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Some results and problems on clique coverings of hypergraphs

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Abstract

For a k-uniform hypergraph F we consider the parameter $\Theta(F)$, the minimum size of a clique cover of the edge set of F. We derive bounds on $\Theta(F)$ for F belonging to various classes of hypergraphs.

KEYWORDS

clique covering, hypergraphs, independent sets, set representation

1 | INTRODUCTION

For a set X, we denote by $X^{(k)}$ the set of all subsets of X of size k. A k-uniform hypergraph (or k-graph) $G = (V, E)$ is an ordered pair of vertices V and edges E, where $E \subseteq V^{(k)}$, that is, the edges consist of subsets of vertices of size k. Since the set V is usually understood from E, we will usually denote G as the set of edges. The complement of a k-uniform graph G is the graph $\overline{G}$ defined as $\overline{G} = V^{(r)} \setminus G$. That is, $\overline{G}$ is the set of r-tuples in $V^{(r)}$ that are not in G. Unless specified, all the edges (inside or outside our host hypergraph) will have uniformity corresponding to the hypergraph studied at the moment. In all the arguments we dropped the floor and ceiling functions whenever these are not necessary.

Let G be a k-uniform hypergraph, a *set representation* of G on a set T is a system of subsets $\{S_v \subset T : v \in V(G)\}$ with the property that

$$\{v_1, v_2, \ldots, v_k\} \in G \text{ if and only if } \bigcap_{i=1}^{k} S_{v_i} \neq \varnothing.$$

The *representation number* $\Theta(G)$ of G is the smallest cardinality of a set T which admits a set representation.

We define the *clique covering number* $\mathrm{cc}(G)$ of a k-graph G as the minimum integer m such that there exist complete graphs $C_1, ..., C_m$ satisfying $G = \bigcup_{i=1}^{m} C_i$, that is, every edge is covered by a clique and no clique contains an edge that is not from G. It is well known that the representation number $\Theta(G)$ is equal to $\mathrm{cc}(G)$ (see, e.g., [8, 9] or Section A for a proof). The parameter Θ in graphs (i.e., 2-graphs) has been the subject of interest for a number of researchers (see, e.g., [1–3]). In particular it was proved in [1] that $\Theta(G) \leq c_1 d^2 \log n$ for any graph G on n vertices with maximum degree on its complement $\Delta(\overline{G}) \leq d$. On the other hand, it was proved in [2] that there are graphs G with $\Delta(\overline{G}) \leq d$ and $\Theta(G) \geq c_2 \frac{d^2}{\log d} \log n$.

Here we will be mainly interested in extending these results to k-uniform hypergraphs. Perhaps surprisingly, this turned out to be quite nontrivial, and our results in this direction are far from being optimal for large values of k.

Since most of our results regarding $\Theta(G)$ will be formulated in terms of restrictions on the degree of the complement $\overline{G}$, we will introduce a parameter $\vartheta(G) = \Theta(\overline{G})$ allowing for simpler formulation of results. Clearly, $\vartheta(G)$ is the minimum cardinality of a set T such that there is a system $\{S_v \subseteq T : v \in V(G)\}$ of sets with

$$\{v_1, ..., v_k\} \in G \text{ if and only if } \bigcap_{i=1}^{k} S_{v_i} = \varnothing.$$

Alternatively, one can view $\vartheta(G)$ as the minimum cover of the edges of the complement $\overline{G}$ by independent sets of G. In what follows we will often use the latter definition instead of the former. Moreover, we say that G can be t-represented if there exists a system of independent sets $\{I_j : 1 \leq j \leq t\}$ covering the edges of $\overline{G}$.

2 | RESULTS

Let G be a k-graph. For $S \subset V(G)$, let

$$\deg_G(S) = |\{e \in G : S \subseteq e\}|$$

be the degree of the set S in G and

$$\Delta_i(G) = \max_{|S|=i, S \subseteq V(G)} \deg_G(S)$$

be the maximum degree of an i-tuple of vertices of G. (Consequently, $\Delta_1(G) = \Delta(G)$ is just the usual maximum degree of a vertex.)

We say that a k-graph G is d-*b*alanced if $\Delta_i(G) \leq d^{\frac{k-i}{k-1}}$ for all $1 \leq i \leq k$. In particular, a d-balanced graph has maximum degree d. Our first result gives almost sharp bounds on $\vartheta(G)$ for the family of d-balanced k-graphs. Let

$$b(n, d, k) = \max\{\vartheta(G) : G \text{ is a } d\text{-balanced } k\text{-graph with } n \text{ vertices}\}$$

be the maximum value of $\vartheta(G)$ over all d-balanced k-graphs with n vertices.

Theorem 2.1. *For* $n, d, k \geq 2$, *there exist positive constants* c_1 *and* c_2 *depending only on* k *such that*

$$\frac{c_1 d^{\frac{k}{k-1}}}{\log d} \log\left(\frac{n^{k-1}}{d}\right) \leq b(n, d, k) \leq c_2 d^{\frac{k}{k-1}} \log n$$

holds.

For the general family of bounded degree hypergraphs, we obtain the following bounds. Let $\vartheta(n, d, k)$ be the maximum value of $\vartheta(G)$, where G is a k-graph on n vertices with $\Delta(G) = \Delta_1(G) \leq d$.

Theorem 2.2. *For* $n, d, k \geq 3$, *there exist positive constants* c_1 *and* c_2 *depending only on* k *such that*

$$c_1 \frac{d^{\frac{k}{k-1}}}{\log d} \log\left(\frac{n^{k-1}}{d}\right) \leq \vartheta(n, d, k) \leq c_2 d^{\frac{k}{2}} \log n.$$

Moreover, if k *is even the lower bound can be improved to*

$$\frac{c_1 d^2}{\log d} \log\left(\frac{n}{d}\right) \leq \vartheta(n, d, k).$$

Note that since every 2-graph G with $\Delta(G) \leq d$ is d-balanced, then the bounds of $\vartheta(n, d, k)$ provided in Theorems 2.1 and 2.2 are almost sharp for $k = 2, 3, 4$ and $d \ll n$. Finally, we address similar questions for some other classes of hypergraphs. For example, a linear k-graph is a hypergraph where two distinct edges do not share more than one vertex. Denoting $\vartheta_{lin}(n, k)$ as the maximum value of $\vartheta(G)$, where G runs over all k-uniform linear hypergraphs with n vertices, we show the following:

Theorem 2.3. *For* $k \geq 3$ *and* $n \geq n_0(k)$, *there exist positive constants* c_1 *and* c_2 *depending only on* k *such that*

$$\frac{c_1 n^{\frac{k}{k-1}}}{(\log n)^{\frac{1}{k-1}}} \leq \vartheta_{lin}(n, k) \leq c_2 n^{\frac{k}{k-1}} \log n$$

holds.

