsilon_k}{\|u_i^k\|} \right]^2 \geq \left[ \frac{f(x^k) + \epsilon_k}{\sigma} \right]^2, \end{aligned}$$

where  $f(x) := \max_{1 \leq i \leq m} f_i(x)$ . The first equality in (5.6) holds by (4.1), the third equality by (5.5), and the last inequality follows from the definition of  $f$  and (5.2).

Take  $N$  as in Proposition 5.2. Combining (5.6) with Proposition 5.2, we obtain

$$\text{dist}^2(x^k, C^k) \leq \beta^2 (f(x^k) + \epsilon_k)^2 \leq m(\beta\sigma)^2 \|x^{k+1} - x^k\|^2,$$

for all  $k \geq N$ , completing the proof.  $\square$

PROPOSITION 5.4. *Let  $\{x^k\}$  be the sequence generated by PACA. If  $\{x^k\}$  is infinite, then the sequence  $\{\text{dist}(x^k, C^k)\}$  converges Q-linearly to 0.*

*Proof.* By Proposition 5.1, there exists  $N \in \mathbb{N}$  such that

$$(5.7) \quad \text{dist}^2(x^{k+1}, C^k) \leq \text{dist}^2(x^k, C^k) - \|x^{k+1} - x^k\|^2,$$

for all  $k \geq N$ . Since  $\{\epsilon_k\}$  is monotonically decreasing, we have  $C^k \subset C^{k+1}$ , so that we get from (5.7),

$$(5.8) \quad \text{dist}^2(x^{k+1}, C^{k+1}) \leq \text{dist}^2(x^k, C^k) - \|x^{k+1} - x^k\|^2.$$

Using Proposition 5.3 and (5.8), we get

$$(5.9) \quad \text{dist}(x^{k+1}, C^{k+1}) \leq \sqrt{1 - \frac{1}{m(\beta\sigma)^2}} \text{dist}(x^k, C^k),$$

completing the proof.  $\square$

Next we prove that the sequence  $\{\text{dist}(x^k, C)\}$  converges R-linearly to 0.

PROPOSITION 5.5. *Let  $\{x^k\}$  be the sequence generated by PACA. If  $\{x^k\}$  is infinite, then the sequence  $\{\text{dist}(x^k, C)\}$  converges R-linearly to 0.*

*Proof.* Let

$$(5.10) \quad \lambda := \sqrt{1 - \frac{1}{m(\beta\sigma)^2}},$$

with  $\beta, \sigma$  as in Proposition 5.4. By Proposition 5.4, there exists  $N \in \mathbb{N}$  such that, for  $k \geq N$ ,

$$(5.11) \quad \text{dist}(x^k, C^k) \leq \lambda^{k+1-N} \text{dist}(x^{N-1}, C^{N-1}).$$

Since  $C^k \subset C$  for all  $k$ , we get from (5.11),

$$(5.12) \quad \text{dist}(x^k, C) \leq \text{dist}(x^k, C^k) \leq \lambda^{k+1-N} \text{dist}(x^{N-1}, C^{N-1}),$$

for all  $k \geq N$ . Taking  $k$ -th roots in (5.12) and then limits with  $k \rightarrow \infty$ , we get

$$\lim_{k \rightarrow \infty} [\text{dist}(x^k, C)]^{1/k} = \lim_{k \rightarrow \infty} \lambda^{1-(N+1)/k} [\text{dist}(x^N, C^N)]^{1/k} = \lambda.$$

Since  $\lambda < 1$  by (5.10), we conclude that  $\{\text{dist}(x^k, C)\}$  converges R-linearly to 0.  $\square$

THEOREM 5.6. *Let  $\{x^k\}$  be the sequence generated by PACA starting from an arbitrary point  $x^0 \in \mathbb{R}^n$ . If  $\{x^k\}$  is infinite, then  $\{x^k\}$  converges R-linearly to some point  $x^* \in C$ .*

*Proof.* The result is a direct consequence of Proposition 5.5 and Theorem 2.6(ii).  $\square$

**6. Finite Convergence.** In this section we will prove that under an additional assumption on the sequence of perturbation parameters  $\{\epsilon_k\}$ , PACA enjoys finite convergence, *i.e.*,  $x^k$  solves the CFP for some value of  $k$ .

