Automatica 167 (2024) 111717

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Exact solution for the rank-one structured singular value with repeated complex full-block uncertainty[☆]

Talha Mushtaq [a,*], Peter Seiler [b], Maziar S. Hemati [a]

[a] Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, MN 55455, USA
[b] Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109, USA

ARTICLE INFO

Article history:
Received 10 June 2023
Received in revised form 3 February 2024
Accepted 24 March 2024
Available online 30 May 2024

ABSTRACT

In this note, we present an exact solution for the structured singular value (SSV) of rank-one complex matrices with repeated complex full-block uncertainty. A key step in the proof is the use of Von Neumann's trace inequality. Previous works provided exact solutions for rank-one SSV when the uncertainty contains repeated (real or complex) scalars and/or non-repeated complex full-block uncertainties. Our result with repeated complex full-blocks contains, as special cases, the previous results for repeated complex scalars and/or non-repeated complex full-block uncertainties. The repeated complex full-block uncertainty has recently gained attention in the context of incompressible fluid flows. Specifically, it has been used to analyze the effect of the convective nonlinearity in the incompressible Navier–Stokes equation (NSE). SSV analysis with repeated full-block uncertainty has led to an improved understanding of the underlying flow physics. We demonstrate our method on a turbulent channel flow model as an example.

1. Introduction

This paper focuses on the computation of the structured singular value (SSV) given a feedback-interconnection between a rank-one complex matrix and a block-structured uncertainty. The rank-one SSV is well-studied with some prominent results given in Chen, Fan, and Nett (1994a, 1994b), Young (1994). A standard SSV upper-bound can be formulated as a convex optimization (Packard & Doyle, 1993). This SSV upper-bound is equal to the true SSV for rank-one matrices when the uncertainty consists of repeated (real or complex) scalar blocks and non-repeated, complex full-blocks. This yields an explicit expression for the rank-one SSV with these uncertainty structures (see Theorem 1 and 2 in Young, 1994). Similar results are given in Chen et al. (1994a, 1994b), Fan, Doyle, and Tits (2006).

Our paper builds on this previous literature by providing an explicit solution to the rank-one SSV problem with repeated complex full-block uncertainty. This explicit solution is the main result and is stated as Theorem 3.1 in the paper. A key step in the proof is the use of Von Neumann's trace inequality (Von Neumann, 1962). The repeated complex full-block uncertainty structure contains, as special cases, repeated complex scalar blocks and non-repeated, complex full-blocks. Hence our explicit solution encompasses prior results for these cases.

The repeated complex full-block uncertainty structure has physical relevance in systems such as fluid flows. Specifically, this uncertainty structure has recently been used to provide consistent modeling of the nonlinear dynamics (Bhattacharjee, Mushtaq, Seiler, & Hemati, 2023; Liu, Colm-cille, & Gayme, 2022; Liu & Gayme, 2021; Mushtaq, Bhattacharjee, Seiler, & Hemati, 2023a; Mushtaq, Luhar, & Hemati, 2023). In Section 4, we demonstrate our rank-one solution to analyze a turbulent channel flow model (McKeon, 2017). Our explicit rank-one solution is compared against existing SSV upper and lower bound algorithms (Mushtaq, Bhattacharjee, Seiler, & Hemati, 2023b) that were developed for general (not-necessarily rank-one) systems.

2. Background: Structured singular value

Consider the standard SSV problem for square[1] complex matrices $M \in \mathbb{C}^{m\times m}$ given by the function $\mu : \mathbb{C}^{m\times m} \to \mathbb{R}$ as

☆ This material is based upon work supported by the ARO, United States under Grant Number W911NF-20-1-0156. MSH acknowledges support from the AFOSR, United States under award number FA 9550-19-1-0034, the National Science Foundation under Grant Number CBET-1943988 and ONR, United States under award number N000140-22-1-2029. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Sandip Roy under the direction of Editor André L. Tits.

* Corresponding author.
E-mail addresses: musht002@umn.edu (T. Mushtaq), pseiler@umich.edu (P. Seiler), mhemati@umn.edu (M.S. Hemati).

