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RESEARCH ARTICLE

A bound on the second Betti number of hyperkähler manifolds of complex dimension six

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Abstract
Let M be an irreducible compact hyperkähler manifold of complex dimension six. Under an assumption on the Looijenga–Lunts–Verbitsky decomposition of the cohomology of M, we prove that the second Betti number of M is at most 23.

Keywords Hyperkähler manifolds · Cohomology · Betti numbers

Mathematics Subject Classification 53C26

1 Introduction

An irreducible hyperkähler manifold is a Riemannian manifold of real dimension $4n$ whose holonomy is equal to $\mathrm{Sp}(n)$. The Riemannian metric will be Kählerian with respect to an S^2-family of complex structures, so henceforth we will always use the *complex* dimension, $2n$. Beauville [2] and Guan [7] independently proved that the second Betti number of an irreducible compact hyperkähler manifold of dimension four is bounded above by 23. The Hilbert scheme of two points on a K3 surface has second Betti number 23, so this bound is sharp.

Looijenga and Lunts [11] and Verbitsky [17] showed that the cohomology of a hyperkähler manifold admits an action of $\mathfrak{so}(4, b_2 - 2)$, where b_2 is the second Betti number. In this article we prove that the second Betti number of an irreducible compact hyperkähler manifold of dimension six is also bounded above by 23, under the assumption that only certain irreducible $\mathfrak{so}(4, b_2 - 2)$-representations appear in the Looijenga–Lunts–Verbitsky decomposition; see Theorem 3.4 for the precise statement. Up to deformation, there are currently three known examples of such manifolds:

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the Hilbert scheme of three points on a K3 surface, the generalized Kummer variety (see Beauville [1]), and an example of O'Grady [14]. These examples have second Betti numbers 23, 7, and 8, respectively, so once again our bound is sharp. Moreover, these examples all satisfy the assumption on the Looijenga–Lunts–Verbitsky decomposition of the cohomology.

Using the same ideas, we prove that the second Betti number of an irreducible compact hyperkähler manifold of dimension eight is bounded above by 24, once again under an assumption on the Looijenga–Lunts–Verbitsky decomposition *and* assuming that all odd Betti numbers vanish; see Theorem 4.3 for the precise statement. In this dimension, the only examples currently known are the Hilbert scheme of four points on a K3 surface and the generalized Kummer variety, which have second Betti numbers 23 and 7, respectively. In particular, it is possible that no example exists in dimension eight with second Betti number 24.

Why is it important to bound the second Betti number? The first Pontryagin class $p_1(M)$ determines a homogeneous polynomial of degree $2n-2$ on $\mathrm{H}^2(M,\mathbb{Z})$, given by $\alpha \mapsto \int_M \alpha^{2n-2} p_1(M)$. Huybrechts [8] proved that if the second integral cohomology H^2 and the homogeneous polynomial of degree $2n-2$ on H^2 determined by the first Pontryagin class are fixed, then up to diffeomorphism there are only finitely many irreducible compact hyperkähler manifolds of dimension $2n$ realizing this structure. (Instead, one can fix H^2 and a certain normalization $\tilde{q}$ of the Beauville–Bogomolov quadratic form on H^2 and arrive at the same conclusion; see [8].) A universal bound on the second Betti number in dimension $2n$ would mean that there are finitely many possibilities for H^2 as a $\mathbb{Z}$-module; it would then remain to bound the other data on H^2, to conclude that there are finitely many diffeomorphism types of irreducible compact hyperkähler manifolds of that dimension.

2 Dimension four

Let us recall how to bound the second Betti number in dimension four. Salamon [15] proved that the Betti numbers of a compact hyperkähler manifold of dimension $2n$ satisfy the relation

$$2\sum_{j=1}^{2n}(-1)^j(3j^2-n)\,b_{2n-j} = nb_{2n}.$$

Theorem 2.1 (Beauville [2], Guan [7]) *Let M be an irreducible compact hyperkähler manifold of complex dimension four. Then the second Betti number b_2 of M is at most* 23.

Proof Irreducible hyperkähler manifolds are simply-connected, so $b_1 = 0$. Therefore Salamon's relation for $n = 2$ gives

$$-2b_3 + 20b_2 + 92 = 2b_4.$$

Verbitsky [16] proved that $\mathrm{Sym}^k \mathrm{H}^2(M, \mathbb{R})$ injects into $\mathrm{H}^{2k}(M, \mathbb{R})$ for $k \leqslant n$. In particular, we can write

$$\mathrm{H}^4(M, \mathbb{R}) \cong \mathrm{Sym}^2 \mathrm{H}^2(M, \mathbb{R}) \oplus \mathrm{H}^4_{\mathrm{prim}}(M, \mathbb{R})$$

and

$$b_4 = \binom{b_2 + 1}{2} + b_4',$$

where b_4' denotes the dimension of the primitive cohomology $\mathrm{H}^4_{\mathrm{prim}}(M, \mathbb{R})$. Substituting this into Salamon's relation gives

$$-2b_3 + 20b_2 + 92 = b_2(b_2 + 1) + 2b_4',$$

and therefore

$$-(b_2 + 4)(b_2 - 23) = -b_2^2 + 19b_2 + 92 = 2b_4' + 2b_3.$$

The left-hand side is negative if $b_2 > 23$, whereas the right-hand side is clearly non-negative. Therefore the second Betti number b_2 can be at most 23. □

Example 2.2 Up to deformation, there are two known examples of irreducible compact hyperkähler manifolds of dimension four: the Hilbert scheme $\mathrm{Hilb}^2 S$ of two points on a K3 surface S (see Fujiki [3]) and the generalized Kummer variety $K_2(A)$ of an abelian surface A (see Beauville [1]). Their Hodge diamonds are

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 1 & & 21 & & 1 & \\
0 & & 0 & & 0 & & 0 \\
1 & & 21 & & 232 & & 21 & & 1 \\
0 & & 0 & & 0 & & 0 \\
 & 1 & & 21 & & 1 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 1 & & 5 & & 1 & \\
0 & & 4 & & 4 & & 0 \\
1 & & 5 & & 96 & & 5 & & 1 \\
0 & & 4 & & 4 & & 0 \\
 & 1 & & 5 & & 1 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array},$$

with $b_2 = 23$, $b_3 = 0$, $b_4' = 0$, and $b_2 = 7$, $b_3 = 8$, $b_4' = 80$, respectively. In fact, it follows from the proof above that if $b_2 = 23$ then b_3 and b_4' must both vanish.

