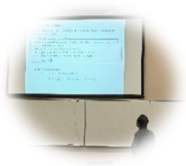
Proceedings of the
Exchange of Mathematical Ideas

2024

Exchange of Mathematical Ideas Conference 2024

Department of Mathematics
Embry-Riddle Aeronautical University
Prescot, AZ
August 12-13, 2024

The Exchange of Mathematical Idea (EMI) conference seeks to enhance collaboration and research stimulation among mathematicians across various disciplines. Organized annually by the mathematics departments at Embry-Riddle Aeronautical University (ERAU) and University of Northern Iowa (UNI), the tenth edition of the conference was held at ERAU, Prescott, Arizona. Supported by NSF (award number 2322922), this year's conference spotlighted Algebra and Analysis and its interconnected domains, featuring expert presenting their research insights.

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Program

Monday, August 12, 2024

Time	Title of the talk and presenter
9:00am	**Opening Remarks: Hisa Tsutsui, Professor, Department of Mathematics, ERAU-Prescott / Doug Mupasiri, Professor and Head, Department of Mathematics, UNI**
9:15am – 9:55am	**Title:** ***The Hyperelliptic Mapping Class Group and the Hyperelliptic Birman Exact Sequence*** **Speaker: Tatsunari Watanabe (ERAU)**
10:00am	**Welcome: Derek Fisher, Chair, Department of Mathematics, ERAU-Prescott** **Zafer Hatahet, Dean Collage of Arts and Sciences, ERAU-Prescott**
10:15am – 10:55am	**Title:** ***Rings of Operators Associated to Quantum Graphs*** **Speaker: Mitch Hamidi (ERAU)**
11:00am – 12:20am	**Title:** ***Elementary submodels and their applications*** **Speaker: Douglas Mupasiri (UNI)**
	Conference Lunch
1:05pm – 1:40pm	**Title:** ***WFP and AFP Rings*** **Speaker: Hisaya Tsutsui (ERAU)**
1:50pm – 2:30pm	**Title:** ***A Deterministic Rotation Method to Color Planar Graphs*** **Speaker: Weiguo Xie (University of Minnesota Duluth)**
	Tea/Coffee Break
2:50pm – 3:30pm	***Title: Edges of a Quantum Graph*** **Speaker: Lara Ismert (ERAU)**
3:40pm – 4:20pm	***Title: Equivariant Learning – an application of representation theory to machine learning*** **Speaker: Anas Chentouf (MIT)**
4:30pm –5:10pm	**Title: Machine Learning Optimization for Keyboard Layout** **Speaker: Christopher Briggs (EPI-USE)**
6:45pm	**Conference Dinner** **Venue: 1927 Salon at Hassayampa Inn, 122 E Gurley St, Prescott, AZ 86301**

Tuesday, August 13, 2024

Time	Title of the talk and presenter
9:00am –9:40am	**Title:** ***Group Rings and Linear Cellular Automata*** **Speaker: Brent Solie (ERAU)**
9:50am – 10:30am	**Title:** ***Behaviors of elements of Thompson's group F*** **Speaker: Tsunekazu Nishinaka (University of Hyogo)**
10:40am – 11:20nn	**Title:** ***Strongly duo modules*** **(*Preliminary report*)** **Speaker: John Beachy (NIU)**
	Conference Lunch
12:05pm – 12:50pm	**Title:** ***Peripheral Poisson Boundary and Jointly Bi-harmonic Functions*** **Speaker: Sayan Das (ERAU)**
1:00pm – 1:40pm	**Title:** ***Problem Solving for Lazy Mathematicians: a dress code puzzle and the magic of constraint solvers*** **Speaker: Nandor Sieben (NAU)**
	Closing Remark: Doug Mupasiri, Professor and Head, Department of Mathematics, UNI

FOREWORD

The International Conference for the Exchange of Mathematical Ideas was started by three founding organizers: Douglas Mupasiri of University of Northern Iowa, Keith Mellinger of University of Mary Washington, and Hisaya Tsutsui of Embry-Riddle Aero-nautical University. The first conference took place at Embry-Riddle Aeronautical University's Prescott campus on May 26, 2012. It had an international audience of 21 participants representing diverse mathematical specialties ranging from noncommutative ring theory to computability theory, cryptography to topology, algebraic number theory to operator theory.

The ethos of the conference is grounded on recognition of the surprising connections that arise between distant fields. That by getting together to describe their research to an audience of non-specialists, researchers often gain new perspective on their own work and find inspiration in the work of others. The EMI is intended to provide a venue for mathematicians to interact in this way. Indeed, collaborations across disciplines sparked at the Exchange have resulted in research productivity, including peer-review journal publications.

Most of all, even though mathematics can be done alone, and often is the product of individual effort, it gains meaning only when shared. We gather to pay homage to this communal aspect of mathematics. We dedicate these proceedings to those who have been with us in the past and those who will join us in the future.

Participants of the 2024 meeting were invited to submit papers to the proceedings. The four selected submissions are published here.

Contents

On singularity of the Poisson boundary inclusion

Sayan Das

1 Introduction

In early 1960's, H. Furstenberg introduced the notion of Poisson boundaries of groups, as a tool to study harmonic analysis on Lie groups and their lattices. Roughly speaking, the Poisson boundary captures asymptotic behavior of random walks on groups (with respect to a probability measure on the group). It was soon realized that the Poisson boundary can play a crucial role in uncovering many *secrets* of groups. For instance, a surprising result of Margulis [Mar75] states that if N is a nontrivial normal subgroup of $SL_3(\mathbb{Z})$, the group of 3×3 matrices with integer entries and determinant 1, then N must have finite index in $SL_3(\mathbb{Z})$! Margulis' proof relied on looking at the Poisson boundary of $SL_3(\mathbb{Z})$. In fact, Margulis' theorem holds more generally (for certain lattices in higher rank lie groups), though the proof strategy remains similar! This result illustrates the power of Poisson boundaries in understanding various rigidity phenomenon involving groups and their actions, and inspired many beautiful results.

Let us now formally introduce the notion of Poisson boundaries. Throughout this article, we will only deal with countable, discrete groups for simplicity.

Let Γ be a countable discrete group, and let μ be a probability measure on the group Γ. We will assume that the measure μ is symmetric ($\mu(g) = \mu(g^{-1})$ for all $g \in \Gamma$), and that the support of the measure generates the whole group Γ.

We say that a bounded function $f \in \ell^\infty(\Gamma)$ is (right) μ-harmonic if $f * mu = f$. Recall that

$$(f * \mu)(g) = \sum_{h \in G} \mu(h) f(gh).$$

The set of all μ-harmonic functions are denoted by $\mathrm{Har}(\mu)$. Unfortunately, product of mu-harmonic functions need not be mu-harmonic, and hence we do not get an algebra. However, Furstenberg proved in [Fur63b] that there exists a probability measure space (B, ν), which admits an action of

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the group Γ, such that $L^\infty(B,\nu) \cong \text{Har}(\mu)$ via a map, called the Poisson transform. More precisely, Furstenberg proved that if we define $\mathcal{P}(f)(g) = \int_B f(gx)d\nu(x)$, for $f \in L^\infty(B,\nu)$, then $\mathcal{P}(f)$ is a μ-harmonic function. Moreover, every μ-harmonic function arises this way. Furthermore, $\mathcal{P}$ is an isometry, and Γ-equivariant, i.e., $\mathcal{P}(g\cdot f) = g\cdot\mathcal{P}(f)$ for all $g\in\Gamma$, and for all $f \in L^\infty(B,\nu)$. Furstenberg also showed that the above probability space is unique (upto isomorphism of Γ-spaces), and called it the Poisson boundary of (Γ,μ).

Let us provide a bit more technical context by invoking the language of random walks, and operator spaces. Given a countable, discrete group G and a probability measure $\mu \in \text{Prob}(G)$, the associated random walk on G is the Markov chain on G whose transition probabilities are given by the measures $\mu * \delta_x$. The Markov operator associated to this random walk is given by $P_\mu f(g) = \sum_{x\in G}\mu(x)f(gx)$, where $f \in \ell^\infty(G)$. The Markov operator is unital and (completely) positive. A function $f \in \ell^\infty(G)$ is μ-harmonic if $\mathcal{P}_\mu(f) = f$. We let $\text{Har}(G,\mu)$ denote the Banach space of μ-harmonic functions. The Poisson boundary [Fur63b] of G with respect to μ is a G-probability space (B,ν), such that we have a natural positivity preserving isometric G-equivariant identification of $L^\infty(B,\nu)$ with $\text{Har}(G,\mu)$ via a Poisson transform. Up to isomorphisms of G-spaces, it is the unique G-probability space such that $L^\infty(B,\nu)$ is isomorphic, as an operator G-space, to $\text{Har}(G,\mu)$.