The paper is organised as follows. In Section 3 we discuss the upper bounds of Theorem 2.1 and 2.2, while in Section 4 we provide their respective lower bounds. Section 5 is devoted to the problem of representing special hypergraphs.

3 | UPPER BOUNDS OF THEOREMS 2.1 AND 2.2

We start with the upper bound of Theorem 2.1. We are going to show that if G is a d-balanced k-graph, then $\vartheta(G) \leq 2^k k^{k+1} d^{\frac{k}{k-1}} \log n$.

Theorem 3.1. *Let $n, d, k \geq 2$ and $c = 2^{k+2}k^{k+1}$. If G is a d-balanced k-graph on n vertices, then $\vartheta(G) \leq cd^{\frac{k}{k-1}} \log n$.*

Proof. We recall that $\vartheta(G)$ is the minimum size of a cover of $\overline{G}$ by independent sets of G. Consider $t = cd^{\frac{k}{k-1}} \log n$ random subsets $W_1, W_2, \ldots, W_t$ of $V := V(G)$, where each W_j is chosen by selecting vertices from V independently and uniformly with probability $p = \frac{1}{2kd^{1/(k-1)}}$. Our aim is to cover $\overline{G}$ by modifying the sets W_j for $1 \leq j \leq t$. For each W_j, construct an independent set as follows: For every edge e in $G[W_j]$ delete all of its vertices from W_j. Let I_j be the set obtained after deleting vertices from every edge in $G[W_j]$, that is,

$$I_j = W_j \setminus \bigcup_{e \in G[W_j]} e.$$

Clearly, I_j is an independent set. We claim that $\{I_1, \ldots, I_t\}$ is a covering of $\overline{G}$ with positive probability.

For a k-tuple $\bar{e} = \{x_1, \ldots, x_k\} \in \overline{G}$, we examine the probability that $\bar{e} \subseteq I_j$ for some $1 \leq j \leq t$. To do that, note that the only reason why a vertex $x_i \in \bar{e}$ could be deleted is if there exists an edge $e \in G[W_j]$ such that $x_i \in e \cap \bar{e}$. That is, $\bar{e} \subseteq I_j$ if $\bar{e} \subseteq W_j$ and $e \cap \bar{e} = \emptyset$ for all $e \in G[W_j]$. Hence, we obtain

$$\mathbb{P}(\bar{e} \subset I_j) \geq p^k \left(1 - \sum_{i=1}^{k-1} \Delta_i(G) p^{k-i} \binom{k}{i}\right) \geq p^k \left(1 - \sum_{i=1}^{k-1} \frac{\binom{k}{k-i}}{(2k)^{k-i}}\right) > \frac{p^k}{4}.$$

Consequently, since all I_j were chosen independently, we obtain that

$$\mathbb{P}\left(\bigwedge_{j=1}^{t} (\bar{e} \not\subseteq I_j)\right) = \prod_{j=1}^{t} \mathbb{P}(\bar{e} \not\subseteq I_j) < \left(1 - \frac{p^k}{4}\right)^t \leq \exp\left(-\frac{p^k t}{4}\right) = \frac{1}{n^k}.$$

Since there are at most $\binom{n}{k}$ nonedges in $\overline{G}$ to cover, the probability that one remains uncovered is at most $\binom{n}{k}\frac{1}{n^k} < 1$. Consequently, with positive probability all edges of $\overline{G}$ are covered by $\bigcup_{j=1}^{t} I_j$. □

The proof of the upper bound of Theorem 2.2 follows similar lines.

Theorem 3.2. *Let $n, d, k \geq 3$, $\delta = \frac{1}{(k-1)2^{k+2}}$ and $c = \frac{2k}{\delta^k}$. If G is a k-graph on n vertices with $\Delta(G) \leq d$, then $\vartheta(G) \leq cd^{\frac{k}{2}} \log n$.*

Proof. As in the proof of Theorem 3.1, we want to cover all the edges of $\overline{G}$ with independent sets of G. To this end, we consider $t = cd^{\frac{k}{2}} \log n$ random subsets $W_1, \ldots, W_t$ of V, where each W_j is chosen by selecting vertices from V independently and uniformly with probability $p = \frac{\delta}{\sqrt{d}}$. Note that the sets $W_1, \ldots, W_t$ are not necessarily independent. Our aim is to modify those sets W_j to independent sets I_j such that $\bigcup_{j=1}^{t} I_j$ covers every k-tuple in $\overline{G}$.

To this end, we consider the auxiliary $(k-1)$-graph H given by $V(H) = V$ and

$$H = \{S \in V^{(k-1)} \,:\, \deg_G(S) \geq \sqrt{d}\},$$

that is, H is the $(k-1)$-graph where the edges are the $(k-1)$-tuples of large degree in G. The next proposition gives an upper bound on the maximum degree of H. □

Proposition 3.3. $\Delta(H) \leq (k-1)\sqrt{d}$.

Proof. Assume the contrary and let v be a vertex with $\deg_H(v) > (k-1)\sqrt{d}$. Then by a counting argument we obtain that

$$\deg_G(v) = \frac{\sum_{v \in S \in V^{(k-1)}} \deg_G(S)}{k-1} \geq \frac{\deg_H(v)\sqrt{d}}{k-1} > d,$$

which contradicts the fact that $\Delta(G) \leq d$. □

Now we describe how to construct the independent sets I_j. For each $1 \leq j \leq t$, we sequentially remove vertices from W_j satisfying the following cleaning strategy:

3.1 | Cleaning strategy

Let $X = W_j$. While there is an edge in $G[X]$ perform one of the two operations:

Operation (i) If there is an edge $g \in G[X]$ containing precisely one $(k-1)$-tuple $S \in H$, then we remove the vertex not in S from X, that is, the vertex in the singleton set $g \backslash S$.