3 Dimension six

In higher dimensions, the injection $\mathrm{Sym}^k \mathrm{H}^2(M, \mathbb{R}) \hookrightarrow \mathrm{H}^{2k}(M, \mathbb{R})$ is insufficient to produce a bound on the second Betti number. Instead we employ the following refinement.

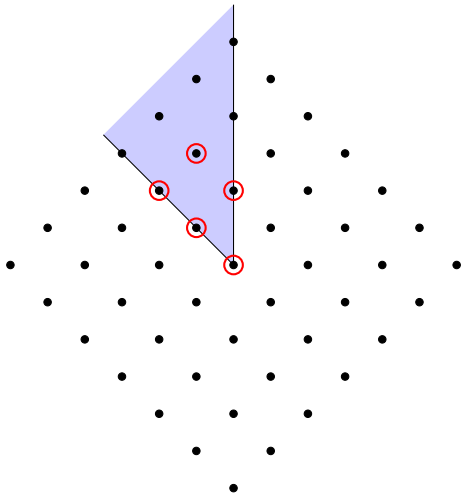

Fig. 1 The Hodge diamond in dimension six

Theorem 3.1 (Looijenga and Lunts [11], Verbitsky [17]) *Let M be an irreducible compact hyperkähler manifold of dimension $2n$ with second Betti number b_2. Then there is an action of $\mathfrak{so}(4, b_2-2)$ on the real cohomology $\bigoplus_{k=0}^{4n} \mathrm{H}^k(M, \mathbb{R})$, and hence of $\mathfrak{so}(b_2+2, \mathbb{C})$ on the complex cohomology $\bigoplus_{k=0}^{4n} \mathrm{H}^k(M, \mathbb{C})$.*

Remark 3.2 This action is generated by Lefschetz operators: for each Kähler class $[\omega]$ the operators $L_{[\omega]}$ and $\Lambda_{[\omega]}$ generate an $\mathfrak{sl}(2, \mathbb{C})$-action on the complex cohomology, and the amalgamation of all these actions yields the $\mathfrak{so}(b_2+2, \mathbb{C})$-action.

We can decompose $\bigoplus_{k=0}^{4n} \mathrm{H}^k(M, \mathbb{C})$ into irreducible representations for this $\mathfrak{so}(b_2+2, \mathbb{C})$-action. Their highest weights are related to Hodge bi-degrees; indeed, the Hodge diamond is the projection onto a plane of the (higher-dimensional) weight lattice of $\mathfrak{so}(b_2+2, \mathbb{C})$. We can choose positive roots so that the dominant Weyl chamber projects onto the shaded octant of the Hodge diamond shown in Fig. 1.

The irreducible representation with highest weight vector $1 \in \mathrm{H}^0(M, \mathbb{C})$ is precisely the subring of the cohomology generated by $\mathrm{H}^2(M, \mathbb{C})$. In dimension six, the remainder of the cohomology comes from irreducible representations whose highest weight vectors lie in the Hodge groups that are circled in Fig. 1. By considering all irreducible representations of $\mathfrak{so}(b_2+2, \mathbb{C})$ (see Fulton and Harris [4]), and observing how their highest weights project to the Hodge diamond, we conclude that the only irreducible representations that could appear are those described in Table 1. In the second column of this table the highest weights are given in terms of the fundamental weights. In the fourth column $\mathbb{C}^{b_2+2}$ and $\mathbb{C}$ denote the standard and trivial representations of $\mathfrak{so}(b_2+2, \mathbb{C})$, respectively. Thus V_k is the representation given by the kth exterior power $\Lambda^k\mathbb{C}^{b_2+2}$ of the standard representation of $\mathfrak{so}(b_2+2, \mathbb{C})$. Not shown in the table is that when $b_2+2 = 2m+1$ is odd the largest exterior power $\Lambda^m\mathbb{C}^{b_2+2}$ has highest weight $2\omega_m$. In addition, when $b_2+2 = 2m$ is even the exterior power $\Lambda^{m-1}\mathbb{C}^{b_2+2}$ has highest weight $\omega_{m-1}+\omega_m$ while the middle degree exterior power

Table 1 Irreducible representations of $\mathfrak{so}(b_2+2,\mathbb{C})$ that could occur in the cohomology of M

	Highest weight	Highest weight vector in	$\mathfrak{so}(b_2+2,\mathbb{C})$-module	Dimension
$U_\bullet$		$\mathrm{H}^{2,1}(M)$	Half-spin representations	
V_1	ω_1	$\mathrm{H}^{2,2}(M)$	$\mathbb{C}^{b_2+2}$	b_2+2
V_2	ω_2	$\mathrm{H}^{3,1}(M)$	$\Lambda^2\mathbb{C}^{b_2+2}$	$\binom{b_2+2}{2}$
V_3	ω_3	$\mathrm{H}^{3,1}(M)$	$\Lambda^3\mathbb{C}^{b_2+2}$	$\binom{b_2+2}{3}$
V_4	ω_4	$\mathrm{H}^{3,1}(M)$	$\Lambda^4\mathbb{C}^{b_2+2}$	$\binom{b_2+2}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
V_k	ω_k	$\mathrm{H}^{3,1}(M)$	$\Lambda^k\mathbb{C}^{b_2+2}$	$\binom{b_2+2}{k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$W_\bullet$		$\mathrm{H}^{3,2}(M)$	Half-spin representations	
T	0	$\mathrm{H}^{3,3}(M)$	$\mathbb{C}$	1