Under natural conditions on the measure μ, the boundary (B,ν) possesses a number of remarkable properties. It is an amenable G-space [Zim78], it is doubly ergodic with isometric coefficients [GW16], and it is strongly asymptotically transitive [Jaw94]. The boundary has therefore become a powerful tool for studying rigidity properties for groups and their probability measure preserving actions [Mar75, Zim80, BS06, BM02, BF20].

In light of the aforementioned successful application of the Poisson boundary to rigidity properties in group theory, Alain Connes suggested (see [Jo00, Section 4]) that developing a theory of the Poisson boundary in the setting of operator algebras would be the first step toward studying rigidity phenomenons associated with group von Neumann algebras. We refer the readers to the Preliminaries section for a definition of group von Neumann algebras. Roughly speaking, given a countable discrete group Γ, there is a natural algebra of operators (on the Hilbert space $\ell^2(\Gamma)$), denoted by $L(\Gamma)$. It turns out that non isomorphic groups can give rise to isomorphic group von Neumann algebras. However, Connes conjectured in [Co82] that if $\Gamma = SL_3(\mathbb{Z})$, and if Λ is any group, such that $L(\Gamma) \cong L(\Lambda)$, then $\Gamma \cong \Lambda$! This problem, called Connes' rigidity conjecture, has so far been intractable. As emphasized by C. Houdayer in his ICM survey [Ho22, Section 5], the study of (noncommutative) Poisson boundaries is expected to be particularly relevant for approaching Connes' rigidity conjecture. Further evidence for this is witnessed by the significant role that Poisson boundaries play in [CP13, CP17, Pet15], where a related rigidity conjecture of Connes was investigated.

Motivated by this, the author and Prof. Peterson initiated the study of noncommutative Poisson boundaries in [DP22]. Roughly speaking, since the group Γ acts on the boundary (B, ν), we can form a "larger" von Neumann algebra, denoted by $L^\infty(B, \nu) \rtimes \Gamma$, called the group measure space construction. We refer the readers to the Preliminaries section for a short introduction to the group measure space construction. Our starting point was to investigate the properties of the inclusion $L(\Gamma) \subseteq L^\infty(B, \nu) \rtimes \Gamma$.

The study of the Poisson boundary inclusion $L(\Gamma) \subseteq L^\infty(B, \nu) \rtimes \Gamma := B_\mu$ proved to be quite fruitful in unlocking several properties of the von Neumann algebra $L(\Gamma)$ in [DP22]. For instance, using this inclusion, we provided a new proof of Ge's theorem that every continuous derivation on $L(\Gamma)$ is inner. We also answered a question of Popa by utilizing this inclusion. Hence, it is imperative to understand this inclusion better, for further future applications of Poisson boundaries to the study of von Neumann algebras. In particular, a natural question is what is the normalizer of $L(\Gamma)$ inside B_μ,? In this short article, we will show that the Poisson boundary inclusion is singular, i.e., if a unitary element of B_μ normalizes $L(\Gamma)$, then it must lie inside $L(\Gamma)$.

2 Preliminaries

2.1 von Neumann algebras

The study of algebra of operators on a Hilbert space $\mathcal{H}$ was initiated by John von Neumann in the mid 1920's to provide a mathematical foundation of quantum mechanics. Indeed, Stone and von Neumann's work in the mid 1920's and early 1930's showed that Heisenberg's matrix mechanics was equivalent to Schrödinger's wave mechanics. This pivotal work, which was based on the study of self-adjoint operators on a Hilbert space, provided a coherent mathematical description of quantum mechanics.

Recall that a linear map (operator) $T : \mathcal{H} \to \mathcal{H}$ is continuous if and only if T is bounded; i.e., $||T||_\infty := sup||T\xi||_{\mathcal{H}} < \infty$, where the *supremum* is taken over all units vectors $\xi \in \mathcal{H}$. Exploiting the self-duality of Hilbert spaces, one defines the adjoint of a bounded linear operator T to be the unique linear operator T^* satisfying $\langle T^*\xi, \eta\rangle = \langle \xi, T\eta\rangle$ for all $\xi, \eta \in \mathcal{H}$. A bounded linear operator T is called self-adjoint if $T = T^*$. The set of all bounded linear operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$. If

$\mathcal{H}$ is finite dimensional, then $\mathcal{B}(\mathcal{H})$ is just the algebra of $n \times n$ matrices over $\mathbb{C}$, where n is the dimension of $\mathcal{H}$.

For his applications to quantum mechanics, von Neumann needed to consider the "topology of pointwise convergence" (as opposed to the topology of uniform convergence afforded by the norm $|| \cdot ||$), called the *Strong Operator Topology*, henceforth abbreviated as SOT. A net of bounded linear operators $\{T_\alpha\}$ converges in SOT to a bounded linear operator T if and only if $||(T_\alpha - T)\xi|| \to 0$ for all $\xi \in \mathcal{H}$.

In 1926 von Neumann made the stunning discovery that if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is an algebra that is closed under the adjoint operation and contains the identity operator I, then $\mathcal{M}$ is closed under the Strong Operator Topology if and only if $\mathcal{M}$ is equal to its own double commutant $\mathcal{M}''$. (Here for a set $X \subseteq \mathcal{B}(\mathcal{H})$ we define its commutant by $X' = \{T \in \mathcal{B}(\mathcal{H}) : Tx = xT \text{ for all } x \in X\}$). A unital, $*$-closed subalgebra of $\mathcal{B}(\mathcal{H})$ will be called a *von Neumann* algebra if $\mathcal{M} = \mathcal{M}''$, or equivalently if $\mathcal{M} = \overline{\mathcal{M}}^{SOT}$.

2.2 Group von Neumann algebras

Let Γ be a countable, discrete group. The left regular representation of Γ, $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ is defined as

$$\lambda_g(\delta_h) = \delta_{gh}, \text{ for all } g, h \in \Gamma.$$

Here δ_x denotes the function $\delta_x(y) = 1$ if $x \neq y$, and $\delta_x(x) = 1$. Note that $\{\delta_g\}_{g\in\Gamma}$ forms a basis of the Hilbert space $\ell^2(\Gamma)$.

We will need to consider a topology of "pointwise convergence" for bounded operators on $\ell^2(\Gamma)$, called the strong operator topology (denoted SOT). Given a net of bounded operators $\{T_\alpha\} \subseteq \mathbb{B}(\ell^2(\Gamma))$, we say that T_α converges in SOT to a bounded operator $T \in \mathbb{B}(\ell^2(\Gamma))$, if $||(T_\alpha - T)\xi|| \to 0$ for each $\xi \in \ell^2(\Gamma)$.

The group von Neumann algebra of Γ, denoted by $L(\Gamma)$, is defined as [MvN43]

$$L(\Gamma) = \overline{\text{span}\{\lambda_g : g \in \Gamma\}}^{SOT}.$$

So, the group von Neumann algebra is just the closure of the complex group ring $\mathbb{C}[\Gamma]$ in the strong operator topology inside $\mathbb{B}(\ell^2(\Gamma))$.

We say that Γ is i.c.c. (or that Γ has the infinite conjugacy class property) if every nontrivial group element of Γ has infinitely many (distinct) conjugates. That is, for all $g \in \Gamma \setminus \{e\}$, the set $\{hgh^{-1} : h \in \Gamma\}$ is infinite. Examples of i.c.c. groups include $\mathbb{F}_2$, $SL_3(\mathbb{Z})$, the infinite symmetric group S_∞ etc. If Γ is i.c.c., then $L(\Gamma)$ is *highly noncommutative*; infact the center of $L(\Gamma)$ is trivial, i.e., $\mathcal{Z}(L(\Gamma)) = \mathbb{C}$.

2.3 Group measure space

Let Γ be a countable, discrete group, and let (X, ν) be a probability measure space. Assume that we have a group action $\Gamma \curvearrowright^{\alpha} X$ preserving null sets, i.e., if $\nu(A) = 0$ then $\nu(\alpha_g(A)) = 0$. Given such an action, Murray and von Neumann found a natural way to associate an algebra of operators, denoted by $L^\infty(X) \rtimes \Gamma$ that captures the group action [MvN37]. This is called the group measure space construction. As a von Neumann algebra, $L^\infty(X) \rtimes \Gamma$ is generated by a copy of $L^\infty(X)$, and the unitary operators $\{\lambda_g\}_{g\in\Gamma}$ (acting on the Hilbert space $L^2(X) \otimes \ell^2(\Gamma)$) such that $\lambda_g f \lambda_g^* = \alpha_g(f)$, for all $g \in \Gamma$ and for all $f \in L^\infty(X)$. There exists a canonical projection $\mathbb{E}_{L^\infty} : L^\infty(X) \rtimes \Gamma \to L^\infty(X)$, called the conditional expectation.