Our next result is somewhat remarkable, because it states that if the sequence  $\{x^k\}$  is infinite then the sequence  $\{\epsilon_k\}$  of perturbation parameters must be summable. Now, the sequence  $\{\epsilon_k\}$  is an exogenous one, which can be freely chosen as long as it decreases to 0. If we select it so that it is not summable, then it turns out that the sequence  $\{x^k\}$  cannot be infinite, and hence, in view of the stopping criterion, there exists some  $k \in \mathbb{N}$  such that  $x^k$  solves CFP.

We give now an informal argument which explains this phenomenon. We have proved that  $x^k$  approaches the perturbed set  $C^k$  at a certain speed (say, linearly), independently of how fast  $\epsilon_k$  decreases to zero. On the other hand, the perturbed sets  $C^k$  keep increasing, approaching the target set  $C$  from the inside, with a speed determined by the  $\epsilon_k$ 's. If  $\epsilon_k$  goes to 0 slowly (say, sublinearly), then, at a certain point,  $x^k$  will get very close to  $C^k$ , while  $C^k$  is still well inside  $C$ . At this point,  $x^k$  gets trapped in  $C$ , so that it solves CFP, the algorithm suddenly stops, and we get finite convergence. [Figure 1](#) depicts such behavior.

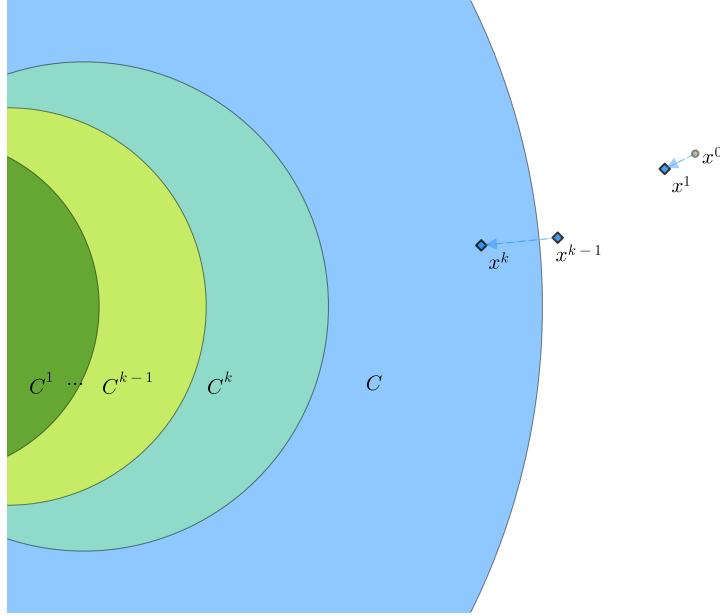


FIG. 1. Sketch of PACA finite convergence.

We mention that, as far as we know, this is the first circumcenter-based method which is proved to be finitely convergent. The methods introduced in [\[19, 20\]](#) achieve superlinear convergence assuming that the convex sets have smooth boundaries; our convergence analysis requires no smoothness hypothesis.

**LEMMA 6.1.** *Let  $\{x^k\}$  be the sequence generated by PACA. Under the Slater condition given in [Definition 2.1](#), if the sequence  $\{x^k\}$  is infinite, then  $\sum_{k=0}^{\infty} \epsilon_k < \infty$ .*

*Proof.* Let  $\delta_k := \text{dist}(x^k, C^k)$ . By [\(5.9\)](#) in [Proposition 5.4](#) and [\(5.10\)](#), we have

$$(6.1) \quad \delta_{k+1} \leq \lambda \delta_k \leq \dots \leq \lambda^{k+1-N} \delta_N,$$

for all  $k \geq N$ . Let  $z^k$  be the closest point to  $x^k$  in  $C^k$ , so that  $\delta_k = \|x^k - z^k\|$ . Since  $\{x^k\}$  is infinite, for all  $k$  there exists  $j(k) \in \{1, \dots, m\}$  such that  $f_{j(k)}(x^k) > 0$ . On the other hand,  $f_{j(k)}(z^k) + \epsilon_k \leq 0$ , because  $z^k \in C^k$ . Using the subgradient inequality, we have

$$(6.2) \quad f_{j(k)}(z^k) \geq f_{j(k)}(x^k) + \left( u_{j(k)}^k \right)^\top (z^k - x^k).$$

Adding  $\epsilon_k$  on both sides of [\(6.2\)](#), we obtain

$$(6.3) \quad 0 \geq f_{j(k)}(z^k) + \epsilon_k \geq f_{j(k)}(x^k) + \epsilon_k + \left( u_{j(k)}^k \right)^\top (z^k - x^k),$$