Operation (ii) Otherwise, if all the edges in $G[X]$ contain either zero or more than one $(k-1)$-tuple from H, then we select an arbitrary edge $g \in G[X]$ and delete an arbitrary vertex from it.

Set I_j to be the resulting X obtained after the process is over.

Clearly, the set I_j does not contain any edge from G and thus it is independent. It remains to show that with positive probability any edge from the complement $\overline{G}$ is covered by $\bigcup_{j=1}^{t} I_j$.

For a k-tuple $\bar{e} = \{x_1, \ldots, x_k\} \in \overline{G}$, we want to estimate the probability that $\bar{e} \subseteq I_j$ for some $1 \leq j \leq t$. The following lemma gives us a lower bound on the probability.

Lemma 3.4. *For $\bar{e} \in \overline{G}$ and $1 \leq j \leq t$,*

$$\mathbb{P}(\bar{e} \subseteq I_j) \geq \frac{1}{2}\left(\frac{\delta}{\sqrt{d}}\right)^k.$$

Proof. In what follows, we will say that the set X *crosses* the pair (Y, Z) if X has nonempty intersection with both Y and Z. Set $\bar{e} = \{x_1, \ldots, x_k\}$. We start by defining some auxiliary events.

For a set $S \subseteq \bar{e}$ with $S \notin H$ and $|S| \leq k-1$, let

$$A_S = \{\nexists R \subseteq W_j - \bar{e}, R \cup S \in G\} \tag{1}$$

be the event that there is no edge $g \in G$ crossing the pair $(\bar{e}, W_j - \bar{e})$ with $\bar{e} \cap g = S$. Similarly, for a set $S \subseteq \bar{e}$ with $|S| \leq k - 2$, let

$$B_S = \{\nexists R \subseteq W_j - \bar{e}, R \cup S \in H\} \tag{2}$$

be the event stating that there is no $(k-1)$-tuple $h \in H$ crossing the pair $(\bar{e}, W_j - \bar{e})$ with $\bar{e} \cap h = S$.

The next claim gives a sufficient condition for when $\bar{e}$ is covered by I_j. □

Proposition 3.5. *If $\bar{e} \in \overline{G}$ is a k-tuple satisfying*

(a) *for every $S \subseteq \bar{e}$ with $|S| \leq k - 1$ and $S \notin H$, the event A_S holds,*
(b) *for every $S \subseteq \bar{e}$ with $|S| \leq k - 2$, the event B_S holds,*
(c) *$\bar{e} \subseteq W_j$,*

then $\bar{e} \subseteq I_j$.

Proof. Suppose that $\bar{e} \nsubseteq I_j$. This means that a vertex of $\bar{e}$ was deleted while performing operations (i) and (ii), that is, in our cleaning process we deleted a vertex from an edge $g \in G$ with $g \cap \bar{e} \neq \varnothing$. Since (a) holds, we obtain that $g \cap \bar{e}$ is an edge of H. Moreover, by (b), we have that $g \cap \bar{e}$ is the only $(k-1)$-tuple of g in H. Therefore, we deleted a vertex of g in the operation (i). By the definition of operation (i), we obtain that the deleted vertex was in $g - \bar{e}$, which contradicts our assumption that the deleted vertex belongs to $\bar{e}$. □

As a consequence of Proposition 3.5, one can estimate the probability of $\bar{e} \subseteq I_j$ by

$$\begin{aligned} \mathbb{P}(\bar{e} \subseteq I_j) &\geq \mathbb{P}\left(\{\bar{e} \subseteq W_j\} \wedge \bigwedge_{S \in \binom{\bar{e}}{\leq k-1} \setminus H} A_S \wedge \bigwedge_{S \in \binom{\bar{e}}{\leq k-2}} B_S\right) \\ &= p^k \mathbb{P}\left(\bigwedge_{S \in \binom{\bar{e}}{\leq k-1} \setminus H} A_S \wedge \bigwedge_{S \in \binom{\bar{e}}{\leq k-2}} B_S\right). \end{aligned} \tag{3}$$

To compute the probability of the intersection of events A_S and B_S, we will estimate the probability of the complementary events A_S^c and B_S^c. Set $\varepsilon = \frac{1}{2^{k+2}}$ and recall that $\delta = \frac{1}{(k-1)2^{k+2}}$. We split the computations into cases depending on the size of S:

Case 1. $|S| \leq k - 2$.

By using the definitions of A_S in (1) we obtain that

$$\begin{aligned} \mathbb{P}\left(A_S^c\right) = \mathbb{P}\left(\bigvee_{R \cup S \in G} \{R \subseteq W_j - \bar{e}\}\right) &\leq \textstyle\sum_{R \cup S \in G} \mathbb{P}(R \subseteq W_j - \bar{e}) \\ &\leq p^{k-|S|} \deg_G(S) \leq p^2 d = \delta^2 \leq \varepsilon. \end{aligned} \tag{4}$$

Proposition 3.3 and the definition of B_S in (2) give us that

$$\mathbb{P}\left(B_S^c\right) = \mathbb{P}\left(\bigvee_{R \cup S \in H} \{R \subseteq W_j - \bar{e}\}\right) \le \sum_{R \cup S \in H} \mathbb{P}(R \subseteq W_j - \bar{e}) \le p^{k-1-|S|}\deg_H(S) \le p\Delta(H) \le (k-1)\delta = \epsilon. \tag{5}$$

Case 2. $|S| = k - 1$ and $S \notin H$.