If X is just a single point, then we recover the group von Neumann algebra, as defined above. In fact, the group measure space construction always contains a canonical copy of $L(\Gamma)$, generated by the operators $\{\lambda_g\}_{g\in\Gamma}$. It turns out that every element $x \in L^\infty(X) \rtimes \Gamma$ can be uniquely represented in a "Fourier series" as $x = \sum_{g\in\Gamma} f_g \lambda_g$, where $f_g \in L^\infty(X)$. This description is often useful in working with the group measure space construction- though we will not need it in this article, and hence we do not elaborate it further.

2.4 Noncommutative Poisson boundaries

Let (B, ν) denote the (classical) Poisson boundary of (Γ, μ), as in the Introduction. We define $B_\mu = L^\infty(B, \nu) \rtimes \Gamma$ to be the noncommutative Poisson boundary.

Let us now describe the notion of Harmonic functions in this context. The noncommutative Markov operator $\mathcal{P}_\mu : \mathcal{B}(\ell^2(\Gamma)) \to \mathcal{B}(\ell^2(\Gamma))$ is defined as

$$\mathcal{P}_\mu(x) = \sum_{g\in\Gamma} \mu(g)\rho_g x \rho_g^*, \text{ where } x \in \mathcal{B}(\ell^2(\Gamma)).$$

In the above equation, ρ denotes the right regular representation of Γ (i.e., $\rho_g(\delta_h) = \delta_{hg^{-1}}$).

If $f \in \ell^\infty(\gamma)$, then we can think of f as a multiplication operator on $\ell^2(\Gamma)$. It turns out that $\mathcal{P}_\mu(f) = f * \mu$ under the above identification. So, the noncommutative Markov operator is a generalization of the classical Markov operator.

We define the space of Harmonic operators as

$$\text{Har}(\mathcal{P}_\mu) = \{T \in \mathcal{B}(\ell^2(\Gamma)) : \mathcal{P}_\mu(T) = T\}.$$

Once again, the space of harmonic operators is not closed under multiplication. Just like in the commutative case, we again have $\mathcal{B}_\mu \cong \text{Har}(\mathcal{P}_\mu)$, where the isomorphism is in the sense of operator systems. We do not elaborate this point further here, but refer the interested reader to [DP22].

We also introduce the conjugate map, which generalizes the notion of left convolution. The conjugate Markov operator $\mathcal{P}_\mu^o : \mathcal{B}(\ell^2(\Gamma)) \to \mathcal{B}(\ell^2(\Gamma))$ is defined as

$$\mathcal{P}_\mu^o(x) = \sum_{g\in\Gamma} \mu(g)\lambda_g x \lambda_g^*, \text{ where } x \in \mathcal{B}(\ell^2(\Gamma)).$$

In the above equation, λ denotes the left regular representation of Γ. If $f \in \ell^\infty(\gamma)$, then $\mathcal{P}_\mu^o(f) = \mu * f$.

We let ζ denote the state on B_μ given by $\zeta(b) = \int_B \mathbb{E}_{L^\infty(B)}(b)d\nu$, for $b \in B_\mu$. The measure ν is μ-stationary, i.e., $\mu * \nu = \nu$. This fact in the noncommutative regime translates to $\zeta \circ P_\mu^o = \zeta$.

The classical Poisson transform $\mathcal{P} : L^\infty(B,\nu) \to \text{Har}(\mu) \subseteq \ell^\infty(\Gamma)$ is given by

$$P(f)(g) = \int_B f(gx)d\nu(x).$$

It is well known that the classical Poisson transform is an isometry (see for example [Jaw94]). We will now define the extension of the classical Poisson to the noncommutative Poisson boundary as follows.

Following [DP22] we define the Poisson transform $\tilde{\mathcal{P}} : L^\infty(B,\nu) \rtimes \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ by the equation

$$\langle \tilde{\mathcal{P}}(b)\delta_g, \delta_h\rangle = \zeta(\lambda_h^* b \lambda_g) \text{ for all } b \in L^\infty(B,\nu) \rtimes \Gamma.$$

It can be shown that $\tilde{\mathcal{P}}|_{L^\infty(B)} = \mathcal{P}$, i.e., $\tilde{\mathcal{P}}$ is indeed an extension of the classical Poisson transform. Hence, we will be lazy, and denote $\tilde{\mathcal{P}}$ by $\mathcal{P}$ in the future, and call this map the Poisson transform. We summarize the crucial properties of the Poisson transform which were proved in [DP22] in the next proposition.

Proposition 2.1. *Let $\mathcal{P} : L^\infty(B,\nu) \rtimes \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ denote the Poisson transform as above. Then we have:*

i) The Poisson transform is $L(\Gamma)$-bimodular, i.e., $\mathcal{P}(xby) = x\mathcal{P}(b)y$ for all $x, y \in L(\Gamma)$ and for all $b \in B_\mu$. In particular, $\mathcal{P}(\lambda_g^* b \lambda_h) = \lambda_g^* \mathcal{P}(b)\lambda_h$ for all $g, h \in \Gamma$ and $b \in B_\mu$.

ii) The Poisson transform is an isometry, i.e., $||\mathcal{P}(b)|| = ||b||$ for all $b \in B_\mu$.

Finally, we mention a remarkable rigidity result that bi-harmonic functions are constant [DP22, Theorem 3.1].

Theorem 2.2 (Double Ergodicity Theorem). *Let $T \in \mathcal{B}(\ell^2(\Gamma))$, with $\mathcal{P}_\mu(T) = \mathcal{P}_\mu^o(T) = T$ (i.e. T is bi-harmonic). Then $T \in \mathcal{Z}(L(\Gamma))$. In particular, if Γ is a i.c.c, then T is a constant.*

3 Singularity of the Poisson boundary inclusion

Throughout this section Γ denotes a countable, discrete i.c.c. group. Let μ be a symmetric, generating probability measure on Γ, i.e., $\mu(g) = \mu(g^{-1})$ for all $g \in \Gamma$ and $\langle$ support$(\mu)\rangle = \Gamma$. Let (B, ν) denote the classical Poisson boundary. We denote the corresponding noncommutative Poisson boundary by $\mathcal{B}_\mu = L^\infty(B, \nu) \rtimes \Gamma$. We will denote by ζ the faithful normal state on B_μ given by $\zeta(x) = \int_B \mathbb{E}_{L^\infty(B)}(x) d\nu$.

Let $\mathcal{N}_{\mathcal{B}_\mu}(\mathcal{M})$ denote the normalizer of $\mathcal{M}$ inside the Poisson boundary $\mathcal{B}_\varphi$. We will show that the inclusion $\mathcal{M} \subset \mathcal{N}_{\mathcal{B}_\mu}(\mathcal{M})$ is a singular inclusion. For convenience, we will denote $\mathcal{N}_{\mathcal{B}_\mu}(\mathcal{M})$ simply by $\mathcal{N}(\mathcal{M})$.

Theorem 3.1. *$\mathcal{N}_{\mathcal{B}_\mu}(\mathcal{M}) = \mathcal{M}$.*

Proof. Let $\mathcal{P}_\mu^o(x) = \sum_g \mu(g)\lambda_g x \lambda_g^*$ for all $x \in \mathcal{B}_\mu$. Let $u \in \mathcal{N}(\mathcal{M})$. Then $\mathcal{P}_\mu^o(u)u^* = \sum_i \mu(g)\lambda(g)(u\lambda_g^* u^*) \in \mathcal{M}$, as u normalizes $\mathcal{M}$.