Since  $f_{j(k)}(x^k) > 0$ , we get from (6.3)

$$-\epsilon_k \geq \left( u_{j(k)}^k \right)^\top (z^k - x^k).$$

In view of the above inequality, it holds that

$$(6.4) \quad \epsilon_k \leq \left| \left( u_{j(k)}^k \right)^\top (z^k - x^k) \right| \leq \|u_{j(k)}^k\| \|z^k - x^k\| = \|u_{j(k)}^k\| \delta_k.$$

From (6.4) and (5.2), we have

$$(6.5) \quad \delta_k \geq \frac{\epsilon_k}{\|u_{j(k)}^k\|} \geq \frac{\epsilon_k}{\sigma}.$$

Now (6.1) and (6.5) imply

$$(6.6) \quad \epsilon_k \leq \sigma \delta_N \lambda^{k+1-N},$$

for all  $k \geq N$ . Let  $\zeta = \sum_{k=0}^{N-2} \epsilon_k$ . From (6.6), we obtain

$$\sum_{k=0}^{\infty} \epsilon_k = \zeta + \sum_{k=N-1}^{\infty} \epsilon_k \leq \zeta + \sigma \delta_N \sum_{k=N-1}^{\infty} \lambda^{k+1-N} = \zeta + \frac{\sigma \delta_N}{1-\lambda} < \infty,$$

establishing the result.  $\square$

**THEOREM 6.2.** *Let  $\{x^k\}$  be the sequence generated by PACA. Under the Slater condition given in Definition 2.1, if the sequence  $\{\epsilon_k\}$  decreases to 0 and  $\sum_{k=0}^{\infty} \epsilon_k = \infty$ , then PACA has finite termination, i.e., there exists some index  $k$  such that  $x^k$  solves CFP.*

*Proof.* Immediate from Lemma 6.1.  $\square$

We observe that there are many choices for the sequence  $\{\epsilon_k\}$  which satisfy the assumptions of Theorem 6.2, for instance,  $\epsilon_k = \nu k^{-r}$ , for any  $\nu > 0$  and any  $r \in (0, 1]$ .

**7. Numerical experiments.** In order to investigate the behavior of PACA, we compare our proposed algorithm with two other approximate projections based algorithms: CRM on the Pierra's reformulation with approximate projection (denoted as CARMprod) (see [4]), the Simultaneous subgradient projection method (SSPM) of [43], and the Modified Cyclic subgradient projection (MCSP) by [35].

Taking into account Theorem 6.2, we set the perturbation parameter for PACA defining  $\epsilon_k := \hat{\epsilon}(k) = \frac{1}{k}$  (algorithm PACA1) and  $\epsilon_k := \bar{\epsilon}(k) = \frac{1}{\sqrt{k}}$  (algorithm PACA2). Therefore, we ensure that PACA1 and PACA2 have finite termination.

We comment that SSPM is somehow similar to PACA: the difference relies on the way one set  $\alpha_k$ , instead of the one PACA uses. For PACA,  $\alpha_k$  is given in (3.5), which arises from a circumcenter (see Lemma 4.2). For SSPM, we set  $\alpha_k = 1$  and for MCSP,  $\alpha_k = \frac{1}{k}$ . Meanwhile, MCSP just computes its iterates visiting all sets by means of perturbed subgradients computed as in (3.3).

Likewise to PACA, SSPM and MCSP also enforce a perturbation parameter, and thus SSPM1 and MCSP1 employ  $\epsilon_k := \hat{\epsilon}(k)$ , while SSPM2 and MCSP2 use  $\epsilon_k := \bar{\epsilon}(k)$ .

These methods are applied to the problem of finding a point in the intersection of  $m$  ellipsoids, *i.e.*, in (1.1), each  $C_i := \xi_i$  is regarded as an ellipsoid given by

$$\xi_i := \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, \text{ for } i = 1, 2, \dots, m\},$$

with  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  defined as  $f_i(x) := x^\top A_i x + 2x^\top b^i - c_i$ . For each  $i = 1, \dots, m$ , we have that  $A_i \in \mathbb{R}^{n \times n}$  is symmetric positive definite,  $b^i$  is a vector, and  $c_i$  is a positive scalar.

The ellipsoids are generated in accordance to [19, sect. 5], so that the sets have not only nonempty intersection, but also a Slater point. Moreover, the subdifferential of each  $f_i$  is a singleton given by  $\partial f_i(x) = 2(A_i x + b^i)$ , since each  $f_i$  is differentiable.