By the definition of A_S in (1) and by the fact that $S \notin H$ implies that $\deg_G(S) \le \sqrt{d}$, we obtain that

$$\mathbb{P}\left(A_S^c\right) = \mathbb{P}\left(\bigvee_{R \cup S \in G} \{R \subseteq W_j - \bar{e}\}\right) \le \sum_{R \cup S \in G} \mathbb{P}(R \subseteq W_j - \bar{e}) \le p^{k-|S|}\deg_G(S) \le p\sqrt{d} = \delta \le \epsilon. \tag{6}$$

Finally, since there exist at most 2^k choices of $S \subseteq \bar{e}$, by putting (3)–(6) together we obtain that

$$\mathbb{P}(\bar{e} \subseteq I_j) \ge p^k\left(1 - \sum_{S \in \binom{\bar{e}}{\le k-1} \setminus H} \mathbb{P}\left(A_S^c\right) - \sum_{S \in \binom{\bar{e}}{\le k-2}} \mathbb{P}\left(B_S^c\right)\right) \ge p^k(1 - 2^{k+1}\epsilon) \ge \frac{p^k}{2} = \frac{1}{2}\left(\frac{\delta}{\sqrt{d}}\right)^k.$$

□

Now the remaining part of the proof of Theorem 3.2 is straightforward. By Lemma 3.4, the probability that $\bar{e}$ is not covered by $\bigcup_{j=1}^t I_j$ is given by

$$\mathbb{P}\left(\bigwedge_{j=1}^t (\bar{e} \not\subseteq I_j)\right) = \prod_{j=1}^t \mathbb{P}(\bar{e} \not\subseteq I_j) < \left(1 - \frac{1}{2}\left(\frac{\delta}{\sqrt{d}}\right)^k\right)^t \le \exp\left(-\frac{t}{2}\left(\frac{\delta}{\sqrt{d}}\right)^k\right) = \frac{1}{n^k}.$$

Since there are at most $\binom{n}{k}$ nonedges in $\overline{G}$ to cover, the probability that one remains uncovered is at most $\binom{n}{k}\frac{1}{n^k} < 1$. Consequently, with positive probability all edges of $\overline{G}$ are covered by $\bigcup_{j=1}^t I_j$. □

4 | LOWER BOUNDS OF THEOREMS 2.1 AND 2.2

In this section, we prove the lower bound of Theorems 2.1 and 2.2. The next proposition shows the lower bound of Theorems 2.1 and 2.2 when k is odd.

Theorem 4.1. *For $n, d, k \ge 2$ with $n \ge dk$, there exists a d-balanced k-graph G such that*

$$\vartheta(G) \ge c\frac{d^{\frac{k}{k-1}}}{\log d}\log\left(\frac{n^{k-1}}{d}\right)$$

for a positive constant c depending only on k.

The proof of Theorem 4.1 follows from a counting argument using the next two auxiliary lemmas. Recall that a t-representation of a k-graph H is a system $\mathcal{I} = \{I_j \subseteq [n] : 1 \le j \le t\}$ covering the edges of the complement $\overline{H}$. For a graph H, let $f(H, t)$ be the number of distinct t-representations for a graph H. If some graph H does not admit a t-representation, we say that $f(H, t) = 0$. For integers $n, \alpha, t \ge 1$, define

$$f(n, \alpha, t) = \sum_{\substack{H \text{ is a } k\text{-graph} \\ \alpha(H) \le \alpha}} f(H, t)$$

as the number of all possible t-representations of the family of graphs H on the vertex set $[n]$ with independence number $\alpha(H) \le \alpha$.

Lemma 4.2. *For $\alpha, t \ge 1$ and sufficiently large n, $f(n, \alpha, t) \le \left(2\binom{n}{\alpha}\right)^t$.*

Proof. Let $\mathcal{I} = \{I_1, \ldots, I_t\}$ be a set system covering $\overline{H}$ for some H. Since I_j is an independent set of H and $\alpha(H) \le \alpha$, we obtain that $|I_j| \le \alpha$. Hence, the number of ways to choose distinct systems $\mathcal{I}$ is bounded by

$$f(n, \alpha, t) \le \left(\sum_{i=1}^{\alpha} \binom{n}{i}\right)^t \le \left(2\binom{n}{\alpha}\right)^t.$$

□

Given integers m, d, k with $d \le m$, we define $\mathcal{F}_{m,d}^{(k)}$ to be a family of k-partite k-graphs F on a fixed set of vertices $V = \bigcup_{i=1}^{k} V_i$ with $|V_i| = m$ for every $1 \le i \le k$ satisfying:

(i) All edges $e \in F$ are transversal to the partition $V_1 \cup \ldots \cup V_k$, that is, $|e \cap V_i| = 1$ for all $1 \le i \le k$.
(ii) F is linear, that is, distinct edges intersects in at most one vertex.
(iii) $\Delta(F) \le d$.

The following lemma gives a lower bound on the size of $\mathcal{F}_{m,d}^{(k)}$.

Lemma 4.3. *Let k, d, m be integers with $k \ge 3$ and $m \ge m_0(k)$. Then,*

$$\left|\mathcal{F}_{m,d}^{(k)}\right| \ge \left(\frac{m^{k-1}}{d}\right)^{\frac{md}{2k^2}}$$

holds.

Proof. Set $s = \frac{dm}{2k^2}$. We will construct a graph F from $\mathcal{F}_{m,d}^{(k)}$ by successively choosing edges $e_1, \ldots, e_s$ transversal to the partition $V_1 \cup \cdots \cup V_k$. Suppose that for $\ell < s$ we already constructed $F_\ell = \{e_1, \ldots, e_\ell\}$ satisfying:

(i) e_i is transversal to $V_1 \cup \cdots \cup V_k$ for all $1 \le i \le \ell$.
(ii) F_ℓ is linear.
(iii) $\Delta(F_\ell) \le d$.