$\Lambda^m\mathbb{C}^{b_2+2}$ is not irreducible; instead it decomposes into two irreducible representations of equal dimensions, $\Lambda^m\mathbb{C}^{b_2+2} = \Lambda^m_+\mathbb{C}^{b_2+2} \oplus \Lambda^m_-\mathbb{C}^{b_2+2}$, with highest weights $2\omega_{m-1}$ and $2\omega_m$, respectively. Note that tensor representations occur in even degrees in cohomology, while half-spin representations occur in odd degrees. The highest weights and dimensions of the latter are not needed for our arguments.

Example 3.3 We can calculate the dimensions of the weight spaces of these representations. The highest weight vector of V_1 lies in $\mathrm{H}^{2,2}(M)$. Acting on this with Lefschetz operators $L_{[\omega]}$ gives us classes in $\mathrm{H}^{4,2}(M)$, $\mathrm{H}^{3,3}(M)$, $\mathrm{H}^{2,4}(M)$, and $\mathrm{H}^{4,4}(M)$, and indeed we find that V_1 will sit inside the Hodge diamond in the following manner (where we have indicated the dimension of $V_1^{p,q}$ for each p, q):

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & 0 & & 0 & & & & & \\
 & & 0 & & 0 & & 0 & & & & \\
 & 0 & & 0 & & 0 & & 0 & & & \\
0 & & 0 & & 1 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & b_2-2 & & 1 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 1 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & & & \\
 & & & 0 & & 0 & & & & & \\
 & & & & 0 & & & & & &
\end{array}.
$$

The weight decomposition of the exterior power $V_k = \Lambda^k V_1$ for $k \geqslant 2$ is derived from the weight decomposition of V_1. Because the Hodge bi-degrees are derived from the weights, we can thereby determine the dimensions of the $V_k^{p,q}$s. For example,

$$\dim V_2^{2,2} = \dim V_1^{2,2} \dim V_1^{3,3}.$$

(Note that weight zero corresponds to Hodge bi-degree $(3, 3)$; if we shift by $(3, 3)$ then the bi-degrees would become additive.) We find that V_2 will sit inside the Hodge diamond as

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & 0 & & 0 & & & & & \\
 & & 0 & & 0 & & 0 & & & & \\
 & 0 & & 0 & & 0 & & 0 & & & \\
 0 & & 1 & & b_2-2 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & b_2-2 & & \binom{b_2-2}{2}+2 & & b_2-2 & & 0 \quad 0, \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
 0 & & 1 & & b_2-2 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & & & \\
 & & & 0 & & 0 & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

V_3 will sit inside the Hodge diamond as

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & 0 & & 0 & & & & & \\
 & & 0 & & 0 & & 0 & & & & \\
 & 0 & & 0 & & 0 & & 0 & & & \\
 0 & & b_2-2 & & \binom{b_2-2}{2}+1 & & b_2-2 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & \binom{b_2-2}{2}+1 & & \binom{b_3-2}{3}+2(b_2-2) & & \binom{b_2-2}{2}+1 & & 0 \quad 0, \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
 0 & & b_2-2 & & \binom{b_2-2}{2}+1 & & b_2-2 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & & & \\
 & & & 0 & & 0 & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

Table 2 Dimensions of $V_\bullet$ in degrees 4 and 6

	Dimension	Dimension of $V_\bullet \cap \mathrm{H}^4(M, \mathbb{C})$	Dimension of $V_\bullet \cap \mathrm{H}^6(M, \mathbb{C})$
V_1	$b_2 + 2$	$\binom{b_2}{0} = 1$	$\binom{b_2}{1} = b_2$
V_2	$\binom{b_2+2}{2}$	$\binom{b_2}{1} = b_2$	$\binom{b_2}{2} + \binom{b_2}{0} = \frac{b_2^2 - b_2 + 2}{2}$
V_3	$\binom{b_2+2}{3}$	$\binom{b_2}{2} = \frac{b_2(b_2-1)}{2}$	$\binom{b_2}{3} + \binom{b_2}{1} = \frac{b_2(b_2^2-3b_2+8)}{6}$
V_4	$\binom{b_2+2}{4}$	$\binom{b_2}{3} = \frac{b_2(b_2-1)(b_2-2)}{6}$	$\binom{b_2}{4} + \binom{b_2}{2} = \frac{b_2(b_2-1)(b_2^2-5b_2+18)}{24}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
V_k	$\binom{b_2+2}{k}$	$\binom{b_2}{k-1}$	$\binom{b_2}{k} + \binom{b_2}{k-2}$

and V_4 will sit inside the Hodge diamond as

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
0 & & \binom{b_2-2}{2} & & \binom{b_2-2}{2} & & 0 \\
 & 0 & & \binom{b_2-2}{3} + b_2 - 2 & & 0 & \\
0 & & \binom{b_2-2}{3} + b_2 - 2 & & \binom{b_2-2}{3} + b_2 - 2 & & 0 \\
 & 0 & & \binom{b_2-2}{4} + 2\binom{b_2-2}{2} + 1 & & 0 & \\
0 & & \binom{b_2-2}{3} + b_2 - 2 & & \binom{b_2-2}{3} + b_2 - 2 & & 0 \\
 & 0 & & \binom{b_2-2}{3} + b_2 - 2 & & 0 & \\
0 & & \binom{b_2-2}{2} & & \binom{b_2-2}{2} & & 0 \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}.$$

Table 2 gives the dimensions of the intersections of these representations with $\mathrm{H}^4(M, \mathbb{C})$ and $\mathrm{H}^6(M, \mathbb{C})$.