[1] We present the square complex matrix case to improve readability of the paper and minimize notation. The general rectangular complex matrix case can be handled by introducing some additional notation.

https://doi.org/10.1016/j.automatica.2024.111717
0005-1098/

(Packard & Doyle, 1993)

$$\mu(M) = (\min \|\Delta\| : \det(I_m - M\Delta) = 0)^{-1} \tag{1}$$

where $\Delta \in \mathbb{C}^{m\times m}$ is the structured uncertainty, I_m is an $m \times m$ identity, $\det(\cdot)$ is the determinant and $\|\cdot\|$ is the induced 2-norm which is equal to the maximum singular value. Then, $\mu(M)$ is the SSV of M. For the trivial case where $M = 0$, the minimization in (1) has no feasible point and $\mu(0) = 0$. In this paper, we will focus on the case where M is rank-one, i.e., $M = uv^{\mathrm{H}}$ for some $u, v \in \mathbb{C}^m$. Then, using the matrix determinant lemma, the minimization problem in (1) can be equivalently written as (Chen et al., 1994a; Young, 1994)

$$\mu(M) = \left(\min \|\Delta\| : v^{\mathrm{H}}\Delta u = 1\right)^{-1}. \tag{2}$$

Hence, for any structured Δ, the determinant constraint in (1) can be converted into an equivalent scalar constraint when M is rank-one. This scalar constraint is a special case of *affine parameter variation* problem for polynomials with perturbed coefficients (Qiu & Davison, 1989; Young, 1994). We will present solution for (2) when $\Delta \in \mathbf{\Delta}$, where $\mathbf{\Delta}$ is a set of repeated complex full-block uncertainties defined as

$$\begin{aligned}\mathbf{\Delta} := \Big\{ &\Delta = \mathrm{diag}(I_{r_1} \otimes \Delta_1, \ldots, I_{r_n} \otimes \Delta_n) \\ &: \Delta_i \in \mathbb{C}^{k_i \times k_i}\Big\} \subset \mathbb{C}^{m\times m}.\end{aligned} \tag{3}$$

This set is comprised of n blocks such that the i^{th} block, i.e., $I_{r_i} \otimes \Delta_i$, corresponds to a full $k_i \times k_i$ matrix repeated r_i times. Any uncertainty $\Delta \in \mathbf{\Delta}$ reduces to the complex uncertainties commonly found in the SSV literature:

(1) When $k_i = 1$ then Δ_i is a scalar, denoted as δ_i. In this case, the i^{th} block in (3) corresponds to a repeated complex scalar, i.e., $I_{r_i} \otimes \Delta_i = \delta_i I_{r_i}$,
(2) When $r_i = 1$ then the i^{th} block in (3) corresponds to a (non-repeated) complex full-block, i.e., $I_{r_i} \otimes \Delta_i = \Delta_i$.

Explicit rank-one solutions of $\mu(M)$ for these special cases are well-known (Chen et al., 1994a; Young, 1994). However, the current SSV literature does not present any explicit rank-one solutions of $\mu(M)$ for the repeated complex full-block case, which is a more general set of complex uncertainties, i.e., for any $\Delta \in \mathbf{\Delta}$. These uncertainty structures have physical importance in engineering systems such as fluid flows (Bhattacharjee et al., 2023; Liu et al., 2022; Liu & Gayme, 2021; Mushtaq et al., 2023a), where they have been exploited to provide physically consistent approximations of the convective nonlinearity in the Navier–Stokes equations (NSE). Therefore, in the next section, we will present an explicit rank-one solution of $\mu(M)$ for any $\Delta \in \mathbf{\Delta}$. It is important to note that the solutions presented in this paper are not limited to fluid problems and can be used for any other system that has $\Delta \in \mathbf{\Delta}$.

3. Repeated complex full-block uncertainty (Main result)

Consider the problem in (2) for any $\Delta \in \mathbf{\Delta}$. We can partition $u, v \in \mathbb{C}^m$ compatibly with the n blocks of $\Delta \in \mathbf{\Delta}$:

$$u = \begin{bmatrix} u_1^{\mathrm{H}} & \ldots & u_n^{\mathrm{H}}\end{bmatrix}^{\mathrm{H}},\; v = \begin{bmatrix} v_1^{\mathrm{H}} & \ldots & v_n^{\mathrm{H}}\end{bmatrix}^{\mathrm{H}} \tag{4}$$

where $u_i, v_i \in \mathbb{C}^{k_i r_i}$. Note that $m = \sum_{i=1}^n r_i k_i$. Since, the i^{th} block is $I_{r_i} \otimes \Delta_i$, we can further partition u_i, v_i based on the repeated structure:

$$u_i = \begin{bmatrix} u_{i,1}^{\mathrm{H}} & \ldots & u_{i,r_i}^{\mathrm{H}}\end{bmatrix}^{\mathrm{H}},\; v_i = \begin{bmatrix} v_{i,1}^{\mathrm{H}} & \ldots & v_{i,r_i}^{\mathrm{H}}\end{bmatrix}^{\mathrm{H}} \tag{5}$$

where each $u_{i,j}, v_{i,j} \in \mathbb{C}^{k_i}$. Based on this partitioning, define the following matrices (for $i = 1, \ldots, n$):

$$Z_i = \sum_{j=1}^{r_i} u_{i,j} v_{i,j}^{\mathrm{H}} \in \mathbb{C}^{k_i \times k_i}. \tag{6}$$

Lemma 3.1. *Let $M = uv^{\mathrm{H}}$ be given with $u, v \in \mathbb{C}^m$ and define Z_i as in (6). Then, for any $\Delta \in \mathbf{\Delta}$, we have*

$$\det(I_m - M\Delta) = 1 - \sum_{i=1}^n \mathrm{Tr}\,(Z_i\Delta_i). \tag{7}$$

Proof. Using the matrix determinant lemma, we have

$$\det(I_m - M\Delta) = 1 - v^{\mathrm{H}}\Delta u. \tag{8}$$

Now, using the block-structure of $\Delta \in \mathbf{\Delta}$ and the corresponding partitioning of (u, v), we can rewrite (8) as

$$\begin{aligned} 1 - v^{\mathrm{H}}\Delta u &= 1 - \sum_{i=1}^n v_i^{\mathrm{H}}\left(I_{r_i} \otimes \Delta_i\right) u_i \\ &= 1 - \sum_{i=1}^n \left[\sum_{j=1}^{r_i} v_{i,j}^{\mathrm{H}}\Delta_i u_{i,j}\right]. \end{aligned} \tag{9}$$

Note that the term in brackets is a scalar and hence equal to its trace. Thus, use the cyclic property of the trace as

$$\begin{aligned} \sum_{j=1}^{r_i} \mathrm{Tr}\left[v_{i,j}^{\mathrm{H}}\Delta_i u_{i,j}\right] &= \sum_{j=1}^{r_i} \mathrm{Tr}\left[u_{i,j}v_{i,j}^{\mathrm{H}}\Delta_i\right] \\ &= \mathrm{Tr}\left[Z_i\Delta_i\right]. \end{aligned} \tag{10}$$

Combine (8), (9) and (10) to obtain the stated result. □

Lemma 3.1 is used to provide an explicit solution for rank-one SSV with repeated complex full-blocks. This is stated next as Theorem 3.1.

Theorem 3.1. *Let $M = uv^{\mathrm{H}}$ be given with $u, v \in \mathbb{C}^m$ and define Z_i as in (6). Then,*

$$\mu(M) = \sum_{i=1}^n \sum_{j=1}^{k_i} \sigma_j(Z_i), \tag{11}$$

where $\sigma_j(Z_i)$ is the j^{th} singular value of Z_i.

Proof. Define $c = \sum_{i=1}^n \sum_{j=1}^{k_i} \sigma_j(Z_i)$ to simplify notation. The proof consists of 2 directions: $(i)\ \mu(M) \geq c$ and $(ii)\ \mu(M) \leq c$.

$(i)\ \mu(M) \geq c$: Let $Z_i = U_i\Sigma_i V_i^{\mathrm{H}}$ be the singular value decomposition (SVD) of Z_i. Note that $\Sigma_i = \mathrm{diag}(\sigma_1(Z_i), \ldots, \sigma_{k_i}(Z_i))$. Then, define $\overline{\Delta} \in \mathbf{\Delta}$ with the blocks $\overline{\Delta}_i = \frac{1}{c}V_iU_i^{\mathrm{H}}$ $(i = 1, \ldots, n)$. Thus, by Lemma 3.1, we have

$$\det(I_m - M\overline{\Delta}) = 1 - \sum_{i=1}^n \mathrm{Tr}\left[Z_i\overline{\Delta}_i\right]. \tag{12}$$

Now, substitute the SVD of Z_i in (12) and use the cyclic property of trace:

$$\begin{aligned} \det(I - M\overline{\Delta}) &= 1 - \sum_{i=1}^n \mathrm{Tr}\left[\Sigma_i V_i^{\mathrm{H}}\overline{\Delta}_i U_i\right] \\ &= 1 - \frac{1}{c}\sum_{i=1}^n \mathrm{Tr}\left[\Sigma_i\right] = 0. \end{aligned} \tag{13}$$

Hence $\overline{\Delta}$ causes singularity and $\|\overline{\Delta}\|_2 = \frac{1}{c}$. Thus, the minimum $\|\Delta\|$ in (2) must satisfy $\|\Delta\| \leq \frac{1}{c}$ and consequently, $\mu(M) \geq c$.