A similar proof shows that

$$(\mathcal{P}_\mu^o)^n(u)u^* \in \mathcal{M} \text{ for all } u \in \mathcal{N}(\mathcal{M}), \text{ which implies that}$$
$$\frac{1}{N}\sum_{n=1}^{N}(\mathcal{P}_\mu^o)^n(u)u^* \in \mathcal{M} \text{ for all } u \in \mathcal{N}(\mathcal{M}) \text{ where } N \in \mathbb{N}.$$

Fix $u \in \mathcal{N}(\mathcal{M})$ and let z be a WOT-limit point of $\{\frac{1}{N}\sum_{n=1}^{N}(\mathcal{P}_\mu^o)^n(u)u^*\}_N$. By the double ergodicity theorem 2.2, we conclude that $z \in \mathcal{Z}(\mathcal{M}) = \mathbb{C}$, as $\mathcal{M} = \mathcal{L}(\Gamma)$ is a factor, since Γ is i.c.c. Hence we have

$$\begin{aligned} z = \zeta(z) &= \lim_{N\to\infty} \zeta(\frac{1}{N}\sum_{n=1}^{N}(\mathcal{P}_\mu^o)^n(u)) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\zeta((\mathcal{P}_\mu^o)^n(u)) \\ &= \zeta(u), \text{ as } \zeta \circ \mathcal{P}_\mu^o = \zeta. \end{aligned}$$

Since $\frac{1}{N}\sum_{n=1}^{N}(\mathcal{P}_\mu^o)^n(u)u^* \in \mathcal{M}$ for all $u \in \mathcal{N}(\mathcal{M})$, we conclude that

$$\zeta(u)u^* \in \mathcal{M}. \tag{1}$$

If $\zeta(u) \neq 0$, then equation 1 implies that $u \in \mathcal{M}$. Thus, we conclude, if $u \in \mathcal{N}(\mathcal{M}) \setminus \mathcal{M}$ then $\zeta(u) = 0$. Let $g, h \in \Gamma$. Note that $\lambda_g^* u \lambda_h \in \mathcal{N}(\mathcal{M}) \setminus \mathcal{M}$ as well. Thus,

$$\zeta(\lambda_g^* u \lambda_h) = 0 \text{ for all } g, h \in \Gamma. \tag{2}$$

Recall that for $z \in \mathcal{B}_\mu$, $\zeta(z) = \langle \mathcal{P}(z)\delta_e, \delta_e \rangle$, where $\mathcal{P} : \mathcal{B}_\mu :\to \mathcal{B}(\ell^2(\Gamma))$ denotes the Poisson transform. Also recall that $\mathcal{P}$ is $\mathcal{M}$-bimodular. Hence for all $g, h \in \Gamma$ we have

$$\begin{aligned} 0 = \zeta(\lambda_g^* u \lambda_h) &= \langle \mathcal{P}(\lambda_g^* u \lambda_h)\delta_e, \delta_e \rangle = \langle \lambda_g^* P(u) \lambda_h \delta_e, \delta_e \rangle \text{ by bimodularity of } \mathcal{P} \\ &= \langle \mathcal{P}(u)\delta_h, \delta_g \rangle \text{ for all } g, h \in \Gamma. \end{aligned}$$

This implies $\mathcal{P}(u) = 0$, as $\{\delta_g\}_{g \in \Gamma}$ is dense in $\ell^2(\Gamma)$. However, that is a contradiction, as $\|\mathcal{P}(u)\| = \|u\| = 1$, since $\mathcal{P}$ is an isometry. Hence we conclude $u \in \mathcal{M}$, which shows $\mathcal{N}_{\mathcal{B}_\mu}(\mathcal{M}) = \mathcal{M}$. □

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WFP and AFP Rings

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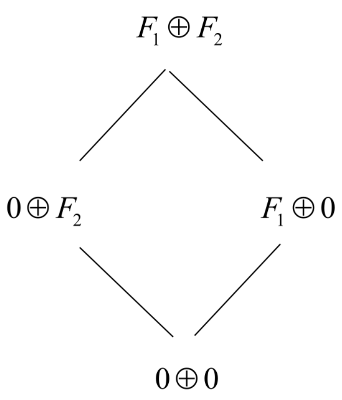
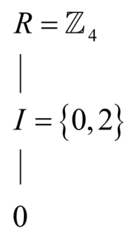
A proper ideal L of a ring R is called weakly prime if for ideals I and J of R, $0 \neq IJ \subseteq L$ implies $I \subseteq L$, or $J \subseteq L$. Rings in which every proper ideal is weakly prime is called fully weakly prime rings (WFP). Rings in which every nonzero proper ideal is prime is called almost fully prime rings (AFP). Necessary and sufficient conditions for a ring to be WFP /AFP are discussed and the relation between those classes of rings are considered.

AFP was introduced in Tsutsui [2], and WFP was introduced in Hirano-Poon-Tsutsui [1]. In this paper, we briefly summarize the two papers and consider the relation between two classes of the rings. A part of the contents with further details may be submitted to elsewhere as a part of our on-going work.

Examples: It is evident that AFP rings are WFP. Perhaps the simplest but non-simple two examples of AFP rings are a direct sum of two fields F_1 and F_2; and the ring with exactly one nonzero proper ideal which is square zero (e.g., $(\mathbb{Z}_4, +, \cdot)$):





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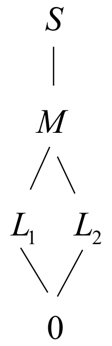
On the other hand, the example below shows that the ring S is WFP, but not an AFP ring:

$R = \mathbb{Z}_4, I = \{0,2\}$.

$S = \left\{ \begin{bmatrix} a & i \\ 0 & a \end{bmatrix} \middle| a \in R, i \in I \right\}$.

$M = \left\{ \begin{bmatrix} a & i \\ 0 & a \end{bmatrix} \middle| a \in I, i \in I \right\}$



Notice that $L_1 = \left\{ \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \middle| i \in I \right\}$, and $L_2 = \left\{ \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix} \middle| j \in I \right\}$ are not prime but weakly prime.

Proposition 1: Both AFP rings and WFP rings have at most two maximal ideals.

Proof: Suppose there are three maximal ideals M, N, L. If $MN = 0$, then $MN \subseteq L$, a contradiction. On the other hand, $0 \neq MN \subseteq M \cap N$ also gives a contradiction.

Proposition 2: If a ring R has exactly two maximal ideals, then R is AFP if and only if R is WFP. In this case, R is isomorphic to a direct sum of two simple rings.

Proof: Suppose R has exactly two maximal ideals N and L. If R is WFP, then $MN = NM = 0$. But $M \cap N \subseteq (M \cap N)(M + N) \subseteq MN$. Hence $R \approx R / M \oplus R / N$.

We refer to the proofs of Theorem 1 and Theorem 2 in Hirano-Poon-Tsutsui [1].

Theorem 1: A ring R is WFP if and only if every ideals I and J, either $IJ = I$, $IJ = J$, or $IJ = 0$.

Theorem 2: Every ideal of a WFP rings is either idempotent or square zero, and every non-trivial idempotent ideal is prime.

We refer to the proofs of Theorem 3 and Theorem 4 in Tsutsui [2].

For AFP rings whose set of ideals is not linearly ordered, we have:

Theorem 3: A ring *R* is AFP if and only if

(a) *R* is a fully idempotent ring that has exactly two minimal ideals,

(b) each minimal ideal of *R* is contained in every nonminimal ideal of *R*, and

(c) the set of all nonminimal ideals of *R* is linearly ordered.

For AFP rings whose set of ideals is linearly ordered, we have:

Theorem 4: A ring *R* is AFP if and only if *R* has a square zero minimal (nonzero) ideal and every ideal of *R* except the minimal one is idempotent.

Theorem 5: Let *R* be a ring whose prime radical is zero. Then *R* is AFP if and only if *R* is WFP.

Theorem 6: Suppose that a ring *R* is commutative with exactly one maximal ideal and it is idempotent. Then

1. *R* is AFP.

2. *R* is WFP.

3. *R* is fully prime.

4, *R* is a field.

By the structure of ideals of AFP rings described in Theorem 2 and 3, we have:

Theorem 7: Suppose that a ring *R* is commutative with exactly one maximal ideal *M* and $M^2 = 0$. Then *R* is AFP if and only if *R* is a field or a ring with exactly one nonzero proper ideal *M*.

Unlike the case of AFP stated in Theorem 7, we have shown that such a WFP ring may have more than one nonzero proper ideals. We refer to the proof of Theorem 8 in Hirano-Poon-Tsutsui [1].

Theorem 8: Let *R* be a commutative ring with a square zero ideal *M*. If ch(*R*/*M*) = 0, then *R* is isomorphic to $\left(R/M\right)*M = \left\{(r+M,m) \mid r \in R, m \in M\right\}$ whose multiplication defined as $(r_1 + M, m_1)(r_2 + M, m_2) = (r_1 r_2 + M, r_1 m_2 + m_2 r_1)$.

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ON THE HYPERELLIPTIC BIRMAN EXACT SEQUENCE

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Abstract. This survey presents key results on the hyperelliptic Birman exact sequence associated with the hyperelliptic mapping class group. We use the relative completion of these groups and show that the sequence does not split for genus $g \geq 3$, utilizing the graded Lie algebras arising from the relative completions.