With these preliminaries out of the way, we can prove our main result.

Theorem 3.4 *Let M be an irreducible compact hyperkähler manifold of complex dimension six. Of the possible irreducible representations of $\mathfrak{so}(b_2+2, \mathbb{C})$ with highest weight vectors in $\mathrm{H}^{2,2}(M)$ and $\mathrm{H}^{3,1}(M)$ in the Looijenga–Lunts–Verbitsky decomposition of the cohomology of M, assume that only V_1, V_2, and V_3 can appear (i.e., assume that $V_4, V_5, \ldots$ do not appear). Then the second Betti number b_2 of M is at most* 23.

Proof When $n = 3$ Salamon's relation gives

$$18b_4 - 48b_3 + 90b_2 + 210 = 3b_6.$$

Decompose the complex cohomology of M into irreducible representations of $\mathfrak{so}(b_2 + 2, \mathbb{C})$, as above. Suppose that V_1 occurs with multiplicity c, V_2 occurs with

multiplicity d, V_3 occurs with multiplicity e, and the trivial representation $T = \mathbb{C}$ occurs with multiplicity f. The contributions of V_1, V_2, and V_3 to $\mathrm{H}^4(M, \mathbb{C})$ are of dimensions 1, b_2, and $b_2(b_2-1)/2$, respectively. Including $\mathrm{Sym}^2\mathrm{H}^2(M, \mathbb{C})$ and multiplicities, we deduce that

$$b_4 = \binom{b_2+1}{2} + c + db_2 + e\left(\frac{b_2(b_2-1)}{2}\right).$$

Similarly, the contributions of V_1, V_2, V_3, and $T = \mathbb{C}$ to $\mathrm{H}^6(M, \mathbb{C})$ are of dimensions b_2, $(b_2^2 - b_2 + 2)/2$, $b_2(b_2^2 - 3b_2 + 8)/6$, and 1, respectively. Including $\mathrm{Sym}^3\mathrm{H}^2(M, \mathbb{C})$ and multiplicities, we deduce that

$$b_6 = \binom{b_2+2}{3} + cb_2 + d\left(\frac{b_2^2 - b_2 + 2}{2}\right) + e\left(\frac{b_2(b_2^2 - 3b_2 + 8)}{6}\right) + f.$$

Substituting the formulae for b_4 and b_6 into Salamon's relation (and multiplying by 2) gives

$$\begin{aligned} &36\left(\binom{b_2+1}{2} + c + db_2 + e\left(\frac{b_2(b_2-1)}{2}\right)\right) - 96b_3 + 180b_2 + 420 \\ &\quad = 6b_6 \\ &\quad = 6\left(\binom{b_2+2}{3} + cb_2 + d\left(\frac{b_2^2 - b_2 + 2}{2}\right) + e\left(\frac{b_2(b_2^2 - 3b_2 + 8)}{6}\right) + f\right), \end{aligned}$$

and after simplifying and rearranging we obtain

$$\begin{aligned} &-(b_2+6)\left(b_2 - \frac{21+\sqrt{721}}{2}\right)\left(b_2 - \frac{21-\sqrt{721}}{2}\right) \\ &\quad = -b_2^3 + 15b_2^2 + 196b_2 + 420 \\ &\quad = 6c(b_2-6) + 3d(b_2^2 - 13b_2 + 2) + eb_2(b_2^2 - 21b_2 + 26) + 6f + 96b_3. \end{aligned}$$

The left-hand side is negative if $b_2 \geqslant 24 > (21+\sqrt{721})/2 \approx 23.9257$. On the other hand, c, d, e, f, and b_3 are all non-negative, so the right-hand side will be non-negative for $b_2 \geqslant 24$ (indeed, for $b_2 \geqslant 20$). Therefore the second Betti number b_2 can be at most 23. □

Remark 3.5 The contributions of V_4 to $\mathrm{H}^4(M, \mathbb{C})$ and $\mathrm{H}^6(M, \mathbb{C})$ have dimensions

$$\frac{b_2(b_2-1)(b_2-2)}{6} \quad \text{and} \quad \frac{b_2(b_2-1)(b_2^2-5b_2+18)}{24},$$

respectively. For each occurrence of V_4 in the decomposition of the cohomology of M we would need to add an additional term

$$6\,\frac{b_2(b_2-1)(b_2^2-5b_2+18)}{24} - 36\,\frac{b_2(b_2-1)(b_2-2)}{6} = \frac{b_2(b_2-1)(b_2^2-29b_2+66)}{4}$$

to the right-hand side of the last displayed equation of the proof. If $b_2 = 24, 25$, or 26 then this term would be negative and the proof would break down. However, we could still conclude that $b_2 \geqslant 27$ is impossible.