$(ii)\ \mu(M) \leq c$: Let $\Delta \in \mathbf{\Delta}$ be given with $\|\Delta\| < \frac{1}{c}$. Von Neumann's trace inequality (Von Neumann, 1962) gives:

$$|\mathrm{Tr}\left[Z_i\Delta_i\right]| \leq \sum_{j=1}^{k_i} \sigma_j(Z_i)\sigma_j(\Delta_i) \tag{14}$$

where $|\cdot|$ is the absolute value. Note that $\|\Delta\| < \frac{1}{c}$ implies that each block satisfies the same bound: $\sigma_j(\Delta_i) < \frac{1}{c}$. Hence, (14) implies

$$|\mathrm{Tr}[Z_i\Delta_i]| < \frac{1}{c}\sum_{j=1}^{k_i}\sigma_j(Z_i). \tag{15}$$

Next, using Lemma 3.1 and the inequality in (15), we get

$$\det(I_m - M\Delta) > 1 - \frac{1}{c}\sum_{i=1}^{n}\left[\sum_{j=1}^{k_i}\sigma_j(Z_i)\right] = 0. \tag{16}$$

Hence, any $\Delta \in \boldsymbol{\Delta}$ with $\|\Delta\| < \frac{1}{c}$ cannot cause $(I_m - M\Delta)$ to be singular. Thus, the minimum $\|\Delta\|$ in (2) must satisfy $\|\Delta\| \geq \frac{1}{c}$ and consequently, $\mu(M) \leq c$. □

Remark 3.1. For the special cases $r_i = 1$ and $k_i = 1$, the solution (11) yields $\mu(M) = \sum_{i=1}^{n}\|u_i\|_2\|v_i\|_2$ and $\mu(M) = \sum_{i=1}^{n}|v_i^{\mathrm{H}}u_i|$. These special cases correspond to solutions presented in previous works for non-repeated, complex full-block and repeated complex scalar uncertainties, respectively (Chen et al., 1994a; Young, 1994).

Remark 3.2. The proof for the lower bound of $\mu(M)$ provides a constructive solution to the structured uncertainty for rank-one systems, i.e., optimal $\Delta \in \boldsymbol{\Delta}$ can be constructed by computing Δ_i from the SVD of Z_i.

4. Illustration of results

In this section, we demonstrate our SSV solution method for repeated complex full-blocks using a rank-one approximation of the turbulent channel flow model. The work in Liu and Gayme (2021) shows that fluid flow models, in general, can be approximated as uncertain systems (i.e., flow dynamics are of the form in (17)), which can then be utilized to perform the SSV analysis. Thus, using the analysis, we can compute worst-case gains (SSV) associated with the flow disturbances to obtain physical insights into the energetic motion as well as compute mode shapes to extract flow instability mechanisms (Bhattacharjee et al., 2023; Liu et al., 2022; Liu, Shuai, Rath, & Gayme, 2023; Mushtaq et al., 2023a; Mushtaq, Luhar, & Hemati, 2023). In our example, we will limit our discussion to the computation of the worst-case gains to compare our rank-one SSV against the general upper and lower-bound algorithms that have been developed for (not necessarily rank-one) systems with repeated complex full-block uncertainties. The upper and lower-bounds are computed using *Algorithm 1* (Upper-Bounds) and *Algorithm 3* (Lower-Bounds) in Mushtaq et al. (2023b), which are based on Method of Centers (Boyd & Vandenberghe, 2004) and Power-Iteration (Packard & Doyle, 1993), respectively. Generally, these algorithms can be used for higher rank problems (see for example Mushtaq et al., 2023a and Bhattacharjee et al., 2023). Additionally, we will compare the computational times between each of the methods to demonstrate the computational scaling of the rank-one SSV solution.

