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# Structured singular value of a repeated complex full-block uncertainty

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## Abstract

The structured singular value (SSV), or  $\mu$ , is used to assess the robust stability and performance of an uncertain linear time-invariant system. Existing algorithms compute upper and lower bounds on the SSV for structured uncertainties that contain repeated (real or complex) scalars and/or nonrepeated complex full-blocks. This paper presents algorithms to compute bounds on the SSV for the case of repeated complex full-blocks. This specific class of uncertainty is relevant for the input-output analysis of many convective systems, such as fluid flows. Specifically, we present a power iteration to compute the SSV lower bound for the case of repeated complex full-blocks. This generalizes existing power iterations for repeated complex scalars and nonrepeated complex full-blocks. The upper bound can be formulated as a semi-definite program (SDP), which we solve using a standard interior-point method to compute optimal scaling matrices associated with the repeated full-blocks. Our implementation of the method only requires gradient information, which improves the computational efficiency of the method. Finally, we test our proposed algorithms on an example model of incompressible fluid flow. The proposed methods provide less conservative bounds as compared to prior results, which ignore the repeated full-block structure.

## KEYWORDS

method of centers, repeated complex full-blocks, structured singular value, structured uncertainty

## 1 | INTRODUCTION

The structured singular value (SSV), or  $\mu$ , is a useful metric for assessing the robust stability and performance of an uncertain linear time-invariant system with a structured uncertainty.<sup>1–3</sup> The SSV is inversely related to the smallest structured uncertainty that destabilizes the uncertain system. Roughly, the SSV is the “gain” of the system with respect to the structured uncertainty and its inverse provides a stability margin.<sup>4,5</sup> It is known that exactly computing the SSV is NP hard.<sup>6,7</sup> Thus, it is a common practice to instead compute upper and lower bounds on the SSV. The upper bound provides a sufficient condition for robust stability and the lower bound for instability, respectively.<sup>2,4,5,8,9</sup> However, for some specific uncertainty structures, as noted in prior works,<sup>2,10,11</sup> the convex upper bound equals the SSV. Thus, for these cases, the exact SSV can be computed through the convex upper bound.

**Abbreviations:** I/O, input-output; PCF, plane Couette flow; SSV, structured singular value.

Much of the previous work has focused on structured uncertainties with a mixture of repeated (real or complex) scalars and/or nonrepeated complex full-block uncertainties (see Section 2).<sup>2,4,5</sup> For these common uncertainty structures, one can use the methods described in prior works to compute the upper and lower bound.<sup>8,9,12,13</sup> The current paper focuses on a new uncertainty structure: repeated complex full-blocks. This particular class of uncertainties consists of a single complex full-block repeated multiple times. This repeated structure naturally arises in fluid dynamics and other convective systems. Recently, SSV has emerged as a means of performing a structured input-output analysis of transitional shear flows to study instability mechanisms.<sup>14–17</sup> However, Liu et al.<sup>14–16</sup> utilize MATLAB's Robust Control Toolbox, which does not allow for repeated full-blocks. The only cases handled by MATLAB are nonrepeated, complex full-blocks and the repeated (real or complex) scalars.<sup>18</sup> Therefore, the numerical results in these works<sup>14–16</sup> replace the repeated complex full-block structure with a nonrepeated one, which yields conservative SSV bounds. In addition to conservatism in the bounds, accounting for the repeated uncertainty structure is important for revealing physical instability mechanisms, as will become clear in the results we present later.

In this paper, we present algorithms to compute upper and lower bounds on the SSV for a repeated complex full-block uncertainty (see Sections 3 and 4). The upper bound is computed using an interior point algorithm known as the method of centers.<sup>19,20</sup> Our implementation only uses gradient (and not Hessian) information. This improves computational efficiency, which is important for any large dimensioned system, such as the fluid flow example presented in our paper. The lower bound is computed by generalizing the existing power iteration algorithm described by Packard et al.<sup>2,13</sup> We demonstrate the proposed algorithms on the plane Couette flow model<sup>14</sup> and a simple academic example. Furthermore, we compare the SSV bounds computed from the proposed algorithms with existing methods that approximate the repeated structure with a nonrepeating one. We show that the proposed algorithms not only reduce the conservatism of the bounds but also highlight the importance of incorporating the correct uncertainty structure for interpreting the underlying physical system/phenomena (see Section 5).

The symbols  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{R}^n$ ,  $\mathbb{C}^n$ , and  $\mathbb{C}^{n \times m}$  denote the sets of real numbers, complex numbers, real vectors of dimension  $n$ , complex vectors of dimension  $n$  and complex matrices of dimension  $n \times m$ , respectively. The  $n \times n$  identity and zero matrices are denoted by  $I_n$  and  $0_n$ , respectively.  $M^H$  and  $\bar{\sigma}(M)$  are the Hermitian transpose and maximum singular value of a matrix  $M \in \mathbb{C}^{n \times m}$ . We use  $\|\cdot\|_2$  to denote the 2-norm for vectors and the induced 2-to-2 norm for matrices. Note that  $\|\cdot\|_2 = \bar{\sigma}(\cdot)$  for matrices. Also,  $\|\cdot\|_F$  denotes the Frobenius norm. For  $M \in \mathbb{C}^{n \times n}$ ,  $\text{Tr}(M)$  and  $\rho(M)$  are the trace and spectral radius. The notations  $\otimes$  and  $\text{diag}(\cdot)$  denote the Kronecker product and block diagonal matrices, respectively. The imaginary unit is denoted as  $i = \sqrt{-1}$ . For  $c \in \mathbb{C}$ ,  $\text{Re}(c)$ ,  $\text{Im}(c)$  and  $\text{conj}(c)$  denote the real and imaginary parts of  $c$ , and the complex conjugate of  $c$ , respectively.

## 2 | BACKGROUND: STRUCTURED SINGULAR VALUE, $\mu$

We briefly review the structured singular value  $\mu$  and its connection to robust stability of dynamical systems.<sup>1,2,4,21</sup> First consider the case for matrices. Specifically, let  $M \in \mathbb{C}^{n \times m}$  be given along with a set of (possibly structured) complex matrices  $\Delta \subseteq \mathbb{C}^{m \times n}$ .

**Definition 1.** The structured singular value,  $\mu_\Delta$ , is defined as

$$\mu_\Delta(M) := \frac{1}{\min(\bar{\sigma}(\Delta) : \Delta \in \Delta, \det(I_n - M\Delta) = 0)}. \quad (1)$$

If there does not exist  $\Delta \in \Delta$  such that  $\det(I_n - M\Delta) = 0$ , then define  $\mu_\Delta(M) = 0$ .

Note that  $\mu_\Delta(M)$  depends on both the matrix  $M$  and the set of matrices  $\Delta$ . We will typically omit the subscript  $\Delta$  for simplicity when the set of matrices is clear.

The SSV is inversely related to the smallest  $\Delta \in \Delta$  that causes  $I_n - M\Delta$  to be singular. Singularity means there exists a nonzero vector  $y \in \mathbb{C}^n$  such that  $y = M\Delta y$ . This is equivalent to the existence of nonzero vectors  $u \in \mathbb{C}^m$  and  $y \in \mathbb{C}^n$  such that  $y = Mu$  and  $u = \Delta y$ , which provides a feedback interpretation of  $\mu_\Delta(M)$  (see Remark 3.4 in Packard and Doyle<sup>2</sup>). Furthermore, the SSV simplifies in two special cases:<sup>2</sup>

- (i)  $\mu(M) = \bar{\sigma}(M)$  for full-block uncertainties,  $\Delta = \mathbb{C}^{m \times n}$ ,
- (ii)  $\mu(M) = \rho(M)$  for repeated scalar uncertainties  $\Delta = \{\delta I_v : \delta \in \mathbb{C}\}$ , where  $n, m = v$ .

There are many known results for structured uncertainties  $\Delta$  that contain block-diagonal concatenation of any number of full-blocks and repeated scalars.<sup>1,2,4,21</sup> It is worth noting that if  $\Delta_1 \subset \Delta_2$  then

$$\mu_{\Delta_1}(M) \leq \mu_{\Delta_2}(M). \quad (2)$$

This follows from the definition of the SSV in (1). This yields the following bound for any matrix  $M$  and block structure  $\Delta \subseteq \mathbb{C}^{m \times n}$ :

$$\mu_{\Delta}(M) \leq \bar{\sigma}(M). \quad (3)$$

Next, consider the case for LTI systems. Specifically, let  $M(s)$  be a transfer function matrix of a multiple-input and multiple-output (MIMO) LTI system and  $\Delta$  be a set of structured LTI uncertainties. The SSV can be used to assess robustness of a feedback loop involving  $M(s)$  and  $\Delta(s)$ . In particular, assume the feedback loop is nominally stable, that is, stable for  $\Delta(s) = 0$ . Define the set of bounded, structured uncertainties as  $\mathbb{B}_{\Delta} := \{\Delta(s) \in \Delta : \|\Delta\|_{\infty} \leq 1\}$ . Then, the feedback loop is stable for all  $\Delta \in \mathbb{B}_{\Delta}$  if and only if  $\max_{\omega} \mu(M(i\omega)) < 1$ , where  $\omega$  is the temporal frequency.<sup>1,2,4,21</sup> This is an adaptation of the small-gain condition for the set of structured uncertainties  $\mathbb{B}_{\Delta}$ . The SSV computations for LTI systems are often reduced to the SSV computations for a complex matrix  $M(i\omega)$  on a grid of frequencies.