The computational experiments were performed on an Intel Xeon W-2133 3.60GHz with 32GB of RAM running Ubuntu 20.04 using Julia v1.9 [27], and are fully available at <https://github.com/lrsantos11/CRM-CFP>.

Figure 2 and Tables 1 and 2 summarize the results for different number of ellipsoids ( $m$ ) and dimensions ( $n$ ). Figure 2 is performance profile from Dolan and Moré [38], a well known tool for comparing multiple algorithms on problem sets based on a performance metric and offers result visualization for benchmark experiments. The best algorithm is the one with higher percentage of problems solved within a given factor of the best time. In our case, we use wall-clock time (in seconds) as the performance measure. Table 1 shows the average wall-clock time per dimension and number of ellipsoids. Table 2 shows the statistics of all the experiments with Ellipsoids intersection considering wall-clock time. Our numerical results indicate the PACA framework as the winner, with PACA2 being the fastest algorithm.

Note that the numerical experiments also portray the finite convergence inherent to PACA, MCSP and SSPM, as the solutions found by all the algorithms PACA1, PACA2, MCSP1, MCSP2, SSPM1 and SSPM2 are interior.

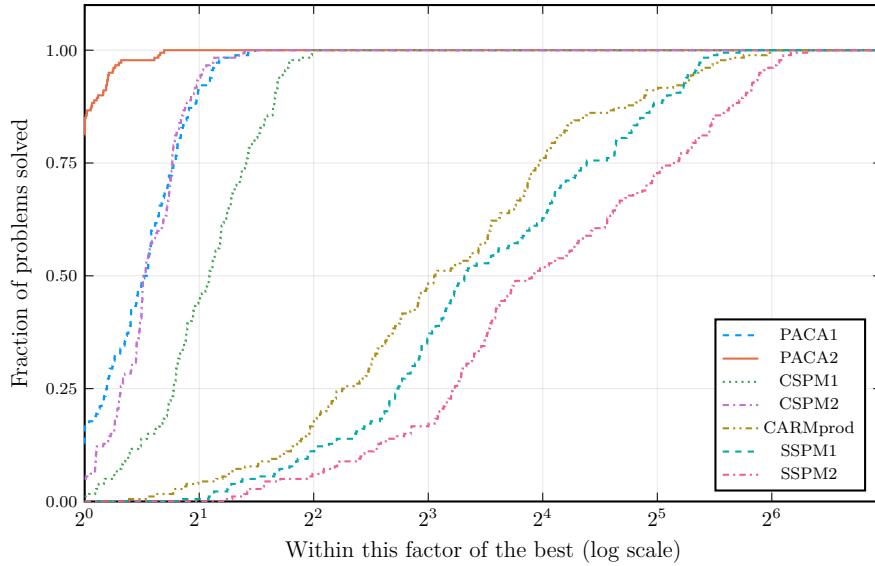


FIG. 2. Performance profile of experiments with Ellipsoids intersection considering wall-clock time (s).

**8. Concluding remarks.** In this work, we introduced a novel algorithm called Perturbed Approximate Circumcenter Algorithm (PACA) to address the Convex Feasibility Problem (CFP). The proposed algorithm ensures finite convergence under a Slater condition, making it unique in the landscape of circumcenter schemes for CFP. In addition to yielding finite convergence, this method leverages projections onto perturbed halfspaces, which are explicitly computed in contrast to obtaining general orthogonal projections onto convex set. Numerical experiments further showcase PACA's efficiency compared to existing methods.

TABLE 1  
The average of wall-clock time ( $\times 10^{-4}$  s) per dimension ( $n$ ) and number of Ellipsoids ( $m$ ).

$n$	$m$	PACA1	PACA2	CSPM1	CSPM2	CARMprod	SSPM1	SSPM2
20	5	1.75	1.31	2.30	2.85	10.04	6.57	8.02
20	10	1.49	1.17	2.49	1.83	17.21	11.45	17.33
20	20	4.52	3.29	7.44	5.18	61.85	49.65	78.60
50	5	1.78	1.81	2.54	2.77	7.92	7.39	9.29
50	10	3.58	3.05	5.62	4.68	21.04	23.67	34.54
50	20	10.50	7.17	13.78	9.84	90.27	106.14	161.45
100	5	20.67	20.21	26.94	21.72	55.49	86.51	94.81
100	10	46.47	37.32	45.07	35.62	228.22	245.30	316.85
100	20	339.67	278.73	283.40	363.67	8813.35	3443.71	3758.55

TABLE 2  
Statistics of all the experiments with Ellipsoids intersection considering wall-clock time ( $\times 10^{-4}$  s).

	mean	median	min	max
PACA1	47.83	3.80	0.23	5447.21
PACA2	39.34	2.94	0.30	4340.08
MCSPM1	43.29	6.12	0.51	4096.26
MCSPM2	49.79	4.05	0.41	5589.35
CARMprod	1033.93	21.95	2.12	160615.00
MSSPM1	442.27	30.42	1.06	53225.90
MSSPM2	497.72	41.24	1.57	54165.50

**Acknowledgments.** The authors are grateful to the anonymous referees and the handling editor for their valuable comments and suggestions which helped to improve the quality of the paper.