More generally, the contributions of V_k to $\mathrm{H}^4(M, \mathbb{C})$ and $\mathrm{H}^6(M, \mathbb{C})$ have dimensions

$$\binom{b_2}{k-1} \quad \text{and} \quad \binom{b_2}{k} + \binom{b_2}{k-2},$$

respectively. For each occurrence of V_k in the decomposition of the cohomology of M we would need to add an additional term

$$6\binom{b_2}{k} + 6\binom{b_2}{k-2} - 36\binom{b_2}{k-1} = \frac{6b_2(b_2-1)\cdots(b_2-k+3)}{k!}\left(b_2^2 - (8k-3)\,b_2 + (8k^2-16k+2)\right)$$

to the right-hand side of the last displayed equation of the proof. Calculating the roots of the quadratic factor, we see that if

$$b_2 \geqslant \frac{8k-3+\sqrt{32k^2+16k+1}}{2}$$

then this additional term will be non-negative, and we again reach the desired contradiction. Thus allowing $V_1, V_2, \ldots, V_k$ to appear in the Looijenga–Lunts–Verbitsky decomposition of the cohomology of M, for some $k \geqslant 4$, we still obtain an upper bound on b_2, but unfortunately this bound grows roughly linearly with k.

Example 3.6 Up to deformation, there are three known examples of irreducible compact hyperkähler manifolds of dimension six: the Hilbert scheme $\mathrm{Hilb}^3 S$ of three points on a K3 surface S, the generalized Kummer variety $K_3(A)$ of an abelian surface A (see Beauville [1]), and an example M_6 of O'Grady [14]. The Hodge numbers of Hilbert schemes of points on K3 surfaces and of generalized Kummer varieties were calculated by Göttsche and Soergel [5]; for $\mathrm{Hilb}^3 S$ and $K_3(A)$ they are

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & & & & & & \\
 & & & & 0 & & 0 & & & & & & \\
 & & & 1 & & 21 & & 1 & & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & & \\
 & 1 & & 22 & & 253 & & 22 & & 1 & & & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
1 & & 21 & & 253 & & 2004 & & 253 & & 21 & & 1 \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 1 & & 22 & & 253 & & 22 & & 1 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & 1 & & 21 & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & & & \\
 & & & & & 1 & & & & & & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccccccccc}
 & & & & & 1 & & & & & & & \\
 & & & & 0 & & 0 & & & & & & \\
 & & & 1 & & 5 & & 1 & & & & & \\
 & & 0 & & 4 & & 4 & & 0 & & & & \\
 & 1 & & 6 & & 37 & & 6 & & 1 & & & \\
0 & & 4 & & 24 & & 24 & & 4 & & 0 & & \\
1 & & 5 & & 37 & & 372 & & 37 & & 5 & & 1 \\
0 & & 4 & & 24 & & 24 & & 4 & & 0 & & \\
 & 1 & & 6 & & 37 & & 6 & & 1 & & & \\
 & & 0 & & 4 & & 4 & & 0 & & & & \\
 & & & 1 & & 5 & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & & & \\
 & & & & & 1 & & & & & & &
\end{array},$$

with $b_2 = 23$, $b_3 = 0$, $d = 1$, $c = e = f = 0$, and $b_2 = 7$, $b_3 = 8$, $c = 16$, $d = 1$, $e = 0$, $f = 240$, respectively. In particular, $V_4, V_5, \ldots$ do not appear in the Looijenga–Lunts–Verbitsky decomposition of the cohomology of $\mathrm{Hilb}^3 S$ and $K_3(A)$. Indeed, $e = 0$ for both, so V_3 also does not appear.

The Hodge numbers of O'Grady's example M_6 were calculated by Mongardi, Rapagnetta, and Saccà [12]; they are

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & & & & & & \\
 & & & & 0 & & 0 & & & & & & \\
 & & & 1 & & 6 & & 1 & & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & & \\
 & 1 & & 12 & & 173 & & 12 & & 1 & & & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
1 & & 6 & & 173 & & 1144 & & 173 & & 6 & & 1 \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 1 & & 12 & & 173 & & 12 & & 1 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & 1 & & 6 & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & & & \\
 & & & & & 1 & & & & & & &
\end{array},$$

with $b_2 = 8$, $b_3 = 0$. A priori there are two different Looijenga–Lunts–Verbitsky decompositions into irreducible $\mathfrak{so}(10, \mathbb{C})$-representations that could produce this Hodge diamond: either $c = 115$, $d = 6$, $e = 0$, $f = 290$ or $c = 135$, $d = 0$, $e = 1$, $f = 240$. (Note that in neither case do $V_4, V_5, \ldots$ appear.) In fact, Green, Kim, Laza, and Robles [6] have determined that the latter is the correct decomposition, but this is not immediate from the representation theory and it requires geometric arguments.

Remark 3.7 The hypothesis of Theorem 3.4 that $V_4, V_5, \ldots$ do not appear in the decomposition of the cohomology of M was originally introduced somewhat artificially, to make the proof work. Indeed the remark above shows that allowing $V_4, V_5, \ldots$ to appear leads to progressively weaker bounds on b_2. Nevertheless, the hypothesis is satisfied for all known examples in dimension six, as observed above.

A more conceptual justification of the hypothesis is provided by Green, Kim, Laza, and Robles [6], by relating it to a conjecture of Nagai [13] concerning monodromy

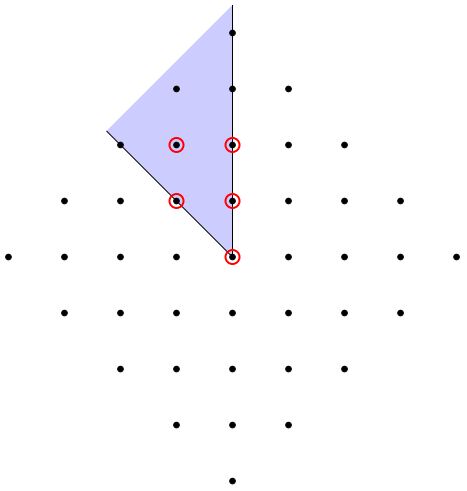

Fig. 2 The Hodge diamond in dimension eight

operators for one-parameter degenerations of hyperkähler manifolds. Green, et al. show that Nagai's conjecture is equivalent to a certain restriction on the highest weight vectors in the Looijenga–Lunts–Verbitsky decomposition of the cohomology in even degrees, and they verify that this restriction (and hence Nagai's conjecture) holds for all known hyperkähler manifolds, in all dimensions. They then recognize that there is a stronger restriction on the highest weight vectors that is more natural, which also holds for all known hyperkähler manifolds. This (conjectural) stronger restriction reduces to our hypothesis in dimension six. See [6] for details.