This paper contributes methods that can be used to compute the SSV for repeated full-block uncertainty

$$\Delta_r := \{\Delta = I_v \otimes \Delta_1 : \Delta_1 \in \mathbb{C}^{m_1 \times m_1}\} \subset \mathbb{C}^{m \times m}, \quad (4)$$

where  $m = vm_1$ . Thus,  $v = 2$  represents the same full-block uncertainty  $\Delta_1$  repeated twice:  $I_2 \otimes \Delta_1 = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_1 \end{bmatrix}$ . The block  $\Delta_1$  is restricted to be square, as is common in the SSV literature, to simplify the presentation. The extension to nonsquare blocks can be made with mainly notational changes. We discuss algorithms in the subsequent sections that compute upper and lower bounds on the  $\mu(M)$  for the uncertainty structure in (4).

### 3 | UPPER BOUND OF STRUCTURED SINGULAR VALUE

This section describes an algorithm that computes an upper bound on  $\mu$  for the uncertainty structure defined in (4). We will describe the upper bound algorithm for the matrix case  $M \in \mathbb{C}^{m \times m}$ . We start by first noting that for each set of uncertainties  $\Delta$ , there is a set of nonsingular “commuting” matrices  $\mathbf{D}$  with the property that  $D\Delta = \Delta D$  for any  $\Delta \in \Delta$  and  $D \in \mathbf{D}$ . For example, the set of  $v$  nonrepeated full-blocks, denoted  $\Delta_{nr} \subset \mathbb{C}^{m \times m}$ , and its corresponding commuting matrices are

$$\Delta_{nr} := \{\Delta = \text{diag}(\Delta_1, \dots, \Delta_v) : \Delta_i \in \mathbb{C}^{m_i \times m_i}\}, \quad (5)$$

$$\mathbf{D}_{nr} := \{\text{diag}(d_1 I_{m_1}, \dots, d_v I_{m_v}) : d_i \in \mathbb{R}, d_i \neq 0\}. \quad (6)$$

The commuting matrices are diagonal when the uncertainty set is nonrepeated. For the repeated full-block structure in (4), the commuting matrices have the following structure:

$$\mathbf{D}_r := \{S \otimes I_{m_1} : S \in \mathbb{C}^{v \times v}, \det(S) \neq 0\}. \quad (7)$$

These commuting matrices are important because  $\det(I - M\Delta) = \det(I - DMD^{-1}\Delta)$ . Thus,  $\mu_{\Delta}(M) = \mu_{\Delta}(DMD^{-1})$ . We can use this to strengthen the upper bound in (3):

$$\mu_{\Delta}(M) \leq \min_{D \in \mathbf{D}} \bar{\sigma}(DMD^{-1}). \quad (8)$$

This is known as the  $D$ -scale upper bound. By setting  $X = D^H D$ , the optimization on the right hand side of (8) can be converted into a semi-definite program (technically a generalized eigenvalue problem) as follows:<sup>2,4</sup>

$$\begin{aligned} & \min_{X=X^H \in \mathbb{C}^{m \times m}, \zeta \in \mathbb{R}} \zeta \\ & \text{subject to: } M^H X M < \zeta X, X > 0. \end{aligned} \quad (9)$$

Then, the upper bound is computed as  $\alpha = (\zeta)^{1/2}$  and the corresponding scale as  $D = X^{1/2}$ . Therefore, there is an implicit constraint that  $\zeta \geq 0$ , which arises naturally during the derivation of constraints in (9) (see Packard and Doyle<sup>2</sup> for details). The optimization problem (9) can be solved using several existing methods such as method of centers, interior-point methods for linear fractional programming, and primal-dual methods.<sup>20,22,23</sup> These methods are efficient for moderate-sized problems but can be computationally costly for larger dimensioned problems. Specifically, primal-dual methods tend to be slower because they require second-order schemes to solve (9). Certainly, there are faster algorithms that utilize a weaker bound, that is,  $\bar{\sigma}(DMD^{-1}) \leq \|DMD^{-1}\|_F$ , which is often sufficient for most large-dimensioned problems. In this case, an upper bound for a given matrix  $M$  becomes

$$\mu_{\Delta}(M) \leq \min_{D \in \mathbf{D}} \|DMD^{-1}\|_F. \quad (10)$$

See Appendix A for a fast algorithm for computing an upper bound of the form (10) for  $D \in \mathbf{D}_r$ , that is, the repeated full-blocks case. However, using a weaker bound yields conservative estimates of the upper bounds, which can result in large gaps between upper and lower bounds. The goal of this paper is to present an efficient algorithm that would yield the least conservative upper bounds for  $\Delta \in \mathbf{\Delta}_r$ . Thus, we will implement the method of centers for upper bound calculations, since it is a relatively fast first-order method with good convergence properties.<sup>19</sup> First, we will briefly summarize an existing upper bound method for the uncertainty structure  $\mathbf{\Delta}_{nr}$ , which we will use later to compare with the upper bounds obtained for  $\mathbf{\Delta}_r$ .

### 3.1 | Standard Osborne's method: Nonrepeated complex full-blocks

Osborne's iteration can be used to efficiently solve the optimization problem in the right-hand side of (10) for specific block structures.<sup>24</sup> For example, a version of Osborne's iteration can be applied to the structure  $\mathbf{\Delta}_{nr}$  with scalings  $\mathbf{D}_{nr}$ . Let  $D_i \in \mathbf{D}_{nr}$  denote a scaling with  $d_j = 1$  for all  $j \neq i$ . For example, if  $i = 1$  then  $d_1$  is a variable and  $d_j = 1$  for  $j \neq 1$ . In addition, partition  $M$  into  $m_i \times m_j$  sub-blocks, denoted  $\hat{M}_{ij}$ , consistent with the block dimensions in  $\mathbf{\Delta}_{nr}$ . Then, the Frobenius norm can be written as

$$\|D_i M D_i^{-1}\|_F^2 = \sum_{r=1, r \neq i}^v \frac{1}{d_i^2} \|\hat{M}_{ri}\|_F^2 + d_i^2 \|\hat{M}_{ir}\|_F^2. \quad (11)$$

The optimal value  $d_i^*$  that minimizes (11) is given by

$$d_i^* = \left( \frac{\sum_{r=1, r \neq i}^v \|\hat{M}_{ri}\|_F^2}{\sum_{r=1, r \neq i}^v \|\hat{M}_{ir}\|_F^2} \right)^{1/4}. \quad (12)$$

Each  $d_i^*$  is computed from (12) using  $M$  and the corresponding matrix  $D^*$  is determined. Then, the cost is obtained as  $\|M^{[2]}\|_F^2$ , where  $M^{[2]} = D^* M D^{*-1}$ . The new  $D$ -scale is then computed from  $M^{[2]}$  and the corresponding new cost is determined. Thus, the iteration proceeds by updating the matrix as  $M^{[k]} = (D^*)^{[k]} M (D^{*-1})^{[k]}$  and computing the corresponding  $(D^*)^{[k]}$  until  $\|M^{[k]}\|_F^2$  has converged. The final  $D$ -scale is denoted by  $D_{nr}^*$  after all the iterations. Osborne showed that the iterative method always converges to the optimal solution of  $\min_{D \in \mathbf{D}} \|DMD^{-1}\|_F$  for the uncertainty  $\mathbf{\Delta}_{nr}$  with  $m_i = 1$ .<sup>24</sup>

### 3.2 | Method of centers: Repeated complex full-blocks

In this section, we discuss the method of centers approach for solving the generalized eigenvalue problem (9) for the case when  $\Delta \in \mathbf{\Delta}_r$  and, consequently,  $D \in \mathbf{D}_r$ . In this case, we have  $X = (S \otimes I_{m_1})^H (S \otimes I_{m_1}) = S^H S \otimes I_{m_1} = R \otimes I_{m_1}$ , where  $R := S^H S$ . Therefore, the generalized eigenvalue problem (GEVP) in (9) becomes

$$\begin{aligned} & \min_{R=R^H \in \mathbb{C}^{v \times v}, \zeta \in \mathbb{R}} \zeta \\ & \text{subject to: } M^H (R \otimes I_{m_1}) M < \zeta (R \otimes I_{m_1}), \quad R > 0. \end{aligned} \quad (13)$$

Since a feasible  $R$  for (13) is scale-invariant (i.e., for a feasible  $R$ , any  $cR$  with  $c > 0$  is also feasible), we will replace the  $R > 0$  constraint in (13) with  $\frac{1}{\gamma}I_v \leq R \leq \gamma I_v$  to prevent solutions from becoming ill-conditioned, where  $\gamma > 0$  and  $\gamma^2$  is the (specified) condition number of  $R$ . Therefore, we numerically implement the following GEVP:

$$\begin{aligned} & \min_{R=R^H \in \mathbb{C}^{v \times v}, \zeta \in \mathbb{R}} \zeta \\ & \text{subject to: } M^H(R \otimes I_{m_1})M < \zeta(R \otimes I_{m_1}), \\ & \quad \frac{1}{\gamma}I_v \leq R \leq \gamma I_v. \end{aligned} \quad (14)$$

The method of centers is an interior-point algorithm that solves for the analytic center of linear matrix inequality (LMI) constraints, given an initial feasible solution.<sup>19,20</sup> Specifically in (14), we are minimizing the largest generalized eigenvalue  $\zeta$  of the matrix pair  $(M^H(R \otimes I_{m_1})M, (R \otimes I_{m_1}))$ . The algorithm utilizes a gradient descent approach, which involves computing the stepping direction towards an optimal  $R$  and the smallest  $\zeta \geq 0$  satisfying the LMI constraints. To this end, the directional derivative is computed using a barrier-function for symmetric positive semi-definite matrices, that is,  $J(R) = -\log \det(R)$ .

