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Comparison formulas for total mean curvatures of Riemannian hypersurfaces

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Abstract: We devise some differential forms after Chern to compute a family of formulas for comparing total mean curvatures of nested hypersurfaces in Riemannian manifolds. This yields a quicker proof of a recent result of the author with Joel Spruck, which had been obtained via Reilly's identities.

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1 Introduction

The *total rth mean curvature* of an oriented $C^{1,1}$ hypersurface Γ in a Riemannian n-manifold M, for $0 \leq r \leq n-1$, is given by

$$\mathcal{M}_r(\Gamma) := \int_\Gamma \sigma_r(\kappa),$$

where $\kappa := (\kappa_1, \ldots, \kappa_{n-1})$ denotes the principal curvatures of Γ, with respect to the choice of orientation, and $\sigma_r: \mathbf{R}^{n-1} \to \mathbf{R}$ is the rth symmetric function; so

$$\sigma_r(\kappa) = \sum_{1 \leq i_1 < \ldots < i_r \leq n-1} \kappa_{i_1} \ldots \kappa_{i_r}.$$

We set $\sigma_0 := 1$, and $\sigma_r := 0$ for $r \geq n$ by convention. Thus $\mathcal{M}_0(\Gamma)$ is the $(n-1)$-dimensional volume, $\mathcal{M}_1(\Gamma)$ is the total mean curvature, and $\mathcal{M}_{n-1}(\Gamma)$ is the total Gauss-Kronecker curvature of Γ. Up to multiplicative constants, these quantities form the coefficients of Steiner's polynomial, and are known as quermassintegrals when Γ is a convex hypersurface in Euclidean space. The following result was established in [1, Thm. 3.1] generalizing earlier work in [2, Thm. 4.7]:

Theorem 1.1. ([1]). *Let* M *be a compact orientable Riemannian n-manifold with boundary components* Γ_1, Γ_0. *Suppose there exists a* $C^{1,1}$ *function* $u: M \to [0,1]$ *with* $\nabla u \neq 0$ *on* M, *and* $u = i$ *on* Γ_i. *Let* $\kappa := (\kappa_1, \ldots, \kappa_{n-1})$ *be principal curvatures of level sets of u with respect to* $e_n := \nabla u / |\nabla u|$, *and let* $e_1, \ldots, e_{n-1}$ *be an orthonormal set of the corresponding principal directions. Then, for* $0 \leq r \leq n-1$,

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$$\mathcal{M}_r(\Gamma_1) - \mathcal{M}_r(\Gamma_0) = (r+1)\int_M \sigma_{r+1}(\kappa) + \int_M \Bigg(- \sum \kappa_{i_1} \dots \kappa_{i_{r-1}} K_{i_r n}$$
$$+ \frac{1}{|\nabla u|} \sum \kappa_{i_1} \dots \kappa_{i_{r-2}} |\nabla u|_{i_{r-1}} R_{i_r i_{r-1} i_r n} \Bigg), \tag{1}$$

where $|\nabla u|_i := \nabla_{e_i} |\nabla u|$, $R_{ijkl} = \langle R(e_i, e_j)e_k, e_l\rangle$ *are components of the Riemann curvature tensor of* M, $K_{ij} = R_{ijij}$ *is the sectional curvature, and the sums range over distinct values of* $1 \le i_1, \dots, i_r \le n-1$, *with* $i_1 < \dots < i_{r-1}$ *in the first sum, and* $i_1 < \dots < i_{r-2}$ *in the second sum.*

In [1], the above theorem was established via Reilly's identities [3]. Here we present a somewhat shorter and conceptually simpler proof using differential forms which we construct after Chern [4], as Borbély [5], [6] had also done earlier. More specifically, we devise a differential $(n-1)$-form Φ_r on M so that $\mathcal{M}_r(\Gamma_i)$ correspond to integration of Φ_r on Γ_i. Then computing the exterior derivative $\mathrm{d}\Phi_r$ yields (1) via Stokes theorem. Various applications of Theorem 1.1 are developed in [2], [7], including total curvature bounds, and rigidity results in Riemannian geometry. See also [1] for more results of this type.