4.1. Example

The spatially-discretized turbulent channel flow model described in McKeon (2017) has the following higher-order dynamical equation:

$$\begin{aligned} E(\kappa_x,\kappa_z)\dot{\phi}(y) &= A(Re,\kappa_x,\kappa_z)\phi(y) + B(\kappa_x,\kappa_z)f(y)\\ \zeta(y) &= C(\kappa_x,\kappa_z)\phi(y)\\ f(y) &= \Delta\zeta(y)\end{aligned} \tag{17}$$

where Re is the Reynolds number, κ_x and κ_z are the streamwise (x) and spanwise (z) direction wavenumbers resulting from the discretization, and the wall-normal direction is given by y. Here, the states $\phi(y) \in \mathbb{C}^{4N}$ and outputs $\zeta(y) \in \mathbb{C}^{9N}$ are given by the following:

$$\begin{aligned}\phi(y) &= [u(y)^{\mathrm{T}}, v(y)^{\mathrm{T}}, w(y)^{\mathrm{T}}, p(y)^{\mathrm{T}}]^{\mathrm{T}},\\ \zeta(y) &= [(\nabla u(y))^{\mathrm{T}}, (\nabla v(y))^{\mathrm{T}}, (\nabla w(y))^{\mathrm{T}}]^{\mathrm{T}}\end{aligned} \tag{18}$$

where $u(y) \in \mathbb{C}^N$, $v(y) \in \mathbb{C}^N$, $w(y) \in \mathbb{C}^N$ and $p(y) \in \mathbb{C}^N$ are streamwise, wall-normal and spanwise velocities, and pressure, respectively. Also, N is the number of collocation points in y to evaluate the system, $\nabla \in \mathbb{C}^{3N\times N}$ is the discrete gradient operator and $E(\kappa_x,\kappa_z) \in \mathbb{C}^{4N\times 4N}$, $A(Re,\kappa_x,\kappa_z) \in \mathbb{C}^{4N\times 4N}$, $B(\kappa_x,\kappa_z) \in \mathbb{C}^{4N\times 3N}$ and $C(\kappa_x,\kappa_z) \in \mathbb{C}^{9N\times 4N}$ are the matrix operators. Readers are referred to the work in McKeon (2017) for details on the construction of matrix operators. It is important to note that Δ for this system has a repeated complex full-block structure that results from the approximate modeling of the quadratic convective nonlinearity as,

$$f(y) = \begin{bmatrix} -u_\xi^{\mathrm{T}} & 0 & 0\\ 0 & -u_\xi^{\mathrm{T}} & 0\\ 0 & 0 & -u_\xi^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\nabla u\\ \nabla v\\ \nabla w\end{bmatrix} = (I_3 \otimes -u_\xi^{\mathrm{T}})\zeta(y) \tag{19}$$

where $f(y) \in \mathbb{C}^{3N}$ is the forcing signal and $u_\xi \in \mathbb{C}^{3N\times N}$ is the velocity gain matrix. Thus, the last row of equations in (17) describes the nonlinear forcing with $\Delta = I_3 \otimes -u_\xi^{\mathrm{T}}$ as the uncertainty matrix. Further details are given in Liu and Gayme (2021) about the Δ modeling. The input–output map of the system in (17) is given by,

$$H(y; Re, \omega, \kappa_x, \kappa_z) = C(\mathrm{i}\omega E - A)^{-1}B, \tag{20}$$

where ω is the temporal frequency. $H(y; Re, \omega, \kappa_x, \kappa_z)$ in (20) is, in general, not a rank-one matrix. However, for demonstration of our method, we will approximate $H(y; Re, \omega, \kappa_x, \kappa_z)$ as a rank-one input–output operator at each of the temporal frequencies ω for a fixed Re, κ_x and κ_z—as is commonly done for such analyses (McKeon, 2017):

$$M_{\omega_i} = \overline{\sigma}_i a_{1_i} b_{1_i}^{\mathrm{H}},\ \ i = 1, \ldots, N_\omega \tag{21}$$

