



# The Nonexistence of Unicorns and Many-Sorted Löwenheim–Skolem Theorems

Benjamin Przybocki<sup>1</sup> , Guilherme Toledo<sup>2</sup> , Yoni Zohar<sup>2</sup> ,  
and Clark Barrett<sup>1</sup>

<sup>1</sup> Stanford University, Stanford, USA

[benjamin.przybocki@gmail.com](mailto:benjamin.przybocki@gmail.com), [barrett@cs.stanford.edu](mailto:barrett@cs.stanford.edu)

<sup>2</sup> Bar-Ilan University, Ramat Gan, Israel

**Abstract.** Stable infiniteness, strong finite witnessability, and smoothness are model-theoretic properties relevant to theory combination in satisfiability modulo theories. Theories that are strongly finitely witnessable and smooth are called *strongly polite* and can be effectively combined with other theories. Toledo, Zohar, and Barrett conjectured that stably infinite and strongly finitely witnessable theories are smooth and therefore strongly polite. They called counterexamples to this conjecture *unicorn theories*, as their existence seemed unlikely. We prove that, indeed, unicorns do not exist. We also prove versions of the Löwenheim–Skolem theorem and the Łoś–Vaught test for many-sorted logic.

## 1 Introduction

Given decision procedures for theories  $\mathcal{T}_1$  and  $\mathcal{T}_2$  with disjoint signatures, is there a decision procedure for  $\mathcal{T}_1 \cup \mathcal{T}_2$ ? In general, the answer is “not necessarily”, but a central question in Satisfiability Modulo Theories (SMT) [3] is: what assumptions on  $\mathcal{T}_1$  and  $\mathcal{T}_2$  suffice for theory combination? This line of research began with Nelson and Oppen’s theory combination procedure [15], which applies when  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are stably infinite, roughly meaning that every  $\mathcal{T}_i$ -satisfiable quantifier-free formula is satisfied by an infinite  $\mathcal{T}_i$ -interpretation for  $i \in \{1, 2\}$ .

The Nelson–Oppen procedure is quite useful, but requires *both* theories to be stably infinite, which is not always the case (e.g., the theories of bit-vectors and finite datatypes are not stably infinite). Thus, sufficient properties of only one of the theories were identified, such as gentleness [7], shininess [20], and flexibility [9]. The most relevant property for our purposes is strong politeness [4, 8, 18, 19]. It is essential to the functioning of the SMT solver cvc5 [1], which is called billions of times per day in industrial production code. A theory is *strongly polite* if it is smooth and strongly finitely witnessable, which are model-theoretic properties we will define later. These properties are more involved than stable infiniteness, so proving a theory to be strongly polite is more difficult. But the advantage of strongly polite theories is that they can be combined with any other decidable theory, including theories that are not stably infinite.

Given the abundance of model-theoretic properties relevant to theory combination, some of which interact in subtle ways, it behooves us to understand the logical relations between them. Recent papers [21, 22] have sought to understand the relations between seven model-theoretic properties—including stable infiniteness, smoothness, and strong finite witnessability—by determining which combinations of properties are possible in various signatures. In most cases, a theory with the desired combination of properties was constructed, or it was proved that none exists. The sole exception was theories that are stably infinite and strongly finitely witnessable but not smooth, dubbed *unicorn theories* and conjectured not to exist. Our main result, Theorem 2, confirms this conjecture.

Besides completing the taxonomy of properties from [21, 22], our result has practical consequences. The nonexistence of unicorns implies that strongly polite theories can be equivalently defined as those that are stably infinite and strongly finitely witnessable. Since it is easier to prove that a theory is stably infinite than to prove that it is smooth, this streamlines the process of proving that a theory is strongly polite. Thus, each time a new theory is introduced, proving that it can be combined with other theories becomes easier.<sup>1</sup> Similarly, our results give a new characterization of shiny theories, which makes it easier to prove that a theory is amenable to the shiny combination procedure (see Corollary 2).

We also believe that our result is of theoretical interest. Theorem 3, which is the main ingredient in the proof of Theorem 2, can be seen as a variant of the upward Löwenheim–Skolem theorem for many-sorted logic, since proving that a theory is smooth amounts to proving that cardinalities of sorts can be increased arbitrarily, including to uncountable cardinals. This result may be of independent interest to logicians studying the model theory of many-sorted logic, and we hope the proof techniques are useful to them as well.

Speaking of proof techniques, our proof is curious in that it uses Ramsey’s theorem from finite combinatorics. This is not the first time Ramsey’s theorem has been used in logic. Ramsey proved his theorem in the course of solving a special case of the decision problem for first-order logic [17]. Ramsey’s theorem also shows up in the Ehrenfeucht–Mostowski construction in model theory [5]. Our proof actually requires a generalization of Ramsey’s theorem, which we prove using the standard version of Ramsey’s theorem.

A major component of the proof of Theorem 2 amounts to proving a many-sorted version of the Löwenheim–Skolem theorem. On the course to proving this, we realized that a proper understanding of this theorem for many-sorted logic appears to be missing from the literature, despite the fact that the SMT-LIB standard [2] is based on many-sorted logic. To fill this gap, we prove generalizations of the Löwenheim–Skolem theorem for many-sorted logic, and use them to prove a many-sorted Łoś–Vaught test, useful for proving theory completeness.

The remainder of this paper is structured as follows. Section 2 provides background and definitions on many-sorted logic and SMT. Section 3 proves the main

---

<sup>1</sup> [21] already proved that stably infinite and strongly finitely witnessable theories can be combined with other theories. Our result gives a new proof (see Corollary 1), and shows that their procedure is not more general than polite combination.

result of this paper, namely the nonexistence of unicorn theories. Section 4 proves new many-sorted variants of the Löwenheim–Skolem theorem. Section 5 concludes and presents directions for future work.<sup>2</sup>

## 2 Preliminaries

### 2.1 Many-Sorted First-Order Logic

We work in many-sorted first-order logic [14]. A *signature*  $\Sigma$  consists of a non-empty set  $\mathcal{S}_\Sigma$  of sorts, a set  $\mathcal{F}_\Sigma$  of function symbols, and a set  $\mathcal{P}_\Sigma$  of predicate symbols containing an equality symbol  $=_\sigma$  for every sort  $\sigma \in \mathcal{S}_\Sigma$ .<sup>3</sup> Every function symbol has an arity  $(\sigma_1, \dots, \sigma_n, \sigma)$  and every predicate symbol an arity  $(\sigma_1, \dots, \sigma_n)$ , where  $\sigma_1, \dots, \sigma_n, \sigma \in \mathcal{S}_\Sigma$  and  $n \geq 0$ . Every equality symbol  $=_\sigma$  has arity  $(\sigma, \sigma)$ . To quantify a variable  $x$  of sort  $\sigma$ , we write  $\forall x : \sigma$ . and  $\exists x : \sigma$ . for the universal and existential quantifiers respectively. Let  $|\Sigma| = |\mathcal{S}_\Sigma| + |\mathcal{F}_\Sigma| + |\mathcal{P}_\Sigma|$ . If a signature contains only sorts and equalities, we say it is *empty*. Two signatures are said to be *disjoint* if they share at most sorts and equality symbols.

We define  $\Sigma$ -terms and  $\Sigma$ -formulas as usual. The set of free variables of sort  $\sigma$  in  $\varphi$  is denoted  $\text{vars}_\sigma(\varphi)$ . For  $S \subseteq \mathcal{S}_\Sigma$ , let  $\text{vars}_S(\varphi) = \bigcup_{\sigma \in S} \text{vars}_\sigma(\varphi)$ . We also let  $\text{vars}(\varphi) = \text{vars}_{\mathcal{S}_\Sigma}(\varphi)$ . A  $\Sigma$ -sentence is a  $\Sigma$ -formula with no free variables.

A  $\Sigma$ -structure  $\mathbb{A}$  interprets each sort  $\sigma \in \mathcal{S}_\Sigma$  as a nonempty set  $\sigma^\mathbb{A}$ , each function symbol  $f \in \mathcal{F}_\Sigma$  as a function  $f^\mathbb{A}$  with the appropriate domain and codomain, and each predicate symbol  $P \in \mathcal{P}_\Sigma$  as a relation  $P^\mathbb{A}$  over the appropriate set, such that  $=_\sigma^\mathbb{A}$  is the identity on  $\sigma^\mathbb{A}$ . A  $\Sigma$ -interpretation  $\mathcal{A}$  is a pair  $(\mathbb{A}, \nu)$ , where  $\mathbb{A}$  is a  $\Sigma$ -structure and  $\nu$  is a function, called an *assignment*, mapping each variable  $x$  of sort  $\sigma$  to an element  $\nu(x) \in \sigma^\mathbb{A}$ , denoted  $x^\mathcal{A}$ . We write  $t^\mathcal{A}$  for the interpretation of the  $\Sigma$ -term  $t$  under  $\mathcal{A}$ , which is defined in the usual way. The entailment relation, denoted  $\models$ , is defined as usual.

Two structures are *elementarily equivalent* if they satisfy the same sentences. We say that  $\mathbb{A}$  is an *elementary substructure* of  $\mathbb{B}$  if  $\mathbb{A}$  is a substructure of  $\mathbb{B}$  and, for all formulas  $\varphi$  and all assignments  $\nu$  on  $\mathbb{A}$ , we have  $(\mathbb{A}, \nu) \models \varphi$  if and only if  $(\mathbb{B}, \nu) \models \varphi$ . Note that if  $\mathbb{A}$  is an elementary substructure of  $\mathbb{B}$ , then they are elementarily equivalent.  $\mathcal{A}$  is an elementary subinterpretation of  $\mathcal{B}$  if  $\mathbb{A}$  is an elementary substructure of  $\mathbb{B}$  and  $\mathcal{A}$ 's assignment is the same as  $\mathcal{B}$ 's assignment.