4 Higher dimensions

When $n = 4$ Salamon's relation gives

$$2b_7 + 16b_6 - 46b_5 + 88b_4 - 142b_3 + 208b_2 + 376 = 4b_8.$$

Thus in dimension eight, b_7 appears with a coefficient of the 'wrong' sign, and we cannot simply imitate the proof of Theorem 3.4. To proceed, we will assume that $b_7 = 0$. In fact, this is equivalent to assuming that *all* odd Betti numbers vanish, as the presence of cohomology in any odd degree will force $\mathrm{H}^7(M, \mathbb{C})$ to be non-vanishing because of the $\mathfrak{so}(b_2 + 2, \mathbb{C})$-action.

The Hodge diamond omitting the odd cohomology is shown in Fig. 2. After removing the cohomology generated by $\mathrm{H}^2(M, \mathbb{C})$, we are left with irreducible representations whose highest weight vectors lie in the circled Hodge groups. The only irreducible representations that could appear are those described in Table 3. Like in dimension six, V_k denotes that kth exterior power $\Lambda^k \mathbb{C}^{b_2+2}$ of the standard representation, while U_1 is given by taking the 2nd symmetric power $\mathrm{Sym}^2 \mathbb{C}^{b_2+2}$ of the

standard representation and removing the trivial direct summand $\mathbb{C}$, and U_k is given by taking the tensor product $\mathbb{C}^{b_2+2} \otimes \Lambda^k \mathbb{C}^{b_2+2}$ of the standard representation with its kth exterior power and removing the direct summands $\Lambda^{k+1}\mathbb{C}^{b_2+2}$ and $\Lambda^{k-1}\mathbb{C}^{b_2+2}$ (this leaves an irreducible representation with highest weight $\omega_1 + \omega_k$, except in a few special cases described in the following remark).

Remark 4.1 In Theorem 4.3 we will only allow irreducible representations U_k and V_k with k small relative to b_2. Nevertheless, for completeness let us clarify that if $b_2 + 2 = 2m + 1$ is odd then $V_m = \Lambda^m \mathbb{C}^{b_2+2}$ has highest weight $2\omega_m$ and U_m has highest weight $\omega_1 + 2\omega_m$. If $b_2 + 2 = 2m$ is even then $V_{m-1} = \Lambda^{m-1}\mathbb{C}^{b_2+2}$ has highest weight $\omega_{m-1} + \omega_m$, U_{m-1} has highest weight $\omega_1 + \omega_{m-1} + \omega_m$, $\Lambda^m \mathbb{C}^{b_2+2} = \Lambda^m_+ \mathbb{C}^{b_2+2} \oplus \Lambda^m_- \mathbb{C}^{b_2+2}$ decomposes into two irreducible representations of equal dimensions with highest weights $2\omega_{m-1}$ and $2\omega_m$, and $\mathbb{C}^{b_2+2} \otimes \Lambda^m \mathbb{C}^{b_2+2} - \Lambda^{m+1}\mathbb{C}^{b_2+2} - \Lambda^{m-1}\mathbb{C}^{b_2+2}$ decomposes into two irreducible representations of equal dimensions with highest weights $\omega_1 + 2\omega_{m-1}$ and $\omega_1 + 2\omega_m$.

Table 4 gives the dimensions of the intersections of these representations with $\mathrm{H}^4(M, \mathbb{C})$, $\mathrm{H}^6(M, \mathbb{C})$, and $\mathrm{H}^8(M, \mathbb{C})$. For instance, to compute these dimensions for U_k we use the description

$$U_k = \mathbb{C}^{b_2+2} \otimes \Lambda^k \mathbb{C}^{b_2+2} - \Lambda^{k+1}\mathbb{C}^{b_2+2} - \Lambda^{k-1}\mathbb{C}^{b_2+2}$$

with $\mathbb{C}^{b_2+2} = \mathbb{C} \oplus \mathbb{C}^{b_2} \oplus \mathbb{C}$ graded by -2, 0, and 2. After an overall shift of 8, this induces the required grading on U_k.

Example 4.2 The representation U_1 is generated by a highest weight vector in

$$U_1 \cap \mathrm{H}^4(M, \mathbb{C}) = U_1 \cap \mathrm{H}^{2,2}(M) \cong \mathbb{C},$$

whereas

$$U_2 \cap \mathrm{H}^4(M, \mathbb{C}) \cong \mathbb{C}^{b_2}$$

with

$$\dim(U_2 \cap \mathrm{H}^{3,1}(M)) = 1 = \dim(U_2 \cap \mathrm{H}^{1,3}(M)) \text{ and } \dim(U_2 \cap \mathrm{H}^{2,2}(M)) = b_2 - 2.$$

Similarly

$$U_3 \cap \mathrm{H}^4(M, \mathbb{C}) \cong \Lambda^2 \mathbb{C}^{b_2}$$

with

$$\dim(U_3 \cap \mathrm{H}^{3,1}(M)) = b_2 - 2 = \dim(U_3 \cap \mathrm{H}^{1,3}(M)) \quad \text{and}$$
$$\dim(U_3 \cap \mathrm{H}^{2,2}(M)) = \binom{b_2 - 2}{2} + 1.$$