2 Basic formulas

As in the statement of Theorem 1.1, we let M be a compact orientable Riemannian n-manifold with boundary $\partial M = \Gamma_1 \cup \Gamma_0$. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the metric on M, with induced norm $|\cdot| := \langle \cdot, \cdot \rangle^{1/2}$, connection ∇, and curvature operator

$$R(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z,$$

for vector fields X, Y, Z on M. The sectional curvature of M with respect to a pair of orthonormal vectors x, y in the tangent space T_pM may be defined as

$$K(x, y) := \langle R(X, Y)X, Y\rangle,$$

where X, Y are local extensions of x, y. With u as in the statement of Theorem 1.1, and for $0 \le t \le 1$, let $\Gamma_t := u^{-1}(t)$ be the level hypersurface of u at height t. Since u is $C^{1,1}$, Γ_t is twice differentiable almost everywhere by Rademacher's theorem. At every such point p of Γ_t, let e_i, $i = 1, \dots, n$, be the orthonormal frame mentioned above, i.e.,

$$e_n := \frac{\nabla u}{|\nabla u|},$$

and $e_1, \dots, e_{n-1}$ form a set of orthonormal principal directions of Γ_t at p. Furthermore we assume that e_i is *positively oriented*, i.e.,

$$\mathrm{dvol}_M(e_1, \dots, e_n) = 1, \tag{2}$$

where dvol_M denotes the volume form of M. We call e_i a *principal frame* associated to (level sets of) u. Let θ^i be the corresponding dual one forms on T_pM given by

$$\theta^i(e_j) = \delta^i_j, \tag{3}$$

where δ^i_j is the Kronecker function. Note that e_i may be extended to a C^1 orthonormal frame $\overline{e}_i$ in a neighborhood of p in M so that $\overline{e}_n = e_n$ and thus $\overline{e}_1, \dots, \overline{e}_{n-1}$ remain tangent to Γ_t (though they may no longer be principal directions). The corresponding connection 1-forms on T_pM are then given by

$$\omega^i_j(\cdot) := \langle \nabla_{(\cdot)} \overline{e}_j, e_i\rangle = -\langle e_j, \nabla_{(\cdot)} \overline{e}_i\rangle = -\omega^j_i(\cdot),$$

for $1 \le i, j \le n$. Since e_i, $i = 1, \dots, n-1$ are principal directions, and $\overline{e}_n = e_n$ is the normal of Γ_t,

$$\omega^i_n(e_j) = \langle \nabla_{e_j} \overline{e}_n, e_i\rangle = \delta^i_j \kappa_i, \quad 1 \le i, j \le n-1, \tag{4}$$

where κ_i are the principal curvatures of Γ_t with respect to e_n. We also record that,

$$\omega_n^i(e_n) = \frac{\langle \nabla_{e_n} \nabla u, e_i \rangle}{|\nabla u|} = \frac{\langle \nabla_{e_i} \nabla u, e_n \rangle}{|\nabla u|} = \frac{\langle \nabla_{e_i} \nabla u, \nabla u \rangle}{|\nabla u|^2} = \frac{|\nabla u|_i}{|\nabla u|}, \quad 1 \le i \le n-1, \tag{5}$$

where $|\nabla u|_i = \nabla_{e_i} |\nabla u|$, and the second equality is due to the symmetry of the Hessian of u. Next, we compute ω_i^j for $i, j \neq n$. We may assume that $\bar{e}_1, \dots, \bar{e}_{n-1}$ are parallel translations of $e_1, \dots, e_{n-1}$ on Γ_t, i.e., $\overline{\nabla}_{e_i} \bar{e}_j = 0$, for $1 \le i, j \le n-1$ where $\overline{\nabla} := \nabla^\top$ is the induced connection on Γ_t. Then $\omega_i^j(e_k) = \langle \overline{\nabla}_{e_k} \bar{e}_j, e_i \rangle = 0$, for $1 \le i, j, k \le n-1$. Furthermore, we may assume that $\bar{e}_1, \dots, \bar{e}_{n-1}$ are parallel translated along integral curves of e_n. Then $\nabla_{e_n} \bar{e}_i = 0$ for $1 \le i \le n-1$, which yields $\omega_i^j(e_n) = 0$, for $1 \le i, j \le n-1$. So we record that

$$\omega_i^j = 0, \quad 1 \le i, j \le n-1. \tag{6}$$

Cartan's structure equations state that

$$d\theta^i = \sum_{j=1}^n \theta^j \wedge \omega_j^i \qquad \text{and} \qquad d\omega_j^i = \Omega_j^i - \sum_{k=1}^n \omega_j^k \wedge \omega_k^i, \tag{7}$$

where Ω_j^i are the curvature 2-forms given by

$$\Omega_j^i(e_\ell, e_k) := -\left\langle R(e_\ell, e_k)e_j, e_i \right\rangle = \left\langle R(e_\ell, e_k)e_i, e_j \right\rangle =: R_{\ell kij}.$$