Next, we will compute the derivative of  $J(R)$ . Let  $r_{ij} \in \mathbb{C}$  denote the  $(i, j)$  entry of  $R$ . Since  $R$  is Hermitian, the diagonal entries are real, that is,  $r_{ii} \in \mathbb{R}$ . Note that the derivative of the barrier function is calculated with respect to the real and imaginary parts of each  $(i, j)$  element of  $R$ . Therefore, each matrix variable in (14) is decomposed as a summation in terms of its basis as  $R = \sum_{i,j} r_{ij} R_{ij}$ , where  $R_{ij}$  is the standard basis for  $\mathbb{R}^{v \times v}$ . Then, the barrier function and its derivative with respect to  $r_{ij}$  are given by

$$J(R) = -\log \det(L_1) - \log \det(L_2) - \log \det(L_3), \quad (15)$$

$$\begin{aligned} \frac{\partial J(R)}{\partial r_{ij}} &= -\zeta \text{Tr}((R_{ij} \otimes I_{m_1})^T L_1^{-1}) \\ &\quad + \text{Tr}((R_{ij} \otimes I_{m_1})^T M L_1^{-1} M^H) \\ &\quad + \text{Tr}(R_{ij}^T L_2^{-1}) - \text{Tr}(R_{ij}^T L_3^{-1}), \end{aligned} \quad (16)$$

where  $L_1 = \zeta(R \otimes I_{m_1}) - M^H(R \otimes I_{m_1})M$ ,  $L_2 = \gamma I_v - R$  and  $L_3 = R - \frac{1}{\gamma}I_v$ . To further simplify the expression in (16), it will be useful to block partition a given matrix  $H \in \mathbb{C}^{m \times m}$ , where  $(H)_{ij} \in \mathbb{C}^{m_1 \times m_1}$  denotes the  $(i, j)$  block for all  $i, j = 1, \dots, v$ .<sup>\*</sup> Thus,  $\text{Tr}((R_{11} \otimes I_{m_1})^T L_1^{-1}) = \text{Tr}((L_1^{-1})_{11})$ , which can be generalized to any  $(i, j)$ , that is,  $\text{Tr}((R_{ij} \otimes I_{m_1})^T L_1^{-1}) = \text{Tr}((L_1^{-1})_{ij})$ . The other terms in (16) can be simplified in a similar manner and we eventually obtain the following expression:

$$\begin{aligned} \frac{\partial J(R)}{\partial r_{ij}} &= -\zeta \text{Tr}((L_1^{-1})_{ij}) + \text{Tr}((M L_1^{-1} M^H)_{ij}) \\ &\quad + (L_2^{-1})_{ij} - (L_3^{-1})_{ij}. \end{aligned}$$

Thus, the derivative  $\Phi_R := \partial J / \partial R$  can be expressed as

$$\Phi_R = -\zeta \Gamma(L_1^{-1}) + \Gamma(M L_1^{-1} M^H) + L_2^{-1} - L_3^{-1},$$

where  $\Gamma : \mathbb{C}^{m \times m} \rightarrow \mathbb{C}^{v \times v}$  is a block-trace operator such that the  $(i, j)$  entry of  $\Gamma(H)$  is equal to  $\text{Tr}((H)_{ij})$ . An overall summary of the proposed method for upper bound calculation using the method of centers is provided in Algorithm 1.

It is possible to compute the convergence properties of the algorithm using a second-order primal dual method, which utilizes the Hessian of the LMIs. However, second-order methods are computationally expensive, especially when the system has a large dimension. For practical purposes, it is computationally efficient to first calculate the lower bounds  $\beta$  using the power-iteration (see Section 4 for details) and then compute the upper bounds  $\alpha$ . Despite the inherent convergence issues of the power-iteration,<sup>8,13</sup> it is always possible to obtain a perturbation, which would result in a valid lower bound of SSV. Then, the gap between the upper and lower bound can be studied to assess the viability of the solution. Therefore, we terminate our algorithm when the upper bounds  $\alpha$  are within a certain desired ratio of the lower bounds  $\beta$ , that is,  $\frac{\alpha}{\beta} \leq p$ , where  $p > 1$  is the chosen bound of the ratio. For example, we can choose  $p = 1.05$  as the desired ratio for our algorithm to get the bounds within 5% of one another. It is important to note that for the cases where the upper

**Algorithm 1.** Upper bound: Method of centers

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1: (Initialization) Choose any feasible  $\theta \ll 1$ ,  $\epsilon \ll 1$  and  $r_{\text{cond}} > 0$ . Set  $R = \text{diag}((d_1^*)^2, \dots, (d_v^*)^2)$ ,  $\alpha = \bar{\sigma}((R \otimes I_{m_1})^{1/2} M (R \otimes I_{m_1})^{-1/2})$  and  $\lambda = \alpha + \epsilon$ . Choose a suitable  $p > 1$  and maximum number of iterations  $k_m$ .
2: while  $\frac{\alpha}{\beta} \geq p$  &  $k \leq k_m$  do
3:   Set  $\lambda = (1 - \theta)\alpha + \theta\lambda$  and  $l = 1$ .
4:   while  $l \leq 2$  do
5:      $L_1 = \lambda^2(R \otimes I_{m_1}) - M^H(R \otimes I_{m_1})M$ ,  $L_2 = \gamma I_v - R$  and  $L_3 = R - \frac{1}{\gamma} I_v$ .
6:      $\Phi_R = \Gamma(M L_1^{-1} M^H) - \lambda^2 \Gamma(L_1^{-1}) + L_2^{-1} - L_3^{-1}$ .
7:     Obtain the step-size  $\delta$  through a line search.
8:     Set  $R = R - \delta \Phi_R$ ,  $l = l + 1$ .
9:   end while
10:  Set  $D = (R \otimes I_{m_1})^{1/2}$ ,  $k = k + 1$ .
11:  Then,  $\alpha = \sqrt{\lambda_{\max}(D^{-H} M^H (R \otimes I_{m_1}) M D^{-1})}$ .
12: end while
13: The upper bound:  $\alpha$ 

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bounds fail to satisfy  $p$ , we take the next best upper bound that will result in a ratio closest to  $p$ . Certainly, if the gap is too large, for example,  $2p < \frac{\alpha}{\beta}$ , then either the lower bound has not converged or possibly the upper bound is not exact. Additionally, a simple initial estimate of  $R$  for Algorithm 1 is  $R = \text{diag}((d_1^*)^2, \dots, (d_v^*)^2)$ , where  $d_i^*$  is computed from the Osborne's iteration, which we will use in Section 5 for the results.

It should be noted that a variant of Algorithm 1 can be conceived for  $\Delta \in \Delta_{\text{nr}}$  by restricting  $R$  to be diagonal with real entries.

## 4 | LOWER BOUND OF STRUCTURED SINGULAR VALUE

In this section, we give details on the computation of SSV lower bound for  $\Delta \in \Delta_r$  using the generalized power iteration algorithm. The algorithm follows the same steps as the standard power iteration commonly used for complex uncertainties given in Packard and Doyle<sup>2</sup> but with slightly modified equations. We will show that the generalized version reduces to the standard algorithm for the commonly used complex uncertainties as a special case. Thus, the standard power iteration for the repeated scalars and full-block uncertainties is described first so the extension to the generalized version will be clear.

### 4.1 | Standard power iteration: Repeated scalars and full blocks

This section briefly summarizes the SSV power iterations for complex uncertainties described in Packard and Doyle.<sup>2</sup> We will consider a problem with a given  $M \in \mathbb{C}^{m \times m}$  and a block structure with one repeated scalar and one full-block:

$$\Delta := \left\{ \Delta = \begin{bmatrix} \delta_1 I_{m_1} & 0 \\ 0 & \Delta_2 \end{bmatrix} : \delta_1 \in \mathbb{C}, \Delta_2 \in \mathbb{C}^{m_2 \times m_2} \right\},$$

where, for consistency among the dimensions, we have  $m = m_1 + m_2$ . The power iteration will be described for this particular block structure. The generalization to other uncertainty block structures with arbitrary numbers of repeated scalars or full-blocks will be clear.

Note that any particular  $\Delta \in \Delta$  such that  $\det(I_n - M\Delta) = 0$  yields a lower bound  $\mu(M) \geq \frac{1}{\bar{\sigma}(\Delta)}$ . The exact value of  $\mu(M)$  is computed by finding the “smallest”  $\Delta \in \Delta$  such that  $\det(I_n - M\Delta) = 0$ . The determinant condition is equivalent to finding  $\Delta \in \Delta$  and nonzero vectors  $y \in \mathbb{C}^m$  and  $u \in \mathbb{C}^m$  such that  $y = Mu$  and  $u = \Delta y$ . The power iteration is an efficient method to find uncertainties  $\Delta \in \Delta$  that satisfy the determinant condition. The power iteration does not, in general, find the smallest uncertainty and hence it only yields a lower bound on  $\mu(M)$ . However, these lower bounds are often

**Algorithm 2.** Lower bound: power iteration

- 1: (Initialization) Choose the number of iterations  $k_m$  and set  $k = 0$ . Select some unit-norm vectors  $b^{[0]}, w^{[0]} \in \mathbb{C}^m$  and  $a^{[0]} = z^{[0]} = 0 \in \mathbb{C}^m$ .
- 2: **while**  $k < k_m$  **do**
- 3:   (17a):  $\beta := \|Mb^{[k]}\|_2$  and  $a^{[k+1]} := Mb^{[k]}/\beta$ .
- 4:   (17b): Use  $(a^{[k+1]}, w^{[k]})$  to compute  $z^{[k+1]}$ .
- 5:   (17c):  $\beta := \|M^H z^{[k+1]}\|_2$  and  $w^{[k+1]} := M^H z^{[k+1]}/\beta$ .
- 6:   (17d): Use  $(a^{[k+1]}, w^{[k+1]})$  to compute  $b^{[k+1]}$ .
- 7:   Set  $k = k + 1$ .
- 8: **end while**
- 9: Use  $a^{[k_m]}, b^{[k_m]}$  and  $\beta$  to compute  $u, y$  and  $\Delta$ .

accurate in practice.<sup>2</sup> Moreover, the particular uncertainty returned by the power iteration can be studied further for insight.