Table 3 Irreducible representations of $\mathfrak{so}(b_2+2,\mathbb{C})$ that that could occur in the cohomology of M

	Highest weight	Highest weight vector in	$\mathfrak{so}(b_2+2,\mathbb{C})$-module	Dimension
U_1	$2\omega_1$	$\mathrm{H}^{2,2}(M)$	$\mathrm{Sym}^2\mathbb{C}^{b_2+2}-\mathbb{C}$	$\frac{(b_2+4)(b_2+1)}{2}$
U_2	$\omega_1+\omega_2$	$\mathrm{H}^{3,1}(M)$	$\mathbb{C}^{b_2+2}\otimes\Lambda^2\mathbb{C}^{b_2+2}-\Lambda^3\mathbb{C}^{b_2+2}-\mathbb{C}^{b_2+2}$	$\frac{(b_2+4)(b_2+2)b_2}{3}$
U_3	$\omega_1+\omega_3$	$\mathrm{H}^{3,1}(M)$	$\mathbb{C}^{b_2+2}\otimes\Lambda^3\mathbb{C}^{b_2+2}-\Lambda^4\mathbb{C}^{b_2+2}-\Lambda^2\mathbb{C}^{b_2+2}$	$\frac{(b_2+4)(b_2+2)(b_2+1)(b_2-1)}{8}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
U_k	$\omega_1+\omega_k$	$\mathrm{H}^{3,1}(M)$	$\mathbb{C}^{b_2+2}\otimes\Lambda^k\mathbb{C}^{b_2+2}-\Lambda^{k+1}\mathbb{C}^{b_2+2}-\Lambda^{k-1}\mathbb{C}^{b_2+2}$	$(b_2+2)\binom{b_2+2}{k}-\binom{b_2+2}{k+1}-\binom{b_2+2}{k-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
V_1	ω_1	$\mathrm{H}^{3,3}(M)$	$\mathbb{C}^{b_2+2}$	b_2+2
V_2	ω_2	$\mathrm{H}^{4,2}(M)$	$\Lambda^2\mathbb{C}^{b_2+2}$	$\binom{b_2+2}{2}$
V_3	ω_3	$\mathrm{H}^{4,2}(M)$	$\Lambda^3\mathbb{C}^{b_2+2}$	$\binom{b_2+2}{3}$
V_4	ω_4	$\mathrm{H}^{4,2}(M)$	$\Lambda^4\mathbb{C}^{b_2+2}$	$\binom{b_2+2}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
V_k	ω_k	$\mathrm{H}^{4,2}(M)$	$\Lambda^k\mathbb{C}^{b_2+2}$	$\binom{b_2+2}{k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	0	$\mathrm{H}^{4,4}(M)$	$\mathbb{C}$	1

Table 4 Dimensions of $U_\bullet$ and $V_\bullet$ in degrees 4, 6, and 8

	Dimension of $\cap \mathrm{H}^4(M,\mathbb{C})$	Dimension of $\cap \mathrm{H}^6(M,\mathbb{C})$	Dimension of $\cap \mathrm{H}^8(M,\mathbb{C})$
U_1	$\binom{b_2}{0}=1$	$b_2\binom{b_2}{0}=b_2$	$b_2\binom{b_2}{1}-\binom{b_2}{2}=\frac{(b_2+1)b_2}{2}$
U_2	$\binom{b_2}{1}=b_2$	$b_2\binom{b_2}{1}=b_2^2$	$b_2\binom{b_2}{2}+b_2\binom{b_2}{0}-\binom{b_2}{3}=\frac{b_2(b_2^2+2)}{3}$
U_3	$\binom{b_2}{2}=\frac{b_2(b_2-1)}{2}$	$b_2\binom{b_2}{2}=\frac{b_2^2(b_2-1)}{2}$	$b_2\binom{b_2}{3}+b_2\binom{b_2}{1}-\binom{b_2}{4}-\binom{b_2}{0}=\frac{(b_2+1)(b_2-1)(b_2^2-2b_2+8)}{8}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
U_k	$\binom{b_2}{k-1}$	$b_2\binom{b_2}{k-1}$	$b_2\binom{b_2}{k}+b_2\binom{b_2}{k-2}-\binom{b_2}{k+1}-\binom{b_2}{k-3}$
V_1		$\binom{b_2}{0}=1$	$\binom{b_2}{1}=b_2$
V_2		$\binom{b_2}{1}=b_2$	$\binom{b_2}{2}+\binom{b_2}{0}=\frac{b_2^2-b_2+2}{2}$
V_3		$\binom{b_2}{2}=\frac{b_2(b_2-1)}{2}$	$\binom{b_2}{3}+\binom{b_2}{1}=\frac{b_2(b_2^2-3b_2+8)}{6}$
V_4		$\binom{b_2}{3}=\frac{b_2(b_2-1)(b_2-2)}{6}$	$\binom{b_2}{4}+\binom{b_2}{2}=\frac{b_2(b_2-1)(b_2^2-5b_2+18)}{24}$
$\vdots$		$\vdots$	$\vdots$
V_k		$\binom{b_2}{k-1}$	$\binom{b_2}{k}+\binom{b_2}{k-2}$

In general

$$U_k \cap \mathrm{H}^4(M, \mathbb{C}) \cong \Lambda^{k-1}\mathbb{C}^{b_2}$$

with

$$\dim(U_k \cap \mathrm{H}^{3,1}(M)) = \binom{b_2 - 2}{k-2} = \dim(U_k \cap \mathrm{H}^{1,3}(M)) \quad \text{and}$$
$$\dim(U_k \cap \mathrm{H}^{2,2}(M)) = \binom{b_2 - 2}{k-1} + \binom{b_2 - 2}{k-3}.$$

We can now prove our result in dimension eight.