#### REFERENCES

- [1] F. J. ARAGÓN ARTACHO, R. CAMPOY, AND M. K. TAM, *The Douglas–Rachford algorithm for convex and nonconvex feasibility problems*, Math Meth Oper Res, 91 (2020), <https://doi.org/10.1007/s00186-019-00691-9>.
- [2] F. J. ARAGÓN ARTACHO, Y. CENSOR, AND A. GIBALI, *The cyclic Douglas–Rachford algorithm with  $r$ -sets-Douglas–Rachford operators*, Optimization Methods and Software, 34 (2018), pp. 875–889, <https://doi.org/10.1080/10556788.2018.1504049>.
- [3] G. H. M. ARAÚJO, *Circumcentering outer-approximate projections and reflections for the convex feasibility problem*, master's thesis, EMAp/Fundação Getúlio Vargas, Rio de Janeiro, BR, Mar. 2021.
- [4] G. H. M. ARAÚJO, R. AREFIDAMGHANI, R. BEHLING, Y. BELLO-CRUZ, A. IUSEM, AND L.-R. SANTOS, *Circumcentering approximate reflections for solving the convex feasibility problem*, Fixed Point Theory and Algorithms for Sciences and Engineering, 2022 (2022), p. 30, <https://doi.org/10.1186/s13663-021-00711-6>, <https://arxiv.org/abs/2105.00497>.
- [5] R. AREFIDAMGHANI, *Circumcentered-Reflection Methods for the Convex Feasibility Problem and the Common Fixed-Point Problem for Firmly Nonexpansive Operators*, PhD thesis, IMPA, Rio de Janeiro, 2022.
- [6] R. AREFIDAMGHANI, R. BEHLING, Y. BELLO-CRUZ, A. N. IUSEM, AND L.-R. SANTOS, *The circumcentered-reflection method achieves better rates than alternating projections*, Comput. Optim. Appl., 79 (2021), pp. 507–530, <https://doi.org/10.1007/s10589-021-00275-6>, <https://arxiv.org/abs/2007.14466>.
- [7] R. AREFIDAMGHANI, R. BEHLING, A. N. IUSEM, AND L.-R. SANTOS, *A circumcentered-reflection method for finding common fixed points of firmly nonexpansive operators*, Journal of Applied and Numerical Optimization, 5 (2023), pp. 299–320, <https://doi.org/10.23952/jano.5.2023.3.02>, <https://arxiv.org/abs/2203.02410>.