1. Introduction

In this survey paper, we present a key result concerning the hyperelliptic mapping class group and introduce an important tool, the relative completion of a discrete group. The full details of the result and its complete proof can be found in the author's work [6]. Let $S_{g,n}$ denote an oriented topological surface of genus g with n punctures. The mapping class group $\Gamma_{g,n}$ of $S_{g,n}$ consists of the isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ that fix the punctures pointwise. Let $\pi_1(S_{g,n})$ be the fundamental group of $S_{g,n}$.

Let σ represent a hyperelliptic involution of $S_g := S_{g,0}$. The hyperelliptic mapping class group Δ_g is defined as the centralizer of the isotopy class of $[\sigma]$ in $\Gamma_g := \Gamma_{g,0}$. Moreover, let $\Delta_{g,n}$ denote the fiber product $\Delta_g \times_{\Gamma_g} \Gamma_{g,n}$. Then, we have the following short exact sequence:

$$1 \to \pi_1(S_g) \to \Delta_{g,1} \to \Delta_g \to 1. \tag{1}$$

This sequence is called the hyperelliptic Birman exact sequence, and it is an analogue of the Birman exact sequence for Γ_g.

Sssume $2g - 2 + n > 0$. As an orbifold, the moduli space of curves of type (g, n) is denoted by $\mathcal{M}_{g,n}$, with its hyperelliptic locus $\mathcal{H}_{g,n}$ being a closed smooth substack. Let $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ be the universal complete curve, and $\pi : \mathcal{C}_{\mathcal{H}_{g,n}} \to \mathcal{H}_{g,n}$ its restriction to $\mathcal{H}_{g,n}$, the universal hyperelliptic curve. This curve admits the hyperelliptic involution J, with tautological sections $x_1, \ldots, x_n$ and their hyperelliptic conjugates $J \circ s_1, \ldots, J \circ s_n$.

The universal punctured hyperelliptic curve $\pi^o : \mathcal{H}_{g,n+1} \to \mathcal{H}_{g,n}$ is the complement of the tautological sections in $\mathcal{C}_{g,n}$. The homotopy exact sequence of its orbifold fundamental group is:

$$1 \to \pi_1(C^o) \to \pi_1^{\mathrm{orb}}(\mathcal{H}_{g,n+1}) \to \pi_1^{\mathrm{orb}}(\mathcal{H}_{g,n}) \to 1, \tag{2}$$

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where C^o is an n-punctured complex curve.

Theorem 1. *For $g \geq 3$ and $n \geq 0$, the sequence (2) does not split.*

As a consequence, the hyperelliptic Birman exact sequence does not split:

Corollary 2. *For $g \geq 3$ and $n \geq 0$, the sequence*

$$1 \to \pi_1(S_{g,n}) \to \Delta_{g,n+1} \to \Delta_{g,n} \to 1$$

does not split.

The readers can find the detail of the result and its complete proof in the author's work [6].

2. The Surface Group $\pi_1(S_{g,n})$

Let S_g be a compact oriented topological surface of genus g, and let P be a set of n distinct points in S_g. Define $S_{g,n} = S_g - P$, an oriented surface of genus g with n punctures. The fundamental group $\pi_1(S_{g,n}, p)$, based at p, consists of homotopy classes of loops in $S_{g,n}$. Changing the base point to q induces a natural isomorphism:

$$\pi_1(S_{g,n}, p) \cong \pi_1(S_{g,n}, q),$$

unique up to conjugation by an element of $\pi_1(S_{g,n}, p)$, so we omit the base point from the notation.

Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be the standard generators of $\pi_1(S_g)$, and $\gamma_1, \ldots, \gamma_n$ the homotopy classes of loops around the punctures. The minimal presentation of $\pi_1(S_{g,n})$ is:

$$\pi_1(S_{g,n}) \cong \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 \rangle .$$

For $n > 0$, $\pi_1(S_{g,n})$ is free on $2g + n - 1$ generators.

By the Hurewicz theorem, the natural map $\pi_1(S_{g,n}) \to H_1(S_{g,n}, \mathbb{Z})$ induces an isomorphism between the abelianization of $\pi_1(S_{g,n})$ and the homology group $H_1(S_{g,n}, \mathbb{Z})$. Denote the images of α_j and β_j in $H_1(S_g, \mathbb{Z})$ by a_j and b_j for $j = 1, \ldots, g$. The abelianization $H_1(S_g, \mathbb{Z})$ is free of rank $2g$.

2.1. Symplectic group.

The symplectic group $\mathrm{Sp}(2g; \mathbb{Z})$ is defined as

$$\mathrm{Sp}(2g; \mathbb{Z}) = \left\{ M \in \mathrm{GL}(2g; \mathbb{Z}) \mid M^T J M = J \right\},$$

where $J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$ and I_g is the $g \times g$ identity matrix.

The homology group $H := H_1(S_g, \mathbb{Z})$ is a symplectic space of rank $2g$, equipped with the non-degenerate bilinear alternating intersection pairing $\langle \ , \ \rangle$. A symplectic basis for H is given by $a_1, \ldots, a_g, b_1, \ldots, b_g$, and the automorphism group of H preserving $\langle \ , \ \rangle$ is isomorphic to $\mathrm{Sp}(2g; \mathbb{Z})$:

$$\mathrm{Aut}(H, \langle \ , \ \rangle) \cong \mathrm{Sp}(2g; \mathbb{Z}).$$

3. Mapping Class Groups

Assume $2g-2+n>0$. The mapping class group of $S_{g,n}$, denoted $\Gamma_{g,n}$, is the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ that fix the punctures pointwise:

$$\Gamma_{g,n} := \mathrm{Diff}^+(S_{g,n})/\sim,$$

where $\sim$ denotes isotopy. By the classification of surfaces, $\Gamma_{g,n}$ is independent of the choice of the punctures P. For $n=0$, we write $\Gamma_g := \Gamma_{g,0}$. Filling in a puncture induces a surjective "forgetful map" $\mathcal{F}\mathrm{orget} : \Gamma_{g,n+1} \to \Gamma_{g,n}$. Composing n such maps gives a surjection $\Gamma_{g,n} \to \Gamma_g$.

3.1. **Dehn twists.** The group Γ_g is finitely generated by mapping class elements known as Dehn twists. Let d be a simple closed curve in S_g with a tubular neighborhood N. A Dehn twist T_d is a left-handed twist around d, fixing the boundary of N. The isotopy class of T_d is independent of the choice of N or the representative of d within its isotopy class, so we denote its isotopy class in Γ_g by T_d.

A simple closed curve d is called separating if cutting S_g along d disconnects the surface; otherwise, d is nonseparating. For $g=1$, Γ_g is generated by the Dehn twists about two nonseparating curves on the torus. For $g \geq 2$, Γ_g is generated by the isotopy classes of Dehn twists around $2g+1$ nonseparating curves on S_g [3, Thm. 4.14]. Moreover, Γ_g is finitely presented [3, Thm. 5.3].

3.2. **Symplectic representation of $\Gamma_{g,n}$.** Each mapping class $[\phi]$ in Γ_g induces an automorphism $\phi_* : H \to H$ that is independent of the representative chosen. This automorphism preserves the intersection pairing $\langle\ ,\ \rangle$, leading to the representation

$$\rho_g : \Gamma_g \to \mathrm{Sp}(2g;\mathbb{Z}).$$

For $g \geq 1$, ρ_g is surjective [3, Thm. 6.4]. Composing with the forgetful map $\Gamma_{g,n} \to \Gamma_g$, we obtain the symplectic representation of $\Gamma_{g,n}$:

$$\rho_{g,n} : \Gamma_{g,n} \to \mathrm{Sp}(2g;\mathbb{Z}).$$

3.3. **Torelli groups.** The Torelli group, denoted $T_{g,n}$, is defined as the kernel of the symplectic representation $\rho_{g,n}$:

$$T_{g,n} := \ker \rho_{g,n}.$$

This leads to the following exact sequence:

$$1 \to T_{g,n} \to \Gamma_{g,n} \overset{\rho_{g,n}}{\to} \mathrm{Sp}(2g;\mathbb{Z}) \to 1.$$

The Torelli group is a subgroup of $\Gamma_{g,n}$ with infinite index, implying that it does not necessarily share the basic properties of $\Gamma_{g,n}$.

3.4. **The Birman exact sequences.** There is a natural injection $\mathcal{P}\mathrm{ush} : \pi_1(S_g) \hookrightarrow \Gamma_{g,1}$ called the push map. Combining with the surjection $\mathcal{F}\mathrm{orget} : \Gamma_{g,1} \to \Gamma_g$, we obtain the sequence

$$1 \to \pi_1(S_g) \to \Gamma_{g,1} \to \Gamma_g \to 1. \tag{3}$$

This sequence is exact and it is called the Birman exact sequence.