To describe the power iteration, consider vectors  $a, z, b, w \in \mathbb{C}^m$ . Partition these vectors compatibly with the block structure, for example,  $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  with  $b_1 \in \mathbb{C}^{m_1}$  and  $b_2 \in \mathbb{C}^{m_2}$ . The power iteration is defined based on the following set of equations for some  $\beta > 0$ :

$$\beta a = Mb, \quad (17a)$$

$$z_1 = \frac{w_1^H a_1}{|w_1^H a_1|} w_1, \quad z_2 = \frac{\|w_2\|_2}{\|a_2\|_2} a_2, \quad (17b)$$

$$\beta w = M^H z, \quad (17c)$$

$$b_1 = \frac{a_1^H w_1}{|a_1^H w_1|} a_1, \quad b_2 = \frac{\|a_2\|_2}{\|w_2\|_2} w_2. \quad (17d)$$

These equations arise from the optimality conditions for the SSV and are related to the concept of principle direction alignment (see details in Packard and Doyle<sup>2</sup>). Here, we will simply show that any solution of these equations yields a lower bound on  $\mu(M)$ . First note that (17d) implies that  $b_1 = q_1 a_1$  with  $q_1 := \frac{a_1^H w_1}{|a_1^H w_1|} \in \mathbb{C}$  and  $|q_1| = 1$ . Equation (17d) also gives  $\|b_2\|_2 = \|a_2\|_2$ . Hence, there is a  $Q_2 \in \mathbb{C}^{m_2 \times m_2}$  with  $\bar{\sigma}(Q_2) = 1$  such that  $b_2 = Q_2 a_2$ . Finally, define  $u := b$ ,  $y := \beta a$  and  $\Delta := \frac{1}{\beta} \text{diag}(q_1 I_{m_1}, Q_2)$ . It can be verified from (17a) that  $y = Mu$ . Moreover,  $u = \Delta y$  and  $\bar{\sigma}(\Delta) = \frac{1}{\beta}$  by construction. Hence,  $\Delta \in \Delta$  satisfies the determinant condition and yields the lower bound  $\mu(M) \geq \frac{1}{\bar{\sigma}(\Delta)} = \beta$ .

The power iteration attempts to solve (17) by iterating through the various relations therein. The procedure is summarized in Algorithm 2. The algorithm, as stated, runs for a fixed number of  $k_m$  iterations. However, more advanced stopping conditions can be used, for example, terminating when the various vectors have small updates as measured in the Euclidean norm. Although  $b^{[0]}, w^{[0]}$  can be chosen randomly, a more specific choice would be to use the right singular vector associated with  $\bar{\sigma}(D_{\text{nr}}^* M (D_{\text{nr}}^*)^{-1})$ , where  $D_{\text{nr}}^*$  is obtained using the standard Osborne's iterations.<sup>13</sup>

This power iteration simplifies in two special cases:

- (i)  $\Delta = \mathbb{C}^{m \times m}$ : As noted above,  $\mu(M) = \bar{\sigma}(M)$  in this case. The power iteration relations in (17) become

$$\beta a = Mb, \quad z = \frac{\|w\|_2}{\|a\|_2} a, \quad \beta w = M^H z, \quad b = \frac{\|a\|_2}{\|w\|_2} w.$$

If  $b$  and  $w$  are initialized to be unit norm, then all vectors are unit norm throughout the iteration. Hence  $z = a$  and  $b = w$ , so the relations further simplify to

$$\beta a = Mb, \quad \beta b = M^H a.$$

We can iterate on these equations starting from an initial unit norm vector  $b$ . This corresponds to the standard power iteration for computing  $\bar{\sigma}(M)$ .

- (ii)  $m := v$  and  $\Delta = \{\delta I_v : \delta \in \mathbb{C}\}$ : As noted above,  $\mu(M) = \rho(M)$  in this case. The power iteration relations in (17) simplify to

$$\beta a = Mb, \quad z = \frac{w^H a}{|w^H a|} w, \quad \beta w = M^H z, \quad b = \frac{a^H w}{|a^H w|} a.$$

Iterating these relations yields a power iteration to find the eigenvalue corresponding to the spectral radius. The iteration also yields the corresponding right  $b$  and left  $z^H$  eigenvectors.

## 4.2 | Generalized power iteration: Repeated complex full-blocks

This subsection describes a generalization of the SSV power iteration to handle repeated complex full-blocks. Again, we consider the problem with  $M \in \mathbb{C}^{m \times m}$  and a structured uncertainty with one  $m_1 \times m_1$  full-block repeated  $v$  times as in (4). A lower bound on  $\mu(M)$  is obtained by finding  $\Delta \in \Delta$  and nonzero vectors  $y \in \mathbb{C}^m$  and  $u \in \mathbb{C}^m$  such that  $y = Mu$  and  $u = \Delta y$ .

It will be useful to define the following reshaping operation  $L_{m_1} : \mathbb{C}^{vm_1} \rightarrow \mathbb{C}^{m_1 \times v}$  such that  $y = [y_1^H \dots y_v^H]^H \in \mathbb{C}^{vm_1}$  maps to  $L_{m_1}(y) = [y_1, \dots, y_v]$ . This operation restacks the partitioned vector  $y \in \mathbb{C}^{vm_1}$  into a matrix. The inverse  $L_{m_1}^{-1}$  will convert the matrix back to a column vector. This notation is useful to handle matrix-vector products for  $\Delta \in \Delta$ . Specifically, let  $\Delta = I_v \otimes \Delta_1$  with  $\Delta_1 \in \mathbb{C}^{m_1 \times m_1}$ . The relation  $u = \Delta y$  is equivalent to  $L_{m_1}(u) = \Delta_1 L_{m_1}(y)$ .

We need one additional operation to define the generalized power iteration. Consider vectors  $a, z, b, w \in \mathbb{C}^{vm_1}$ . Let  $G$  be a matrix of any dimension with the following SVD:

$$G = U \Sigma V^H = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^H. \quad (18)$$

Define  $\mathbf{Q}(G) := U_1 V_1^H$  and note that  $\bar{\sigma}(\mathbf{Q}(G)) = 1$ . The power iteration is defined based on the following set of equations for some  $\beta > 0$ :

$$\beta a = Mb, \quad (19a)$$

$$L_{m_1}(z) = \mathbf{Q}(L_{m_1}(a)L_{m_1}(w)^H) L_{m_1}(w), \quad (19b)$$

$$\beta w = M^H z, \quad (19c)$$

$$L_{m_1}(b) = \mathbf{Q}(L_{m_1}(w)L_{m_1}(a)^H) L_{m_1}(a). \quad (19d)$$

Any solution of these equations yields a lower bound on  $\mu(M)$ . To show this, define  $u := b$ ,  $y := \beta a$  and  $\Delta := I_v \otimes \frac{1}{\beta} \mathbf{Q}(L_{m_1}(w)L_{m_1}(a)^H)$ . Then (19a) and (19d) are equivalent to  $y = Mu$  and  $u = \Delta y$ . Moreover,  $\bar{\sigma}(\Delta) = \frac{1}{\beta}$  by construction. Hence  $\Delta \in \Delta$  satisfies the determinant condition and yields the lower bound  $\mu(M) \geq \beta$ . A power iteration can be used to find a solution by iterating through equations (19a)–(19d) as outlined in Algorithm 3. Note that the comments on initialization and stopping criterion for Algorithm 2 applies for Algorithm 3 as well. In cases where the power iteration does not converge, the perturbations  $\Delta_1 = I_v \otimes \mathbf{Q}(L_{m_1}(a)L_{m_1}(w)^H)$  and  $\Delta_2 = I_v \otimes \mathbf{Q}(L_{m_1}(w)L_{m_1}(a)^H)$  can be used to obtain a valid lower bound as  $\beta = \max(\rho(\Delta_1^H M), \rho(\Delta_2 M))$ .

Equations (19b) and (19c) generalize the cases in the previous subsection:

- (i)  $v = 1$ : In this case, the block structure (4) is just a single full-block uncertainty. The stacking operations are just  $L_{m_1}(z) = z$ ,  $L_{m_1}(a) = a$ ,  $L_{m_1}(w) = w$ , and  $L_{m_1}(b) = b$ . Thus, an SVD of  $L_{m_1}(a)L_{m_1}(w)^H = aw^H$  is given by  $U_1 = \frac{a}{\|a\|_2}$ ,  $V_1 = \frac{w}{\|w\|_2}$ , and  $\hat{\Sigma} = \|a\|_2 \|w\|_2$ . Equation (19b) is thus equivalent to  $z = \frac{\|w\|_2}{\|a\|_2} a$ , which corresponds to the full-block update in (17b).
- (ii)  $m_1 = 1$ : In this case, the block structure (4) is a scalar uncertainty repeated  $v$  times. The stacking operations are just  $L_{m_1}(z) = z^T$ ,  $L_{m_1}(a) = a^T$ ,  $L_{m_1}(w) = w^T$ , and  $L_{m_1}(b) = b^T$ . Thus, the stacking operation is a transpose

**Algorithm 3.** Lower bound: generalized power iteration

- 1: (Initialization) Choose the number of iterations  $k_m$  and set  $k = 0$ . Select some unit-norm vectors  $b^{[0]}, w^{[0]} \in \mathbb{C}^m$  and  $a^{[0]} = z^{[0]} = 0 \in \mathbb{C}^m$ .
- 2: **while**  $k < k_m$  **do**
- 3:   (19a):  $\beta := \|Mb^{[k]}\|_2$  and  $a^{[k+1]} := Mb^{[k]}/\beta$ .
- 4:   (19b):  $z_L := \mathbf{Q}(L_{m_1}(a^{[k+1]})L_{m_1}(w^{[k]})^H)$   
 $L_{m_1}(w^{[k]})$  and  $z^{[k+1]} = L_{m_1}^{-1}(z_L)$
- 5:   (19c):  $\beta := \|M^H z^{[k+1]}\|_2$  and  $w^{[k+1]} := M^H z^{[k+1]}/\beta$ .
- 6:   (19d):  $b_L := \mathbf{Q}(L_{m_1}(w^{[k+1]})L_{m_1}(a^{[k+1]})^H)$   
 $L_{m_1}(a^{[k+1]})$  and  $b^{[k+1]} = L_{m_1}^{-1}(b_L)$ .
- 7:   Set  $k = k + 1$ .
- 8: **end while**
- 9: Use  $a^{[k_m]}, b^{[k_m]}, w^{[k_m]}$  and  $\beta$  to compute  $u, y$  and  $\Delta$ .