- [8] H. BAUSCHKE, J.-Y. BELLO-CRUZ, T. T. A. NGHIA, H. M. PHAN, AND X. WANG, *The rate of linear convergence of the Douglas–Rachford algorithm for subspaces is the cosine of the Friedrichs angle*, Journal of Approximation Theory, 185 (2014), pp. 63–79, <https://doi.org/10.1016/j.jat.2014.06.002>.
- [9] H. BAUSCHKE, J.-Y. BELLO-CRUZ, T. T. A. NGHIA, H. M. PHAN, AND X. WANG, *Optimal Rates of Linear Convergence of Relaxed Alternating Projections and Generalized Douglas–Rachford Methods for Two Subspaces*, Numer. Algorithms, 73 (2016), pp. 33–76, <https://doi.org/10.1007/s11075-015-0085-4>.
- [10] H. BAUSCHKE AND J. M. BORWEIN, *On the convergence of von Neumann’s alternating projection algorithm for two sets*, Set-Valued Analysis, 1 (1993), pp. 185–212, <https://doi.org/10.1007/BF01027691>.
- [11] H. BAUSCHKE AND J. M. BORWEIN, *On Projection Algorithms for Solving Convex Feasibility Problems*, SIAM Review, 38 (1996), pp. 367–426, <https://doi.org/10.1137/S0036144593251710>.
- [12] H. BAUSCHKE AND M. N. DAO, *On the Finite Convergence of the Douglas–Rachford Algorithm for Solving (Not Necessarily Convex) Feasibility Problems in Euclidean Spaces*, SIAM J. Optim., 27 (2017), pp. 507–537, <https://doi.org/10.1137/16M1071079>.
- [13] H. BAUSCHKE, H. OUYANG, AND X. WANG, *On circumcenters of finite sets in Hilbert spaces*, Linear and Nonlinear Analysis, 4 (2018), pp. 271–295, <https://arxiv.org/abs/1807.02093>.
- [14] H. BAUSCHKE, H. OUYANG, AND X. WANG, *Circumcentered methods induced by isometries*, Vietnam Journal of Mathematics, 48 (2020), pp. 471–508, <https://doi.org/10.1007/s10013-020-00417-z>, <https://arxiv.org/abs/1908.11576>.
- [15] H. BAUSCHKE, H. OUYANG, AND X. WANG, *On circumcenter mappings induced by nonexpansive operators*, Pure and Applied Functional Analysis, 6 (2021), pp. 257–288, <https://arxiv.org/abs/1811.11420>.
- [16] H. BAUSCHKE, H. OUYANG, AND X. WANG, *On the linear convergence of circumcentered isometry methods*, Numer. Algor, 87 (2021), pp. 263–297, <https://doi.org/10.1007/s11075-020-00966-x>, <https://arxiv.org/abs/1912.01063>.
- [17] H. BAUSCHKE, H. OUYANG, AND X. WANG, *Best approximation mappings in Hilbert spaces*, Math. Program., 195 (2022), pp. 855–901, <https://doi.org/10.1007/s10107-021-01718-y>, <https://arxiv.org/abs/2006.02644>.
- [18] H. BAUSCHKE, C. WANG, X. WANG, AND J. XU, *On the Finite Convergence of a Projected Cutter Method*, J Optim Theory Appl, 165 (2015), pp. 901–916, <https://doi.org/10.1007/s10957-014-0659-7>.
- [19] R. BEHLING, Y. BELLO-CRUZ, A. IUSEM, D. LIU, AND L.-R. SANTOS, *A successive centralized circumcenter reflection method for the convex feasibility problem*, Comp. Appl. Math., 87 (2024), pp. 83–116, <https://doi.org/10.1007/s10589-023-00516-w>, <https://arxiv.org/abs/2212.06911>.
- [20] R. BEHLING, Y. BELLO-CRUZ, A. N. IUSEM, AND L.-R. SANTOS, *On the centralization of the circumcentered-reflection method*, Math. Program., 205 (2024), pp. 337–371, <https://doi.org/10.1007/s10107-023-01978-w>, <https://arxiv.org/abs/2111.07022>.
- [21] R. BEHLING, Y. BELLO-CRUZ, H. LARA-URDANETA, H. OVIEDO, AND L.-R. SANTOS, *Circumcentric directions of cones*, Optim. Lett., 17 (2023), pp. 1069–1081, <https://doi.org/10.1007/s11590-022-01923-4>, <https://arxiv.org/abs/2112.08314>.
- [22] R. BEHLING, Y. BELLO-CRUZ, AND L.-R. SANTOS, *Circumcentering the Douglas–Rachford method*, Numer. Algorithms, 78 (2018), pp. 759–776, <https://doi.org/10.1007/s11075-017-0399-5>, <https://arxiv.org/abs/1704.06737>.
- [23] R. BEHLING, Y. BELLO-CRUZ, AND L.-R. SANTOS, *On the linear convergence of the circumcentered-reflection method*, Oper. Res. Lett., 46 (2018), pp. 159–162, <https://doi.org/10.1016/j.orl.2017.11.018>, <https://arxiv.org/abs/1711.08651>.
- [24] R. BEHLING, Y. BELLO-CRUZ, AND L.-R. SANTOS, *The block-wise circumcentered-reflection method*, Comput. Optim. Appl., 76 (2020), pp. 675–699, <https://doi.org/10.1007/s10589-019-00155-0>, <https://arxiv.org/abs/1902.10866>.
- [25] R. BEHLING, Y. BELLO-CRUZ, AND L.-R. SANTOS, *Infeasibility and error bound imply finite convergence of alternating projections*, SIAM J. Optim., 31 (2021), pp. 2863–2892, <https://doi.org/10.1137/20M1358669>, <https://arxiv.org/abs/2008.03354>.
- [26] R. BEHLING, Y. BELLO-CRUZ, AND L.-R. SANTOS, *On the Circumcentered-Reflection Method for the Convex Feasibility Problem*, Numer. Algorithms, 86 (2021), pp. 1475–1494, <https://doi.org/10.1007/s11075-020-00941-6>, <https://arxiv.org/abs/2001.01773>.
- [27] J. BEZANSON, A. EDELMAN, S. KARPINSKI, AND V. B. SHAH, *Julia: A Fresh Approach to Numerical Computing*, SIAM Review, 59 (2017), pp. 65–98, <https://doi.org/10.1137/141000671>.
- [28] A. CEGIELSKI, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, no. 2057 in Lecture Notes in Mathematics, Springer Verlag, New York, 2012.
- [29] Y. CENSOR, W. CHEN, AND H. PAJOOHESH, *Finite Convergence of a Subgradient Projections Method with Expanding Controls*, Appl Math Optim, 64 (2011), pp. 273–285, <https://doi.org/10.1007/s00245-011-9139-8>.
- [30] G. CIMMINO, *Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari*, La Ricerca Scientifica, 9 (1938), pp. 326–333.
- [31] P. L. COMBETTES, *The Convex Feasibility Problem in Image Recovery*, in Advances in Imaging and Electron Physics, P. W. Hawkes, ed., vol. 95, Elsevier, San Diego, Jan. 1996, pp. 155–270, <https://doi.org/10.1016/>