4. Hyperelliptic Mapping Class Groups

We study a particular subgroup of the mapping class group that preserves a symmetry on S_g. This symmetry is induced by an orientation-preserving diffeomorphism σ of order 2 on S_g, which fixes exactly $2g+2$ points. We refer to σ as a hyperelliptic involution of S_g, and it can be visualized as shown in Figure 1.

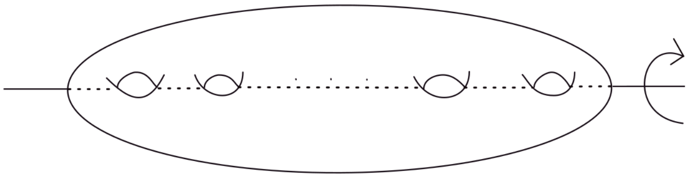

Figure 1. A hyperelliptic involution of S_g, rotation by π

Fix a hyperelliptic involution σ of S_g.

Definition 4.1. The hyperelliptic mapping class group Δ_g of S_g is defined as:

$$\Delta_g := \text{the centralizer of the isotopy class of } \sigma \text{ in } \Gamma_g.$$

We define the hyperelliptic mapping class group $\Delta_{g,n}$ as the fiber product of Δ_g and $\Gamma_{g,n}$ over Γ_g:

$$\Delta_{g,n} := \Delta_g \times_{\Gamma_g} \Gamma_{g,n},$$

where the surjection $\Gamma_{g,n} \to \Gamma_g$ is the forgetful map and $\Delta_g \to \Gamma_g$ is the natural inclusion.

4.1. **The moduli space of hyperelliptic curves.** We refer to a compact Riemann surface as a complex curve. Assume that $2g-2+n>0$. The pair (S,P) represents an n-pointed smooth compact oriented surface S of genus g. The Teichmüller space of (S,P) is denoted by $\mathfrak{X}_{g,n}$. When $n=0$, we denote $\mathfrak{X}_{g,0}$ as $\mathfrak{X}_g$, which is the set of isotopy classes of orientation-preserving diffeomorphisms $h : S \to C$ from S to a complex curve C. This space is a contractible complex analytic manifold of dimension $3g-3+n$.

The mapping class group $\Gamma_{g,n}$ acts on $\mathfrak{X}_{g,n}$ via biholomorphisms, defined by its action on the markings:

$$\lambda : h \mapsto h \circ \lambda^{-1},\ \lambda \in \Gamma_{g,n},\ h \in \mathfrak{X}_{g,n}.$$

It is well known that this action is properly discontinuous and virtually free. As an orbifold, the moduli space of n-pointed smooth projective curves of genus g is defined as the orbifold quotient of $\mathfrak{X}_{g,n}$ by $\Gamma_{g,n}$:

$$\mathcal{M}_{g,n} = (\Gamma_{g,n} \backslash \mathfrak{X}_{g,n})^{\text{orb}}.$$

It is a basic fact that the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{M}_{g,n})$ is isomorphic to $\Gamma_{g,n}$.

Now, let $\sigma : S \to S$ be a fixed hyperelliptic involution of S. Define $\mathfrak{Y}_g$ as the set of points in $\mathfrak{X}_g$ that are fixed by σ:

$$\mathfrak{Y}_g = \mathfrak{X}_g^{\sigma}.$$

This set consists of markings $[h : S \to C]$ such that $h\sigma h^{-1} \in \text{Aut}(C)$. According to a result by Earle [2], $\mathfrak{Y}_g$ is biholomorphic to the Teichmüller space $\mathfrak{X}_{0,2g+2}$, making it connected and contractible with dimension $2g-1$.

Note that the stabilizer of $\mathfrak{Y}_g$ is the hyperelliptic mapping class group Δ_g. Since there is a unique conjugacy class of hyperelliptic involutions in Γ_g, it follows that the hyperelliptic locus $\mathfrak{X}_g^{\mathrm{hyp}}$ within $\mathfrak{X}_g$ is given by:

$$\mathfrak{X}_g^{\mathrm{hyp}} = \bigcup_{\lambda \in \Gamma_g/\Delta_g} \lambda(\mathfrak{Y}_g) = \bigcup_{\lambda \in \Gamma_g/\Delta_g} \mathfrak{X}_g^{\lambda\sigma\lambda^{-1}}.$$

As an orbifold, the moduli space of smooth projective hyperelliptic curves of genus g is given by the orbifold quotient of $\mathfrak{Y}_g$ by Δ_g:

$$\mathcal{H}_g = (\Delta_g \backslash \mathfrak{Y}_g)^{\mathrm{orb}}.$$

The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{H}_g)$ is isomorphic to Δ_g.

4.2. **The hyperelliptic Birman exact sequence.** By pulling back the Birman exact sequence for Γ_g (see (3)) along the natural inclusion $\Delta_g \hookrightarrow \Gamma_g$, we obtain the exact sequence:

$$1 \to \pi_1(S_g) \to \Delta_{g,1} \to \Delta_g \to 1, \tag{4}$$

which gives the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S_g) & \longrightarrow & \Delta_{g,1} & \longrightarrow & \Delta_g & \longrightarrow & 1 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(S_g) & \longrightarrow & \Gamma_{g,1} & \longrightarrow & \Gamma_g & \longrightarrow & 1.
\end{array}$$

Theorem 4.2 ([6, Cor. 4])**.** *If $g \geq 3$, the hyperelliptic Birman exact sequence (4) does not split.*

4.3. **Hyperelliptic Torelli group.** The obstruction to the splitting of the hyperelliptic Birman exact sequence arises from the intersection of Δ_g and T_g.

Definition 4.3. The hyperelliptic Torelli group $T\Delta_g$ is defined as:

$$T\Delta_g := \Delta_g \cap T_g.$$

It remains unknown whether $T\Delta_g$ is finitely generated. However, the following significant result by Brendle, Margalit, and Putman provides insight into its structure.

Theorem 4.4 ([1, Brendle-Margalit-Putman])**.** *For $g \geq 2$, the group $T\Delta_g$ is generated by Dehn twists about symmetric separating curves.*

Remark 4.5. When $g = 2$, any two simple separating curves intersect at least 4 times (see Figure 2). However, for $g \geq 3$, there exist disjoint symmetric separating curves, as illustrated in Figure 3, which give rise to commuting Dehn twists within $T\Delta_g$.

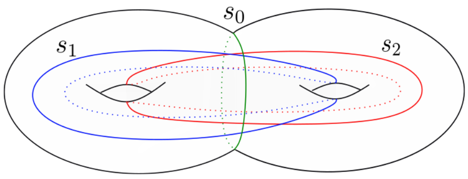

FIGURE 2. Symmetric separating curves in S_2

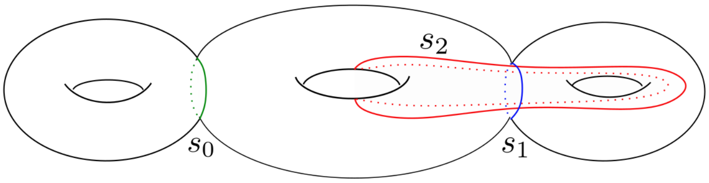

FIGURE 3. Symmetric separating curves in S_3

5. Relative completion of $\Delta_{g,n}$

Relative completion of a discrete group can be viewed as a linearization. It is controlled by cohomology and thus computable to some extent. The readers can find a detailed introduction of relative completion in [4].

Definition 5.1. Let G be a group and R a reductive group over $\mathbb{Q}$. Suppose that $\rho : G \to R$ is a representation with Zariski-dense image. The relative completion of G with respect to ρ is an extension of R by a prounipotent $\mathbb{Q}$-group $\mathcal{U}$. This fits into the following commutative diagram:

$$\begin{array}{ccccccccc} & & \ker(\rho) & \longrightarrow & G & & & & \\ & & \downarrow & & \downarrow \tilde{\rho} & \searrow^{\rho} & & & \\ 1 & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{G} & \longrightarrow & R & \longrightarrow & 1, \end{array}$$

satisfying the following universal property: if G' is a proalgebraic $\mathbb{Q}$-group that is also an extension of R by a prounipotent $\mathbb{Q}$-group U', such that ρ factors through $G' \to R$ with Zariski-dense image in G', then there exists a unique morphism $\phi_{G'} : \mathcal{G} \to G'$ of proalgebraic groups over $\mathbb{Q}$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\rho}} & \mathcal{G} \\ \downarrow & \swarrow_{\phi_{G'}} & \downarrow \\ G' & \longrightarrow & R \end{array}$$

commutes.