Now we intend to add an edge $e_{\ell+1}$ to construct the next graph $F_{\ell+1} = F_\ell \cup \{e_{\ell+1}\}$. For each $1 \le i \le k$, we have

$$\sum_{x \in V_i} \deg_{F_\ell}(x) = \ell < s = \frac{dm}{2k^2}.$$

Consequently, if $X_i = \{x \in V_i \ : \ \deg_{F_\ell}(x) = d\}$, then we have

$$|X_i| \le \frac{m}{2k^2}. \tag{7}$$

To choose an edge $e_{\ell+1}$ we need to satisfy $|e_i \cap e_{\ell+1}| \le 1$ for every $1 \le i \le \ell$. Given e_i, there are at most $\binom{k}{2} m^{k-2}$ k-tuples $f \in V_1 \times \cdots \times V_k$ such that $|e_i \cap f| \ge 2$. Therefore, since $d \le m$, there are at most

$$\ell \binom{k}{2} m^{k-2} < s \binom{k}{2} m^{k-2} \le \frac{dm^{k-1}}{4} \le \frac{m^k}{4}$$

k-tuples of $V_1 \times \cdots \times V_k$ that violate condition (ii) of $\mathcal{F}^{(k)}_{m,d}$. Since the addition of any k-tuple containing an element of X_i violates condition (iii) of $\mathcal{F}^{(k)}_{m,d}$, we obtain by (7) that there are at least

$$\prod_{i=1}^{k} |V_i \setminus X_i| - \frac{m^k}{4} \ge m^k \left(\left(1 - \frac{1}{2k^2}\right)^k - \frac{1}{4} \right) \ge m^k \left(1 - \frac{1}{2k} - \frac{1}{4}\right) \ge \frac{m^k}{2}$$

valid choices for $e_{\ell+1}$.

Thus there exist at least $(m^k/2)^s$ sequences of edges $e_1, \ldots, e_s$ forming a graph $F = \{e_1, \ldots, e_s\}$ satisfying conditions (i)–(iii) of $\mathcal{F}^{(k)}_{m,d}$. Since the same graph can be obtained by at most $s!$ of these sequences, we have

$$\left| \mathcal{F}^{(k)}_{m,d} \right| \ge \frac{\left(\frac{m^k}{2}\right)^s}{s!} \ge \left(\frac{m^k}{2s}\right)^s \ge \left(\frac{m^{k-1}}{d}\right)^{\frac{md}{2k^2}}. \qquad \square$$

Proof of Theorem 4.1. We will construct a family of k-graphs $\mathcal{H}$ on n vertices of maximum degree d and small independence number. To this end, set $p = (d/2)^{\frac{1}{k-1}}$ and consider a partition $[n] = V_1 \cup \cdots \cup V_k$ with each $|V_i| = n/k$. For each $1 \le i \le k$, let H_i be the k-graph with vertex set V_i consisting of the union of n/kp vertex disjoint cliques $K^{(k)}_p$ of size p. Let

$$\mathcal{H} = \left\{ F \cup \bigcup_{i=1}^{k} H_i : \ F \in \mathcal{F}^{(k)}_{n/k, d/2} \right\}$$

be the family of k-graphs obtained by adding a k-graph $F \in \mathcal{F}^{(k)}_{n/k,d/2}$ with k-partition $V_1 \cup \cdots \cup V_k$ to the union $\bigcup_{i=1}^{k} H_i$. By Lemma 4.3, we have

$$|\mathcal{H}| = \left|\mathcal{F}^{(k)}_{n/k,d/2}\right| \geq \left(\frac{2}{d}\binom{n}{k}^{k-1}\right)^{\frac{dn}{4k^3}}. \tag{8}$$

We claim that $H = F \cup \bigcup_{i=1}^{k} H_i$ is d-balanced for every $H \in \mathcal{H}$. First, note that if $x \in V(H)$, then $x \in V_i$ for some $1 \leq i \leq k$ and consequently

$$\deg_H(x) = \deg_{H_i}(x) + \deg_F(x) \leq \binom{p-1}{k-1} + \frac{d}{2} \leq \frac{d}{2} + \frac{d}{2} = d.$$

Thus $\Delta(H) \leq d$.

Now let $S \in [n]^{(\ell)}$ for $2 \leq \ell \leq k-1$. Note that $S \subseteq e$ for an edge $e \in \bigcup_{i=1}^{k} H_i$ if and only if S is a subset of vertices of some $K_p^{(k)}$. Consequently, if S is a subset of the vertex set of some $K_p^{(k)}$, then $S \subseteq V_i$ for some $1 \leq i \leq k$. Thus, by condition (i) of $\mathcal{F}^{(k)}_{n/k,d/2}$ we have that $\deg_F(S) = 0$. Hence,

$$\deg_H(S) \leq \binom{p-\ell}{k-\ell} \leq d^{\frac{k-\ell}{k-1}}.$$

Otherwise, if S is not a subset of any $K_p^{(k)}$ then

$$\deg_H(S) = \deg_F(S) \leq 1,$$

since F is linear. Therefore, H is a d-balanced k-graph.

Finally, we turn our attention to the independence number of $H \in \mathcal{H}$. Note that $\bigcup_{i=1}^{k} H_i$ is a union of n/p vertex disjoint copies of $K_p^{(k)}$. Thus, since $\bigcup_{i=1}^{k} H_i \subseteq H$, we have that

$$\alpha(H) \leq \alpha\left(\bigcup_{i=1}^{k} H_i\right) = (k-1)\frac{n}{p} \tag{9}$$

for every $H \in \mathcal{H}$.

Let t be the minimum integer such that it is possible to t-represent any element $H \in \mathcal{H}$, that is, t is the minimum integer such that for any $H \in \mathcal{H}$, there exists a system of independent sets $\{I_j : 1 \leq j \leq t\}$ with the property that every edge in $\overline{H}$ is covered by some I_j. Lemma 4.2 applied with (9) gives us that there exists at most

$$\left(2\binom{n}{(k-1)n/p}\right)^t \leq (2ep)^{(k-1)nt/p}$$

ways to t-represent the family $\mathcal{H}$.