Note that $R_{\ell kij} = -R_{k\ell ij}$. We also set

$$K_{ij} := K(e_i, e_j) = R_{ijij}. \tag{8}$$

Finally we record some basic formulas from exterior algebra which will be used in the next section. If λ is a k-form, and ϕ is an ℓ-form, then

$$\lambda \wedge \phi(e_1, \dots, e_{k+\ell}) = \sum \varepsilon(i_1 \dots i_{k+\ell})\, \lambda(e_{i_1}, \dots, e_{i_k})\, \phi(e_{i_{k+1}}, \dots, e_{i_{k+\ell}}) \tag{9}$$

where the sum ranges over $1 \le i_1, \dots, i_{k+\ell} \le k+\ell$, with $i_1 < \dots < i_k$, and $i_{k+1} < \dots < i_{k+\ell}$; furthermore, $\varepsilon(i_1 \dots i_n) := 1$, or -1 depending on whether $i_1 \dots i_n$ is an even or odd permutation of $1 \dots n$ respectively. Note that

$$\varepsilon(i_1 \dots i_{r-1} n i_{r+1} \dots i_{n-1}) = (-1)^{n-1-r} \varepsilon(i_1 \dots i_{n-1}), \tag{10}$$

since $\varepsilon(i_1 \dots i_{n-1}) = \varepsilon(i_1 \dots i_{n-1} n)$. The following identities will also be useful

$$\begin{aligned} d(\theta^1 \wedge \dots \wedge \theta^k) &= \sum \varepsilon(i_1 \dots i_k)\, d\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_k} \\ &= (-1)^{k-1} \sum \varepsilon(i_1 \dots i_k)\, \theta^{i_1} \wedge \dots \wedge \theta^{i_{k-1}} \wedge d\theta^{i_k}, \end{aligned} \tag{11}$$

where the sums range over $1 \le i_1, \dots, i_k \le k$ with $i_2 < \dots < i_k$ in the first sum, and $i_1 < \dots < i_{k-1}$ in the second sum.

3 Proof of Theorem 1.1

Let θ^i be the dual 1-forms, and ω_j^i be the connection forms corresponding to the principal frame e_i of u discussed in the last section. For $0 \le r \le n-1$, we define the $(n-1)$-forms

$$\Phi_r := \sum \varepsilon(i_1 \dots i_{n-1})\, \omega_n^{i_1} \wedge \dots \wedge \omega_n^{i_r} \wedge \theta^{i_{r+1}} \wedge \dots \wedge \theta^{i_{n-1}},$$

where the sum ranges over $1 \le i_1, \dots, i_{n-1} \le n-1$ with $i_1 < \dots < i_r$, and $i_{r+1} < \dots < i_{n-1}$. For $r = n-1$, this form appears in Chern [4], and later in Borbély [6] (where it is denoted as "Φ_0" and "Φ" respectively). The form Φ_1 has also been used by Borbély in [5]. One quickly checks, using (3), (4), and (9), that

$$\Phi_r(e_1, \dots, e_{n-1}) = \sigma_r(\kappa), \tag{12}$$

which is the main feature of these forms. Recall that $\Gamma_t := u^{-1}(t)$ is the level hypersurface of u at height t, for $0 \le t \le 1$. Let $\Phi_r|_{\Gamma_t}$ denote the pull back of Φ_r via the inclusion map $\Gamma_t \to M$. Since $\Phi_r|_{\Gamma_t}$ is an $(n-1)$-form on Γ_t, it is a multiple of the volume form of Γ_t, which is given by

$$\mathrm{dvol}_{\Gamma_t}(e_1, \ldots, e_{n-1}) := \mathrm{dvol}_M(e_n, e_1, \ldots, e_{n-1}) = \varepsilon(n1 \ldots n-1) = (-1)^{n-1}. \tag{13}$$

Note that here we have used the assumption (2) that e_i is positively oriented. So it follows from (12) and (13) that

$$\Phi_r|_{\Gamma_t} = (-1)^{n-1}\sigma_r(\kappa)\, d\mathrm{vol}_{\Gamma_t}. \tag{14}$$

This shows that Φ_r depends only on e_n, not the choice of $e_1, \ldots, e_{n-1}$ (which also follows from transformation rules for ω_n^i and θ^i under a change of frame $e_i \to e_i'$ with $e_n = e_n'$; see [5, p. 269]). In addition, (14) shows that

$$\mathcal{M}_r(\Gamma_t) = \int\limits_{\Gamma_t} \sigma_r(\kappa) := \int\limits_{\Gamma_t} \sigma_r(\kappa)\, d\mathrm{vol}_{\Gamma_t} = (-1)^{n-1} \int\limits_{\Gamma_t} \Phi_r.$$