Given a  $\Sigma$ -structure  $\mathbb{A}$ , let  $\mathcal{S}_{\geq \aleph_0}^\mathbb{A} = \{\sigma \in \mathcal{S}_\Sigma : |\sigma^\mathbb{A}| \geq \aleph_0\}$  and  $\mathcal{S}_{< \aleph_0}^\mathbb{A} = \mathcal{S}_\Sigma \setminus \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$ . We similarly define  $\mathcal{S}_{\geq \aleph_0}^\mathcal{A}$  and  $\mathcal{S}_{< \aleph_0}^\mathcal{A}$  for a  $\Sigma$ -interpretation  $\mathcal{A}$ .

A  $\Sigma$ -theory  $\mathcal{T}$  is a set of  $\Sigma$ -sentences, called the *axioms* of  $\mathcal{T}$ . We write  $\vdash_{\mathcal{T}} \varphi$  instead of  $\mathcal{T} \vdash \varphi$ . Structures satisfying  $\mathcal{T}$  are called  $\mathcal{T}$ -models, and interpretations satisfying  $\mathcal{T}$  are called  $\mathcal{T}$ -interpretations. We say a  $\Sigma$ -formula is  $\mathcal{T}$ -satisfiable if it is satisfied by some  $\mathcal{T}$ -interpretation, and we say two  $\Sigma$ -formulas are  $\mathcal{T}$ -equivalent if every  $\mathcal{T}$ -interpretation satisfies one if and only if it satisfies the other.  $\mathcal{T}$  is

<sup>2</sup> Due to lack of space, some proofs are omitted. They can be found in the arXiv version of this paper [16].

<sup>3</sup> When specifying a signature, we often omit the equality symbols, and include them implicitly. We also omit  $\sigma$  from  $=_\sigma$  when it does not cause confusion.

*complete* if for every sentence  $\varphi$ , we have  $\vdash_{\mathcal{T}} \varphi$  or  $\vdash_{\mathcal{T}} \neg\varphi$ .  $\mathcal{T}$  is *consistent* if there is no formula  $\varphi$  such that  $\vdash_{\mathcal{T}} \varphi$  and  $\vdash_{\mathcal{T}} \neg\varphi$ . If  $\Sigma_1$  and  $\Sigma_2$  are disjoint, let  $\Sigma_1 \cup \Sigma_2$  be the signature with the union of their sorts, function symbols, and predicate symbols. Given a  $\Sigma_1$ -theory  $\mathcal{T}_1$  and a  $\Sigma_2$ -theory  $\mathcal{T}_2$ , the  $(\Sigma_1 \cup \Sigma_2)$ -theory  $\mathcal{T}_1 \cup \mathcal{T}_2$  is the theory whose axioms are the union of the axioms of  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

The following theorem, proved in [14], is a many-sorted variant of the first-order compactness theorem.

**Theorem 1 (Compactness Theorem [14]).** *A set of  $\Sigma$ -formulas  $\Gamma$  is satisfiable if and only if every finite subset of  $\Gamma$  is satisfiable.*

We say that a  $\Sigma$ -theory  $\mathcal{T}$  has *built-in Skolem functions* if for all formulas  $\psi(\vec{x}, y)$ , there is  $f \in \mathcal{F}_{\Sigma}$  such that  $\vdash_{\mathcal{T}} \forall \vec{x}. (\exists y. (\psi(\vec{x}, y)) \rightarrow \psi(\vec{x}, f(\vec{x})))$ .<sup>4</sup> The following is a many-sorted variant of Lemma 2.3.6 of [12]. The proof is almost identical to that of the single-sorted case from [12].

**Lemma 1.** *If  $\mathcal{T}$  is a  $\Sigma$ -theory for a countable  $\Sigma$ , then there is a countable signature  $\Sigma^* \supseteq \Sigma$  and  $\Sigma^*$ -theory  $\mathcal{T}^* \supseteq \mathcal{T}$  with built-in Skolem functions.*

We state a many-sorted generalization of the Tarski–Vaught test, whose proof is also similar to the single-sorted case [12, Proposition 2.3.5].

**Lemma 2 (The Tarski–Vaught Test).** *Suppose  $\mathbb{A}$  is a substructure of  $\mathbb{B}$ . Then,  $\mathbb{A}$  is an elementary substructure of  $\mathbb{B}$  if and only if  $(\mathbb{B}, \nu) \models \exists v. \varphi(\vec{x}, v)$  implies  $(\mathbb{A}, \nu) \models \exists v. \varphi(\vec{x}, v)$  for every formula  $\varphi(\vec{x}, v)$  and assignment  $\nu$  over  $\mathbb{A}$ .*

## 2.2 Model-Theoretic Properties

**Definition 1.** *Let  $\Sigma$  be a many-sorted signature,  $S \subseteq \mathcal{S}_{\Sigma}$ , and  $\mathcal{T}$  a  $\Sigma$ -theory.*

- $\mathcal{T}$  is *stably infinite* with respect to  $S$  if for every  $\mathcal{T}$ -satisfiable quantifier-free formula  $\varphi$ , there is a  $\mathcal{T}$ -interpretation  $\mathcal{A}$  satisfying  $\varphi$  with  $|\sigma^{\mathcal{A}}| \geq \aleph_0$  for every  $\sigma \in S$ .
- $\mathcal{T}$  is *stably finite* with respect to  $S$  if for every quantifier-free  $\Sigma$ -formula  $\varphi$  and  $\mathcal{T}$ -interpretation  $\mathcal{A}$  satisfying  $\varphi$ , there is a  $\mathcal{T}$ -interpretation  $\mathcal{B}$  satisfying  $\varphi$  such that  $|\sigma^{\mathcal{B}}| \leq |\sigma^{\mathcal{A}}|$  and  $|\sigma^{\mathcal{B}}| < \aleph_0$  for every  $\sigma \in S$ .
- $\mathcal{T}$  is *smooth* with respect to  $S$  if for every quantifier-free formula  $\varphi$ ,  $\mathcal{T}$ -interpretation  $\mathcal{A}$  satisfying  $\varphi$ , and function  $\kappa$  from  $S$  to the class of cardinals such that  $\kappa(\sigma) \geq |\sigma^{\mathcal{A}}|$  for every  $\sigma \in S$ , there is a  $\mathcal{T}$ -interpretation  $\mathcal{B}$  satisfying  $\varphi$  with  $|\sigma^{\mathcal{B}}| = \kappa(\sigma)$  for every  $\sigma \in S$ .

Next, we define *arrangements*. Given a set of sorts  $S \subseteq \mathcal{S}_{\Sigma}$ , finite sets of variables  $V_{\sigma}$  of sort  $\sigma$  for each  $\sigma \in S$ , and equivalence relations  $E_{\sigma}$  on  $V_{\sigma}$ , the *arrangement*  $\delta_V$  on  $V = \bigcup_{\sigma \in S} V_{\sigma}$  induced by  $E = \bigcup_{\sigma \in S} E_{\sigma}$  is

$$\bigwedge_{\sigma \in S} \left[ \bigwedge_{xE_{\sigma}y} (x = y) \wedge \bigwedge_{x \overline{E_{\sigma}} y} \neg(x = y) \right],$$

<sup>4</sup> Intuitively:  $\mathcal{T}$  has enough function symbols to witness all existential formulas.

where  $\overline{E_\sigma}$  is the complement of  $E_\sigma$ .

**Definition 2.** Let  $\Sigma$  be a many-sorted signature,  $S \subseteq \mathcal{S}_\Sigma$  a finite set, and  $\mathcal{T}$  a  $\Sigma$ -theory. Then  $\mathcal{T}$  is *strongly finitely witnessable* with respect to  $S$  if there is a computable function  $wit$  from the quantifier-free formulas into themselves such that for every quantifier-free formula  $\varphi$ :

- (i)  $\varphi$  and  $\exists \vec{w}. wit(\varphi)$  are  $\mathcal{T}$ -equivalent, where  $\vec{w} = vars(wit(\varphi)) \setminus vars(\varphi)$ ; and
- (ii) given a finite set of variables  $V$  and an arrangement  $\delta_V$  on  $V$ , if  $wit(\varphi) \wedge \delta_V$  is  $\mathcal{T}$ -satisfiable, then there is a  $\mathcal{T}$ -interpretation  $\mathcal{A}$  satisfying  $wit(\varphi) \wedge \delta_V$  such that  $\sigma^{\mathcal{A}} = vars_\sigma(wit(\varphi) \wedge \delta_V)^{\mathcal{A}}$  for every  $\sigma \in S$ .

## 2.3 Notation

$\mathbb{N}$  denotes the set of non-negative integers. Given  $m, n \in \mathbb{N}$ , let  $[m, n] := \{\ell \in \mathbb{N} : m \leq \ell \leq n\}$  and  $[n] := [1, n]$ . Given a set  $X$ , let  $P_n(X) := \{Y \subseteq X : |Y| = n\}$ ,  $X^n := \{(x_1, \dots, x_n) : x_i \in X \text{ for all } i \in [n]\}$ , and  $X^* := \bigcup_{n \in \mathbb{N}} X^n$ . For any  $x$ , we denote  $(x_1, \dots, x_n)$  by  $(x)^{\oplus n}$ . Given a tuple of tuples  $(\vec{x}_1, \dots, \vec{x}_n)$ , where  $\vec{x}_i \in X^*$  for all  $i$ , we will often treat it as an element of  $X^*$  by flattening the tuple.