**TABLE 1** Different algorithms used for the results.

| Uncertainty structure | $\Delta \in \Delta_{nr}$          | $\Delta \in \Delta_r$                     |
|-----------------------|-----------------------------------|-------------------------------------------|
| Upper bound algorithm | Osborne's iteration (Section 3.1) | Method of centers (Algorithm 1)           |
| Lower bound algorithm | Power iteration (Algorithm 2)     | Generalized power iteration (Algorithm 3) |

(but not conjugation) of the column vector to a row vector. This yields:

$$L_{m_1}(a)L_{m_1}(w)^H = a^T(w^T)^H = w^H a. \quad (20)$$

This is a scalar and an SVD of this product is given by  $U_1 = \frac{w^H a}{|w^H a|}$ ,  $V_1 = 1$ , and  $\hat{\Sigma} = |w^H a|$ . Step (19b) is thus equivalent to  $z = \frac{|w^H a|}{|w^H a|} w$ . This corresponds to the repeated scalar block update in (17b).

## 5 | RESULTS

We consider a fluid-flow problem wherein the uncertainty has a repeated full-block structure as in (4). The SSV bounds are computed for the true uncertainty structure (i.e.,  $\Delta \in \Delta_r$ ) using the proposed methods. We compare those bounds with the ones obtained by treating the uncertainty to be nonrepeating (i.e.,  $\Delta \in \Delta_{nr}$ ), which is an approximation of the true uncertainty. The motivation behind this comparison is to highlight the differences that arise due to this approximation, and how those differences can alter the subsequent interpretation of the physical system/phenomena. The algorithms used for different cases are summarized in Table 1. Furthermore, we showcase the gap between the upper and lower bounds for the two sets of results. In addition, all of the above mentioned aspects have been repeated for a simple academic example.

### 5.1 | Example Model-1: Incompressible plane Couette flow

We will demonstrate our proposed algorithms on the same spatially discretized incompressible plane Couette flow (PCF) model initially used to investigate SSV—with nonrepeated full-blocks—in Liu and Gayme.<sup>14</sup> PCF is a simple shear-driven flow between two parallel plates, wherein the lower plate is held stationary and the upper plate moves with a fixed speed  $U_\infty$ . The PCF example is chosen as a demonstration in this study, but the proposed methods are equally applicable to other systems where repeated full-block uncertainties arise.<sup>14,15,25,26</sup>

The input–output (I/O) map of the forced perturbation dynamics about a steady baseflow is a frequency response matrix defined as

$$M = C_V(\kappa_x, \kappa_z)(i\omega I_{2S} - A(\kappa_x, \kappa_z))^{-1}B(\kappa_x, \kappa_z), \quad (21)$$

where  $\omega$  is the temporal frequency,  $\kappa_x$  and  $\kappa_z$  are the wavenumbers from discretization in  $x$  and  $z$  directions using Fourier modes, and  $A(\kappa_x, \kappa_z, Re) \in \mathbb{C}^{2s \times 2s}$ ,  $B(\kappa_x, \kappa_z) \in \mathbb{C}^{2s \times m}$  and  $C_V(\kappa_x, \kappa_z) \in \mathbb{C}^{n \times 2s}$  are the system operators, respectively. Additionally,  $A$  is a function of the Reynolds number  $Re = U_\infty h / \nu$ , where  $h$  is the distance between the two plates and  $\nu$  is the kinematic viscosity of the fluid. Then, for  $M \in \mathbb{C}^{n \times m}$ , we simply have the relation  $\eta = Mf$  between the system inputs ( $f(y, t) \in \mathbb{C}^m$ ) and outputs ( $\eta(y, t) \in \mathbb{C}^n$ ) defined as

$$f = \begin{bmatrix} f_x(y, t) \\ f_y(y, t) \\ f_z(y, t) \end{bmatrix}, \quad \eta(y, t) = \begin{bmatrix} \nabla u_x(y, t) \\ \nabla u_y(y, t) \\ \nabla u_z(y, t) \end{bmatrix},$$

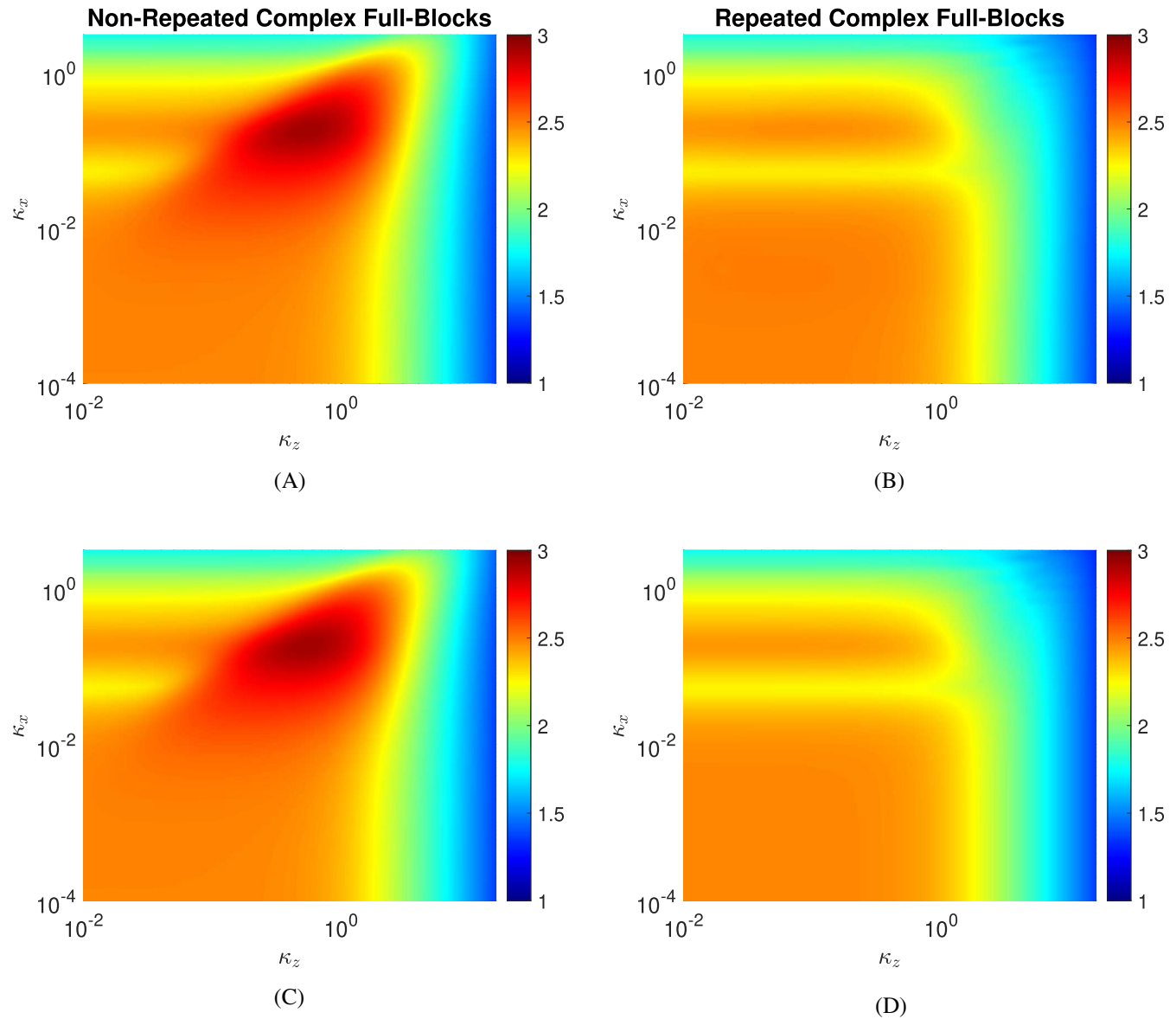
where  $\nabla$  is the discrete gradient operator,  $u_x$ ,  $u_y$  and  $u_z$  represent flow perturbation velocities and  $f_x$ ,  $f_y$  and  $f_z$  represent input forcing, in  $x$ ,  $y$  and  $z$  directions, respectively. The forcing signal  $f$  is a pseudo-linear approximation of the quadratic convective nonlinear term in the incompressible PCF model. This is given by  $f = \Delta \eta$  with  $\Delta = I_3 \otimes \Delta_1$ , where  $\Delta_1$  is considered an unknown matrix approximation of the velocity vectors (see Liu and Gayme<sup>14</sup> for more details). Therefore, the uncertainty for this system is of the form shown in (4) with a rectangular block  $\Delta_1 \in \mathbb{C}^{m_1 \times n_1}$  repeated three times ( $v = 3$ ). Thus, the SSV bounds for the PCF model indicate the sensitivity of flow at each  $\kappa_x$  and  $\kappa_z$  to this forcing, which is an indication of flow's potential for transition to turbulence.<sup>27</sup> Large bound values indicate that the system in (21) has a higher tendency to transition, and vice versa, which is a consequence of a variation of the small-gain condition for structured uncertainties (see Section 2). For additional details on the model formulation and discretization, we refer the reader to prior works.<sup>14,28</sup>