- S1076-5670(08)70157-5.
- [32] P. L. COMBETTES, *Hilbertian convex feasibility problem: Convergence of projection methods*, Appl Math Optim, 35 (1997), pp. 311–330, <https://doi.org/10.1007/BF02683333>.
- [33] P. L. COMBETTES, *Strong Convergence of Block-Iterative Outer Approximation Methods for Convex Optimization*, SIAM J. Control Optim., 38 (2000), pp. 538–565, <https://doi.org/10.1137/S036301299732626X>.
- [34] G. CROMBEZ, *Finding common fixed points of a class of paracontractions*, Acta Mathematica Hungarica, 103 (2004), pp. 233–241, <https://doi.org/10.1023/B:AMHU.0000028410.94541.fb>.
- [35] A. R. DE PIERRO AND A. N. IUSEM, *A finitely convergent “row-action” method for the convex feasibility problem*, Appl Math Optim, 17 (1988), pp. 225–235, <https://doi.org/10.1007/BF01448368>.
- [36] N. DIZON, J. HOGAN, AND S. LINDSTROM, *Circumcentered reflections method for wavelet feasibility problems*, ANZIAMJ, 62 (2022), pp. C98–C111, <https://doi.org/10.21914/anziamj.v62.16118>, <https://arxiv.org/abs/2005.05687>.
- [37] N. D. DIZON, J. A. HOGAN, AND S. B. LINDSTROM, *Circumcentering Reflection Methods for Non-convex Feasibility Problems*, Set-Valued Var. Anal, 30 (2022), pp. 943–973, <https://doi.org/10.1007/s11228-021-00626-9>.
- [38] E. D. DOLAN AND J. J. MORÉ, *Benchmarking optimization software with performance profiles*, Mathematical Programming, 91 (2002), pp. 201–213, <https://doi.org/10.1007/s101070100263>.
- [39] J. DOUGLAS AND H. H. RACHFORD JR., *On the numerical solution of heat conduction problems in two and three space variables*, Transactions of the American Mathematical Society, 82 (1956), pp. 421–421, <https://doi.org/10.1090/S0002-9947-1956-0084194-4>.
- [40] M. FÄLT AND P. GISELSSON, *Optimal convergence rates for generalized alternating projections*, in 2017 IEEE 56th Annual Conference on Decision and Control (CDC), Dec. 2017, pp. 2268–2274, <https://doi.org/10.1109/CDC.2017.8263980>.
- [41] M. FÄLT AND P. GISELSSON, *Generalized Alternating Projections on Manifolds and Convex Sets*, Journal of Nonsmooth Analysis and Optimization, 4 (2023), p. 33, <https://doi.org/10.46298/jnsao-2023-7139>, <https://arxiv.org/abs/2101.07286>.
- [42] Y. HU, C. LI, AND X. YANG, *On Convergence Rates of Linearized Proximal Algorithms for Convex Composite Optimization with Applications*, SIAM J. Optim., 26 (2016), pp. 1207–1235, <https://doi.org/10.1137/140993090>.
- [43] A. N. IUSEM AND L. MOLEDO, *A finitely convergent method of simultaneous subgradient projections for the convex feasibility problem*, Matemática Aplicada e Computacional, 5 (1986), pp. 169–184.
- [44] S. KACZMARZ, *Angenäherte Auflösung von Systemen linearer Gleichungen*, Bull. Int. Acad. Pol. Sci. Lett. Class. Sci. Math. Nat. A, 35 (1937), pp. 355–357.
- [45] K. C. KIWIEL, *Block-iterative surrogate projection methods for convex feasibility problems*, Linear Algebra and its Applications, 215 (1995), pp. 225–259, [https://doi.org/10.1016/0024-3795\(93\)00089-1](https://doi.org/10.1016/0024-3795(93)00089-1).