Theorem 4.3 *Let M be an irreducible compact hyperkähler manifold of complex dimension eight whose odd Betti numbers all vanish. Consider the part of the cohomology of M not generated by $\mathrm{H}^2(M, \mathbb{C})$. Of the possible irreducible representations of $\mathfrak{so}(b_2 + 2, \mathbb{C})$ in its Looijenga–Lunts–Verbitsky decomposition, assume that only $U_1, U_2, U_3, V_1, V_2, V_3, V_4, V_5$, and T can appear. Then the second Betti number b_2 of M is at most* 24.

Proof The proof follows the same ideas as that of Theorem 3.4. When $n = 4$ and all odd Betti numbers vanish, Salamon's relation gives

$$16b_6 + 88b_4 + 208b_2 + 376 = 4b_8.$$

The part of the complex cohomology of M generated by $\mathrm{H}^2(M, \mathbb{C})$ contributes $\binom{b_2+1}{2}$, $\binom{b_2+2}{3}$, and $\binom{b_2+3}{4}$ to b_4, b_6, and b_8, respectively. Writing the remainders of these Betti numbers as b_4', b_6', and b_8', Salamon's relation becomes

$$\begin{aligned} -4\binom{b_2+3}{4} + 16\binom{b_2+2}{3} + 88\binom{b_2+1}{2} + 208b_2 + 376 & \\ = 4b_8' - 16b_6' - 88b_4' & \\ = 4(b_8' - 4b_6' - 22b_4'). & \end{aligned}$$

After simplifying and factoring, the left-hand side becomes

$$-\frac{1}{6}(b_2+3)(b_2+8)\left(b_2 - \frac{21+\sqrt{817}}{2}\right)\left(b_2 - \frac{21-\sqrt{817}}{2}\right),$$

which is negative if $b_2 \geqslant 25 > (21 + \sqrt{817})/2 \approx 24.7916$. It remains to show that the right-hand is non-negative for $b_2 \geqslant 25$.

First consider the contribution of V_k to the right-hand side. Each occurrence of V_k in the Looijenga–Lunts–Verbitsky decomposition contributes

$$\begin{aligned}&\binom{b_2}{k}+\binom{b_2}{k-2}-4\binom{b_2}{k-1}\\&\quad=\frac{b_2(b_2-1)\cdots(b_2-k+3)}{k!}\left(b_2^2-(6k-3)\,b_2+(6k^2-12k+2)\right)\end{aligned}$$

to $b'_8-4b'_6-22b'_4$. This contribution will be non-negative if

$$b_2\geqslant\frac{6k-3+\sqrt{12k^2+12k+1}}{2}.$$

In particular, if $k=5$ we require $b_2\geqslant 23$, whereas if $k=6$ we would require $b_2\geqslant 28>(33+\sqrt{505})/2\approx 27.7361$.

Next consider the contribution of U_k to the right-hand side. Each occurrence of U_k in the Looijenga–Lunts–Verbitsky decomposition contributes

$$\begin{aligned}&b_2\binom{b_2}{k}+b_2\binom{b_2}{k-2}-\binom{b_2}{k+1}-\binom{b_2}{k-3}-4b_2\binom{b_2}{k-1}-22\binom{b_2}{k-1}\\&\quad=\frac{b_2(b_2-1)\cdots(b_2-k+4)(b_2-k+2)(b_2+4)}{(k+1)(k-1)!}\\&\qquad\cdot\left(b_2^2-(6k+3)\,b_2+(6k^2-12k-16)\right)\end{aligned}$$

to $b'_8-4b'_6-22b'_4$. This contribution will be non-negative if

$$b_2\geqslant\frac{6k+3+\sqrt{12k^2+84k+73}}{2}.$$

In particular, if $k=3$ we require $b_2\geqslant 21>(21+\sqrt{433})/2\approx 20.9043$, whereas if $k=4$ we would require $b_2\geqslant 26>(27+\sqrt{601})/2\approx 25.7577$.

Finally, the contribution of the trivial representation T to $b'_8-4b'_6-22b'_4$ is just 1, so always positive.

In conclusion, if we allow the irreducible representations U_1, U_2, U_3, V_1, V_2, V_3, V_4, V_5, and T to appear in the Looijenga–Lunts–Verbitsky decomposition of the complex cohomology of M, then for $b_2\geqslant 25$ the right-hand side $4(b'_8-4b'_6-22b'_4)$ of Salamon's relation will be non-negative, whereas the left-hand side will be negative. This contradiction proves that the second Betti number b_2 can be at most 24. □

Remark 4.4 As the proof shows, if we allow U_4 to appear then we can conclude that b_2 can be at most 25, and if we allow V_6 to appear then we can conclude that b_2 can be at most 27. Allowing U_k and V_k to appear for larger k, we still obtain upper bounds on b_2, but these bounds grow roughly linearly with k.

Remark 4.5 Following the same steps in higher dimensions, the pattern appears to be that in dimension $2n$, the polynomial in b_2 on the left-hand side has largest root

$(21 + \sqrt{433 + 96n})/2$. Given an irreducible compact hyperkähler manifold whose odd Betti numbers all vanish, we expect one can show that its second Betti number must satisfy $b_2 \leqslant (21 + \sqrt{433 + 96n})/2$, under some restrictions on the irreducible representations appearing in the Looijenga–Lunts–Verbitsky decomposition of its cohomology. The author has not rigorously verified this, though this direction has been pursued by Kurnosov [10], and indeed, Kim and Laza [9] later identified sufficient restrictions to guarantee that a bound of this form holds. Their restrictions are again related to Nagai's conjecture; see [9] and Green, et al. [6] for details.

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