We will use a  $50 \times 90 \times 50$  grid of  $n_{\kappa_x} \times n_{\kappa_z} \times n_\omega$  to compute the SSV bounds, where  $n_{\kappa_x}$ ,  $n_{\kappa_z}$  and  $n_\omega$  are total grid points for  $\kappa_x$ ,  $\kappa_z$  and  $\omega$ , respectively. We use logarithmically spaced values  $\kappa_x \in [10^{-4}, 10^{0.48}]$ ,  $\kappa_z \in [10^{-2}, 10^{1.2}]$  and  $\omega \in [-10^{0.5}, 10^{0.5}]$  for all the results in this section. Note that we consider negative temporal frequencies as the system matrices are complex-valued and the corresponding frequency response is not symmetric about the  $\omega = 0$  line. The state dimension of the system is  $s = 30$ , and the input and output dimensions are  $m = 3s = 90$  and  $n = 9s = 270$ , respectively. Then,  $m_1 = s = 30$  and  $n_1 = 3s = 90$  for  $\Delta_1$ . The operating Reynolds number for the system is set to  $Re = 358$ . The system is nominally stable, that is, the eigenvalues of  $A(\kappa_x, \kappa_z, Re)$  are in the open left-half plane for the parameter values chosen here. Algorithm 1 is initialized with  $R = \text{diag}((d_1^*)^2, \dots, (d_v^*)^2)$  using the Osborne's iteration,  $p = 1.05$ ,  $k_m = 500$ ,  $\theta = 10^{-3}$ ,  $\gamma = 10^6$  and  $\epsilon = 2 \times 10^{-4}$ . Algorithms 2 and 3 are initialized by setting  $w^{[0]}$  and  $b^{[0]}$  to be the right singular vector associated with  $\bar{\sigma}(D_{nr}^* M(i\omega) (D_{nr}^*)^{-1})$ , where  $D_{nr}^*$  is obtained using the standard Osborne's iterations on  $M(i\omega)$ . Additionally, the total number of iterations given by  $k_m$  are set to 60 for both the power iterations. Since  $M$  in (21) is a frequency response operator, we will compute the “best” upper ( $\alpha_{\max}$ ) and lower ( $\beta_{\max}$ ) bounds at each  $(\kappa_x, \kappa_z)$  pair by choosing the maximum  $\alpha$  and  $\beta$  over a spectrum of frequencies  $\omega$ .

MATLAB's `parfor` command is used to compute  $\alpha_{\max}$  and  $\beta_{\max}$  values using parallel computing for  $n_{\kappa_x} \times n_{\kappa_z}$  grid at each  $n_\omega$ . The computations were performed on a desktop computer with 3.61 GHz 12-th Gen Intel(R) Core(TM) i7-12700K processor with 12 cores and 16 GB RAM. The computation times for Algorithms 1 and 3 were approximately 4 h and 22 min, respectively. On the other hand, Osborne's iteration and Algorithm 1 took about 2 min and 4 min, respectively, to compute all the  $\alpha_{\max}$  and  $\beta_{\max}$  values.

The results are depicted in Figure 1. Comparing the results shown in Figure 1A,B, we deduce that  $\alpha_{\max}$  values computed using Algorithm 1 are smaller overall than the  $\alpha_{\max}$  values computed using the Osborne's iteration. The distributions of the  $\alpha_{\max}$  values over the wavenumber pair grid are also markedly different. There is a prominent peak in Figure 1A for the largest  $\alpha_{\max}$  value at  $\kappa_x = 0.1956$  and  $\kappa_z = 0.5778$ . This peak is not present in Figure 1B. Instead, there are two areas with similar  $\alpha_{\max}$  values, which are separated by a narrow ‘valley’ in between. Therefore, approximating a repeating full-block uncertainty with a nonrepeating one in this case not only leads to conservative upper bound estimates, but also results in a local maximum that does not necessarily represent actual system behavior. A similar argument follows for the lower bounds computed using the two power iteration variants, as shown in Figure 1C,D. Additionally, the largest  $\alpha_{\max}$  value in Figure 1B corresponds to the negative spectrum of temporal frequency grid, which provides further insight into the most sensitive direction for instability of the PCF model in (21).

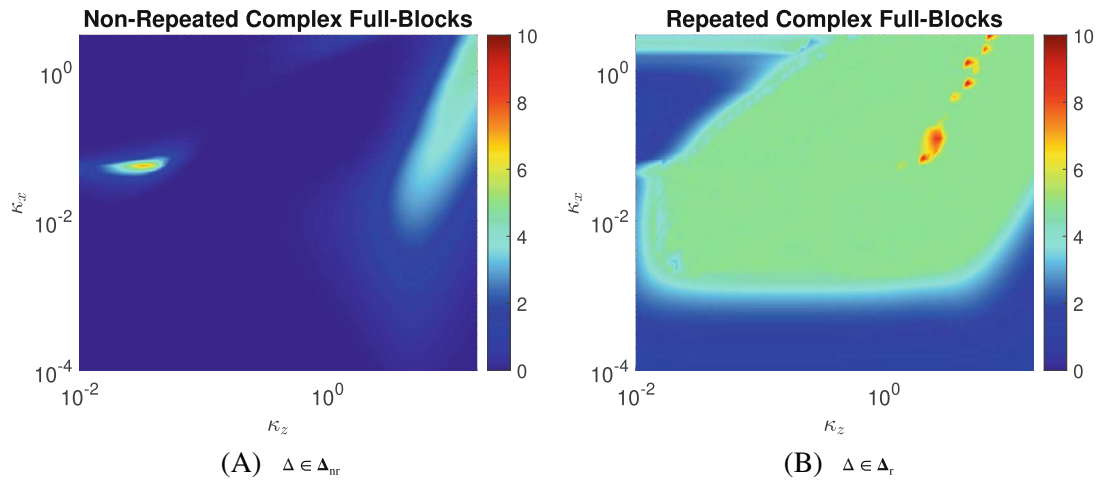
The gaps between  $\alpha_{\max}$  and  $\beta_{\max}$  are shown in Figure 2A,B which indicate that  $\beta_{\max}$  values are within 5% of  $\alpha_{\max}$  values for approximately 99.8% and 98.9% of wavenumber pairs in Figure 2A,B, respectively. This means that the true SSV values lie within a small interval for a large subset of the wavenumber pairs considered for both repeated and nonrepeated full-blocks. A summary of the gaps in both sets of bounds is provided in Table 2. Although the stopping ratio between



**FIGURE 1** The  $\alpha_{\max}$  and  $\beta_{\max}$  results over the wavenumber pair  $(\kappa_x, \kappa_z)$  grid. The top row plots represent the upper bounds  $\alpha_{\max}$  and the bottom row plots represent the lower bounds  $\beta_{\max}$ . (A)  $\log_{10}(\alpha_{\max})$  for  $\Delta \in \Delta_{\text{nr}}$ : Osborne's iteration (upper bounds), (B)  $\log_{10}(\alpha_{\max})$  for  $\Delta \in \Delta_r$ : Algorithm 1 (upper bounds), (C)  $\log_{10}(\beta_{\max})$  for  $\Delta \in \Delta_{\text{nr}}$ : Algorithm 2 (lower bounds), (D)  $\log_{10}(\beta_{\max})$  for  $\Delta \in \Delta_r$ : Algorithm 3 (lower bounds).

upper and lower bounds for Algorithm 1 is set to 1.05, we still end up with a percentage difference greater than 5% at some wavenumber pairs (see Figure 2B). However, the maximum gap is 9.33% for only one wavenumber pair and the rest of the wavenumber pairs have an average gap of 6.5% at the hotspots in Figure 2B. The relatively large gap can be attributed to one of the three reasons: (i) The  $D$ -scale upper bound is not necessarily equal to  $\mu$ , (ii) the upper bound algorithm fails to converge to the optimal  $D$ -scale, and/or (iii) the power iteration fails to converge to the true value of  $\mu$ . It is possible that the repeated complex full-blocks are a special case, where  $\mu$  is equal to its corresponding  $D$ -scale upper bound. In this case, issue (i) would not be the source of the gap. We will explore this conjecture in future work.