By Levi's theorem, the exact sequence

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

splits, and hence there is an isomorphism $\mathcal{G} \cong \mathcal{U} \rtimes R$.

Let $\mathcal{D}_{g,n}$ denote the relative completion of $\Delta_{g,n}$ with respect to the representation $\Delta_{g,n} \to \mathrm{Sp}_{2g}(\mathbb{Q})$, and let $\mathcal{V}_{g,n}$ be the prounipotent radical of $\mathcal{D}_{g,n}$. This gives rise to the exact sequence:

$$1 \to \mathcal{V}_{g,n} \to \mathcal{D}_{g,n} \to \mathrm{Sp}_{2g}(\mathbb{Q}) \to 1.$$

5.1. **The Key Exact Sequences of Completions.** Let $\mathcal{P}$ be the unipotent completion of $\pi_1(S_g)$ over $\mathbb{Q}$. There is a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & \overset{x}{\curvearrowleft} & & & \\
1 & \longrightarrow & \pi_1(S_g) & \longrightarrow & \Delta_{g,1} & \longrightarrow & \Delta_g & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \tilde{\rho} & \overset{\tilde{x}}{\curvearrowleft} & \downarrow \tilde{\rho} & & \\
1 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{D}_{g,1} & \longrightarrow & \mathcal{D}_g & \longrightarrow & 1 \\
 & & \| & & \uparrow & \overset{\tilde{x}_{\mathcal{V}}}{\curvearrowleft} & \uparrow & & \\
1 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{V}_{g,1} & \longrightarrow & \mathcal{V}_g & \longrightarrow & 1,
\end{array}$$

where all rows are exact.

Key Property of Relative Completion: A splitting of $\Delta_{g,1} \to \Delta_g$ induces a splitting of $\mathcal{D}_{g,1} \to \mathcal{D}_g$ and, consequently, a splitting of $\mathcal{V}_{g,1} \to \mathcal{V}_g$.

The Lie algebra $\mathfrak{v}_{g,n}$ of $\mathcal{V}_{g,n}$ admits a weight filtration $W_\bullet \mathfrak{v}_{g,n}$ arising from Hodge theory. Using the Lie algebra structure of the associated graded Lie algebra $\mathrm{Gr}^W_\bullet\, \mathfrak{v}_{g,n}$, we can show that the Lie algebra surjection

$$\mathrm{Gr}^W_\bullet\, \mathfrak{v}_{g,1} \to \mathrm{Gr}^W_\bullet\, \mathfrak{v}_g$$

does not admit any Sp_{2g}-equivariant Lie algebra section. If $\mathcal{V}_{g,1} \to \mathcal{V}_g$ admits a section, it would induce such a Lie algebra section. Therefore, the projection $\Delta_{g,1} \to \Delta_g$ does not split.

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A Deterministic Rotation Method to Color Planar Graphs

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Abstract

The Four Color Theorem was first posed by Francis Guthrie in 1852, which indicates that any planar graph can be colored with four or fewer such that no two adjacent regions have the same color. Both proving this claim and obtaining a four coloring for an arbitrary graph are nontrivial tasks. We have developed several algorithms based on a systematic rotation method that are entirely deterministic which we have demonstrated to work on a very large number of diverse graphs (over 280 million).

Key Words: Four Color Problem, cubic map, Kempe chains, deterministic rotation method, coloring algorithms

AMS Subject Classification: 05C15.

1 Introduction

The Four Color Problem was first introduced by Francis Guthrie in 1852 (see [9]), which states that the regions of any plane graph can be colored with four colors such that no two regions sharing a boundary line have the same color. Although despite the best efforts of many mathematicians, a proof did not materialize for over a century. There are also some researchers who used computer technology to assist their work effort such as in 1976 made by Wolfgang Haken and Kenneth Appel [1], and later by Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas to pursue their own proof [8]. However, they were “machine-checkable proof”, which were less practical to check by human readers. It would still be very satisfying to obtain a proof that does not use computers. Therefore, we will continue a study of the Four Color Theorem in an effort to gain further insight that may lead to such a proof.

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In addition to proving the Four Color Theorem, we are also interested in algorithms to obtain four colorings. While the methods in [1, 8] can be converted into polynomial time algorithms for four-coloring plane graphs, the algorithms take a very long time in practice. Algorithms for efficiently coloring plane graphs have been explored in [7], reaching a practical efficiency of nearly linear time. However, these algorithms make use of randomness, and furthermore cannot be proven to color any map. Here we will examine several deterministic algorithms for coloring planar graphs which have been previously studied in [10, 11, 12, 13, 14, 15, 16]. We will present several new results and conjectures on these algorithms.

2 Definitions

One of the first notable attempts to prove the Four Color Theorem is due to Alfred Kempe in [5]. Kempe made use of what are now called Kempe chains to replace one proper coloring with another. An *AB Kempe chain* is a maximal connected set of regions of G such that every region has color either A or B. Kempe chains have the property that given a proper region coloring (or a proper partial region coloring), exchanging the colors A and B on an AB Kempe chain results in a new proper (partial) region coloring. While Kempe's proof utilizing Kempe chains was incorrect, the ideas behind it have been used in the study of the Four Color Theorem ever since.

The terminology we present next is motivated by [6], with some alterations based on further work in [12, 15, 16]. In the following definitions, the exterior region ER is the region we are attempting to assign a color. In addition, we assume that only five neighbors of ER are colored. We will also assume that all four colors are on regions adjacent with ER with no two consecutive regions (ignoring uncolored regions) having the same color. We will refer to the colored regions adjacent to ER as *boundary regions*. Figure 1 will be used for reference.

- The boundary region situated between two boundary regions of the same color is called the *top region*. In Figure 1, this is the boundary region labeled R.

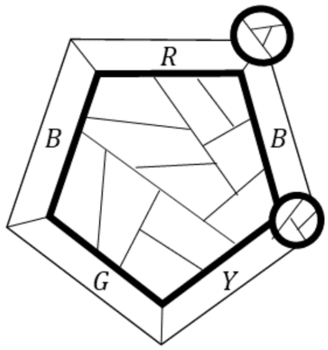

Figure 1: A partially colored graph where the exterior region has 5 colored neighbors, with no two consecutive colored regions of the same color. The label R refers to red color, the label B for blue color, the label G for green color, and the label Y for yellow color. Some of the unmarked regions may be colored, but not those adjacent to the exterior region.

- A Kempe chain containing both the top region and the boundary region two spaces counterclockwise from the top region is called the *left-hand circuit.* If such a Kempe chain does not exist, we will refer to the Kempe chain using these colors and starting at the top region as a *broken left-hand circuit.* In Figure 1, this will be an RG Kempe chain.

- Similarly, a Kempe chain containing both the top region and the boundary region two spaces clockwise from the top region is called the *right-hand circuit* (analagously, *broken right-hand circuit*).

- A Kempe chain beginning at the boundary region counterclockwise to the top region and whose other color is that of the region two spaces clockwise from the top region (in Figure 1, B and Y) is called the *left-hand chain.*

- A Kempe chain beginning at the boundary region clockwise to the top region and whose other color is that of the region two spaces counterclockwise from the top region is called the *right-hand chain.*

- The Kempe chain containing the two boundary regions not directly clockwise or counterclockwise to the top region is called the *end tangent chain*. In this case, this would be a GY Kempe chain. It is possible that these two regions are not connected by a Kempe chain; in this case, we arbitrarily choose a chain using these two colors and containing one of these regions as the end tangent chain.

We will use the following operations. Each operation is a function mapping one partial coloring to another partial coloring of the same regions.

- ℓ: Exchange colors on the left-hand chain
- r: Exchange colors on the right-hand chain
- t_ℓ: Exchange colors on the left-hand circuit (or the broken left-hand circuit)
- t_r: Exchange colors on the right-hand circuit (or the broken right-hand circuit)
- e: Exchange colors on the end tangent chain

For simplicity, we indicate function composition by concatenation. Therefore, for operations σ_1, σ_2, we use $\sigma_2\sigma_1(c)$ to indicate applying σ_1 to c, followed by σ_2.

We follow the conventions established in [16] and say a coloring c is at *impasse* if $c, \ell(c)$, and $r(c)$ each have a left-hand and right-hand circuit. If a coloring is not at impasse, it can easily be used to obtain a color for the exterior region. We also require that ℓ is only be applied to colorings having a left-hand circuit, and similarly r is only applied to colorings having a right-hand circuit. Having established this terminology, we are now ready to give precise definitions of our various algorithms.