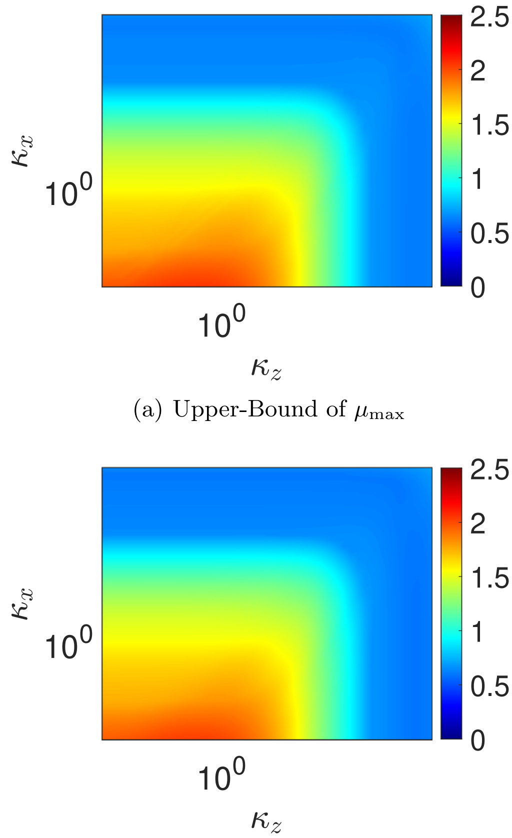
where N_ω are the total number of frequency points, $\overline{\sigma}_i \in \mathbb{R}_{\geq 0}$ is the maximum singular value of a matrix, and $a_{1_i} \in \mathbb{C}^{9N}$ and $b_{1_i} \in \mathbb{C}^{3N}$ are the left and right unitary vectors associated with $\overline{\sigma}_i$, respectively. Then, the rank-one SSV is given by $\mu_{\max} = \max_i \mu(M_{\omega_i})$, where $\mu(M_{\omega_i})$ is computed using (11).

4.2. Numerical implementation

We will compute $\mu_{\max}$ on an $N_\kappa \times N_\kappa \times N_\omega$ grid of space and temporal frequencies. The spatial frequencies (wavenumbers) κ_x and κ_z are both defined on a log-spaced grid of $N_\kappa = 50$ points in the interval $[10^{-1.45}, 10^{2.55}]$. This grid is denoted G_κ. The temporal frequency ω is defined on a grid $G_\omega := \{c_p G_\kappa\}$, where c_p is the wave speed, i.e., speed of the moving base flow (see McKeon, 2017 for details). Wave speeds are chosen as $c_p \in \{5, 10, 15, 18, 22\}$ resulting in $N_\omega = 250$ points in the temporal frequency grid. Additionally, we will fix $Re = 180$ and $N = 60$ for all computations and use MATLAB's `parfor` command to loop over temporal frequencies. Lastly, all percentage errors computed for comparison are relative errors.

4.3. Discussion

We can see in Fig. 1 that $\mu_{\max}$ values are qualitatively and quantitatively similar (within 5%) to the upper-bounds of $\mu_{\max}$ obtained from *Algorithm 1* in Mushtaq et al. (2023b). In fact, $\mu_{\max}$



(a) Upper-Bound of $\mu_{\max}$

(b) Exact Rank-One $\mu_{\max}$

Fig. 1. The plots depict the log_{10} values of $\mu_{\max}$ and its upper-bound. We see that $\mu_{\max}$ solutions are similar to its respective upper-bounds. The lower-bounds of $\mu_{\max}$ (not shown here) are "identical" to the $\mu_{\max}$ solutions, i.e., within 1% of each other.

values are "identical" to the lower-bound values of $\mu_{\max}$ (not shown here), i.e., values match up to 1%. Thus, the algorithms converge to the optimal solutions obtained from our method.

Furthermore, computing $\mu_{\max}$ is relatively fast as compared to obtaining its bounds (see Fig. 2). Each point on the plot in Fig. 2 represents the average[2] CPU time for a single data-point $(\omega, \kappa_x, \kappa_z)$ at each of the state dimensions. All computational times include CPU time for SVD of H to obtain a rank-one approximation. From the plot in Fig. 2, the upper-bound and lower-bound solutions have a time complexity of $\mathcal{O}(N^{2.83})$ and $\mathcal{O}(N^{1.525})$,[3] respectively. Meanwhile, computing $\mu_{\max}$ from our method has a time complexity of $\mathcal{O}(N^{1.28})$.

5. Conclusion

This work presents an exact solution of SSV for rank-one complex matrices with repeated, complex full-block uncertainties. The solution obtained from this method generalizes previous exact solutions for the repeated complex scalar and/or non-repeated complex full-block uncertainties (Chen et al., 1994a; Young, 1994). We illustrated the proposed method on a turbulent channel flow model. In future work, we would like to explore similar arguments to the ones presented here for rank-one complex matrices to compute SSV for general (not necessarily rank-one) complex matrices, especially when $\Delta \in \mathbf{\Delta}$.