## 3 The Nonexistence of Unicorns

We now state our main theorem, which implies that unicorn theories do not exist. Note that since we are motivated by applications to SMT, we hereafter assume all signatures are countable.<sup>5</sup>

**Theorem 2.** Assume that  $\mathcal{T}$  is a  $\Sigma$ -theory, where  $\Sigma$  is countable. If  $\mathcal{T}$  is stably infinite and strongly finitely witnessable, both with respect to  $S \subseteq \mathcal{S}_\Sigma$ , then  $\mathcal{T}$  is smooth with respect to  $S$ .

For our proof, we define a weaker variant of smoothness, that focuses the requirements only for finite cardinals.

**Definition 3.** A  $\Sigma$ -theory  $\mathcal{T}$  is *finitely smooth* with respect to  $S \subseteq \mathcal{S}_\Sigma$  if for every quantifier-free formula  $\varphi$ ,  $\mathcal{T}$ -interpretation  $\mathcal{A}$  with  $\mathcal{A} \models \varphi$ , and function  $\kappa$  from  $\mathcal{S}_{<\aleph_0}^A \cap S$  to the class of cardinals with  $|\sigma^{\mathcal{A}}| \leq \kappa(\sigma) < \aleph_0$  for every  $\sigma \in \mathcal{S}_{<\aleph_0}^A \cap S$ , there is a  $\mathcal{T}$ -interpretation  $\mathcal{B}$  with  $\mathcal{B} \models \varphi$  with  $|\sigma^{\mathcal{B}}| = \kappa(\sigma)$  for every  $\sigma \in \mathcal{S}_{<\aleph_0}^A \cap S$ .

We make use of the following two lemmas.

**Lemma 3.** If  $\mathcal{T}$  is stably infinite and strongly finitely witnessable, both with respect to some set of sorts  $S \subseteq \mathcal{S}_\Sigma$ , then  $\mathcal{T}$  is finitely smooth with respect to  $S$ .

**Lemma 4 ([22, Theorem 3]).** If  $\mathcal{T}$  is strongly finitely witnessable with respect to some set of sorts  $S \subseteq \mathcal{S}_\Sigma$ , then  $\mathcal{T}$  is stably finite with respect to  $S$ .

---

<sup>5</sup> The paper that introduced unicorn theories [21] also made this assumption.

In light of the above two lemmas, the following theorem implies Theorem 2.

**Theorem 3.** *Assume that  $\mathcal{T}$  is a  $\Sigma$ -theory, where  $\Sigma$  is countable. If  $\mathcal{T}$  is stably finite and finitely smooth, both with respect to some set of sorts  $S \subseteq \mathcal{S}_\Sigma$ , then  $\mathcal{T}$  is smooth with respect to  $S$ .*

The remainder of this section is thus dedicated to the proof of Theorem 3.

### 3.1 Motivating the Proof

In this section, we illustrate the proof technique with a simple example. The goal is to motivate the proof of Theorem 3 before delving into the details.

Suppose  $\mathcal{T}$  is a  $\Sigma$ -theory, where  $\mathcal{S}_\Sigma = \{\sigma_1, \sigma_2\}$ ,  $\mathcal{F}_\Sigma = \{f\}$ ,  $f$  has arity  $(\sigma_2, \sigma_1)$ , and the only predicate symbols are equalities. Suppose that  $\mathcal{T}$  is also stably finite and finitely smooth, both with respect to  $S = \mathcal{S}_\Sigma$ . Let  $\varphi$  be a  $\mathcal{T}$ -satisfiable quantifier-free formula and  $\mathcal{A}$  a  $\mathcal{T}$ -interpretation satisfying  $\varphi$ . Let  $\kappa$  be a function from  $S$  to the class of cardinals such that  $\kappa(\sigma) \geq |\sigma^\mathcal{A}|$  for both  $\sigma \in S$ . For concreteness, suppose  $|\sigma_1^\mathcal{A}| = |\sigma_2^\mathcal{A}| = 10$ ,  $\kappa(\sigma_1) = \aleph_0$ , and  $\kappa(\sigma_2) = \aleph_1$ . Our goal is to show that there is a  $\mathcal{T}$ -interpretation  $\mathcal{B}^-$  satisfying  $\varphi$  with  $|\sigma_1^{\mathcal{B}^-}| = \aleph_0$  and  $|\sigma_2^{\mathcal{B}^-}| = \aleph_1$ .<sup>6</sup>

A natural thought is to apply some variant of the upward Löwenheim–Skolem theorem, but this doesn't quite work. As will be seen in Sect. 4, generalizations of the Löwenheim–Skolem theorem to many-sorted logic do not let us control the cardinalities of  $\sigma_1$  and  $\sigma_2$  independently. Nevertheless, let us emulate the standard proof technique for the upward Löwenheim–Skolem theorem.

Here is the most natural way of generalizing the proof of the upward Löwenheim–Skolem theorem to our setting. For simplicity, assume that  $\mathcal{T}$  already has built-in Skolem functions. We introduce  $\aleph_0$  new constants  $\{c_{1,\alpha}\}_{\alpha < \omega}$  and  $\aleph_1$  new constants  $\{c_{2,\alpha}\}_{\alpha < \omega_1}$ . We define a set of formulas  $\Gamma = \{\varphi\} \cup \Gamma_1$ , where

$$\Gamma_1 = \{\neg(c_{i,\alpha} = c_{i,\beta}) : i \in \{1, 2\}; \alpha, \beta < \kappa(\sigma_i); \alpha \neq \beta\}.$$

By Theorem 1 and finite smoothness, there is a  $\mathcal{T}$ -interpretation  $\mathcal{B}$  satisfying  $\Gamma$ : indeed, were that not true, Theorem 1 would guarantee that some finite subset of  $\Gamma$  is unsatisfiable; yet such a set would only demand the existence of finitely many new elements, which can be achieved by making use of finite smoothness. Since  $\mathcal{B} \models \Gamma_1$ , we have  $|\sigma_1^\mathcal{B}| \geq \aleph_0$  and  $|\sigma_2^\mathcal{B}| \geq \aleph_1$ .

Since  $\mathcal{B}$  may be too large, we construct a subinterpretation  $\mathcal{B}^-$  with

$$\begin{aligned}\sigma_1^{\mathcal{B}^-} &= \{c_{1,\alpha}^\mathcal{B}\}_{\alpha < \omega} \cup \{f^\mathcal{B}(c_{2,\alpha}^\mathcal{B})\}_{\alpha < \omega_1} \\ \sigma_2^{\mathcal{B}^-} &= \{c_{2,\alpha}^\mathcal{B}\}_{\alpha < \omega_1}.\end{aligned}$$

And using the assumption that  $\mathcal{T}$  has built-in Skolem functions, we can prove that  $\mathcal{B}^-$  is an elementary subinterpretation of  $\mathcal{B}$ , so  $\mathcal{B}^- \models \Gamma$ ; we can then prove

<sup>6</sup> The reason for the  $-$  superscript in  $\mathcal{B}^-$  will be clear presently.

that  $|\sigma_2^{\mathcal{B}^-}| = \aleph_1$ , but we unfortunately cannot guarantee that  $|\sigma_1^{\mathcal{B}^-}| = \aleph_0$ . This is because  $\mathcal{B}^-$  has not only the  $\aleph_1$  elements  $\{c_{2,\alpha}^{\mathcal{B}}\}_{\alpha < \omega_1}$  of sort  $\sigma_2$ , but also the elements  $\{f^{\mathcal{B}}(c_{2,\alpha}^{\mathcal{B}})\}_{\alpha < \omega_1}$  of sort  $\sigma_1$ . The function symbol  $f$  has created a “spillover” of elements from  $\sigma_2$  to  $\sigma_1$ .

To fix this, we need to ensure that  $|\{f^{\mathcal{B}}(c_{2,\alpha}^{\mathcal{B}})\}_{\alpha < \omega_1}| \leq \aleph_0$ . To that end, define  $\Gamma$  to instead be  $\{\varphi\} \cup \Gamma_1 \cup \Gamma_2$ , where

$$\Gamma_2 = \{f(b) = f(d) : b, d \in \{c_{2,\alpha}\}_{\alpha < \omega_1}\}.$$

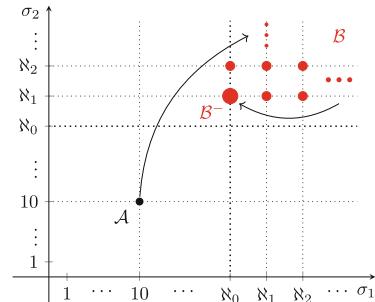
Then, if there is a model  $\mathcal{B}$  satisfying  $\Gamma$ , we have  $|\{f^{\mathcal{B}}(c_{2,\alpha}^{\mathcal{B}})\}_{\alpha < \omega_1}| = 1 \leq \aleph_0$ . To show  $\Gamma$  is  $\mathcal{T}$ -satisfiable, it suffices by the compactness theorem to show that  $\mathcal{T} \cup \Gamma'$  is satisfiable for every finite subset  $\Gamma' \subseteq \Gamma$ . So let  $\Gamma'_1 \subseteq \Gamma_1$  and  $\Gamma'_2 \subseteq \Gamma_2$  be finite subsets. We will construct a  $\mathcal{T}$ -interpretation  $\mathcal{B}'$  such that  $\mathcal{B}' \models \{\varphi\} \cup \Gamma'_1 \cup \Gamma'_2$ . For concreteness, suppose that  $\{c_{1,0}, c_{1,1}, \dots, c_{1,99}\}$  and  $\{c_{2,0}, c_{2,1}, \dots, c_{2,9}\}$  are the new constants that appear in  $\Gamma'_1 \cup \Gamma'_2$ . By finite smoothness, there is a  $\mathcal{T}$ -interpretation  $\mathcal{B}'$  satisfying  $\varphi$  such that  $|\sigma_1^{\mathcal{B}'}| = 100$  and  $|\sigma_2^{\mathcal{B}'}| = 901$ . By the pigeonhole principle, there is a subset  $Y \subseteq \sigma_2^{\mathcal{B}'}$  with  $|Y| \geq 10$  such that  $f^{\mathcal{B}'}$  is constant on  $Y$ ; if 901 pigeons are put in 100 holes, then some hole has at least 10 pigeons (although this is not true for 900 pigeons). Then,  $\mathcal{B}'$  can interpret the constants  $\{c_{1,0}, c_{1,1}, \dots, c_{1,99}\}$  as distinct elements of  $\sigma_1^{\mathcal{B}'}$  and the constants  $\{c_{2,0}, c_{2,1}, \dots, c_{2,9}\}$  as distinct elements of  $Y$ . This proves that  $\Gamma$  is  $\mathcal{T}$ -satisfiable.