3 Algorithms and New Results

In order to simplify the generation of graphs and application of these operations, we found it easier to work with vertex colorings of *maximal planar* graphs, or plane graphs where each region is bounded by a triangle. Since the plane dual of a maximal planar graph is a cubic map, this is an equivalent problem. Here we will predominantly use the region coloring terminology.

The following algorithms will describe approaches to coloring regions with exactly five colored neighbors as in Figure 1. As such, we must ensure that each region has a maximum of five colored neighbors when we attempt to color it. Therefore, we always begin by obtaining a *smallest-last* ordering of the regions graph as described in [7] (where the discussion is in terms of vertices). This ordering attains the property that when regions are colored in this order, no more than 5 colored neighbors will be adjacent to each region as we attempt to color it. If the region has at most 4 colored neighbors, or otherwise if it has 5 colored neighbors with an arrangement of colors different from that in Figure 1, we use methods such as those in [3] to color the region. To color regions with 5 colored neighbors situatated as in Figure 1, we use the algorithms described in this section.

Alogirthm 0: The Basic Rotation Algorithm. Algorithm 0 is the primary algorithm studied in [10, 11, 14]. In terms of the operations described here, Algorithm 0 uses only operations ℓ and t_ℓ. We begin with a coloring c_0. If the coloring c_i has a left-hand circuit, we obtain $c_{i+1} = \ell(c_i)$. Otherwise, we apply t_ℓ to c_i and return this coloring, as this results in a coloring where the boundary regions only use 3 different colors. This algorithm can be summarized as $c_i = \ell^i(c_0)$. Algorithm 0 terminates if and only if $\ell^i(c_0)$ has no left-hand circuit for some i.

In practice, Algorithm 0 has been quite effective. We have now tested over $280,000,000$ graphs having up to $34,110$ regions generated using the methods described in [7]. Of the graphs tested, over 99.998% of graphs were successfully 4-colored using Algorithm 0. In addition, the following result on Heawood's historical counterexample in [4] to Kempe's color-swapping argument, here referred to as the Heawood map, has been previously proven by Wieguo Xie in [11, 14]:

Theorem 3.1. *The Heawood map can be* 4*-colored by Algorithm* 0*.*

However, there is a historical map and coloring c_0 for which Algorithm 0 does not terminate. This map is the Errera map, introduced by Alfred Errera in [2]. As reported by Kittell in [6], $\ell^{20}(c_0) = c_0$. The coloring c_0 is illustrated in Figure 2 as presented in [6].

In addition, certain maps related to the Errera map cannot be colored by Algorithm 0. The remaining algorithms will all explore distinct ways to modify Algorithm 0 to account

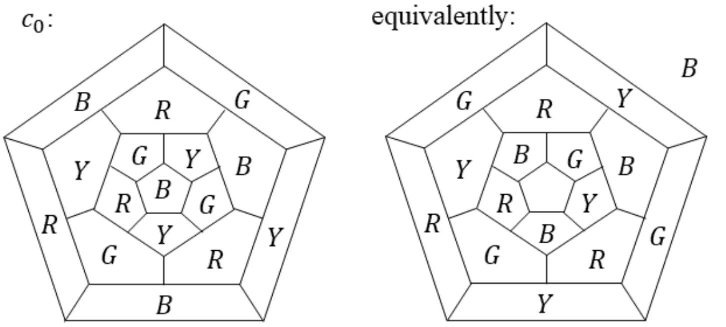

Figure 2: A coloring of the Errera map for which Algorithm 0 does not terminate, with the uncolored region drawn as the exterior region. An equivalent drawing with the uncolored region as the pentagon at the center is also drawn.

for the Errera map and related maps.

Alogrithm 1: Rotation with Errera Fix. Since Algorithm 0 is simple and works for many maps, we would like to use it when possible, and only resort to other methods when necessary. Therefore, in Algorithm 1, we run Algorithm 0 for a large number of times. If the algorithm does not successfully terminate, or if we detect that we have returned to our original coloring, we use the additional operations e and $e\ell$.

It was shown in [16] that for the Errera map and certain variations that this successfully resolves impasse and allows for a coloring of the given region. In fact, of the graphs for which Algorithm 0 failed, all but 25 graphs were 4-colorable by Algorithm 1. An example of a graph for which Algorithm 1 does not resolve impasse is given in Figure 3.

Algorithm 2: Rotation with Multistart. For Algorithm 2, we use multistart to find a region with 5 colored neighbors that can be more easily colored by Algorithm 0. Let R_1 be the region we are attempting to color. We use ℓ up to 20 times to color R_1. If this fails, then we apply ℓ until the top region has at most 4 colored neighbors. We call the top region R_2. We assign to R_1 the color of R_2, and then uncolor R_2. Finally, we use up to 20 applications of ℓ to color R_2. If we either cannot find a neighbor of R_1 that has only four

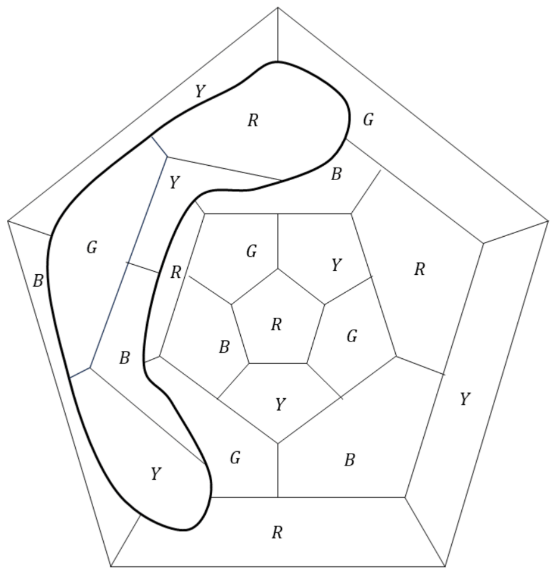

Figure 3: A graph for which Algorithm 1 does not resolve impasse.

colored neighbors or cannot color R_2 after this exchange, then Algorithm 2 does not find a coloring.

Every graph in our testing suite was able to be successfully colored using Algorithm 2. In particular we have the following result, which was also presented in [13]:

Theorem 3.2. *The Errera map can be* 4*-colored by Algorithm 2.*

In addition, we have the following:

Theorem 3.3. *The graph in Figure 3 can be* 4*-colored by Algorithm* 2*.*

In addition to the multistart algorithm described above, we have also designed and implemented a more general multistart algorithm that can change the region to be colored to any other region with at most 5 colored neighbors. However, this necessitates obtaining a new smallest-last ordering and restarting the coloring process each time we run into an obstacle, reducing the efficiency of this algorithm.

Algorithm 3: Alternating e and ℓ. This last approach is motivated by Algorithm 1. In Algorithm 1, we use ℓ many times; if this fails we try either e or $e\ell$. In both cases the final colorings are $e\ell^n(c_0)$ for some n. Algorithm 3 applies color exchanges in order $\{e, \ell, r, e, \ell, e, \ell, r, e, \ell, ...\}$. This systematically tests whether each coloring $\ell^n(c_0)$ and $e\ell^n(c_0)$ is at impasse. More simply, we apply ℓ repeatedly, checking whether e resolves impasse each time. When a coloring that is not at impasse is encountered, a coloring of the exterior region is produced.

This has also not failed to color any of the graphs in our testing suite. In particular,

Theorem 3.4. *The graph in Figure 3 can be* 4*-colored by Algorithm* 3*.*

4 Concluding Remarks

As stated above, neither Algorithm 2 nor Algorithm 3 have failed to color a graph in our simulations. This leads us to some conjectures. The first conjecture is based on the success in the more general version of Algorithm 2.

Conjecture 4.1. *Let M be a cubic map. Then for some smallest-last ordering α of the regions, Algorithm* 0 *can be used to resolve impasse in each region with* 5 *colored neighbors as in Figure 1.*

It should be noted that by the Four Color Theorem there exists a four-coloring of the regions, which can be used to determine a region order that can be greedily four-colored. Whether this ordering can be restricted to a smallet-last ordering is unknown, hence the conjecture. Our next conjecture is based on the success of Algorithm 3.

Conjecture 4.2. *Let c be a coloring like that in Figure 1. Then for some n, the coloring $\ell^n(c)$ or $e\ell^n(c)$ is not at impasse.*

A proof of this fact would be a proof of the Four Color Theorem; therefore, we would anticipate such a proof to be quite challenging. Even if this conjecture is false, it would could be quite enlightening to study a counterexample.

Data Availability Statements The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations On behalf of all authors, the corresponding author states that there is no conflict of interest.

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