[2] The CPU times are averaged over 10 data-points. We used an ASUS ROG M15 laptop with Intel 2.6 GHz i7-10750H CPU with 6 cores, 16 GB RAM, and an RTX 2070 Max-Q GPU for run time computations.

[3] Time complexities are computed from curve fitting the data in Fig. 2.

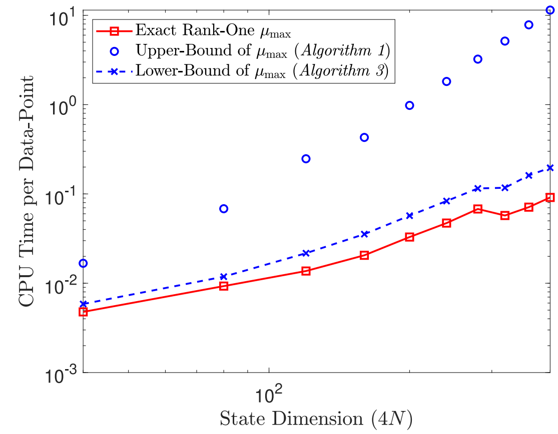


Fig. 2. The plots show the computational run time for $\mu_{\max}$, and upper and lower-bound calculations of $\mu_{\max}$.

References

Bhattacharjee, D., Mushtaq, T., Seiler, P. J., & Hemati, M. (2023). Structured input-output analysis of compressible plane couette flow. AIAA Paper 2023-1984.

Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge University Press.

Chen, J., Fan, M. K., & Nett, C. N. (1994a). Structured singular values and stability analysis of uncertain polynomials, Part 1: the generalized μ. *Systems & Control Letters*, *23*(1), 53–65.

Chen, J., Fan, M. K., & Nett, C. N. (1994b). Structured singular values and stability analysis of uncertain polynomials, Part 2: A missing link. *Systems & Control Letters*, *23*(2), 97–109.

Fan, M. K., Doyle, J. C., & Tits, A. L. (2006). Robustness in the presence of parametric uncertainty and unmodeled dynamics. In *Advances in computing and control* (pp. 363–367). Springer.

Liu, C., Colm-cille, P. C., & Gayme, D. F. (2022). Structured input–output analysis of stably stratified plane couette flow. *Journal of Fluid Mechanics*, *948*, A10.

Liu, C., & Gayme, D. F. (2021). Structured input–output analysis of transitional wall-bounded flows. *Journal of Fluid Mechanics*, *927*.

Liu, C., Shuai, Y., Rath, A., & Gayme, D. F. (2023). A structured input-output approach to characterizing optimal perturbations in wall-bounded shear flows. http://dx.doi.org/10.48550/arXiv.2303.10498, arXiv:2303.10498.

McKeon, B. (2017). The engine behind (wall) turbulence: perspectives on scale interactions. *Journal of Fluid Mechanics*, *817*, P1.

Mushtaq, T., Bhattacharjee, D., Seiler, P. J., & Hemati, M. (2023a). Structured input-output tools for modal analysis of a transitional channel flow. AIAA Paper 2023-1805.

Mushtaq, T., Bhattacharjee, D., Seiler, P., & Hemati, M. S. (2023b). Structured singular value of a repeated complex full block uncertainty. *Int. J. Robust Nonlinear Control*, in press.

Mushtaq, T., Luhar, M., & Hemati, M. (2023). Structured input-output analysis of turbulent flows over riblets. AIAA Paper 2023-3446.

Packard, A., & Doyle, J. (1993). The complex structured singular value. *Automatica*, *29*(1), 71–109.

Qiu, L., & Davison, E. (1989). A simple procedure for the exact stability robustness computation of polynomials with affine coefficient perturbations. *Syst. Control Lett.*, *13*(5), 413–420.

Von Neumann, J. (1962). *Some matrix-inequalities and metrization of matrix-space. Tomsk Univ. Rev. 1, 286–300 (1937). Reprinted in Collected Works*. Pergamon Press.

Young, P. M. (1994). The rank one mixed μ problem and 'Kharitonov-type' analysis. *Automatica*, *30*(12), 1899–1911.