We illustrate the top level structure of the proof idea in Fig. 1, applied to the working example. The  $x$  axis represents cardinalities of interpretations of  $\sigma_1$ , and the  $y$  axis does the same for  $\sigma_2$ . Starting from the interpretation  $\mathcal{A}$  with  $|\sigma_1^{\mathcal{A}}| = |\sigma_2^{\mathcal{A}}| = 10$ , we construct some interpretation  $\mathcal{B}$ , represented by the array of red dots as there is some degree of uncertainty regarding the precise cardinalities of its domains, with  $|\sigma_1^{\mathcal{B}}| \geq \aleph_0$  and  $|\sigma_2^{\mathcal{B}}| \geq \aleph_1$ . From  $\mathcal{B}$  we hope to construct  $\mathcal{B}^-$ , which has  $|\sigma_1^{\mathcal{B}^-}| = \aleph_0$  and  $|\sigma_2^{\mathcal{B}^-}| = \aleph_1$ : the latter can be achieved using techniques similar to the many-sorted Löwenheim-Skolem theorems (see Sect. 4 below), while the former requires the aforementioned pigeonhole principle arguments.

The above proof sketch illustrates the main ideas behind the proof of Theorem 3. The generalization to more sorts and function symbols requires some extra bookkeeping. More interestingly, the generalization to functions of arity greater than one requires a version of Ramsey’s theorem, which is a generalization of the pigeonhole principle.

### 3.2 Ramsey’s Theorem and Generalizations

In this section, we state Ramsey’s theorem and a generalization of it.



**Fig. 1.** How we move from interpretation to interpretation

Ramsey's theorem is sometimes stated in terms of coloring the edges of hypergraphs, but for our purposes it is more convenient to state it as follows. In the following lemma, the notations  $P_n(X)$  and  $[k]$  are defined as in Sect. 2.3.

**Lemma 5 (Ramsey's theorem [17, Theorem B]).** *For any  $k, n, m \in \mathbb{N}$ , there is an  $R(k, n, m) \in \mathbb{N}$  such that for any set  $X$  with  $|X| \geq R(k, n, m)$  and function  $f : P_n(X) \rightarrow [k]$ , there is a subset  $Y \subseteq X$  with  $|Y| \geq m$  such that  $f$  is constant on  $P_n(Y)$ .*

Note that in Ramsey's theorem, the set  $[k]$  can be replaced by any set of cardinality  $k$ .

We want to generalize Ramsey's theorem to functions  $f : X^n \rightarrow [k]$ . The most natural generalization would state that there is a large subset  $Y \subseteq X$  such that  $f$  is constant on  $Y^n$ . But this generalization is false, as the following example shows.

*Example 1.* Let  $X = \mathbb{Z}$ , and let  $f : X^2 \rightarrow [2]$  be given by

$$f(m, n) = \begin{cases} 1 & \text{if } m < n \\ 2 & \text{otherwise.} \end{cases}$$

Then,  $f(m, n) \neq f(n, m)$  for all  $m, n \in X$  with  $m \neq n$ . Thus, there is no subset  $Y \subseteq X$  with  $|Y| \geq 2$  such that  $f$  is constant on  $Y^2$ .

To avoid counterexamples like this, our generalization needs to consider the order of the arguments of  $f$ . This motivates the following definition.

**Definition 4.** *Let  $(X, <)$  be a totally ordered set, and let  $\vec{x} = (x_1, \dots, x_n)$  and  $\vec{y} = (y_1, \dots, y_n)$  be elements of  $X^n$ . We write  $\vec{x} \sim \vec{y}$  if for every  $1 \leq i < j \leq n$  we have*

$$\begin{aligned} x_i < x_j &\iff y_i < y_j \quad \text{and} \\ x_i = x_j &\iff y_i = y_j. \end{aligned}$$

Observe that  $\sim$  is an equivalence relation on  $X^n$  with finitely many equivalence classes.<sup>7</sup>

Now we can state our first generalization of Ramsey's theorem.

**Lemma 6.** *For any  $k, n, m \in \mathbb{N}$ , there is an  $R^*(k, n, m) \in \mathbb{N}$  such that for any totally ordered set  $(X, <)$  with  $|X| \geq R^*(k, n, m)$  and function  $f : X^n \rightarrow [k]$ , there is a subset  $Y \subseteq X$  with  $|Y| \geq m$  such that  $f$  is constant on each  $\sim$ -equivalence class of  $Y^n$ .*

Next, we further generalize Ramsey's theorem to multiple functions  $f_1, \dots, f_r$ .

---

<sup>7</sup> The number of equivalence classes is given by the ordered Bell numbers (<https://oeis.org/A000670>).

**Lemma 7.** *For any  $k, m \in \mathbb{N}$  and  $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ , there is a number  $R^{**}(k, \vec{n}, m) \in \mathbb{N}$ , such that for any totally ordered set  $(X, <)$  with  $|X| \geq R^{**}(k, \vec{n}, m)$  and functions  $f_i : X^{n_i} \rightarrow [k]$  for  $i \in [r]$ , there is a subset  $Y \subseteq X$  with  $|Y| \geq m$ , such that  $f_i$  is constant on each  $\sim$ -equivalence class of  $Y^{n_i}$  for all  $i \in [r]$ .*

### 3.3 The Proof of Theorem 3

Fix a  $\Sigma$ -theory  $\mathcal{T}$  and a set of sorts  $S \subseteq \mathcal{S}_\Sigma$ . Assume that  $\Sigma$  is countable. Suppose that  $\mathcal{T}$  is stably finite and finitely smooth, both with respect to  $S$ . Let  $\varphi$  be a  $\mathcal{T}$ -satisfiable quantifier-free formula and  $\mathcal{A}$  a  $\mathcal{T}$ -interpretation satisfying  $\varphi$ . Let  $\kappa$  be a function from  $S$  to the class of cardinals such that  $\kappa(\sigma) \geq |\sigma^\mathcal{A}|$  for every  $\sigma \in S$ .

Write  $S = \{\sigma_1, \sigma_2, \dots\}$  and, without loss of generality, assume  $\kappa(\sigma_1) \leq \kappa(\sigma_2) \leq \dots$ . For notational convenience, we write all  $\Sigma$ -terms in the form  $t(\vec{x}_1, \vec{x}_2, \dots)$ ,<sup>8</sup> where  $\vec{x}_i$  is a tuple of variables of sort  $\sigma_i$ . If  $\kappa(\sigma_i) < \aleph_0$  for all  $i$ , then we are done by the fact  $\mathcal{T}$  is finitely smooth. Otherwise, let  $\ell$  be the largest natural number such that  $\kappa(\sigma_\ell) < \aleph_0$  if there is such a number, and let  $\ell = 0$  otherwise.

The proof of Theorem 3 proceeds in two steps. First, we construct a set of formulas  $\Gamma$  such that  $\varphi \in \Gamma$  and prove that there is a  $\mathcal{T}$ -interpretation  $\mathcal{B}$  satisfying  $\Gamma$ . Second, we prove that  $\mathcal{B}$  has an elementary subinterpretation  $\mathcal{B}^-$  such that  $|\sigma_i^{\mathcal{B}^-}| = \kappa(\sigma_i)$  for all  $i$ . Since  $\varphi \in \Gamma$ , it will follow that  $\mathcal{T}$  is smooth.

The assumption that  $\mathcal{T}$  is stably finite and finitely smooth is used to construct  $\mathcal{T}$ -interpretations of the following form, which will be useful for a compactness argument.