Note that if the system of independent sets $\{I_j : 1 \leq j \leq t\}$ represents a graph H, then the edges of $\overline{H}$ are covered by H. Since I_j is an independent set, this implies that $\overline{H} = \bigcup_{j=1}^{t} I_j$. That is, the system $\{I_j : 1 \leq j \leq t\}$ fully determines the edges of $\overline{H}$. Thus, two graphs $H, H' \in \mathcal{H}$ cannot be represented by the same system $\{I_j : 1 \leq j \leq t\}$ and we conclude that

$$(cp)^{(k-1)nt/p} \geq |\mathcal{H}| \geq \left(\frac{2}{d}\left(\frac{n}{k}\right)^{k-1}\right)^{\frac{dn}{4k^3}}.$$

Hence, by the fact that $p = (d/2)^{\frac{1}{k-1}}$ we obtain that

$$t \geq c\frac{d^{\frac{k}{k-1}}}{\log d}\log\left(\frac{n^{k-1}}{d}\right)$$

for a positive constant c depending only on k. That is, there exists a d-balanced k-graph $H \in \mathcal{H}$ such that

$$\vartheta(H) \geq c\frac{d^{\frac{k}{k-1}}}{\log d}\log\left(\frac{n^{k-1}}{d}\right) \qquad \square$$

For k even, we can further improve the lower bound for k-graphs of bounded maximum degree d.

Theorem 4.4. *For $n, d, k \geq 2$ and k even, there exists k-graph G with $\Delta(G) \leq d$ such that*

$$\vartheta(G) \geq c\frac{d^2}{\log d}\log\left(\frac{2n}{kd}\right)$$

for a positive constant c.

Proof. Suppose that $k = 2\ell$ for some integer ℓ. Theorem 4.1 says that there exists a 2-graph F on n/ℓ vertices with $\Delta(F) \leq d$ such that

$$\vartheta(F) \geq c\frac{d^2}{\log d}\log\left(\frac{n}{\ell d}\right).$$

We will construct a k-graph G with $\Delta(G) \leq d$ satisfying the inequality of the statement as follows: Let $V(F) = [n/\ell]$. For each $i \in [n/\ell]$, let $V_i = \{v_{i,1}, \ldots, v_{i,\ell}\}$ be a set consisting of ℓ copies of the vertex i. We define $V(G) = \bigcup_{i=1}^{n/\ell} V_i$ and

$$E(G) = \{V_i \cup V_j \ : \ \{i, j\} \in F\}.$$

That is, the edges of G are the 2ℓ-tuples of the form $V_i \cup V_j$ where i and j are adjacent in F. We will prove that G is our desired graph. First note that if $x \in V_i$ for some $i \in [n/\ell]$, then $\deg_G(x) = \deg_F(i)$. Thus $\Delta(G) = \Delta(F) \leq d$.

Now suppose that G can be t-represented and let $\mathcal{I} = \{I_j : 1 \le j \le t\}$ be the system of independent sets covering the edges of $\overline{G}$. For each $1 \le j \le t$, we consider the subset $\tilde{I}_j \subseteq V(F)$ given by

$$\tilde{I}_j = \{i \in [n/\ell] \ : \ V_i \subseteq I_j\}.$$

That is, $\tilde{I}_j$ consists of all vertices $i \in [n/\ell]$ such that V_i is fully contained in I_j. Note that $\tilde{I}_j$ is an independent set. Indeed, suppose to the contrary that there exist $i, i' \in \tilde{I}_j$ adjacent in F. Then, on the one hand, by the definition of $\tilde{I}_j$, we have that $V_i \cup V_{i'} \subseteq I_j$. On the other hand, by the definition of G, we have that $V_i \cup V_{i'} \in G$, which contradicts the fact that I_j is independent.

We claim that $\tilde{\mathcal{I}} = \{\tilde{I}_j \ : \ 1 \le j \le t\}$ is a system of independent sets in F which covers the edges of $\overline{F}$. In particular, this proves that $\vartheta(G) \ge \vartheta(F)$. Let $\{i, i'\} \in \overline{F}$ be a nonadjacent pair of vertices in $V(F)$. Then, by the construction of G, the 2ℓ-tuple $V_i \cup V_{i'}$ is an edge of $\overline{G}$. Since $\{I_j \ : \ 1 \le j \le t\}$ covers the edges of $\overline{G}$, there exists I_j such that $V_i \cup V_{i'} \subseteq I_j$. Thus $\{i, i'\} \subseteq \tilde{I}_j$ and consequently $\{\tilde{I}_j \ : \ 1 \le j \le t\}$ covers $\overline{F}$. Therefore, we obtained a k-graph G with $\Delta(G) \le d$ such that

$$\vartheta(G) \ge \vartheta(F) \ge c\frac{d^2}{\log d}\log\left(\frac{2n}{kd}\right). \qquad \square$$

5 | SPECIAL HYPERGRAPHS

While for k-graphs of bounded degree with k large we still have a significant gap between lower and upper bounds, for Steiner systems one can obtain more precise bounds. Given $1 < \ell < k < n$, a *partial* Steiner (n, k, ℓ)-system is a k-graph $S \subseteq [n]^{(k)}$ such that every ℓ-tuple $P \in [n]^{(\ell)}$ is contained in at most one edge of S. An (n, k, ℓ)-system in which every ℓ-tuple $P \in [n]^{(\ell)}$ is contained in precisely one edge of S is called a *full* Steiner (n, k, ℓ)-system. While it is easy to check the existence of a partial (n, k, ℓ)-system, the existence of a full (n, k, ℓ)-system for all admissible parameters $1 < \ell < k$ and $n \ge n_0(\ell, k)$ was established only recently in [4, 6]. In this section we are going to provide bounds for the representation number of partial and full Steiner (n, k, ℓ)-systems for $\ell = 2$.

For $1 < \ell < k < n$, we define

$$s(n, k, \ell) = \max\{\vartheta(S) : S \text{ is a partial } (n, k, \ell)\text{-system}\}$$

as the maximum value of $\vartheta(S)$, where S runs over all partial (n, k, ℓ)-systems. Similarly, we define $s^*(n, k, \ell)$ as the maximum value of $\vartheta(S)$, where S runs over all full (n, k, ℓ)-systems.