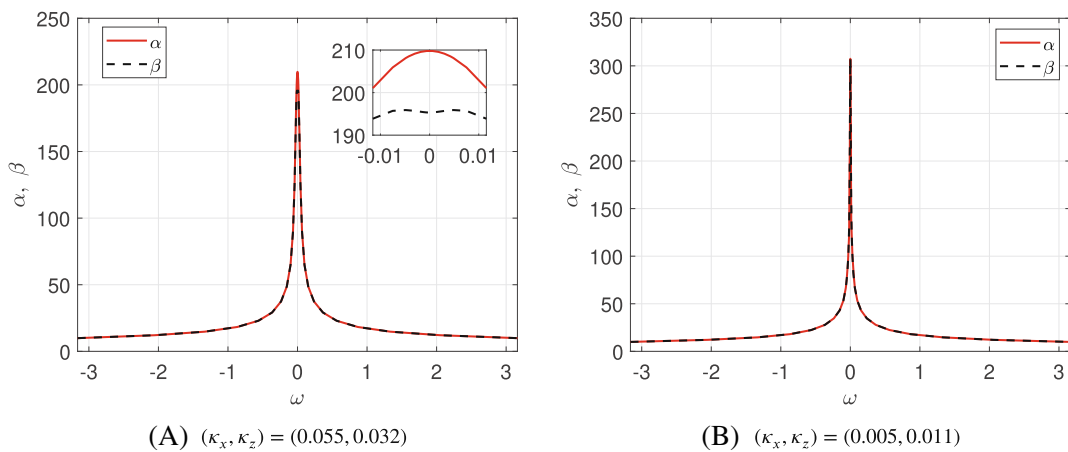
To further investigate the bounds, we plot  $\alpha$  and  $\beta$  over the temporal frequency at chosen wavenumber pairs  $(\kappa_x, \kappa_z)$ . These results for the nonrepeated and repeated full-blocks are shown in Figures 3 and 4, respectively. The wavenumber pairs chosen are the ones corresponding to the largest and smallest gap between  $\alpha_{\max}$  and  $\beta_{\max}$  over the wavenumber pair grid. The result in Figure 3A showcases the bounds for the nonrepeated full blocks at  $(\kappa_x, \kappa_z) = (0.055, 0.032)$ , where the gap between  $\alpha_{\max}$  and  $\beta_{\max}$  is the largest at 7.09%. The zoomed-in plot in Figure 3A highlights a single global peak



**FIGURE 2** The percentage difference between  $\alpha_{\max}$  and  $\beta_{\max}$  values over the wavenumber pair  $(\kappa_x, \kappa_z)$  grid. The stopping ratio between upper and lower bounds for Algorithm 1 was set to 1.05, which means that all the computed upper bounds must be within 5% of the lower bounds. Therefore, the majority of percentage differences in (B) are  $\leq 5\%$ . The only upper bounds that failed to achieve the stopping criterion are given by the red hotspots.

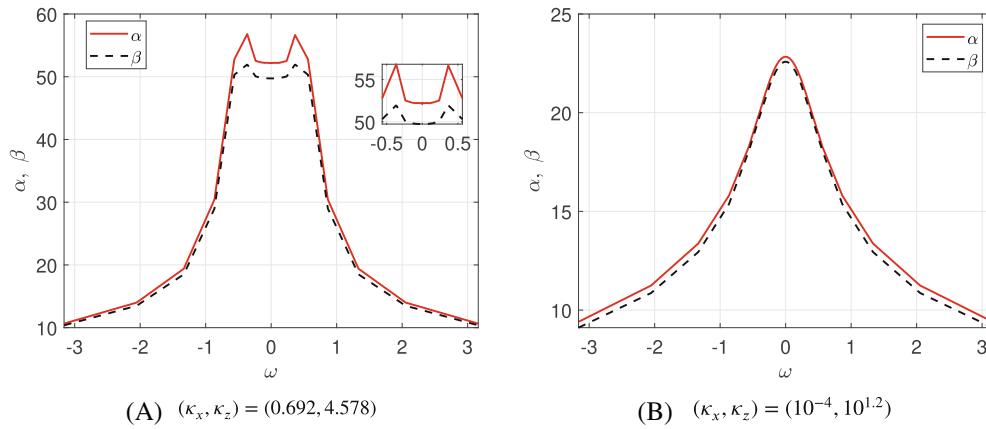
**TABLE 2** Summary of the gaps between  $\alpha_{\max}$  and  $\beta_{\max}$  for the Couette flow model.

| Uncertainty structure               | $\Delta \in \Delta_{nr}$                | $\Delta \in \Delta_r$             |
|-------------------------------------|-----------------------------------------|-----------------------------------|
| Maximum gap, $(\kappa_x, \kappa_z)$ | 7.09%, (0.055, 0.032)                   | 9.33%, (0.692, 4.578)             |
| Minimum gap, $(\kappa_x, \kappa_z)$ | $3.2 \times 10^{-5}\%$ , (0.005, 0.011) | 1.14%, ( $10^{-4}$ , $10^{1.2}$ ) |
| Average gap                         | 0.46%                                   | 3.8%                              |



**FIGURE 3** The  $\alpha$  and  $\beta$  results over the temporal frequency ( $\omega$ ) grid for  $\Delta \in \Delta_{nr}$ . The (A) and (B) correspond to wavenumber pairs where the gap between  $\alpha_{\max}$  and  $\beta_{\max}$  are the largest and smallest, respectively.

in  $\alpha$  at  $\omega \approx 0$ , while there are two local peaks in  $\beta$ , located almost symmetrically about the  $\omega = 0$  line at  $\omega = \pm 0.005$ . In the case of smallest gap between  $\alpha_{\max}$  and  $\beta_{\max}$  for the nonrepeated full-blocks, which occurs at  $(\kappa_x, \kappa_z) = (0.005, 0.011)$ , the bounds are virtually identical (see Figure 3A). On the other hand, both the bounds are qualitatively similar for the repeated full-blocks case, as shown in Figure 4. The largest gap between  $\alpha_{\max}$  and  $\beta_{\max}$  in this case occurs at  $(\kappa_x, \kappa_z) = (0.692, 4.578)$ , and both  $\alpha$  and  $\beta$  have two local peaks that occur at  $\omega = \pm 0.365$  (see Figure 4A). Although these peaks are symmetric about the  $\omega = 0$  line, the peak  $\alpha$  values in Figure 4A at  $\omega = -0.365$  and  $\omega = 0.365$  are 56.799 and 56.651, respectively.



**FIGURE 4** The  $\alpha$  and  $\beta$  results over the temporal frequency ( $\omega$ ) grid for  $\Delta \in \Delta_r$ . The (A) and (B) correspond to wavenumber pairs where the gap between  $\alpha_{\max}$  and  $\beta_{\max}$  are the largest and smallest, respectively. (A)  $(\kappa_x, \kappa_z) = (0.692, 4.578)$ , (B)  $(\kappa_x, \kappa_z) = (10^{-4}, 10^{1.2})$ .

## 5.2 | Simple academic example

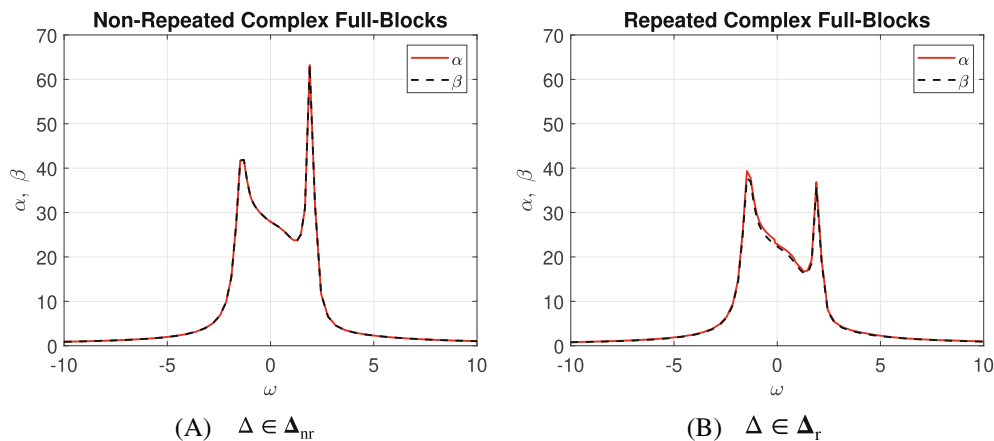
We now demonstrate the proposed algorithms on a MIMO LTI system with the frequency response matrix given by  $M = C(i\omega I_4 - A)^{-1}B$ , where  $\omega$  is the temporal frequency and the randomly generated state-space matrices  $A, B, C \in \mathbb{C}^{4 \times 4}$  are as follows:

$$A = \begin{bmatrix} 0.720 - i0.663 & -0.602 - i0.684 & -1.937 - i0.792 & -1.021 - i0.153 \\ 0.059 - i1.875 & -1.103 + i0.350 & -0.728 + i0.164 & -0.135 + i2.021 \\ 0.071 + i0.114 & 0.948 + i0.237 & -1.493 + i0.491 & 1.486 - i0.025 \\ -0.647 - i0.260 & -0.272 + i0.829 & -0.709 + i0.908 & -0.506 + i0.276 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.738 - i0.773 & 1.271 + i0.118 & 1.152 + i0.494 & -0.764 - i0.400 \\ -0.166 + i0.896 & 0.504 + i1.761 & 0.291 - i0.516 & 0.425 - i0.028 \\ -1.103 + i0.449 & -1.408 - i0.195 & 0.067 - i1.287 & -0.595 + i0.316 \\ 1.308 - i0.744 & 0.358 + i0.728 & -0.174 + i0.665 & -1.489 - i0.094 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.255 + i0.101 & 1.681 + i0.048 & -0.386 - i0.051 & 0.633 - i0.874 \\ -1.827 + i1.132 & -0.267 - i0.846 & -0.863 + i0.840 & 0.244 + i1.447 \\ 1.877 + i0.179 & -1.124 + i0.752 & 1.014 + i0.731 & -1.502 + i0.431 \\ -0.803 + i1.056 & 0.002 - i0.284 & 1.029 - i0.801 & -0.444 + i0.543 \end{bmatrix}.$$

The nominal system is stable as all the eigenvalues of  $A$  are in the open left-half plane. The uncertainty for this model is chosen as  $\Delta = I_2 \otimes \Delta_1$  with  $\Delta_1 \in \mathbb{C}^{2 \times 2}$ . Numerical implementation of the algorithms are as described in Section 5.1. We take 200 logarithmically spaced points for  $\omega \in [-10^{1.5}, 10^{1.5}]$ . The results for  $\alpha$  and  $\beta$  for both the nonrepeated and repeated cases are shown in Figure 5. In terms of qualitative similarities, there are two peaks—one for  $\omega < 0$  and the other for  $\omega > 0$ —in each bound in both the cases, and the bounds are not symmetric about the  $\omega = 0$  line. However, approximating the repeated full-block structure with a nonrepeated one leads to very conservative bounds at some temporal frequencies. For example, the  $\alpha$  value in Figure 5A is approximately 1.7 times that of the  $\alpha$  value in Figure 5B at  $\omega = 1.896$ . A similar set of comments applies to the lower bounds  $\beta$  at  $\omega = 1.896$ . This means that the true value of  $\mu$  at this frequency in the nonrepeated case is roughly 1.7 times that of the repeated case. It is also noteworthy that the global peaks of the bounds in Figure 5A are at  $\omega > 0$ , whereas the global peaks in Figure 5B are at  $\omega < 0$ . Therefore, similar to the fluid-flow example, neglecting the repeated structure of the uncertainty cannot only lead to conservative bounds, which translates into conservative stability-margin estimates, but also might lead to inaccurate conclusions about the temporal behavior of the system.