**Lemma 8.** *There is a  $\mathcal{T}$ -interpretation  $\mathcal{B}$  satisfying  $\varphi$  such that  $|\sigma_i^\mathcal{B}| = \kappa(\sigma_i)$  for all  $i \leq \ell$ , and  $|\sigma_i^\mathcal{B}|$  is arbitrarily large but finite for all  $i > \ell$ .*

*Proof.* First, apply stable finiteness to get a  $\mathcal{T}$ -interpretation  $\mathcal{A}'$  satisfying  $\varphi$  such that  $|\sigma_i^{\mathcal{A}'}| \leq |\sigma_i^\mathcal{A}|$  and  $|\sigma_i^{\mathcal{A}'}| < \aleph_0$  for all  $i$ . Then, apply finite smoothness to  $\mathcal{A}'$  with  $\kappa'$  given by  $\kappa'(\sigma_i) = \kappa(\sigma_i)$  for all  $i \leq \ell$  and  $\kappa'(\sigma_i)$  arbitrarily large but finite for all  $i > \ell$ .  $\square$

It will be convenient to work with a theory with built-in Skolem functions, so we use Lemma 1 to get a  $\Sigma^*$ -theory  $\mathcal{T}^* \supseteq \mathcal{T}$ , where  $\Sigma^* \supseteq \Sigma$  and  $\Sigma^*$  is countable. To construct our set of formulas  $\Gamma$ , we introduce  $\kappa(\sigma_i)$  new constants  $\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}$  of sort  $\sigma_i$  for each  $i$ . We consider these constants to be part of an even larger signature  $\Sigma' \supseteq \Sigma^*$ . In what follows, we construct sentences and interpretations over  $\Sigma'$ . Impose an arbitrary total order on each  $\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}$  to be used for the  $\sim$  relation. For the definition below, recall that given a set  $X$ , we define  $X^* = \bigcup_{n \in \mathbb{N}} X^n$ .

<sup>8</sup> Even if  $S$  is infinite, the denoted term is still finite since each term only has a finite number of variables occurring in it.

**Definition 5.** We define a set of formulas  $\Gamma = \{\varphi\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ , where

$$\begin{aligned}\Gamma_1 &= \{\neg(c_{i,\alpha} = c_{i,\beta}) : 1 \leq i \leq |S|; \alpha, \beta < \kappa(\sigma_i); \alpha \neq \beta\} \\ \Gamma_2 &= \left\{ t \left( \overrightarrow{c_1}, \dots, \overrightarrow{c_i}, \overrightarrow{b_{i+1}}, \overrightarrow{b_{i+2}}, \dots \right) = t \left( \overrightarrow{c_1}, \dots, \overrightarrow{c_i}, \overrightarrow{d_{i+1}}, \overrightarrow{d_{i+2}}, \dots \right) : \right. \\ &\quad \left. t \text{ is a } \Sigma^* \text{-term of sort } \sigma_i; i > \ell; \overrightarrow{c_k}, \overrightarrow{b_k}, \overrightarrow{d_k} \in (\{c_{k,\alpha}\}_{\alpha < \kappa(\sigma_k)})^* \right. \\ &\quad \left. \text{for all } k; \overrightarrow{b_j} \sim \overrightarrow{d_j} \text{ for all } j > i \right\} \\ \Gamma_3 &= \left\{ \forall x : \sigma_i. \bigvee_{\alpha < \kappa(\sigma_i)} x = c_{i,\alpha} : i \leq \ell \right\}.\end{aligned}$$

Note that the disjunctions in  $\Gamma_3$  are finite given the condition  $i \leq \ell$ .

**Lemma 9.** There is a  $\mathcal{T}^*$ -interpretation  $\mathcal{B}$  such that  $\mathcal{B} \vDash \Gamma$ .

This lemma forms the core of the argument. By the compactness theorem, it suffices to prove that for any finite subset  $\Gamma' \subseteq \Gamma$ , there is a  $\mathcal{T}^*$ -interpretation  $\mathcal{B}'$  such that  $\mathcal{B}' \vDash \Gamma'$ . The tricky part is making  $\mathcal{B}'$  satisfy  $\Gamma' \cap \Gamma_2$ . The strategy is to use Lemma 8 to construct a model  $\mathcal{B}'$  in which  $|\sigma_{i+1}^{\mathcal{B}'}|$  is very large in terms of  $|\sigma_i^{\mathcal{B}'}|$  for each  $i > \ell$ . Lemma 7 will ensure that there is some way of interpreting the constants  $\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}$  so that  $\mathcal{B}' \vDash \Gamma' \cap \Gamma_2$ .

We are now ready to prove Theorem 3.

*Proof (Theorem 3).* By Lemma 9, there is a  $\mathcal{T}^*$ -interpretation  $\mathcal{B}$  such that  $\mathcal{B} \vDash \Gamma$ . Let

$$B = \{t^{\mathcal{B}}((\overrightarrow{c_1})^{\mathcal{B}}, (\overrightarrow{c_2})^{\mathcal{B}}, \dots) : t \text{ is a } \Sigma^*\text{-term}; \overrightarrow{c_i} \in (\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)})^* \text{ for all } i\}.$$

For every  $f \in \mathcal{F}_{\Sigma}$ , the set  $B$  is closed under  $f^{\mathcal{B}}$ . Thus, we can define  $\mathcal{B}^-$  to be the subinterpretation of  $\mathcal{B}$  obtained by restricting the sorts, functions, and predicates to  $B$ .<sup>9</sup> Since the  $\Sigma^*$ -theory  $\mathcal{T}^*$  has built-in Skolem functions,  $\mathcal{B}^-$  is an elementary subinterpretation of  $\mathcal{B}$  by Lemma 2. We claim  $|\sigma_i^{\mathcal{B}^-}| = \kappa(\sigma_i)$  for all  $i$ .

First,  $\{c_{i,\alpha}^{\mathcal{B}^-}\}_{\alpha < \kappa(\sigma_i)}$  is a set of  $\kappa(\sigma_i)$  distinct elements in  $\sigma_i^{\mathcal{B}^-}$ , because  $\mathcal{B}^- \models \Gamma_1$ . Thus,  $|\sigma_i^{\mathcal{B}^-}| \geq \kappa(\sigma_i)$  for all  $i$ .

Second,  $|\sigma_i^{\mathcal{B}^-}| \leq |\{c_{i,\alpha}\}_{\alpha < \kappa(\sigma_i)}| = \kappa(\sigma_i)$  for all  $i \in [\ell]$ , as  $\mathcal{B}^- \models \Gamma_3$ .

Finally, it remains to show that  $|\sigma_i^{\mathcal{B}^-}| \leq \kappa(\sigma_i)$  for all  $i > \ell$ . Inductively suppose that  $|\sigma_j^{\mathcal{B}^-}| \leq \kappa(\sigma_j)$  for all  $j < i$ . Now, every element of  $\sigma_i^{\mathcal{B}^-}$  is of the form

$$t^{\mathcal{B}}((\overrightarrow{c_1})^{\mathcal{B}}, \dots, (\overrightarrow{c_i})^{\mathcal{B}}, (\overrightarrow{c_{i+1}})^{\mathcal{B}}, (\overrightarrow{c_{i+2}})^{\mathcal{B}}, \dots),$$

where  $t$  is a  $\Sigma^*$ -term of sort  $\sigma_i$ . Since  $\Sigma^*$  is countable, there are at most  $\aleph_0$  choices for  $t$ . We have at most  $\kappa(\sigma_i)$  choices for  $(\overrightarrow{c_1})^{\mathcal{B}}, \dots, (\overrightarrow{c_i})^{\mathcal{B}}$ . Finally, we have finitely many choices for  $(\overrightarrow{c_{i+1}})^{\mathcal{B}}, (\overrightarrow{c_{i+2}})^{\mathcal{B}}, \dots$  up to  $\sim$ -equivalence. Since

<sup>9</sup> In other words,  $\mathcal{B}^-$  is the Skolem hull of  $\bigcup_i \{c_{i,\alpha}^{\mathcal{B}}\}_{\alpha < \kappa(\sigma_i)}$  in  $\mathcal{B}$  [12, p. 180].

$\mathcal{B}^- \models \Gamma_2$ , it follows that there are at most  $\kappa(\sigma_i)$  elements of  $\sigma_i^{\mathcal{B}^-}$ . Therefore,  $\mathcal{B}^-$  is a  $\mathcal{T}^*$ -interpretation satisfying  $\varphi$  with  $|\sigma_i^{\mathcal{B}^-}| = \kappa(\sigma_i)$  for all  $i$ . Taking the reduct of  $\mathcal{B}^-$  to  $\Sigma$  gives the desired  $\mathcal{T}$ -interpretation.  $\square$

### 3.4 Applications to Theory Combination

Since Theorem 2 implies that stably infinite and strongly finitely witnessable theories are strongly polite, we can restate the theorem on strongly polite theory combination with weaker hypotheses. This was already proved in [21] via a different method, but is now obtained as an immediate corollary of Theorem 2.

**Corollary 1.** *Let  $\Sigma_1$  and  $\Sigma_2$  be disjoint countable signatures. Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be  $\Sigma_1$ - and  $\Sigma_2$ -theories respectively, and let  $\varphi_1$  and  $\varphi_2$  be quantifier-free  $\Sigma_1$ - and  $\Sigma_2$ -formulas respectively. Suppose  $\mathcal{T}_1$  is stably infinite and strongly finitely witnessable, both with respect to  $\mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2}$ , and let  $V = \text{vars}_{\mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2}}(\text{wit}(\varphi_1))$ . Then,  $\varphi_1 \wedge \varphi_2$  is  $(\mathcal{T}_1 \cup \mathcal{T}_2)$ -satisfiable if and only if there is an arrangement  $\delta_V$  on  $V$  such that  $\text{wit}(\varphi_1) \wedge \delta_V$  is  $\mathcal{T}_1$ -satisfiable and  $\varphi_2 \wedge \delta_V$  is  $\mathcal{T}_2$ -satisfiable.*

We can also use our results to give a new characterization of shiny theories, which allows us to restate shiny combination theorem with weaker hypotheses.