Consequently, by Stokes theorem, for the left hand side of (1) we have

$$\mathcal{M}_r(\Gamma_1) - \mathcal{M}_r(\Gamma_0) = (-1)^{n-1} \int\limits_{\partial M} \Phi_r = (-1)^{n-1} \int\limits_{M} d\Phi_r. \tag{15}$$

Here we have used the assumption that $u|_{\Gamma_1} > u|_{\Gamma_0}$, which ensures that e_n points outward on Γ_1 and inward on Γ_0 with respect to M. Furthermore, since Φ_r depends only on e_n and u is $C^{1,1}$, it follows that Φ_r is Lipschitz (in local coordinates). Hence $\mathrm{d}\Phi_r$ is integrable, and the use of Stokes theorem here is justified.

Next we compute $\mathrm{d}\Phi_r$. Since $\omega_n^{i_1} \wedge \ldots \wedge \omega_n^{i_r}$ is an r-form, the product rule for exterior differentiation yields that

$$\begin{aligned} \mathrm{d}\Phi_r = (-1)^r \sum \varepsilon(i_1 \ldots i_{n-1})\, \omega_n^{i_1} \wedge \ldots \wedge \omega_n^{i_r} \wedge \mathrm{d}\big(\theta^{i_{r+1}} \wedge \ldots \wedge \theta^{i_{n-1}}\big) \\ + \sum \varepsilon(i_1 \ldots i_{n-1})\, d\Big(\omega_n^{i_1} \wedge \ldots \wedge \omega_n^{i_r}\Big) \wedge \theta^{i_{r+1}} \wedge \ldots \wedge \theta^{i_{n-1}}, \end{aligned} \tag{16}$$

where the sums still range over $i_1 < \ldots < i_r$ and $i_{r+1} < \ldots < i_{n-1}$. By (11), the structure equations (7), and (6), the first term in (16) reduces to

$$\begin{aligned} &(-1)^{r+1} \sum \varepsilon(i_1 \ldots i_{n-1})\, \omega_n^{i_1} \wedge \ldots \wedge \omega_n^{i_r} \wedge \omega_n^{i_{r+1}} \wedge \theta^n \wedge \theta^{i_{r+2}} \wedge \ldots \wedge \theta^{i_{n-1}} \\ &= (-1)^{n-1} \sum \varepsilon(i_1 \ldots i_{n-1})\, \omega_n^{i_1} \wedge \ldots \wedge \omega_n^{i_r} \wedge \omega_n^{i_{r+1}} \wedge \theta^{i_{r+2}} \wedge \ldots \wedge \theta^{i_{n-1}} \wedge \theta^n \\ &= (-1)^{n-1}(r+1)\Phi_{r+1} \wedge \theta^n, \end{aligned}$$

where the sums now range over $i_1 < \ldots < i_r$, and $i_{r+2} < \ldots < i_{n-1}$. The factor $(r+1)$ appears in the last line because definition of Φ_{r+1} requires that $i_1 < \ldots < i_{r+1}$. Applying (11) and (7) also to the second term in (16), we obtain

$$\begin{aligned} \mathrm{d}\Phi_r = (-1)^{n-1}(r+1)\Phi_{r+1} \wedge \theta^n + (-1)^{r-1} \sum \varepsilon(i_1 \ldots i_{n-1})\, \omega_n^{i_1} \wedge \\ \ldots \wedge \omega_n^{i_{r-1}} \wedge \Omega_n^{i_r} \wedge \theta^{i_{r+1}} \wedge \ldots \wedge \theta^{i_{n-1}}, \end{aligned} \tag{17}$$

where the sum ranges over $i_1 < \ldots < i_{r-1}$, and $i_{r+1} < \ldots < i_{n-1}$. For $r = 1$, this formula had been computed earlier by Borbély [5, (6)].