**FIGURE 5** The  $\alpha$  and  $\beta$  results over the temporal frequency ( $\omega$ ) grid. Although we consider  $\omega \in [-10^{1.5}, 10^{1.5}]$ , the results are shown for  $\omega \in [-10, 10]$  to better highlight the local behavior of the bounds.

## 6 | CONCLUSIONS

We proposed two algorithms for computing upper and lower bounds of structured singular value for repeated complex full-block uncertainty. Such uncertainty structures naturally arise in models of fluid flows and other convective systems. The proposed algorithms yield bounds that are less conservative as compared to the algorithms that ignore the repeated full-block structure, for example, Osborne's iteration for nonrepeated full-blocks. Thus, properly accounting for the repeated block structure can improve stability-margin estimates and also enable one to draw more representative conclusions regarding the temporal behavior of the system. These points were demonstrated on an example of incompressible plane Couette flow and an academic example. Furthermore, our future work will involve investigating the gap between the  $\mu$  and the convex (or  $D$ -scale) upper bound for a single repeated full-block. This particular case is of interest due to the fact that  $\mu$  is equal to its upper bound for a single full block and also for a single repeated complex scalar.

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## CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

## DATA AVAILABILITY STATEMENT

The data that support the findings of this study are openly available in GitHub at [https://github.com/talhamushtaq7/Structured\\_Singular\\_Value\\_for\\_Incompressible\\_Flows.git](https://github.com/talhamushtaq7/Structured_Singular_Value_for_Incompressible_Flows.git).

## ENDNOTES

\*For  $m = v$ ,  $(H)_{ij} \in \mathbb{C}$  is the  $(i, j)$  scalar element of  $H$ .

<sup>†</sup> $f_2(\bar{s}_{ij})$  is nonconvex for some combinations of the coefficients  $\{c_0, \dots, c_5\}$ . Therefore, the solution is only guaranteed to converge to a local optimum.

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## APPENDIX . GENERALIZED OSBORNE

In this section, we will describe a fast algorithm for  $\Delta \in \Delta_r$  and  $M \in \mathbb{C}^{m \times m}$ . The standard Osborne iteration cannot be used for  $\Delta_r$  as  $D \in \mathbf{D}_r$  contains off-diagonal entries. This section describes our generalization of Osborne's method

(GenOsborne) to handle the matrix scales in (7). The proposed GenOsborne algorithm is an iteration that solves the following minimization problem:

$$\min_{D \in \mathbf{D}_r} \|DMD^{-1}\|_F^2, \quad (\text{A1})$$

where  $\mathbf{D}_r$  is defined in (7). To simplify the calculations, we use the square of Frobenius norm in (A1). Let  $s_{ij}$  denote the  $(i, j)$  entry of  $S$  in (7). The Frobenius norm in (A1) yields a cumbersome expression that has various  $s_{ij} \in \mathbb{C}$  entries multiplying each other. Thus, it is difficult to minimize the function for each  $s_{ij}$ , since each of the scalings are coupled together. To avoid this issue, we iteratively optimize over a single off-diagonal entry and then couple it, similar to the Osborne's iteration. Thus, we first use the standard Osborne's iterations to calculate the optimal diagonal scalings  $s_i^*$  and then use an iterative approach to optimize a single off-diagonal term  $s_{ij} \in \mathbb{C}$  at each iteration and iterate over all possible pairs of  $(i, j)$ , where  $i \neq j$ . We denote the matrices with a single off-diagonal entry  $s_{ij} \in \mathbb{C}$  as  $D_{ij} = S_{ij} \otimes I_{m_1}$ , where  $S_{ij}$  has ones along the diagonal,  $s_{ij}$  in the  $(i, j)$  entry and zero everywhere else.

Let  $M^{[k]}$  be the scaled matrix at step  $k$  of the generalized iteration and  $s_{ij} \in \mathbb{C}$  be the off-diagonal scaling to be optimized. Then, the objective function is:

$$\begin{aligned} f_1(s_{ij}) &= \|D_{ij}M^{[k]}D_{ij}^{-1}\|_F^2 \\ &= c_0 + \text{conj}(c_1s_{ij}) + c_1s_{ij} + c_2\|s_{ij}\|^2 + c_3s_{ij}^2 \\ &\quad + \text{conj}(c_3)(\text{conj}(s_{ij}))^2 + c_4s_{ij}^2(\text{conj}(s_{ij})) \\ &\quad + \text{conj}(c_4)s_{ij}(\text{conj}(s_{ij}))^2 + c_5\|s_{ij}^2\|^2, \end{aligned} \quad (\text{A2})$$

where  $\{c_0, \dots, c_5\} \subseteq \mathbb{C}$  are coefficients that can be computed from the definition of the Frobenius norm. Note that the coefficients depend on the pair  $(i, j)$  and  $M^{[k]}$ . By expressing  $s_{ij} = s_{R_{ij}} + is_{I_{ij}}$ , the objective function  $f_1(s_{ij})$  can be written in the following equivalent form:

$$\begin{aligned} f_2(\bar{s}_{ij}) &= c_0 + 2\text{Re}(c_1)s_{R_{ij}} - 2\text{Im}(c_1)s_{I_{ij}} \\ &\quad + (c_2 + 2\text{Re}(c_3))s_{R_{ij}}^2 + (c_2 - 2\text{Re}(c_3))s_{I_{ij}}^2 \\ &\quad - 4\text{Im}(c_3)s_{R_{ij}}s_{I_{ij}} + 2\text{Re}(c_4)s_{R_{ij}}(s_{R_{ij}}^2 + s_{I_{ij}}^2) \\ &\quad - 2\text{Im}(c_4)s_{I_{ij}}(s_{R_{ij}}^2 + s_{I_{ij}}^2) + c_5(s_{R_{ij}}^2 + s_{I_{ij}}^2)^2, \end{aligned} \quad (\text{A3})$$

where  $\bar{s}_{ij} = [s_{R_{ij}}, s_{I_{ij}}]^T$ . We use the damped newton method (see Algorithm 9.5 in Boyd and Vandenberghe<sup>29</sup>) to solve the minimization problem. Therefore, we obtain the local optimum  $\bar{s}_{ij}^* = \text{argmin}_{\bar{s}_{ij} \in \mathbb{R}^2} f_2(\bar{s}_{ij})$ .<sup>†</sup> Hence, each  $s_{ij}^* = s_{R_{ij}}^* + is_{I_{ij}}^*$  has the corresponding scaling matrix  $D_{ij}^*$ . We perform the following update for  $k \geq 1$ :

$$M^{[k+1]} = D_{ij}^*M^{[k]}(D_{ij}^*)^{-1}. \quad (\text{A4})$$

The iterative algorithm results in the total effective scaling as:

$$D'' = \left( \prod_{\forall i, j, i \neq j} D_{ij}^* \right) D_{\text{nr}}^*, \quad (\text{A5})$$

where  $D_{\text{nr}}^*$  is the optimal diagonal scaling after applying the standard Osborne's iteration. For example, if we choose to optimize the  $s_{12}$  entry then we compute  $s_{12}$  by minimizing (A3). We scale the matrix  $M^{[2]} = D_{12}^*M^{[1]}(D_{12}^*)^{-1}$ , where  $M^{[1]} = D_{\text{nr}}^*M(D_{\text{nr}}^*)^{-1}$ . We use  $M^{[2]}$  and repeat the steps for other  $s_{ij}$  until all  $s_{ij}$  are computed and effectively  $D''$  is obtained. The above approach allows for computing optimal value of each  $s_{ij}$  and then coupling them. Finally, the upper bound

is computed as  $\alpha = \bar{\sigma}(D''M(D'')^{-1})$ . We refer to the entire process of computing  $D''$  described above as the Generalized Osborne algorithm or GenOsborne for short, which is summarized in Algorithm 4.

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**Algorithm 4.** Upper bound: GenOsborne algorithm
 

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- 1: (Initialization) Use the standard Osbornes method on  $M$  to obtain the diagonal scaling matrix  $D_{nr}^*$ . Define  $M^{[1]} = D_{nr}^* M (D_{nr}^*)^{-1}$ . Set  $k = 1$ .
  - 2: **for**  $k = 1$  to  $v(v-1)$  **do**
  - 3:   Set  $(i, j)$
  - 4:   Compute coefficients  $\{c_e\}_{e=0}^5$  for  $(i, j)$  and  $M^{[k]}$ .
  - 5:   Find  $\bar{s}_{ij}^* = \operatorname{argmin}_{\bar{s}_{ij} \in \mathbb{R}^2} f_2(\bar{s}_{ij})$  using the damped newton method and form  $s_{ij}^* = s_{R_{ij}}^* + i s_{I_{ij}}^*$  from  $\bar{s}_{ij}^*$ .
  - 6:   Compute the corresponding  $D_{ij}^*$  and set  $M^{[k+1]} = D_{ij}^* M^{[k]} (D_{ij}^*)^{-1}$ ,
  - 7: **end for**
  - 8: Compute the upper bound  $\alpha = \bar{\sigma}(M^{[k]})$
-