To define shininess, we first need a few other notions. Let  $\Sigma$  be a signature with  $\mathcal{S}_\Sigma$  finite, and let  $S \subseteq \mathcal{S}_\Sigma$ . Write  $S = \{\sigma_1, \dots, \sigma_n\}$ . Then, the  $S$ -size of a  $\Sigma$ -interpretation  $\mathcal{A}$  is given by the tuple  $(|\sigma_1^{\mathcal{A}}|, \dots, |\sigma_n^{\mathcal{A}}|)$ . Such  $n$ -tuples are partially ordered by the product order:  $(x_1, \dots, x_n) \preceq (y_1, \dots, y_n)$  if and only if  $x_i \leq y_i$  for all  $i \in [n]$ . Given a quantifier-free formula  $\varphi$ , let  $\text{minmods}_{\mathcal{T}, S}(\varphi)$  be the set of minimal  $S$ -sizes of  $\mathcal{T}$ -interpretations satisfying  $\varphi$ . It follows from results in [10] that  $\text{minmods}_{\mathcal{T}, S}(\varphi)$  is a finite set of tuples.<sup>10</sup>

Then, we say a  $\Sigma$ -theory  $\mathcal{T}$  is *shiny* with respect to some subset of sorts  $S \subseteq \mathcal{S}_\Sigma$  if  $\mathcal{S}_\Sigma$  is finite,  $\mathcal{T}$  is stably finite and smooth, both with respect to  $S$ , and  $\text{minmods}_{\mathcal{T}, S}$  is computable. Theorem 3 implies that we can replace smoothness by finite smoothness, which may make it easier to prove that some theories are shiny. We can therefore improve the shiny theory combination theorem from [4, Theorem 2] as an immediate corollary of Theorem 3.

**Corollary 2.** *Let  $\Sigma_1$  and  $\Sigma_2$  be disjoint countable signatures, where  $\mathcal{S}_{\Sigma_1}$  and  $\mathcal{S}_{\Sigma_2}$  are finite. Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be  $\Sigma_1$ - and  $\Sigma_2$ -theories respectively, and assume the satisfiability problems for quantifier-free formulas of both  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are decidable. Suppose  $\mathcal{T}_1$  is stably finite and finitely smooth, both with respect to  $\mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2}$ , and  $\text{minmods}_{\mathcal{T}_1, \mathcal{S}_{\Sigma_1} \cap \mathcal{S}_{\Sigma_2}}$  is computable. Then, the satisfiability problem for quantifier-free formulas of  $\mathcal{T}_1 \cup \mathcal{T}_2$  is decidable.*

<sup>10</sup> [4] proves this assuming that  $\mathcal{T}$  is stably finite, using Hilbert's basis theorem. This assumption can be dropped by using the fact that if  $(X, \leq)$  is a well-quasi-order, then so is  $(X^n, \prec)$ , where  $\prec$  is the product order. Here  $X$  is the class of cardinals.

## 4 Many-Sorted Löwenheim–Skolem Theorems

In this section, we state many-sorted generalizations of the Löwenheim–Skolem theorem. Our first results, in Sect. 4.2, hold with no assumptions on the signature. Later, in Sect. 4.3, we state stronger results for restricted signatures, which we then use for a many-sorted variant of the Łoś–Vaught test in Sect. 4.4. But first, in Sect. 4.1, we explain the limitations of relying solely on translations to single-sorted first-order logic.

### 4.1 Lost in Translation

We may transform a many-sorted signature into a single-sorted signature by adding unary predicates signifying the sorts; of course, some restrictions are necessary, distinctness of sorts, etc. This procedure [6, 13, 24] is often used to lift results from single-sorted to many-sorted logic. As one example, standard versions of the downward Löwenheim–Skolem theorem for many-sorted logic, found in [14], are proven using this translation; we can, however, strengthen these results while still using only translations:

**Theorem 4 (Downward).** *Let  $\Sigma$  be a many-sorted signature with  $|\mathcal{S}_\Sigma| < \aleph_0$ . Suppose we have a  $\Sigma$ -structure  $\mathbb{A}$  with  $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$ , a cardinal  $\kappa$  satisfying  $\max\{|\Sigma|, \aleph_0\} \leq \kappa \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}\}$ , and sets  $A_\sigma \subseteq \sigma^\mathbb{A}$  with  $|A_\sigma| \leq \kappa$  for each  $\sigma \in \mathcal{S}_\Sigma$ . Then, there is an elementary substructure  $\mathbb{B}$  of  $\mathbb{A}$  such that  $\sigma^\mathbb{B} = \sigma^\mathbb{A}$  for every  $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$ ,  $\aleph_0 \leq |\sigma^\mathbb{B}| \leq \kappa$  for all  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$ ,  $|\sigma^\mathbb{B}| = \kappa$  for some  $\sigma \in \mathcal{S}_\Sigma$ , and  $A_\sigma \subseteq \sigma^\mathbb{B}$  for all  $\sigma \in \mathcal{S}_\Sigma$ .*

**Theorem 5 (Upward).** *Let  $\Sigma$  be a many-sorted signature with  $|\mathcal{S}_\Sigma| < \aleph_0$ . Suppose we have a  $\Sigma$ -structure  $\mathbb{A}$  with  $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$  and a cardinal  $\kappa \geq \max\{|\Sigma|, \max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\}\}$ . Then, there is a  $\Sigma$ -structure  $\mathbb{B}$  containing  $\mathbb{A}$  as an elementary substructure such that  $\sigma^\mathbb{B} = \sigma^\mathbb{A}$  for all  $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$ ,  $\aleph_0 \leq |\sigma^\mathbb{B}| \leq \kappa$  for all  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$ , and  $|\sigma^\mathbb{B}| = \kappa$  for some sort  $\sigma \in \mathcal{S}_\Sigma$ .*

As convenient as translation arguments are, the above Löwenheim–Skolem theorems seem unsatisfactory, as they only allow us to choose a single cardinal, rather than one for each sort.

### 4.2 Downward, Upward, and Combined Versions

The following are generalizations of the downward and upward Löwenheim–Skolem theorems to many-sorted logic, which are proved by adapting the proofs of the single-sorted case. Notice that we set all infinite domains to the same cardinality, while finite domains preserve their cardinalities.

**Theorem 6 (Downward).** *Fix a first-order many-sorted signature  $\Sigma$ . Suppose we have a  $\Sigma$ -structure  $\mathbb{A}$ , a cardinal  $\kappa$  such that  $\max\{\aleph_0, |\Sigma|\} \leq \kappa \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}\}$ , and sets  $A_\sigma \subseteq \sigma^\mathbb{A}$  with  $|A_\sigma| \leq \kappa$  for each  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$ . Then, there is an elementary substructure  $\mathbb{B}$  of  $\mathbb{A}$  that satisfies  $|\sigma^\mathbb{B}| = \kappa$  and  $\sigma^\mathbb{B} \supseteq A_\sigma$  for every  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$ , and also  $\sigma^\mathbb{B} = \sigma^\mathbb{A}$  for every  $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$ .*

**Theorem 7 (Upward).** Fix a first-order many-sorted signature  $\Sigma$ . Given a  $\Sigma$ -structure  $\mathbb{A}$ , pick a cardinal  $\kappa \geq \max\{|\Sigma|, \aleph_0, \sup\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}\}\}$ . Then, there is a  $\Sigma$ -structure  $\mathbb{B}$  containing  $\mathbb{A}$  as an elementary substructure that satisfies  $|\sigma^\mathbb{B}| = \kappa$  for every  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$ , and also  $\sigma^\mathbb{B} = \sigma^\mathbb{A}$  for every  $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$ .

Theorems 6 and 7 can be combined to yield yet another variant of the Löwenheim–Skolem theorem, which may be called the combined version.

**Corollary 3 (Combined).** Fix a many-sorted signature  $\Sigma$ . Given a  $\Sigma$ -structure  $\mathbb{A}$ , pick a cardinal  $\kappa \geq \max\{|\Sigma|, \aleph_0\}$ . Then, there is a  $\Sigma$ -structure  $\mathbb{B}$  elementarily equivalent to  $\mathbb{A}$  with  $|\sigma^\mathbb{B}| = \kappa$  for every  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\mathbb{A}$ , and  $\sigma^\mathbb{B} = \sigma^\mathbb{A}$  for  $\sigma \in \mathcal{S}_{< \aleph_0}^\mathbb{A}$ .

We illustrate Corollary 3 in Fig. 2.

In black, we represent the cardinalities of the resulting structure, and in red, those of the original one. When they coincide, we use marks split between the two colors. This representation shows a set of sorts in the horizontal axis, and the heights of the marks represent the cardinalities of the respective domains. We clearly separate cardinals larger and smaller than  $\aleph_0$  with a rule. Assume, without loss of generality, that initially  $\sigma_1 \dots \sigma_n$  have finite cardinalities and  $\sigma'_1$  has the least and  $\sigma'_m$  the greatest infinite cardinality.<sup>11</sup> Corollary 3 allows us to pick an infinite cardinal  $\kappa$  in between the least and greatest infinite cardinalities, and set all infinite cardinalities in the interpretation to  $\kappa$ .

The above theorems require that the desired cardinalities of the infinite sorts are all equal. The following example shows that this limitation is necessary.