- [46] K. C. KIWIEL AND B. LOPUCH, *Surrogate Projection Methods for Finding Fixed Points of Firmly Nonexpansive Mappings*, SIAM J. Optim., 7 (1997), pp. 1084–1102, <https://doi.org/10.1137/S1052623495279569>.
- [47] V. I. KOLOBOV, S. REICH, AND R. ZALAS, *Finitely convergent iterative methods with overrelaxations revisited*, J. Fixed Point Theory Appl., 23 (2021), p. 57, <https://doi.org/10.1007/s11784-021-00888-8>.
- [48] V. I. KOLOBOV, S. REICH, AND R. ZALAS, *Finitely convergent deterministic and stochastic iterative methods for solving convex feasibility problems*, Math. Program., 194 (2022), pp. 1163–1183, <https://doi.org/10.1007/s10107-021-01628-z>.
- [49] A. Y. KRUGER, D. R. LUKE, AND N. H. THAO, *Set regularities and feasibility problems*, Math. Program., 168 (2018), pp. 279–311, <https://doi.org/10.1007/s10107-016-1039-x>.
- [50] S. B. LINDSTROM, *Computable centering methods for spiraling algorithms and their duals, with motivations from the theory of Lyapunov functions*, Comput Optim Appl, 83 (2022), pp. 999–1026, <https://doi.org/10.1007/s10589-022-00413-8>, <https://arxiv.org/abs/2001.10784>.
- [51] J. M. ORTEGA AND W. C. RHEINBOLDT, *Iterative Solution of Nonlinear Equations in Several Variables*, Classics in Applied Mathematics, SIAM, Philadelphia, PA, 1 ed., Jan. 2000.
- [52] H. OUYANG, *Circumcenter operators in Hilbert spaces*, master’s thesis, University of British Columbia, Okanagan, CA, 2018, <https://doi.org/10.14288/1.0371095>.
- [53] H. OUYANG, *Circumcentered Methods and Generalized Proximal Point Algorithms*, PhD thesis, University of British Columbia, Kelowna, BC, 2022, <https://doi.org/10.14288/1.0416335>.
- [54] H. OUYANG, *Finite convergence of locally proper circumcentered methods*, Journal of Convex Analysis, 29 (2022), pp. 857–892, <https://arxiv.org/abs/2011.13512>.
- [55] H. OUYANG, *Bregman circumcenters: Monotonicity and forward weak convergence*, Optim Lett, 17 (2023), pp. 121–141, <https://doi.org/10.1007/s11590-022-01881-x>, <https://arxiv.org/abs/2105.02308>.
- [56] H. OUYANG AND X. WANG, *Bregman Circumcenters: Basic Theory*, J Optim Theory Appl, 191 (2021), pp. 252–280, <https://doi.org/10.1007/s10957-021-01937-5>, <https://arxiv.org/abs/2104.03234>.
- [57] G. PIERRA, *Decomposition through formalization in a product space*, Mathematical Programming, 28 (1984), pp. 96–115, <https://doi.org/10.1007/BF02612715>.
- [58] B. T. POLYAK, *Random Algorithms for Solving Convex Inequalities*, in Studies in Computational Mathematics, D. Butnariu, Y. Censor, and S. Reich, eds., vol. 8 of Inherently Parallel Algorithms in Feasibility

- and Optimization and Their Applications, Elsevier, Jan. 2001, pp. 409–422, [https://doi.org/10.1016/S1570-579X\(01\)80024-0](https://doi.org/10.1016/S1570-579X(01)80024-0).
- [59] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton Landmarks in Mathematics and Physics, Princeton Univ. Press, Princeton, NJ, 10 ed., 1997.
  - [60] J. WANG, Y. HU, C. LI, AND J.-C. YAO, *Linear convergence of CQ algorithms and applications in gene regulatory network inference*, Inverse Problems, 33 (2017), p. 055017, <https://doi.org/10.1088/1361-6420/aa6699>.