Theorem 5.1. *Given $k > 1$, there exist positive constants c_1 and c_2 depending only on k such that the following holds:*

(i) *For n sufficiently large,*

$$c_1 \frac{n^{\frac{k}{k-1}}}{(\log n)^{\frac{1}{k-1}}} \le s(n,k,2) \le c_2 n^{\frac{k}{k-1}} \log n$$

(ii) *For infinitely many n,*

$$c_1 \frac{n^{\frac{k}{k-1}}}{(\log n)^{\frac{1}{k-1}}} \le s^*(n,k,2) \le c_2 n^{\frac{k}{k-1}} \log n.$$

Note that Theorem 5.1(i) is just a reformulation of Theorem 2.3. Before we give a proof of the theorem, we are going to introduce two useful results.

Let $K_k^{(\ell)}(m)$ be an m-blowup of the complete ℓ-graph with k vertices, that is, $K_k^{(\ell)}(m)$ is the graph with vertex set $V = \bigcup_{i=1}^k V_i$, where $|V_i| = m$ for $1 \le i \le k$, and is such that for every $1 \le i_1 < \cdots < i_\ell \le k$ and $x_{i_j} \in V_{i_j}$ the ℓ-tuple $\{x_{i_1}, \ldots, x_{i_\ell}\}$ is an edge. The $K_k^{(\ell)}$-decomposition of $K_k^{(\ell)}(m)$ is a system of pairwise edge disjoint copies of $K_k^{(\ell)}$ covering all edges of $K_k^{(\ell)}(m)$. Equivalently, one can view each such decomposition as a k-partite k-graph F with vertex partition $V(F) = \bigcup_{i=1}^k V_i$, where $|V_i| = m$ for $1 \le i \le k$, and such that for every $1 \le i_1 < \cdots < i_\ell \le k$ and vertices $x_{i_j} \in V_{i_j}$ there exists exactly one edge $f \in F$ with $\{x_{i_1}, \ldots, x_{i_\ell}\} \subseteq f$.

In [7], Keevash obtained strong bounds on the number of such decompositions.

Theorem 5.2 (Keevash [7, Theorem 2.8]). *The number of distinct $K_k^{(\ell)}$-decompositions of $K_k^{(\ell)}(m)$ is given by*

$$\left(\left(e^{1-\binom{k}{\ell}} + o(1)\right) m^{k-\ell}\right)^{m^\ell}$$

for sufficiently large m.

We will also need the following result on the existence of (n, k, ℓ)-systems with bounded independence number.

Theorem 5.3 (Grable et al. [5]). *For $k > \ell \ge 2$, there is a positive constant c depending only on k such that the following holds. If $m = q^2$ for q sufficiently large prime power, then there exists a full Steiner $(m, k, 2)$-system S with $\alpha(S) \le c m^{\frac{k-2}{k-1}} (\log m)^{\frac{1}{k-1}}$.*

Proof of Theorem 5.1. To establish the upper bound we first note that any full or partial $(n, k, 2)$-system S is a $\Delta_1(S)$-balanced k-graph. Indeed, observe that for any $2 \le i \le k-1$ we have $\Delta_i(S) \le 1 < \Delta_1(S)^{\frac{k-i}{k-1}}$. Also note that for $x \in V(S)$,

$$\deg(x) \le \sum_{y \in [n] \setminus \{x\}} \deg(\{x, y\}) \le n - 1,$$

since $\deg(\{x, y\}) \le 1$ for all $x, y \in V(S)$. Hence $\Delta_1(S) \le n$ and by the upper bound of Theorem 2.1, there exists positive constant c_2 such that

$$\vartheta(S) \le c_2 \Delta_1(S)^{\frac{k}{k-1}} \log n \le c_2 n^{\frac{k}{k-1}} \log n.$$

Next we prove the lower bounds of $s(n, k, 2)$ and $s^*(n, k, 2)$. We may assume without changing the asymptotics that $m = n/k = q^2$ for q sufficiently large prime. By Theorem 5.3 there exists a full $(m, k, 2)$-system S with $\alpha(S) \leq cm^{\frac{k-2}{k-1}}(\log m)^{\frac{1}{k-1}}$. Consider k vertex disjoint copies $S_1, \ldots, S_k$ of S and let $V_i = V(S_i)$ for $1 \leq i \leq k$. Let $\mathcal{F}$ be the family of k-partite k-graphs F with vertex set $V(F) = \bigcup_{i=1}^{k} V_i$ such that for every $1 \leq i < j \leq k$ and $x_i \in V_i, x_j \in V_j$, there exists exactly one $f \in F$ such that $\{x_i, x_j\} \subseteq f$. As discussed previously, the family $\mathcal{F}$ is in one-to-one correspondence with the $K_k^{(2)}$-decompositions of $K_k^{(2)}(m)$. Thus, by Theorem 5.2, we have

$$|\mathcal{F}| = \left(\left(e^{1-\binom{k}{2}} + o(1)\right)m^{k-2}\right)^{m^2}, \tag{10}$$

for sufficiently large m.

Let $\mathcal{H}$ be the family of graphs defined by

$$\mathcal{H} = \left\{F \cup \bigcup_{i=1}^{k} S_i \,:\, F \in \mathcal{F}\right\}.$$

We observe that every $H \in \mathcal{H}$ is a full $(n, k, 2)$-system. Let $H = F \cup \bigcup_{i=1}^{k} S_i$ for some $F \in \mathcal{F}$ and let $\{x, y\} \subseteq V(H)$. If $\{x, y\} \subseteq V_i$ for some $1 \leq i \leq k$, then since S_i is a full Steiner system, there exists exactly one edge $e \in S_i$ such that $\{x, y\} \subseteq e$. Also if $x \in V_i$ and $y \in V_j$ for $i \neq j$, then by the definition of F, there exists exactly one edge $f \in F$ such that $\{x, y\} \subseteq f$.