*Example 2.* Take the signature  $\Sigma$  with sorts  $S = \{\sigma_1, \sigma_2\}$ , no predicates, and only one function  $f$  of arity  $(\sigma_1, \sigma_2)$ . Take the  $\Sigma$ -structure  $\mathbb{A}$  with:  $\sigma_1^\mathbb{A}$  and  $\sigma_2^\mathbb{A}$  of cardinality  $\aleph_1$ , and  $f^\mathbb{A}$  a bijection. It is then true that  $\mathbb{A} \models \varphi_{inj} \wedge \varphi_{sur}$ , where  $\varphi_{inj} = \forall x : \sigma_1. \forall y : \sigma_1. [[f(x) = f(y)] \rightarrow [x = y]]$  and  $\varphi_{sur} = \forall u : \sigma_2. \exists x : \sigma_1. [f(x) = u]$ , codifying that  $f$  is injective and surjective respectively. Notice then that, although  $\max\{|\Sigma|, \aleph_0\} = \aleph_0$ , there cannot be an elementary substructure  $\mathbb{B}$  of  $\mathbb{A}$  with  $|\sigma_1^\mathbb{B}| = \aleph_0$  and  $|\sigma_2^\mathbb{B}| = \aleph_1$ : for if  $\mathbb{B} \models \varphi_{inj} \wedge \varphi_{sur}$ ,  $f^\mathbb{B}$  must be a bijection between  $\sigma_1^\mathbb{B}$  and  $\sigma_2^\mathbb{B}$ . A similar argument shows that the corresponding generalization of the upwards theorem fails as well.

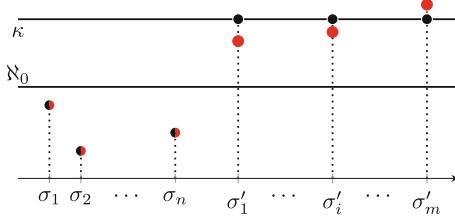


Fig. 2. Illustration of Corollary 3.

<sup>11</sup> For greater clarity, the diagram only depicts the cases where there are finitely many sorts and the signature is countable.

### 4.3 A Stronger Result for Split Signatures

Example 2 relies on “mixing sorts” by using a function symbol with arities spanning different sorts. We can state stronger versions of the many-sorted Löwenheim–Skolem theorems when such mixing of sorts is restricted.

**Definition 6.** A signature  $\Sigma$  is said to be split by  $\Lambda$  into a family of signatures  $\{\Sigma_\lambda : \lambda \in \Lambda\}$  if  $\Lambda$  is a partition of  $\mathcal{S}_\Sigma$ ,  $\mathcal{S}_{\Sigma_\lambda} = \lambda$  for each  $\lambda \in \Lambda$ ,  $\mathcal{F}_\Sigma = \bigcup_{\lambda \in \Lambda} \mathcal{F}_{\Sigma_\lambda}$ , and  $\mathcal{P}_\Sigma = \bigcup_{\lambda \in \Lambda} \mathcal{P}_{\Sigma_\lambda}$ . If  $\Sigma$  is split by  $\Lambda$  and each  $\lambda \in \Lambda$  is a singleton, then we say that  $\Sigma$  is completely split by  $\Lambda$ .

If  $\Sigma$  is split by  $\Lambda$ , then the function/predicate symbols of  $\Sigma_\lambda$  must be disjoint from  $\Sigma_{\lambda'}$  for  $\lambda \neq \lambda'$ . Given a partition  $\Lambda$  of  $\mathcal{S}_\Sigma$  and  $\lambda \in \Lambda$ , let  $\mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda) = \mathcal{S}_{\geq \aleph_0}^\Lambda \cap \lambda$ . We state the downward, upward, and combined theorems for split signatures.

**Theorem 8 (Downward).** Fix a first-order many-sorted signature  $\Sigma$  split by  $\Lambda$ . Suppose we have a  $\Sigma$ -structure  $\mathbb{A}$ , a cardinal  $\kappa_\lambda$  such that  $\max\{\aleph_0, |\Sigma_\lambda|\} \leq \kappa_\lambda \leq \min\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda)\}$  for each  $\lambda \in \Lambda$ , and sets  $A_\sigma \subseteq \sigma^\mathbb{A}$  with  $|A_\sigma| \leq \kappa_\lambda$  for each  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda)$ . Then, there is an elementary substructure  $\mathbb{B}$  of  $\mathbb{A}$  that satisfies  $|\sigma^\mathbb{B}| = \kappa_\lambda$  and  $\sigma^\mathbb{B} \supseteq A_\sigma$  for  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda)$ , and  $\sigma^\mathbb{B} = \sigma^\mathbb{A}$  for  $\sigma \in \mathcal{S}_{< \aleph_0}^\Lambda$ .

**Theorem 9 (Upward).** Suppose  $\Sigma$  is split by  $\Lambda$ . Given a  $\Sigma$ -structure  $\mathbb{A}$ , pick a cardinal  $\kappa_\lambda \geq \max\{|\Sigma_\lambda|, \aleph_0, \sup\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda)\}\}$  for each  $\lambda \in \Lambda$ . Then, there is a  $\Sigma$ -structure  $\mathbb{B}$  containing  $\mathbb{A}$  as an elementary substructure that satisfies  $|\sigma^\mathbb{B}| = \kappa_\lambda$  for  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda)$ , and  $\sigma^\mathbb{B} = \sigma^\mathbb{A}$  for  $\sigma \in \mathcal{S}_{< \aleph_0}^\Lambda$ .

**Corollary 4 (Combined).** Suppose  $\Sigma$  is split by  $\Lambda$ . Given a  $\Sigma$ -structure  $\mathbb{A}$ , pick a cardinal  $\kappa_\lambda \geq \max\{|\Sigma_\lambda|, \aleph_0\}$  for each  $\lambda \in \Lambda$ . Then, there is a  $\Sigma$ -structure  $\mathbb{B}$  elementarily equivalent to  $\mathbb{A}$  with  $|\sigma^\mathbb{B}| = \kappa_\lambda$  for every  $\sigma \in \mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda)$ , and also  $\sigma^\mathbb{B} = \sigma^\mathbb{A}$  for every  $\sigma \in \mathcal{S}_{< \aleph_0}^\Lambda$ .

Corollary 4 is illustrated in Fig. 3. We add sorts  $S'' = \{\sigma''_1, \dots, \sigma''_m\}$ , and assume our signature is split into  $\Sigma_{\lambda_1}$  and  $\Sigma_{\lambda_2}$ , where  $\mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda_1) = \{\sigma'_1, \dots, \sigma'_m\}$  and  $\mathcal{S}_{\geq \aleph_0}^\Lambda(\lambda_2) = S''$  (the sorts with finite cardinalities can belong to either). Then,  $\kappa'$  is the cardinal associated with  $\Sigma_{\lambda_1}$ , and  $\kappa''$  with  $\Sigma_{\lambda_2}$ . Thus, we are able to choose a cardinality for each class of sorts.

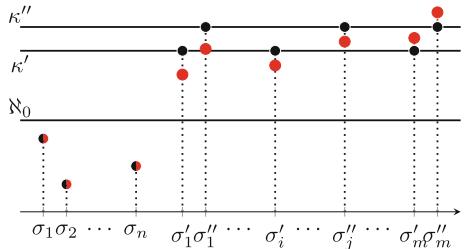


Fig. 3. Illustration of Corollary 4.

### 4.4 An Application: The Łoś–Vaught Test

We describe an application of our Löwenheim–Skolem theorems for theory-completeness: the Łoś–Vaught test. This is particularly relevant to SMT, as if a complete theory  $\mathcal{T}$  has a decidable set of axioms, then it is decidable whether  $\vdash_{\mathcal{T}} \varphi$  [12, Lemma 2.2.8]. The single-sorted Łoś–Vaught is the following.

**Definition 7.** Let  $\Sigma$  be a signature and  $\kappa$  a function from  $\mathcal{S}_\Sigma$  to the class of cardinals. A  $\Sigma$ -theory  $\mathcal{T}$  is  $\kappa$ -categorical if it has exactly one model  $\mathbb{A}$  (up to isomorphism) with the property that  $|\sigma^\mathbb{A}| = \kappa(\sigma)$  for every  $\sigma \in \mathcal{S}_\Sigma$ . If there is only one sort  $\sigma \in \mathcal{S}_\Sigma$ , we abuse notation by using  $\kappa$  to denote the cardinal  $\kappa(\sigma)$ .

**Theorem 10** ([11, 23]). Suppose  $\Sigma$  is single-sorted and  $\mathcal{T}$  is a  $\Sigma$ -theory with only infinite models. If  $\mathcal{T}$  is  $\kappa$ -categorical for some  $\kappa \geq |\Sigma|$ , then  $\mathcal{T}$  is complete.

The Łoś–Vaught test is quite useful, e.g., for the completeness of dense linear orders without endpoints and algebraically closed fields. We generalize it to many sorts. Translating to one-sorted logic and using Theorem 10 gives us:

**Corollary 5.** Let  $\Sigma$  be a signature with  $|\mathcal{S}_\Sigma| < \aleph_0$ . Suppose  $\mathcal{T}$  is a  $\Sigma$ -theory, all of whose models  $\mathbb{A}$  satisfy  $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} \geq \aleph_0$ . Suppose further that for some cardinal  $\kappa \geq |\Sigma|$ ,  $\mathcal{T}$  has exactly one model  $\mathbb{A}$  (up to isomorphism) such that  $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} = \kappa$ . Then,  $\mathcal{T}$  is complete.

This is not the result one would hope for, because it excludes some many-sorted  $\kappa$ -categorical theories, as the following example demonstrates.