By (15), it remains to show that $(-1)^{n-1}\int_M \mathrm{d}\Phi_r$ yields the right hand side of (1). To see this first note that, by (9) and (12),

$$\Phi_{r+1} \wedge \theta^n = \Phi_{r+1} \wedge \theta^n(e_1, \ldots, e_n)\, \mathrm{dvol}_M = \sigma_{r+1}(\kappa)\, d\mathrm{vol}_M.$$

Thus the first term on the right hand side of (17) quickly yields the first integral on the right hand side of (1). To obtain the second integral there, we evaluate the sum in (17) at e_i, which yields

$$\begin{aligned}&\sum \varepsilon(j_1 \dots j_n)\,\varepsilon(i_1 \dots i_{n-1})\,\omega_n^{i_1}(e_{j_1}) \dots \omega_n^{i_{r-1}}(e_{j_{r-1}})\Omega_n^{i_r}(e_{j_r}, e_{j_{r+1}})\theta^{i_{r+1}}(e_{j_{r+2}}) \dots \theta^{i_{n-1}}(e_{j_n})\\ &\quad = \sum \varepsilon(j_1 \dots j_{r+1} i_{r+1} \dots i_{n-1})\,\varepsilon(i_1 \dots i_{n-1})\,\omega_n^{i_1}(e_{j_1}) \dots \omega_n^{i_{r-1}}(e_{j_{r-1}})R_{j_r j_{r+1} i_r n},\end{aligned}$$

where the sums range over $1 \le j_1 \ \dots \ j_n \le n$ with $j_r < j_{r+1}$ by (9), and the range for $1 \le i_1, \dots, i_{n-1} \le n-1$ remains as in (17), i.e., $i_1 < \dots < i_{r-1}$, and $i_{r+1} < \dots < i_{n-1}$. The last sum may be partitioned into $A + B$, where A consists of terms with $j_{r+1} = n$, and B of terms with $j_{r+1} \ne n$. If $j_{r+1} = n$, then $j_1, \dots, j_{r-1} \ne n$, which yields $i_k = j_{k+1}$ for $k = 1, \dots, r-2$ by (4). This in turn forces $j_r = i_r$, as they are the only remaining indices. So by (10) and (8),

$$\begin{aligned}A &= \sum \varepsilon(i_1 \dots i_{r-1} n i_{r+1} \dots i_{n-1})\,\varepsilon(i_1 \dots i_{n-1})\,\kappa_{i_1} \dots \kappa_{i_{r-1}} R_{i_r n i_r n}\\ &= (-1)^{n-r-1} \sum \kappa_{i_1} \dots \kappa_{i_{r-1}} K_{i_r n},\end{aligned}$$

where we still have $i_1 < \dots < i_{r-1}$. This yields the first term in the second integral in (1), after multiplication by the sign factors $(-1)^{r-1}$ from (17) and $(-1)^{n-1}$ from (15), which ensures the desired sign -1. Next, to compute B, note that if $j_{r+1} \ne n$, then $j_r \ne n$ either, since $j_r < j_{r+1}$, which forces $j_k = n$, for some $1 \le k \le r-1$. We may assume $k = r-1$ after reindexing. Then $j_1, \dots, j_{r-2} \ne n$, which yields $i_k = j_k$ for $k = 1, \dots, r-2$ by (4). So by (5)

$$\begin{aligned}B &= \sum \varepsilon(i_1 \dots i_{r-2} n j_r j_{r+1} i_{r+1} \dots i_{n-1})\,\varepsilon(i_1 \dots i_{n-1})\,\kappa_{i_1} \dots \kappa_{i_{r-2}} \frac{|\nabla u|_{i_{r-1}}}{|\nabla u|} R_{j_r j_{r+1} i_r n}\\ &= \sum \varepsilon(i_1 \dots i_{r-2} n i_{r-1} \dots i_{n-1})\,\varepsilon(i_1 \dots i_{n-1})\,\kappa_{i_1} \dots \kappa_{i_{r-2}} \frac{|\nabla u|_{i_{r-1}}}{|\nabla u|} R_{i_{r-1} i_r i_r n}\\ &= (-1)^{n-r} \sum \kappa_{i_1} \dots \kappa_{i_{r-2}} \frac{|\nabla u|_{i_{r-1}}}{|\nabla u|} R_{i_r i_{r-1} i_r n},\end{aligned}$$

where the second equality holds because $\{j_r, j_{r+1}\} = \{i_{r-1}, i_r\}$, since these are the only remaining indices. We may assume then that $j_r = i_{r-1}$, and $j_{r+1} = i_r$, since switching j_r and j_{r+1} does not change the sign of the right hand side of the first equality for B. The sign $(-1)^{n-r}$ in the third equality is due to (10) and switching two indices in the Riemann tensor coefficient. Finally note that the restriction on the range of indices in the last sum is now $i_1 < \dots < i_{r-2}$, since i_{r-1} corresponds to j_{r-1}, and we set $r-1 = k$ during the reindexing above. So B yields the second term in the second integral in (1), after multiplication by $(-1)^{r-1}$ and $(-1)^{n-1}$, as was the case for A, which ensures the desired sign $+1$. This concludes the proof of Theorem 1.1.

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