Set $\alpha = k\alpha(S)$ and let t be the minimum integer such that every graph $H \in \mathcal{H}$ admits a t-representation. By Lemma 4.2 there are $\left(2\binom{n}{\alpha}\right)^t$ ways to t-represent graphs in $\mathcal{H}$. Since every t-representation corresponds to a unique graph in $\mathcal{H}$ and by our assumption that every graph $H \in \mathcal{H}$ has a t-representation, we have by (10) that

$$2^{t\alpha \log n} \geq \left(2\binom{n}{\alpha}\right)^t \geq |\mathcal{H}| = |\mathcal{F}| = \left(\left(e^{1-\binom{k}{2}} + o(1)\right)m^{k-2}\right)^{m^2} \geq 2^{c' n^2 \log n}.$$

Since $\alpha(S) \leq cm^{\frac{k-2}{k-1}}(\log m)^{\frac{1}{k-1}} \leq c'' n^{\frac{k-2}{k-1}}(\log n)^{\frac{1}{k-1}}$, we obtain that

$$t \geq c_1 \frac{n^{\frac{k}{k-1}}}{(\log n)^{\frac{1}{k-1}}},$$

for a positive constant c_1 depending on k. Therefore, there exists a full $(n, k, 2)$-system $H \in \mathcal{H}$ such that $\vartheta(H) \geq c_1 \frac{n^{\frac{k}{k-1}}}{(\log n)^{\frac{1}{k-1}}}$. Since a full $(n, k, 2)$-system is a partial $(n, k, 2)$-system, we obtain that

$$s(n, k, 2) \geq s^*(n, k, 2) \geq c_1 \frac{n^{\frac{k}{k-1}}}{(\log n)^{\frac{1}{k-1}}}.$$

□

6 | CONCLUDING REMARKS

6.1 | Representation of sparse hypergraphs of high uniformity

In Sections 3 and 4 we determined upper and lower bounds for the family of k-graphs on n vertices of bounded degree d. In particular, if we define

$$f_k(d) = \lim_{n\to\infty} \frac{\vartheta(n, d, k)}{\log n},$$

then we established almost sharp bounds for $k = 2, 3, 4$, that is, $\Omega(d^2/\log d) \leq f_2(d) \leq O(d^2)$, $\Omega(d^{3/2}/\log d) \leq f_3(d) \leq O(d^{3/2})$ and $\Omega(d^2/\log d) \leq f_4(d) \leq O(d^2)$.

Unfortunately, for $k \geq 5$, the bounds we have at the moment are worse. More precisely,

$$\Omega\left(\frac{d^2}{\log d}\right) \leq f_k(d) \leq O(d^{k/2})$$

for k even and

$$\Omega\left(\frac{d^{\frac{k}{k-1}}}{\log d}\right) \leq f_k(d) \leq O(d^{k/2})$$

for k odd. We believe that the upper bound can be improved for large k, although maybe a new idea will be necessary. It would be interesting to close the gap between the bounds for $f_k(d)$.

6.2 | Representation of k-partite k-graphs

Let $\mathcal{K}(n, d, k)$ be the family of k-partite k-graphs G on n vertices with bounded maximum degree $\Delta(G) \leq d$. As in the previous case, it would be interesting to find good bounds on $\vartheta(G)$ for $G \in \mathcal{K}(n, d, k)$. Let

$$g_k(d) = \lim_{n\to\infty} \frac{\max\{\vartheta(G) : G \in \mathcal{K}(n, d, k)\}}{\log n}.$$

A similar counting argument as the one used in Section 4 yields that $g_k(d) = \Omega(d)$. For $k = 2$, similar techniques as in Section 3 give the upper bound which is linear in d, that is, $g_2(d) = O(d)$. This leads us to believe that perhaps the same could be true for $k \geq 3$.

Conjecture 6.1. *$g_k(d) = O(d)$ for $k \geq 3$.*

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APPENDIX A

Proposition A.1. *Let F be a k-graph for $k \geq 2$. Then $\Theta(F) = \mathrm{cc}(F)$.*

Proof. Let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be a system of cliques covering all edges of F. To each vertex $v \in V(F)$ assign the set S_v of all cliques containing v. Since $\mathcal{C}$ is a covering of the edges, if $\{v_1, \ldots, v_k\} \in F$, then there exists a clique C_i containing $\{v_1, \ldots, v_k\}$. Thus $C_i \in \bigcap_{i=1}^k S_{v_i}$ and hence $\bigcap_{i=1}^k S_{v_i} \neq \varnothing$. On the other hand, by construction, $S_{v_j} = \{C_i \in \mathcal{C} : v_j \in C_i\}$ and therefore the set $\bigcap_{i=1}^k S_{v_i}$ contains all cliques containing $\{v_1, \ldots, v_k\}$. Consequently, if $\bigcap_{j=1}^k S_{v_j} = \varnothing$, then there is no clique containing $\{v_1, \ldots, v_k\}$, which implies that $\{v_1, \ldots, v_k\} \notin F$. Hence, we proved that $\mathcal{C}$ is a set representation of F and we obtain that

$$\Theta(F) \leq \mathrm{cc}(F).$$

To prove the opposite inequality, let $V(F) = \{v_1, \ldots, v_n\}$ and let t be an integer such that F admits a set representation $\{S_{v_i} \subset [t] : 1 \leq i \leq n\}$. For each $s \in [t]$, we define the set $C(s) = \{v_i : s \in S_{v_i}\}$. Since s belongs to the intersection of any k sets of the form S_v for $v \in C(s)$, we obtain that $F[C(s)]$ is a clique. We claim that $\mathcal{F} = \{F[C(1)], \ldots, F[C(s)]\}$ is a clique covering of F. Indeed, if $\{v_{i_1}, \ldots, v_{i_k}\} \in F$ is an edge of F, then $\bigcap_{j=1}^k S_{v_{i_j}} \neq \varnothing$. Let $s \in \bigcap_{j=1}^k S_{v_{i_j}}$. Then clearly $\{v_{i_1}, \ldots, v_{i_k}\} \in F[C(s)]$. Now if $\{v_{i_1}, \ldots, v_{i_k}\} \notin F$, then $\bigcap_{j=1}^k S_{v_{i_j}} = \varnothing$ and consequently there is no s such that $\{v_{i_1}, \ldots, v_{i_k}\} \subseteq C(s)$. Hence, $\mathcal{F}$ is a clique covering and $\mathrm{cc}(F) \leq \Theta(F)$ follows. □