*Example 3.* Suppose  $\Sigma$  has  $S = \{\sigma_1, \sigma_2\}$ , no predicate symbols, and function symbols 0, 1, +, and  $\times$ , of the expected arities. Let  $\mathcal{T} = \text{ACF}_0 \cup \{\psi_{\geq n}^{\sigma_2} : n \in \mathbb{N}\}$ , where  $\text{ACF}_0$  is the theory of algebraically closed fields of characteristic zero (with respect to  $\sigma_1$ ) and  $\psi_{\geq n}^{\sigma_2} = \exists x_1 : \sigma_1 \cdots \exists x_n : \sigma_1. \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)$ , which asserts that there are at least  $n$  elements of sort  $\sigma_1$ .  $\mathcal{T}$  is  $\kappa$ -categorical, where  $\kappa(\sigma_1) = \aleph_1$  and  $\kappa(\sigma_2) = \aleph_0$ . But  $\mathcal{T}$  is also  $\kappa'$ -categorical, where  $\kappa'(\sigma_1) = \kappa'(\sigma_2) = \aleph_1$ . Thus,  $\mathcal{T}$  has multiple models  $\mathbb{A}$  satisfying  $\max\{|\sigma^\mathbb{A}| : \sigma \in \mathcal{S}_\Sigma\} = \aleph_1$ . Similar reasoning holds for other infinite cardinals, so Corollary 5 does not apply.

For completely split signatures, we prove a more natural Łoś–Vaught test:

**Definition 8.** A  $\Sigma$ -structure  $\mathbb{A}$  is strongly infinite if  $|\sigma^\mathbb{A}| \geq \aleph_0$  for all  $\sigma \in \mathcal{S}_\Sigma$ .

**Theorem 11.** Suppose  $\Sigma$  is completely split into  $\{\Sigma_\sigma : \sigma \in \mathcal{S}_\Sigma\}$ ,  $\mathcal{T}$  is a  $\Sigma$ -theory all of whose models are strongly infinite, and  $\mathcal{T}$  is  $\kappa$ -categorical for some function  $\kappa$  such that  $\kappa(\sigma) \geq |\Sigma_\sigma|$  for every  $\sigma \in \mathcal{S}_\Sigma$ . Then,  $\mathcal{T}$  is complete.

The assumption that  $\Sigma$  is completely split is necessary for Theorem 11:

*Example 4.* Let  $\Sigma$  have sorts  $\sigma_1, \sigma_2$ , and function symbol  $f$  of arity  $(\sigma_1, \sigma_2)$ . Let  $\mathcal{T} = \{\psi_{\geq n}^{\sigma_1} : n \in \mathbb{N}\} \cup \{\psi_{\geq n}^{\sigma_2} : n \in \mathbb{N}\} \cup \{\varphi_{\text{inj}} \vee \forall x : \sigma_1. \forall y : \sigma_1. [f(x) = f(y)]\}$ . In  $\mathcal{T}$ ,  $\sigma_1, \sigma_2$  are infinite, and  $f$  is injective or constant.  $\mathcal{T}$  is  $\kappa$ -categorical for  $\kappa(\sigma_1) = \aleph_1, \kappa(\sigma_2) = \aleph_0$ , but not complete, due to the sentence  $\forall x, y : \sigma_1. f(x) = f(y)$ . This does not contradict Theorem 11, as  $\Sigma$  is not completely split.

## 5 Conclusion

We closed the problem of the existence of unicorn theories and discussed applications to SMT. This included a result similar to the Löwenheim–Skolem theorem, which inspired us to investigate the adaptation of this theorem to many-sorted logic. We also obtained a many-sorted version of the Łoś–Vaught test.

In future work, we plan to investigate whether Theorem 3 can be extended to uncountable signatures. More broadly, we intend to continue studying the relationships among many-sorted model-theoretic properties related to SMT.

**Acknowledgments.** This work was supported in part by the Stanford Center for Automated Reasoning, NSF-BSF grant numbers 2110397 (NSF) and 2020704 (BSF), ISF grant 619/21, and the Colman-Soref fellowship. The first author thanks the organizers of the CURIS research program.

## References

1. Barbosa, H., et al.: cvc5: a versatile and industrial-strength SMT solver. In: TACAS (1). Lecture Notes in Computer Science, vol. 13243, pp. 415–442. Springer, Munich (2022)
2. Barrett, C., Fontaine, P., Tinelli, C.: The SMT-LIB Standard: Version 2.6. Tech. rep., Department of Computer Science, The University of Iowa (2017). <http://smt-lib.org>
3. Barrett, C., Tinelli, C.: Satisfiability modulo theories. In: Clarke, E.M., Henzinger, T.A., Veith, H., Bloem, R. (eds.) Handbook of Model Checking, pp. 305–343. Springer, New York (2018). [https://doi.org/10.1007/978-3-319-10575-8\\_11](https://doi.org/10.1007/978-3-319-10575-8_11), <http://theory.stanford.edu/~barrett/pubs/BT18.pdf>
4. Casal, F., Rasga, J.a.: Many-sorted equivalence of shiny and strongly polite theories. *J. Automat. Reason.* **60**(2), 221–236 (2018)
5. Ehrenfeucht, A., Mostowski, A.: Models of axiomatic theories admitting automorphisms. *Fund. Math.* **43**, 50–68 (1956)
6. Enderton, H.B.: A Mathematical Introduction to Logic. Academic Press, New York (1972)
7. Fontaine, P.: Combinations of theories for decidable fragments of first-order logic. In: Ghilardi, S., Sebastiani, R. (eds.) Frontiers of Combining Systems, pp. 263–278. Springer, Berlin Heidelberg, Berlin, Heidelberg (2009)
8. Jovanović, D., Barrett, C.: Polite theories revisited. Tech. Rep. TR2010-922, Department of Computer Science, New York University (Jan 2010). <http://www.cs.stanford.edu/~barrett/pubs/JB10-TR.pdf>
9. Krstić, S., Goel, A., Grundy, J., Tinelli, C.: Combined satisfiability modulo parametric theories. In: Grumberg, O., Huth, M. (eds.) Tools and Algorithms for the Construction and Analysis of Systems, pp. 602–617. Springer Berlin Heidelberg, Berlin, Heidelberg (2007). [https://doi.org/10.1007/978-3-540-71209-1\\_47](https://doi.org/10.1007/978-3-540-71209-1_47)
10. Kruskal, J.B.: The theory of well-quasi-ordering: a frequently discovered concept. *J. Comb. Theory Ser. A* **13**, 297–305 (1972). [https://doi.org/10.1016/0097-3165\(72\)90063-5](https://doi.org/10.1016/0097-3165(72)90063-5)
11. Łoś, J.: On the categoricity in power of elementary deductive systems and some related problems. *Colloquium Mathematicum* **3**, 58–62 (1954)

12. Marker, D.: Model theory: an introduction, Graduate Texts in Mathematics, vol. 217. Springer-Verlag, New York (2002). <https://doi.org/10.1007/b98860>
13. Monk, J.D.: Mathematical Logic. Springer, New York (1976). <https://doi.org/10.1007/978-1-4684-9452-5>
14. Monzano, M.: Introduction to many-sorted logic. In: Meinke, K., Tucker, J.V. (eds.) Many-sorted Logic and its Applications. Wiley professional computing, Wiley, New York (1993)
15. Nelson, G., Oppen, D.C.: Simplification by cooperating decision procedures. ACM Trans. Program. Lang. Syst. **1**(2), 245–257 (oct 1979). <https://doi.org/10.1145/357073.357079>
16. Przybocki, B., Toledo, G., Zohar, Y., Barrett, C.: The nonexistence of unicorns and many-sorted Löwenheim–Skolem theorems (2024). <https://arxiv.org/abs/2406.18912>
17. Ramsey, F.P.: On a problem of formal logic. Proc. London Math. Soc. (2) **30**(4), 264–286 (1929)
18. Ranise, S., Ringeissen, C., Zarba, C.G.: Combining data structures with nonstably infinite theories using many-sorted logic. In: Gramlich, B. (ed.) 5th International Workshop on Frontiers of Combining Systems - FroCoS'05. Lecture Notes in Artificial Intelligence, vol. 3717, pp. 48–64. Springer, Vienna/Austria (Sep 2005) <https://doi.org/10.1007/11559306>, <https://hal.inria.fr/inria-00000570>
19. Sheng, Y., Zohar, Y., Ringeissen, C., Lange, J., Fontaine, P., Barrett, C.: Polite combination of algebraic datatypes. J. Autom. Reasoning **66**(3), 331–355 (2022). <https://doi.org/10.1007/s10817-022-09625-3>
20. Tinelli, C., Zarba, C.G.: Combining decision procedures for sorted theories. Tech. rep., Berlin, Heidelberg (2004). [https://doi.org/10.1007/978-3-540-30227-8\\_53](https://doi.org/10.1007/978-3-540-30227-8_53)
21. de Toledo, G.V., Zohar, Y., Barrett, C.W.: Combining combination properties: an analysis of stable infiniteness, convexity, and politeness. In: CADE, Lecture Notes in Computer Science, vol. 14132, pp. 522–541. Springer, Rome (2023). [https://doi.org/10.1007/978-3-031-38499-8\\_30](https://doi.org/10.1007/978-3-031-38499-8_30)
22. de Toledo, G.V., Zohar, Y., Barrett, C.W.: Combining finite combination properties: finite models and busy beavers. In: FroCoS, Lecture Notes in Computer Science, vol. 14279, pp. 159–175. Springer, Prague (2023). [https://doi.org/10.1007/978-3-031-43369-6\\_9](https://doi.org/10.1007/978-3-031-43369-6_9)
23. Vaught, R.L.: Applications of the Löwenheim–Skolem–Tarski theorem to problems of completeness and decidability. Nederl. Akad. Wetensch. Proc. **57**, 467–472 (1954)
24. Wang, H.: Logic of many-sorted theories. J. Symbolic Logic **17**(2), 105–116 (1952). <http://www.jstor.org/stable/2266241>